The cardinal and the idempotent number of various monoids of transformations on a finite chain

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Abstract

In this note we consider various classes of monoids of transformations on a finite chain, in particular of transformations that preserve or reverse either the order or the orientation. Being finite monoids we are naturally interested in computing both their cardinals and their idempotent numbers. Fibonacci and Lucas numbers play an essential role in the last computations.

2000 *Mathematics Subject Classification*: Primary 20M17; Secondary 20M20, 05A10.

Keywords: monoid, transformation, order-preserving, orientation-preserving.

Introduction

Let X_n be a finite chain with n elements, say $X_n = \{1 < \cdots < n\}$. We denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations of X_n. The submonoids of $\mathcal{PT}_n$ of all full transformations and of all injective partial transformations are denoted by $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively.

For general background on monoids, we refer the reader to Howie's book [10]. Given $s \in \mathcal{PT}_n$, we denote its *domain* by $\mathrm{Dom}(s)$ and its *image* by $\mathrm{Im}(s)$.

A transformation s in $\mathcal{PT}_n$ is said to be *order-preserving* (resp., *order-reversing*) if $x \leq y$ implies $xs \leq ys$ (resp., $xs \geq ys$), for all $x, y \in \mathrm{Dom}(s)$.

Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all order-preserving partial transformations. As usual, we denote by $\mathcal{O}_n$ the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations that preserve the order. Howie [9] calculated the cardinal and the number of idempotents of $\mathcal{O}_n$ and later jointly with Gomes [8] determined the cardinal of $\mathcal{PO}_n$. More recently, using a different approach, Laradji and Umar [11, 12] also obtained these results as well as the number of idempotents of $\mathcal{PO}_n$. The injective counterpart of $\mathcal{O}_n$ is the inverse monoid $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$ of all injective order-preserving partial transformations, whose cardinal was first calculated by Garba [7] (see also [2]). Obviously $\mathcal{POI}_n$ and $\mathcal{I}_n$ have exactly the same idempotents, which are the 2^n partial identities on X_n.

Wider classes of monoids are obtained when we take transformations that either preserve or reverse the order. In this way, we get the submonoid $\mathcal{POD}_n$ of $\mathcal{PT}_n$ of all partial transformations that preserve or reverse the order, as well as its submonoids $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$ and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$, whose cardinals were calculated by the authors in [4, 5].

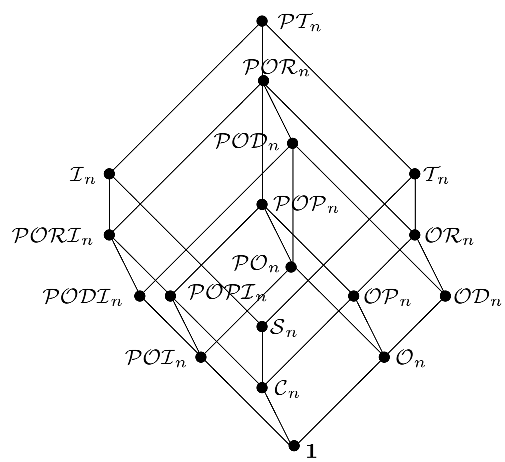
Before mentioning a different class of transformation monoids, we require to recall some other definitions.

[1] This work was developed within the projects POCTI-ISFL-1-143 and PTDC/MAT/69514/2006 of Centro de Álgebra da Universidade de Lisboa, supported by FCT, FEDER and PIDDAC.

Let $a=(a_1,a_2,\ldots,a_t)$ be a sequence of t ($t\geq 0$) elements from the chain X_n. We say that a is *cyclic* (resp., *anti-cyclic*) if there exists no more than one index $i\in\{1,\ldots,t\}$ such that $a_i>a_{i+1}$ (resp., $a_i<a_{i+1}$), where a_{t+1} denotes a_1. Let $s\in\mathcal{PT}_n$ and suppose that $\mathrm{Dom}(s)=\{a_1,\ldots,a_t\}$, with $t\geq 0$ and $a_1<\cdots<a_t$. We say that s is *orientation-preserving* (resp., *orientation-reversing*) if the sequence of its images $(a_1s,\ldots,a_ts)$ is cyclic (resp., anti-cyclic). These notions were first introduced by McAlister [13]. Catarino and Higgins worked these concepts too in [1].

