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THE CARLESON MEASURE AND MEROMORPHIC FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC

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Abstract. For a meromorphic function $f(z)$ defined in the unit disc $D:|z|<1$ on the complex z-plane, $z=x+iy$, we denote its spherical derivative by $f^{\#}(z)$ and introduce the differentiable form $d\mu_f(z)=(1-|z|^2)[f^{\#}(z)]^2\,dx\,dy$. We prove that $f(z)$ has the uniformly bounded characteristic if and only if the measure $\mu_f(z)$ is the Carleson measure. This result answers a question posed by S. Yamashita in Internat. J. Math. Math. Sci. 8, 1985, pp. 477–482.

1. Let $f(z)$ be a meromorphic function defined in the unit disc $D:|z|<1$ on the complex z-plane, $z=x+iy$, and $f^{\#}(z)=|f'(z)|\big[1+|f(z)|^2\big]^{-1}$ its spherical derivative. For an arbitrary $w\in D$ and t, $0<t<1$, we denote $\varphi_w(z)=(z-w)(1-\bar{w}z)^{-1}$ and $\Delta(w,t)=\{z\in D;\,|\varphi_w(z)|<t\}$.

The Nevanlinna characteristic function $T(r,f)$ of the function $f(z)$ is defined by the Ahlfors–Shimizu formula

$$T(r,f)=\frac{1}{\pi}\int_0^r\frac{S(t,f)}{t}\,dt,\qquad 0<r<1,$$

where

$$S(t,f)=\iint_{|z|<t}\big[f^{\#}(z)\big]^2dx\,dy,\qquad 0<t<1.$$

In [5], S. Yamashita proved that the Nevanlinna characteristic $T(r,w,f)$ of the function $f\big(w+(1-|w|)z\big)$, $w\in D$ fixed, is of the form

$$T(r,w,f)=\frac{1}{\pi}\int_0^r\frac{1}{t}\Big[\iint_{D(w,t)}\big[f^{\#}(z)\big]^2dx\,dy\Big]\,dt,\qquad 0<r<1,$$

where $D(w,t)=\{z\in D;\,|z-w|<t(1-|w|)\}$, $0<t<1$. In particular, $T(r,0,f)=T(r,f)$.

Denote $T(1,w,f)=\lim_{r\to 1-0}T(r,w,f)$. If $T(1,f)<+\infty$, then $f(z)$ belongs to the class BC of meromorphic functions with bounded Nevanlinna characteristic. If $\sup_{|z|<1}\big(1-|z|\big)f^{\#}(z)<+\infty$, then $f(z)$ belongs to the class N of normal meromorphic functions in D (cf. [2]).

For any $w\in D$, we denote $f_w(z)=f\big(\varphi_w(z)\big)$, $\varphi_w(z)=(z-w)(1-\bar{w}z)^{-1}$.

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Theorem 1. *A meromorphic function $f(z)$ in D belongs to $BC \cap N$ if and only if*

$$\sup_{|w|<1} T(1, w, f) < +\infty. \tag{1}$$

Proof. Let $f(z) \in \mathrm{BC} \cap N$. Since $f(z) \in \mathrm{BC}$, then, according to [4], p. 351, Theorem 2.1

$$\sup_{|w|<\varrho} T(1, f_w) \le c_\varrho < +\infty \tag{2}$$

for any ϱ, $0 < \varrho < 1$.

It immediately follows from the definition that

$$T(1, f_w) = \frac{1}{\pi} \int_0^1 \Big[\iint_{\Delta(w,t)} \left[f^{\#}(z) \right]^2 dx\, dy \Big] \, dt$$

(cf. [4]).

Since $D(w,t) \subset \Delta(w,t)$ for any $w \in D$ and $0 < t < 1$, then

$$T(1, w, f) \le T(1, f_w) \tag{3}$$

for any $w \in D$.

It follows from (2) and (3) that

$$\sup_{|w|<\varrho} T(1, w, f) \le c_\varrho < +\infty \tag{4}$$

for any ϱ, $0 < \varrho < 1$.

Since $f \in N$, then, according to [5], Theorem 1

$$\sup_{q<|w|<1} T(1, w, f) \le c_q < +\infty \tag{5}$$

for any q, $0 < q < 1$.

