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THE CATALAN CASE OF ARMSTRONG'S CONJECTURE ON SIMULTANEOUS CORE PARTITIONS*

RICHARD P. STANLEY[†] AND FABRIZIO ZANELLO[‡]

Abstract. A beautiful recent conjecture of Armstrong predicts the average size of a partition that is simultaneously an s -core and a t -core, where s and t are coprime. Our goal is to prove this conjecture when $t = s + 1$. These simultaneous $(s, s + 1)$ -core partitions, which are enumerated by Catalan numbers, have average size $\binom{s+1}{3}/2$.

Key words. integer partition, core partition, Catalan number, numerical semigroup, Ferrers diagram, hook length

AMS subject classifications. Primary, 05A15; Secondary, 05A17, 05A19, 20M99

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1. Introduction and some simple cases. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a *partition of size n* , i.e., the λ_i are weakly decreasing positive integers summing to n . We can represent λ by means of its *Young* (or *Ferrers*) *diagram*, which consists of a collection of left-justified rows, where row i contains λ_i cells. To each of these cells B one associates its *hook length*, that is, the number of cells in the Young diagram of λ that are directly to the right or below B (including B itself). Figure 1 represents the Young diagram of the partition $\lambda = (5, 3, 3, 2)$ of size 13; the number inside each cell represents its hook length.

Let s be a positive integer. We say that λ is an s -*core* if λ has no hook of length equal to s (or equivalently, equal to a multiple of s). For instance, from Figure 1 we can see that $\lambda = (5, 3, 3, 2)$ is an s -core for $s = 6$ and for all $s \geq 9$. Finally, λ is an (s, t) -*core* if it is simultaneously an s -core and a t -core.

The theory of (s, t) -cores has been the focus of much interesting research in recent years (see [5, 6, 8] for some of the main results). In particular, when s and t are coprime, there exists only a finite number of (s, t) -core partitions. In fact, there are exactly $\binom{s+t}{s}/(s+t)$ such cores (see [5]), the largest of which has size $(s^2 - 1)(t^2 - 1)/24$ [8]. More generally, a nice result of Anderson [5] provides a bijective correspondence between (s, t) -cores and order ideals of the poset of the positive integers that are not contained in the numerical semigroup generated by s and t , which we write as $P_{(s,t)}$. The partial order on $P_{(s,t)}$ is determined by specifying that a covers b whenever $a - b$ equals either s or t . (Our poset terminology follows [9, Chap. 3].)

For instance, let $s = 3$ and $t = 5$. Then $P_{(3,5)} = \{1, 2, 4, 7\}$, where $7 > 4 > 1$ and $7 > 2$. Figure 2 represents the Hasse diagram of the poset $P_{(3,5)}$, rotated 45° counterclockwise from the usual convention. The order ideals of $P_{(3,5)}$ are the following $\frac{1}{3+5}\binom{3+5}{3} = 7$ subsets: $\emptyset, \{1\}, \{2\}, \{2, 1\}, \{4, 1\}, \{4, 2, 1\}$, and $\{7, 4, 2, 1\}$. Notice that

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[†]Department of Mathematics, MIT, Cambridge, MA 02139-4307 (rstan@math.mit.edu). This author's research was partially supported by the National Science Foundation under grant DMS-1068625.

[‡]Department of Mathematical Sciences, Michigan Tech, Houghton, MI 49931-1295 (zanello@mtu.edu). This author's research was supported by a Simons Foundation grant (274577).

8	7	5	2	1
5	4	2		
4	3	1		
2	1			

FIG. 1. The Young diagram of $\lambda = (5, 3, 3, 2)$. The number inside each cell indicates its hook length.

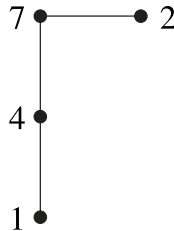


FIG. 2. The Hasse diagram of the poset $P_{(3,5)}$.

from this diagram it is clear that if an element a of $P_{(3,5)}$ belongs to a given order ideal I , then all elements immediately to the right or below a also belong to I .