We denote by $\mathcal{POP}_n$ the submonoid of $\mathcal{PT}_n$ of all orientation-preserving transformations. Adding to $\mathcal{POP}_n$ all orientation-reversing transformations we obtain the submonoid $\mathcal{POR}_n$ of $\mathcal{PT}_n$. Next, we look both at the "full" and the "injective" parts of $\mathcal{POP}_n$ and $\mathcal{POR}_n$. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{POP}_n$ of all its full transformations and, similarly, by $\mathcal{OR}_n$ the submonoid of $\mathcal{POR}_n$ of all its full elements; by $\mathcal{POPI}_n$ the submonoid of $\mathcal{POP}_n$ whose transformations are injective and, finally, by $\mathcal{PORI}_n$ the submonoid of $\mathcal{POR}_n$ whose elements are injective too. The cardinals of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were calculated by McAlister [13] and, independently, by Catarino and Higgins [1] who also computed the number of their idempotents. In [3] Fernandes calculated the cardinal of $\mathcal{POPI}_n$ and the cardinal of $\mathcal{PORI}_n$ was determined by the authors in [4]. In this case, it is easy to show that the idempotents of $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ are also the 2^n idempotents of $\mathcal{I}_n$.

In what follows we denote by $\mathbf{1}$ the trivial monoid, by $\mathcal{S}_n$ the symmetric group and by $\mathcal{C}_n$ the cyclic group of order n. With respect to the inclusion relation, the diagram bellow presents the relationship between the various monoids introduced above



The first section of this note is dedicated to calculating the cardinals of these monoids and in the second section we compute the cardinals of their sets of idempotents. We recall the known results and complete the study by computing the remaining cases.

1 Cardinals

Let $\mathcal{PD}_n$ be the set of all order-reversing partial transformations of X_n and let $\mathcal{ID}_n$ and $\mathcal{D}_n$ be the subsets of all its injective transformations and of all its full transformations, respectively. Clearly, $\mathcal{POD}_n=\mathcal{PO}_n\cup\mathcal{PD}_n$ hence $\mathcal{PODI}_n=\mathcal{POI}_n\cup\mathcal{ID}_n$ and $\mathcal{OD}_n=\mathcal{O}_n\cup\mathcal{D}_n$. Furthermore, $\mathcal{PO}_n\cap\mathcal{PD}_n=\{s\in\mathcal{PT}_n:|\operatorname{Im}(s)|\leq 1\}$, whence $\mathcal{POI}_n\cap\mathcal{ID}_n=\{s\in\mathcal{I}_n:|\operatorname{Im}(s)|\leq 1\}$ and $\mathcal{O}_n\cap\mathcal{D}_n=\{s\in\mathcal{T}_n:|\operatorname{Im}(s)|=1\}$.

Now, consider the following particular order-reversing permutation of order two:

$$h=\begin{pmatrix}1 & 2 & \cdots & n-1 & n\\ n & n-1 & \cdots & 2 & 1\end{pmatrix}.$$

To calculate the cardinals of $\mathcal{POD}_n$, $\mathcal{PODI}_n$ and $\mathcal{OD}_n$, we will make use of the mapping $\varphi:\mathcal{PO}_n\longrightarrow\mathcal{PD}_n$ defined by $(s)\varphi=sh$, for all $s\in\mathcal{PO}_n$. Obviously, this mapping is a bijection and so we have $|\mathcal{PD}_n|=|\mathcal{PO}_n|$. On the other hand, φ maps $\mathcal{O}_n$ onto $\mathcal{D}_n$ and $\mathcal{POI}_n$ onto $\mathcal{ID}_n$, therefore $|\mathcal{D}_n|=|\mathcal{O}_n|$ and $|\mathcal{ID}_n|=|\mathcal{POI}_n|$.

Now, recalling that Howie [9] computed

$$|\mathcal{O}_n| = \binom{2n-1}{n-1}$$

and since $|\mathcal{O}_n \cap \mathcal{D}_n| = n$, the next result follows.