Choosing $q < \varrho$, we obtain from (4) and (5)

$$\sup_{|w|<1} T(1, w, f) \le \max(c_\varrho, c_q) < +\infty,$$

and (1) is proved.

Sufficiency. We put $w = 0$ in (1). Then $T(1,0,f) = T(1,f) < +\infty$, so that $f(z) \in \mathrm{BC}$.

It follows from (1) that

$$\sup_{q<|w|<1} T(1,w,f) \leq \sup_{|w|<1} T(1,w,f) < +\infty \tag{6}$$

for every q, $0 < q < 1$. Combining (6) with [5], Theorem 1, we conclude that $f(z) \in N$.

2. Following S. Yamashita ([4]), a meromorphic function $f(z)$ defined in D is called a function with uniformly bounded characteristic if $\sup_{w\in D} T(1, f_w)$. The class of such functions is denoted by UBC. The inclusion $\mathrm{UBC} \subset \mathrm{BC} \cap N$, proved by S. Yamashita in [4], now follows immediately from our Theorem 1.

3. For a meromorphic function $f(z)$ defined in D we introduce the differentiable form $d\mu_f(z) = (1-|z|^2)\left[f^{\#}(z)\right]^2 dx\,dy$ and the measure $\mu_f(E) = \iint_E d\mu_f(z)$ generated by $d\mu_f(z)$ on a Borel set $E \subset D$. Let

$$Q(\mu_f, w) = \frac{1}{2\bar{u}}\mu_f\big(R(w)\big)\big(1-|w|\big)^{-1},$$

where $R(w) = \{z \in D;\, |w| < |z| < 1, |\arg z - \arg w| < \pi(1-|w|)\}$ for $w \neq 0$, and $R(w) = D$ for $w = 0$. The measure μ_f is called the Carleson measure if $\sup_{w\in D} Q(\mu_f, w) < +\infty$ (cf. [6], p. 38).

Theorem 2. *A meromorphic function $f(z)$ belongs to the class UBC if and only if $f(z)$ is normal and μ_f is the Carleson measure.*

In fact, the necessity in Theorem 2 follows from [4], Theorem 3.1 and [6], Theorem 2. The sufficiency in Theorem 2 is contained in the proof of Theorem 3 from [6], pp. 42–43.

4. In the paper [7], S. Yamashita posed the following problem: Does a meromorphic function $f(z)$ belong to the class UBC if the measure μ_f is the Carleson measure?

The solution of this problem is contained in part 5, Theorem 3.

We prove the following

Lemma. *If the measure μ_f for a meromophic function $f(z)$ in D is the Carleson measure, $f(z)$ is normal (i.e. $f(z) \in N$).*

Proof. Let a, $\frac{1}{4} < a < 1$, be fixed. Then $\frac{1}{4} < |z-w||1-\bar{w}z|^{-1}$ for any $z \in D$, for which $|z| < r = r(a) = \frac{1}{5}(4a-1)$, and for any w with $a < |w| < 1$.

Since $\log\left(|z|^{-1}\right) \le c\left(1-|z|^2\right)$ for $|z| > \frac{1}{4}$ (cf. for instance [1], p. 238), we have

$$\log|1-\bar{w}z||z-w|^{-1} \le c\Big(1-\left(|1-\bar{w}z|^{-1}|z-w|\right)^2\Big) = c\frac{\left(1-|z|^2\right)\left(1-|w|^2\right)}{|1-\bar{w}z|^2}$$

for all z, $|z| < r(a)$, and w, $a < |w| < 1$, with $\frac{1}{4} < a < 1$.