Anderson's result then gives that (s, t) -cores correspond bijectively to the order ideals of $P_{(s,t)}$ by associating the ideal $\{a_1, \dots, a_j\}$, where $a_1 > \dots > a_j$, to the (s, t) -core partition $(a_1 - (j - 1), a_2 - (j - 2), \dots, a_{j-1} - 1, a_j)$. For instance, the $(3, 5)$ -cores are the following seven partitions: \emptyset (corresponding to the order ideal \emptyset of $P_{(3,5)}$), (1) (corresponding to $\{1\}$), (2) (corresponding to $\{2\}$), $(1, 1)$ (corresponding to $\{2, 1\}$), $(3, 1)$ (corresponding to $\{4, 1\}$), $(2, 1, 1)$ (corresponding to $\{4, 2, 1\}$), and $(4, 2, 1, 1)$ (corresponding to $\{7, 4, 2, 1\}$).

The following conjecture of Armstrong, informally stated sometime in 2011 and then recently published in [6], predicts, for any s and t coprime, a surprisingly simple formula for the average size of an (s, t) -core.

CONJECTURE 1.1. For any coprime positive integers s and t , the average size of an (s, t) -core is $(s + t + 1)(s - 1)(t - 1)/24$. Equivalently, the sum of the sizes of all (s, t) -cores is

$$(1) \quad \frac{(s + t + 1)(s - 1)(t - 1)}{24(s + t)} \binom{s + t}{s}.$$

For instance, the seven $(3, 5)$ -cores computed above are of size 0, 1, 2, 2, 4, 4, and 8, with average size 3, as predicted by Armstrong's conjecture.

One of the interesting aspects of this conjecture, besides the partition-theoretic result that it predicts, is the extra combinatorial information that it would imply on numerical semigroups generated by two elements. In fact, even though one would generally expect these semigroups to be very well understood, Armstrong's conjecture had until now resisted all attempts at significant progress.

The main goal of this paper is to show the conjecture in what is probably its most interesting case, namely, that of $(s, s + 1)$ -cores. The number of these cores is the *Catalan number* $C_s := \frac{1}{s+1} \binom{2s}{s}$, and the corresponding posets $P_{(s, s+1)}$ present a particularly nice structure, which will allow us to use induction in the proof.

We now wrap up this first section by briefly discussing Armstrong's conjecture in a few initial cases. For any given s , in principle, the conjecture can be verified computationally for all (s, t) -cores, given how explicitly one can determine these cores by means of Anderson's bijection. In fact, the authors of [6] indicate that Hanusa has verified the conjecture for small values of s , though they provide no details in the paper. (We thank Hanusa for subsequently informing us that he had checked the conjecture on Mathematica for all $(s, ms + 1)$ -cores and $(s, ms - 1)$ -cores, when $s \leq 10$.) We also wish to remark here that, since this paper has appeared as a preprint, lots of work has already been done that has applied or extended our ideas. For instance, for a nice proof of the case $(s, ms + 1)$ of Armstrong's conjecture, for arbitrary s and $m \geq 1$, see [3], while for two interesting works on multiple simultaneous cores, see [4, 11]. Instead, for a complete proof of the analogue of Armstrong's conjecture for the case of self-conjugate (s, t) -cores (also stated in [6]), see [7].

In this section, we will just present a short proof of the case $(3, t)$ of Armstrong's conjecture (the case $(2, t)$ being trivial), which also gives us the opportunity to state a simple but useful fact on arbitrary (s, t) -cores that seems to have not yet been recorded in the literature. We will provide this lemma without proof, since the argument is analogous to the classical proof that if λ is an s -core, then it is also an ms -core, for all $m \geq 1$ (see, e.g., the first author's [10, Exercise 7.60 and its solution on pp. 518–519]). In principle, the use of this lemma would considerably simplify a “brute-force” proof for any given s , and indeed the case $s = 4$ is still relatively quick to prove along the same lines; nonetheless, for higher values of s the computations remain extremely unpleasant.