Theorem 1.1 **[5]** $|\mathcal{OD}_n| = 2\binom{2n-1}{n-1} - n.$ □

As $|\mathcal{POI}_n \cap \mathcal{ID}_n| = |\{s \in \mathcal{I}_n : |\operatorname{Im}(s)| \leq 1\}| = n^2+1$ and Garba [7] (independently, Fernandes [2]) proved, that

$$|\mathcal{POI}_n| = \binom{2n}{n},$$

we deduced the cardinal of $\mathcal{PODI}_n$.

Theorem 1.2 **[4]** $|\mathcal{PODI}_n| = 2\binom{2n}{n} - n^2 - 1.$ □

Taking into account that Gomes and Howie [8] established that

$$|\mathcal{PO}_n| = \sum_{i=1}^{n} \binom{n}{i}\binom{n+i-1}{i} + 1$$

and using the fact that $|\mathcal{PO}_n \cap \mathcal{PD}_n| = n\sum_{i=1}^{n}\binom{n}{i} + 1$, we may compute $|\mathcal{POD}_n|$.

Theorem 1.3 **[5]** $|\mathcal{POD}_n| = \sum_{i=1}^{n} \binom{n}{i}\left(2\binom{n+i-1}{i} - n\right) + 1.$ □

The cardinal of $\mathcal{POP}_n$ was also calculated by the authors.

Theorem 1.4 **[6]** $|\mathcal{POP}_n| = 1 + (2^n-1)n + \sum_{k=2}^{n} k\binom{n}{k}^2 2^{n-k}.$ □

Denote by $\mathcal{PR}_n$ the set of all orientation-reversing partial transformations of X_n. By definition, we have $\mathcal{POR}_n = \mathcal{POP}_n \cup \mathcal{PR}_n$. To obtain the cardinal of $\mathcal{POR}_n$ we use the following result of Catarino and Higgins:

Lemma 1.5 **[1]** *Let a be a cyclic (resp., anti-cyclic) sequence. Then a is also anti-cyclic (resp., cyclic) if and only if a has no more than two distinct values.* □

This fact allows us to conclude that $\mathcal{POP}_n \cap \mathcal{PR}_n = \{s \in \mathcal{POP}_n : |\operatorname{Im}(s)| \leq 2\}$. As the mapping $\Psi : \mathcal{POP}_n \longrightarrow \mathcal{PR}_n$ defined by $(s)\Psi = sh$, for all $s \in \mathcal{POP}_n$, is a bijection, we get $|\mathcal{POP}_n| = |\mathcal{PR}_n|$ and so $|\mathcal{POR}_n| = 2|\mathcal{POP}_n| - |\{s \in \mathcal{POP}_n : |\operatorname{Im}(s)| \leq 2\}|$. Therefore we are able to obtain the cardinal of $\mathcal{POR}_n$.

Theorem 1.6 **[6]** $|\mathcal{POR}_n| = 1 + (2^n-1)n + 2\binom{n}{2}^2 2^{n-2} + \sum_{k=3}^{n} 2k\binom{n}{k}^2 2^{n-k}.$ □

The cardinal of $\mathcal{POPI}_n$, computed by Fernandes [3], is given by the next formula

$$|\mathcal{POPI}_n| = 1 + \frac{n}{2}\binom{2n}{n}.$$

As Ψ maps $\mathcal{POPI}_n$ onto the set of all injective orientation-reversing transformations, we conclude that $|\mathcal{PORI}_n| = 2|\mathcal{POPI}_n| - |\{s \in \mathcal{POPI}_n : |\operatorname{Im}(s)| \leq 2\}|$ and may deduce the following.

Theorem 1.7 [4] $|\mathcal{PORI}_n| = 1 + n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 3)$. □

The cardinals of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were calculated by McAlister [13] and, independently, by Catarino and Higgins [1], who proved that

$$|\mathcal{OP}_n| = n\binom{2n-1}{n-1} - n(n-1) \quad \text{and} \quad |\mathcal{OR}_n| = n\binom{2n}{n} - \frac{n^2}{2}(n^2 - 2n + 5) + n.$$

Just to complete the picture recall that

$$|\mathcal{C}_n| = n,\ |\mathcal{S}_n| = n!,\ |\mathcal{I}_n| = \sum_{k=0}^{n}\binom{n}{k}^2 k!,\ |\mathcal{T}_n| = n^n \quad \text{and} \quad |\mathcal{PT}_n| = (n+1)^n.$$

2 Number of idempotents

For a given monoid M, we denote by $E(M)$ its set of idempotents.