It is known ([4]) that

$$T(r, f_w) = \iint\limits_{|z|<r} \left[f^{\#}(z)\right]^2 \log\left|\frac{1-\bar{w}z}{z-w}\right| dx\,dy, \qquad 0 < r \le 1.$$

We hence have, for any w, $a < |w| < 1$, and a, $\frac{1}{4} < a < 1$, and $r = r(a) = \frac{1}{5}(4a-1)$,

$$T(r, f_w) \le c \sup_{w\in D} \iint\limits_{|z|<1} \frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-\bar{w}z|^2}\left[f^{\#}(z)\right]^2 dx\,dy \le cc_1 < +\infty, \tag{7}$$

where

$$c_1 = \sup_{w\in D} \iint\limits_{|z|<1} \frac{\left(1-|w|^2\right)\left(1-|z|^2\right)}{|1-\bar{w}z|^2}\left[f^{\#}(z)\right]^2 dx\,dy$$

and $c_1 < +\infty$ since μ_f is the Carleson measure (see [1], Lemma 3.3, p. 239).

For every $w \in D$ and any r_1, $0 < r_1 < 1$, the estimate

$$\left(1-|w|^2\right)f^{\#}(w) \le \frac{1}{r_1^2}\left(\exp 2T(r_1, f_w) - 1\right)^{1/2} \tag{8}$$

is proved by S. Yamashita ([8], p. 193, the inequality (3.5)).

Combining (7) and (8), we obtain the inequality

$$\left(1-|w|^2\right)f^{\#}(w) \le \frac{1}{r^2}\left(\exp 2cc_1 - 1\right)^{1/2} = M < +\infty \tag{9}$$

valid for any w, $a < |w| < 1$, and $r = r(a)$.

Since $f^{\#}(z)$ is a positive continuous function in D, we have for any a_1, $a < a_1 < 1$, and any w, $|w| \le a_1$,

$$\left(1-|w|^2\right)f^{\#}(w) \le f^{\#}(w) \le m < +\infty, \tag{10}$$

where $m = \max_{|w|\le a_1} f^{\#}(w)$.

Combining (9) and (10) we get

$$\sup_{w\in D}\left(1-|w|^2\right)f^{\#}(w) \le \max(M, m) < +\infty,$$

that is, $f(z) \in N$. The Lemma is proved.

Remark 1. Our Lemma essentially improves a result in [6], p. 42.

Remark 2. Let $f(z)$ be a holomorphic function in the disk D and let $d\lambda_f(z)$ be the differential form $d\lambda_f(z) = \left(1 - |z|^2\right)\left|f'(z)\right|^2 dx\,dy$. If, in the proof of the Lemma, we put $\left|f'(z)\right|$ instead of $f^{\#}(z)$, we obtain a new proof of the following well-known result: If the measure $\lambda_f(z)$ for a holomorphic function $f(z)$ in D is the Carleson measure, then $\limsup_{|z|\to 1} \left(1 - |z|\right)\left|f'(z)\right| < +\infty$; i.e., it is a Bloch function (cf. for instance [7], p. 481).

5. Theorem 3. *A meromorphic function $f(z)$ belongs to the class* $\cup\mathrm{BC}$ *if and only if μ_f is a Carleson measure.*

This theorem immediately follows from Theorem 2 and the Lemma.

Corollary. *A meromorphic function $f(z)$ belongs to the class* $\cup\mathrm{BC}$ *if and only if*

$$\sup_{w\in D} \iint\limits_{|z|<1} \frac{\left(1 - |w|^2\right)\left(1 - |z|^2\right)}{|1 - \bar{w}z|^2} \left[f^{\#}(z)\right]^2 dx\,dy \le cc_1 < +\infty.$$

6. Theorem 4. *A meromorphic function $f(z)$ belongs to the class* $\cup\mathrm{BC}$ *if and only if*

$$\sup_{w\in D} \iint\limits_{|z|<1} T(1, f_z)\left|\varphi_w'(z)\right|^2 dx\,dy < +\infty.$$

This theorem follows from Theorem 3 and a result in [3], Theorem 4.

7. A well-known result states that a holomorphic function $f(z)$ in D belongs to the class BMOA if and only if the measure λ_f, $d\lambda_f(z) = \left(1-|z|^2\right)\left|f'(z)\right|^2 dx\,dy$, is the Carleson measure (see for instance [7], p. 481). We note that the proof of Theorem 3 presents a new proof of this result if in the proof of Theorem 3 one uses $\left|f'(z)\right|$ instead of $f^{\#}(z)$ and the inequality (3.5.3) in [8], p. 194 instead of (8).

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