LEMMA 1.2. *If a partition λ is an (s, t) -core, then it is also an $(s, s + t)$ -core.*

PROPOSITION 1.3. *Armstrong's conjecture holds for all $(3, t)$ -cores.*

Proof. Let $s = 3$. We will show formula (1) for $t = 3n - 2$, the case $t = 3n - 1$ being entirely similar. Notice that by Lemma 1.2 all $(3, 3n - 2)$ -cores are $(3, 3n + 1)$ -cores. Thus by induction, proving the result is now equivalent to showing that the sum of the sizes of the $(3, 3n + 1)$ -cores that are *not* also $(3, 3n - 2)$ -cores is the difference between the two total sums predicted by Armstrong's conjecture, namely,

$$\Delta(n) = \frac{(3n + 5) \cdot 2 \cdot 3n}{24(3n + 4)} \binom{3n + 4}{3} - \frac{(3n + 2) \cdot 2 \cdot (3n - 3)}{24(3n + 1)} \binom{3n + 1}{3} = \binom{3n + 2}{3}.$$

Figure 3 represents the Hasse diagrams of $P_{(3, 10)}$ and $P_{(3, 13)}$. From these diagrams, we can see that the order ideals of $P_{(3, 13)}$ that are not also in $P_{(3, 10)}$ are exactly the six principal ideals generated by 11, 14, 17, 20, 23, and 10, plus the seven ideals generated by $\{2, 10\}$, $\{5, 10\}$, $\{8, 10\}$, $\{11, 10\}$, $\{14, 10\}$, $\{17, 10\}$, and $\{20, 10\}$.

In a similar fashion, it can be seen that the order ideals of $P_{(3, 3n+1)}$ but not of $P_{(3, 3n-2)}$ are exactly the $n + 2$ principal ideals generated by $3n - 1, 3n + 2, \dots, 6n - 1$, and $3n - 2$, and the $2n - 1$ ideals generated by $\{2, 3n - 2\}, \{5, 3n - 2\}, \dots, \{6n - 4, 3n - 2\}$.

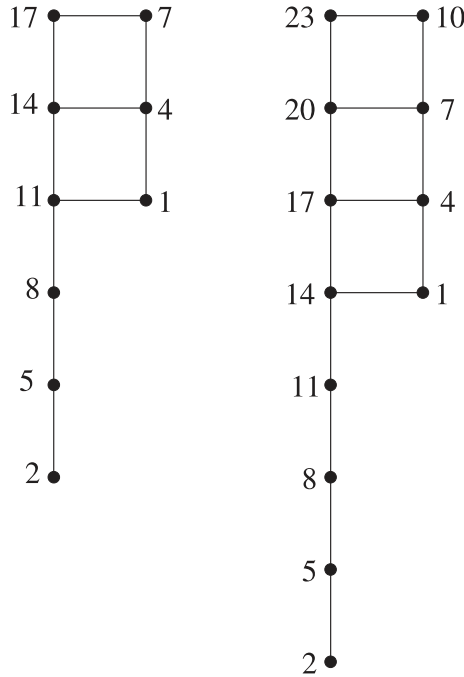


FIG. 3. The Hasse diagrams of $P_{(3,10)}$ (on the left) and $P_{(3,13)}$.

A standard computation now gives that $\Delta(n)$, i.e., the sum of the elements of the above order ideals I minus $\binom{\#I}{2}$, where $\#I$ denotes the cardinality of I , is given by