First we will consider the "ordered case". Let $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$. Let $e \in E(M)$. As the product of two order-preserving transformations or of two order-reversing transformations is an order-preserving transformation, we conclude that e must be order-preserving. Thus $E(\mathcal{OD}_n) = E(\mathcal{O}_n)$ and $E(\mathcal{POD}_n) = E(\mathcal{PO}_n)$.

In [9] Howie showed that

$$|E(\mathcal{O}_n)| = F_{2n},$$

where F_n is the nth Fibonacci number.

Recall that the Fibonacci numbers are recursively defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+1} = F_k + F_{k-1}, \text{ for } k \geq 1.$$

Another interesting set of numbers is the Lucas sequence, which is also recursively defined as follows

$$L_0 = 2, \quad L_1 = 1, \quad L_{k+1} = L_k + L_{k-1}, \text{ for } k \geq 1.$$

Fibonacci and Lucas numbers are intrinsically related. In fact, for any $n \in \mathbb{N}_0$,

$$F_n = \frac{\tau^n - \theta^n}{\tau - \theta} \quad \text{and} \quad L_n = \tau^n + \theta^n,$$

where τ is the golden number and θ is its rational conjugate, that is $\tau = \frac{1+\sqrt{5}}{2}$ and $\theta = \frac{1-\sqrt{5}}{2}$. Moreover, $F_{2n} = F_n L_n$, for any $n \in \mathbb{N}_0$. For further details, see e.g. [14].

In view of the above observations, we conclude that

$$|E(\mathcal{OD}_n)| = |E(\mathcal{O}_n)| = F_{2n} = \frac{\tau^{2n} - \theta^{2n}}{\tau - \theta}.$$

Concerning the correspondent classes of partial transformations, the following formula for the number of idempotents of $\mathcal{PO}_n$ was given by Laradji and Umar [11].

Theorem 2.1 $|E(\mathcal{POD}_n)| = |E(\mathcal{PO}_n)| = (\sqrt{5})^{n-1}(\tau^n - (-\theta)^n) + 1$. □

The table below gives us an idea of the size of the monoids we are dealing with. By E_n we denote the set $E(\mathcal{PO}_n)=E(\mathcal{POD}_n)$.

n	$\lvert\mathcal{PO}_n\rvert$	$\lvert\mathcal{POD}_n\rvert$	$\lvert E_n\rvert$
1	2	2	2
2	8	9	6
3	38	54	21
4	192	323	76
5	1002	1848	276

n	$\lvert\mathcal{PO}_n\rvert$	$\lvert\mathcal{POD}_n\rvert$	$\lvert E_n\rvert$
6	5336	10293	1001
7	28814	56738	3626
8	157184	312327	13126
9	864146	1723692	47501
10	4780008	9549785	171876

Next, we look at the "oriented case". Let $M\in\{\mathcal{OR}_n,\mathcal{PORI}_n,\mathcal{POR}_n\}$ and, again, let $e\in E(M)$. Similarly to what happened in the "ordered case", the product of two orientation-preserving or of two orientation-reversing elements of M is an orientation-preserving transformation, whence e must preserve the orientation. Thus $E(\mathcal{OR}_n)=E(\mathcal{OP}_n)$ and $E(\mathcal{POR}_n)=E(\mathcal{POP}_n)$.

Catarino and Higgins [1] showed that

$$|E(\mathcal{OR}_n)|=|E(\mathcal{OP}_n)|=L_{2n}-(n^2-n+2)=\tau^{2n}+\theta^{2n}-(n^2-n+2).$$

We finish this note by computing the remaining cases, namely the number of idempotents of $\mathcal{POP}_n$ and of $\mathcal{POR}_n$.