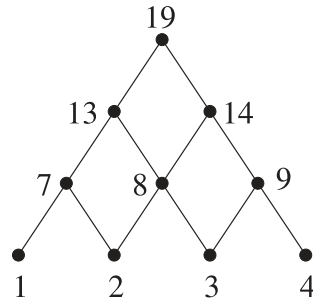
$$\begin{aligned} \Delta(n) &= (2 + 5 + \dots + (3n - 1)) \\ &\quad + \sum_{i=n}^{2n-1} [(2 + 5 + \dots + (3i + 2)) + (1 + 4 + \dots + 3(i - n) + 1)] \\ &\quad + 2n(1 + 4 + \dots + (3n - 2)) + \sum_{i=0}^{2n-2} (2 + 5 + \dots + (3i + 2)) \\ &\quad - \left[\binom{n}{2} + \sum_{i=n}^{2n-1} \binom{2i - n + 2}{2} + \sum_{i=0}^{2n-1} \binom{n + i}{2} \right]. \end{aligned}$$

Showing now that the right-hand side is equal to $\binom{3n+2}{3}$ is a routine task that we omit. This completes the proof. \square

We only remark here that using Lemma 1.2, Armstrong's conjecture can also be verified relatively quickly for $s = 4$, i.e., for all $(4, 2n + 1)$ -cores (though the computations are of course already much more tedious than for $s = 3$). In fact, by formula (1), in this case one has to show that the sum of the sizes of all $(4, 2n + 1)$ -cores equals $S(n) := (4n + 6) \binom{n+3}{4}$. It is easy to check (see also [1]) that, for all $n \geq 7$, the sequence $S(n)$ satisfies the following curious recursive relation:

$$\sum_{i=0}^6 (-1)^i \binom{6}{i} S(n - i) = 0.$$

It would be very interesting to combinatorially explain this identity in the context of $(4, 2n + 1)$ -cores, and thus give an elegant proof of Armstrong's conjecture for $s = 4$.

FIG. 4. The Hasse diagram of T_5 .

2. The Catalan case. The goal of this section is to show Armstrong's conjecture for $(s, s+1)$ -cores. We denote by $T_s := P_{(s, s+1)}$ the corresponding poset according to Anderson's bijection [5]. For simplicity, we will draw the Hasse diagram of T_s from top to bottom; thus, each element of T_s covers the two elements immediately below, and the elements increase by s at each step up and to the left, and by $s+1$ at each step up and to the right. (See Figure 4 for the Hasse diagram of T_5 .)

Let us define the functions

$$g_j := \frac{j(j-1)}{12} \binom{2j}{j},$$

$$f_j := \frac{j^2 + 5j + 2}{8j + 4} \binom{2j+2}{j+1} - 4^j,$$

$$h_j := 2^{2j-1} - \binom{2j+1}{j} + \binom{2j-1}{j-1},$$

where by convention we set $h_0 := 0$. We need the following two identities.

LEMMA 2.1.

$$f_s = \sum_{i=1}^s C_{s-i} (2f_{i-1} + h_{i-1}).$$

Proof. This is the Maple code that verifies the identity (it gives 0 as output):

```
g := j -> binomial(2*j,j)*j*(j-1)/12;
f := j -> binomial(2*j+2,j+1)*(j^2+5*j+2)/(8*j+4)-4^j;
h := j -> 2^(2*j-1)-binomial(2*j+1,j)+binomial(2*j-1,j-1);
C := j -> binomial(2*j,j)/(j+1);
simplify(sum(C(s-i)*(2*f(i-1)+h(i-1)),i=2..s)-f(s));
```

LEMMA 2.2.

$$g_s = \sum_{i=1}^s 2C_{s-i}g_{i-1} + 2(s-i+1)C_{s-i}f_{i-1} \\ + (s-i+3)C_{s-i}h_{i-1} + (i-1)C_{s-i}C_{i-1} - h_{s-i}h_{i-1}.$$

Proof. This is the Maple code that verifies the identity (it gives 0 as output):

```
g := j -> binomial(2*j,j)*j*(j-1)/12;
f := j -> binomial(2*j+2,j+1)*(j^2+5*j+2)/(8*j+4)-4^j;
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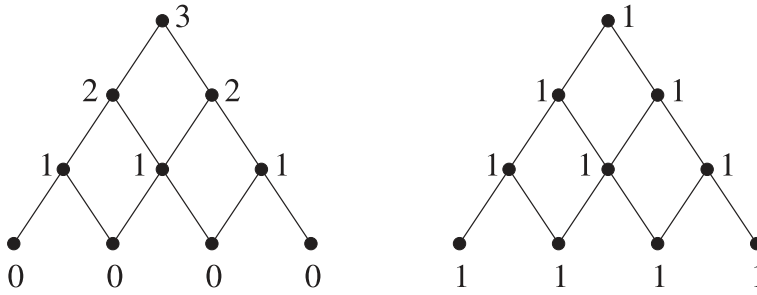


FIG. 5. The poset T_5 with weights $\rho(a)$ on the left and $\tau(a)$ on the right.

$$\begin{aligned}
 h &:= j \rightarrow 2^{(2*j-1)} \cdot \text{binomial}(2*j+1, j) + \text{binomial}(2*j-1, j-1); \\
 C &:= j \rightarrow \text{binomial}(2*j, j) / (j+1); \\
 \text{simplify} &(\text{sum}(2*C(s-i)*g(i-1) + 2*(s-i+1)*C(s-i)*f(i-1) + (s-i+3)*C(s-i)*h(i-1) + (i-1) \\
 &*C(s-i)*C(i-1) - h(s-i)*h(i-1), i=2..s-1) + 2*C(0)*g(s-1) + 2*(1)*C(0)*f(s-1) + (3)*C(0) \\
 &*h(s-1) + (s-1)*C(0)*C(s-1) - g(s)); \quad \square
 \end{aligned}$$

THEOREM 2.3. *Armstrong's conjecture holds for all $(s, s + 1)$ -cores.*

Proof. For any given s , and for any weight function $w : T_s \rightarrow \mathbb{Z}$, define the two functions

$$\begin{aligned}
 f_s(w) &:= \sum_{I \in J(T_s)} \sum_{a \in I} w(a), \\
 g_s(w) &:= \sum_{I \in J(T_s)} \left(\sum_{a \in I} w(a) - \binom{\#I}{2} \right) = f_s(w) - \sum_{I \in J(T_s)} \binom{\#I}{2},
 \end{aligned}$$

where as usual $J(P)$ denotes the set of order ideals of a poset P .

We consider three weight functions on T_s . The weight σ is the “standard weight” that associates, with each element of T_s , itself as a weight; i.e., $\sigma(a) = a$ for all $a \in T_s$. The weight τ is identically 1; i.e., $\tau(a) = 1$ for all $a \in T_s$. Finally, ρ records the ranks of the elements of T_s , when we see this latter as a ranked poset whose minimal elements have rank 0. Figure 5 represents the Hasse diagrams of T_5 , where the elements are being weighted according to τ and ρ .

Showing Armstrong’s conjecture for $(s, s + 1)$ -cores in the form of formula (1) is tantamount to proving that

$$g_s(\sigma) = g_s = \frac{s(s-1)}{12} \binom{2s}{s}.$$

Notice that the elements of rank 0 of T_s are $1, 2, \dots, s - 1$. We can partition $J(T_s)$ as $J(T_s) = \bigcup_i J_i(T_s)$, where $J_i(T_s)$ is the set of those order ideals of T_s whose least element that they do *not* contain is i . Notice that either $1 \leq i \leq s - 1$, or we are considering order ideals whose least missing element i (if any) has positive rank. With some abuse of notation, in this latter case we set by convention $i := s$, so that we can write