Theorem 2.2 $|E(\mathcal{POR}_n)|=|E(\mathcal{POP}_n)|=\sum_{j=1}^{n}\binom{n}{j}[L_{2j}-(j^2-j+2)]+1=\sum_{j=1}^{n}\binom{n}{j}[\tau^{2j}+\theta^{2j}-(j^2-j+2)]+1.$

PROOF. For $s\in\mathcal{PT}_n$, we define $\mathrm{Fix}(s)=\{x\in\mathrm{Dom}(s):(x)s=x\}$. An element $s\in\mathcal{PT}_n$ is idempotent if and only if $\mathrm{Im}(s)\subseteq\mathrm{Fix}(s)$. Also, for each nonempty subset A of X_n, the number of idempotents of $\mathcal{POP}_n$ with domain A coincides with $|E(\mathcal{OP}_{|A|})|$. Therefore

$$\begin{aligned}|E(\mathcal{POR}_n)|=|E(\mathcal{POP}_n)| &= \textstyle\sum_{j=1}^{n}\binom{n}{j}|E(\mathcal{OP}_j)|+1\\ &= \textstyle\sum_{j=1}^{n}\binom{n}{j}[L_{2j}-(j^2-j+2)]+1\\ &= \textstyle\sum_{j=1}^{n}\binom{n}{j}\left[\tau^{2j}+\theta^{2j}-(j^2-j+2)\right]+1,\end{aligned}$$

as required. □

Now, let E_n denote the set $E(\mathcal{POP}_n)=E(\mathcal{POR}_n)$. We apply the last formula to compute some examples.

n	$\lvert\mathcal{POP}_n\rvert$	$\lvert\mathcal{POR}_n\rvert$	$\lvert E_n\rvert$
1	2	2	2
2	9	9	6
3	61	64	23
4	449	549	96
5	3161	4566	402

n	$\lvert\mathcal{POP}_n\rvert$	$\lvert\mathcal{POR}_n\rvert$	$\lvert E_n\rvert$
6	21145	34711	1643
7	136529	243944	6526
8	862209	1622025	25280
9	5369617	10402858	96011
10	33133481	65219931	359288

To conclude with a full picture recall that $|E(\mathcal{C}_n)|=|E(\mathcal{S}_n)|=1$ and $|E(\mathcal{POI}_n)|=|E(\mathcal{PODI}_n)|=|E(\mathcal{PORI}_n)|=|E(\mathcal{I}_n)|=2^n$, also $|E(\mathcal{T}_n)|=\sum_{j=1}^{n}\binom{n}{j}j^{n-j}$ and $|E(\mathcal{PT}_n)|=\sum_{j=0}^{n}\binom{n}{j}(j+1)^{n-j}$.

References

[1] Paula M. Catarino and Peter M. Higgins, The monoid of orientation-preserving mappings on a chain, *Semigroup Forum* **58** (1999), 190-206.

[2] Vítor H. Fernandes, Semigroups of order-preserving mappings on a finite chain: a new class of divisors, *Semigroup Forum* **54** (1997), 230-236.

[3] Vítor H. Fernandes, The monoid of all injective orientation-preserving partial transformations on a finite chain, *Comm. Algebra* **28** (2000), 3401-3426.

[4] Vítor H. Fernandes, Gracinda M. S. Gomes and Manuel M. Jesus, Presentations for some monoids of injective partial transformations on a finite chain, *Southeast Asian Bull. Math.* **28** (2004), 903-918.

[5] Vítor H. Fernandes, Gracinda M. S. Gomes and Manuel M. Jesus, Congruences on monoids of order-preserving or order-reversing transformations on a finite chain, *Glasgow Math. J.* **47** (2005), 413-424.

[6] Vítor H. Fernandes, Gracinda M. S. Gomes and Manuel M. Jesus, Congruences on monoids of transformations preserving the orientation on a finite chain, *J. Algebra* **321** (2009), 743-757.

[7] G. U. Garba, Nilpotents in semigroups of partial one-to-one order-preserving mappings, *Semigroup Forum* **48** (1994), 37-49.

[8] Gracinda M. S. Gomes and John M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992), 272-282.

[9] John M. Howie, Product of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* **17** (1971) 223–236.

[10] John M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, 1995.

[11] A. Laradji and A. Umar, Combinatorial results for semigroups of order-preserving partial transformations, *J. Algebra* **278** (2004), 342-359.

[12] A. Laradji and A. Umar, Combinatorial results for semigroups of order-preserving full transformations, *Semigroup Forum* **72** (2006), 51-62.

[13] Donald McAlister, Semigroups generated by a group and an idempotent, *Comm. Algebra* **26** (1998), 515-547.

[14] Fred S. Roberts, Applied Combinatorics, Prentice-Hall, Inc., Englewood Cliffs, 1984.

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