$$J(T_s) = \bigcup_{i=1}^s J_i(T_s).$$

Notice that, given i , the elements I of $J_i(T_s)$ *must* contain all of $1, 2, \dots, i - 1$, *cannot* contain any element covering i (this is an empty condition for $i = s$), and

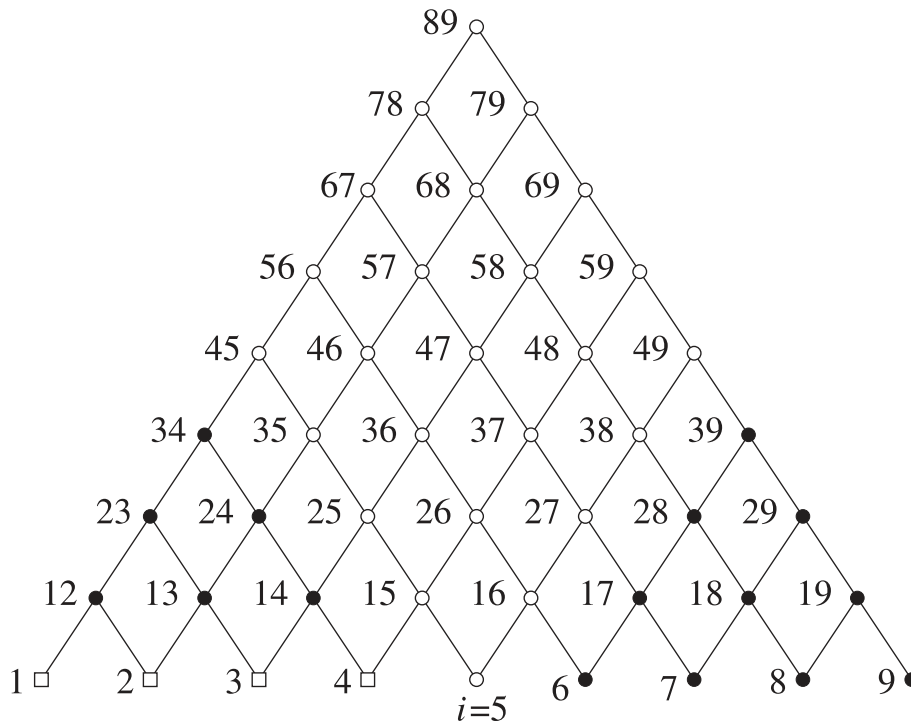


FIG. 6. The possible elements of the order ideals $I \in J_5(T_{10})$. Elements that must appear in I are indicated by squares, that cannot appear by open circles, and that may or may not appear by solid circles.

may or may not contain any other element. Figure 6 gives the Hasse diagram of T_{10} ; for $i = 5$, it indicates by squares the elements of T_{10} that must belong to any given order ideal $I \in J_5(T_{10})$, by open circles the elements that cannot be in I , and by solid circles the elements that may or may not be in I .

It follows that any given order ideal $I \in J_i(T_s)$ can be partitioned into the disjoint union of two order ideals, say I_1 and I_2 , plus the elements $1, 2, \dots, i-1$. Notice that, in the Hasse diagram of T_s , I_1 belongs to a poset that is isomorphic to T_{i-1} and sits to the left of i (starting in rank one), and I_2 belongs to a poset that is isomorphic to T_{s-i} and sits to the right of i . The posets T_1 and T_0 , which arise when $i = 1$, $i = 2$, $i = s-1$, or $i = s$, are empty. (See again Figure 6 for the case $n = 10$ and $i = 5$.)

Given this, it is a simple exercise to show that the sum of the elements of T_s that belong to a given order ideal $I = I_1 \cup I_2 \cup \{1, 2, \dots, i-1\} \in J(T_s)$ is given by

$$\sum_{a \in I} \sigma(a) = \sum_{a \in I_1} w(a) + \sum_{a \in I_2} w(a) + \binom{i}{2},$$

where the weight function w is defined as

$$w := \sigma + (s+1)\tau + (s-i+1)\rho$$

over I_1 , and by

$$w := \sigma + i\tau + i\rho$$

over I_2 . Further, notice that, given i , we can choose the order ideals $I_1 \in J(T_{i-1})$

and $I_2 \in J(T_{s-i})$ independently. Therefore, the elements $a \in I_1$ will appear a total of C_{s-i} times in the order ideals I of T_s , and similarly, the elements $a \in I_2$ will appear a total of C_{i-1} times in the order ideals I of T_s .

Therefore, the contribution of any given i to the desired function $g_s(\sigma)$ is given by

$$(2) \quad m(i, s) - \sum_{I_1 \in J(T_{i-1}), I_2 \in J(T_{s-i})} \binom{\#I_1 + \#I_2 + i - 1}{2},$$

where we have

$$\begin{aligned} m(i, s) &:= \sum_{I_1 \in J(T_{i-1})} C_{s-i} \left(\sum_{a \in I_1} w(a) + \binom{i}{2} \right) + \sum_{I_2 \in J(T_{s-i})} C_{i-1} \sum_{a \in I_2} w(a) \\ &= C_{s-i}(f_{i-1}(\sigma) + (s+1)f_{i-1}(\tau) + (s-i+1)f_{i-1}(\rho)) \\ &\quad + C_{s-i}C_{i-1} \binom{i}{2} + C_{i-1}(f_{s-i}(\sigma) + if_{s-i}(\tau) + if_{s-i}(\rho)). \end{aligned}$$

Let us now consider, again for a fixed i , the term that is being subtracted in formula (2). Notice that

$$\begin{aligned} \binom{\#I_1 + \#I_2 + i - 1}{2} &= \binom{\#I_1}{2} + \binom{\#I_2}{2} \\ &\quad + (i-1)\#I_1 + (i-1)\#I_2 + (\#I_1)(\#I_2) + \binom{i-1}{2}. \end{aligned}$$

Thus, once we sum over all I_1 and I_2 , similar considerations to the above on the number of such order ideals give us that

$$\begin{aligned} &\sum_{I_1 \in J(T_{i-1}), I_2 \in J(T_{s-i})} \binom{\#I_1 + \#I_2 + i - 1}{2} \\ &= \sum_{I_1 \in J(T_{i-1})} C_{s-i} \left(\binom{\#I_1}{2} + (i-1)\#I_1 \right) \\ &\quad + \sum_{I_2 \in J(T_{s-i})} C_{i-1} \left(\binom{\#I_2}{2} + (i-1)\#I_2 \right) \\ &\quad + \left(\sum_{I_1 \in J(T_{i-1})} \#I_1 \right) \left(\sum_{I_2 \in J(T_{s-i})} \#I_2 \right) + C_{s-i}C_{i-1} \binom{i-1}{2}. \end{aligned}$$

Essentially by definition, we have $\sum_{I_1 \in J(T_{i-1})} \#I_1 = f_{i-1}(\tau)$ and, likewise, $\sum_{I_2 \in J(T_{s-i})} \#I_2 = f_{s-i}(\tau)$. Also, it is a known fact (see e.g. [2]) that the function $f_j(\tau)$ appearing in the above formula for $m(i, s)$ satisfies

$$f_j(\tau) = 2^{2j-1} - \binom{2j+1}{j} + \binom{2j-1}{j-1}.$$

As for determining $f_j(\rho)$, by employing the above decomposition of the order ideals I and summing over all i , with a similar argument we can see that

$$f_j(\rho) = \sum_{i=1}^s C_{s-i}(f_{i-1}(\rho) + f_{i-1}(\tau)) + C_{i-1}f_{s-i}(\rho),$$

which, by rearranging the indices, yields

$$f_s(\rho) = \sum_{i=1}^s C_{s-i}(2f_{i-1}(\rho) + f_{i-1}(\tau)).$$

Therefore, by induction, if we apply Lemma 2.1 with $f_j = f_j(\rho)$ and $h_j = f_j(\tau)$, we promptly get the following formula for $f_j(\rho)$:

$$f_j(\rho) = \frac{j^2 + 5j + 2}{8j + 4} \binom{2j + 2}{j + 1} - 4^j.$$

Finally, notice that $g_{i-1}(\sigma) = f_{i-1}(\sigma) - \sum_{I_1 \in J(T_{i-1})} \binom{\#I_1}{2}$, and similarly for $g_{s-i}(\sigma)$.

Therefore, by formula (2) and the subsequent formula for $m(i, s)$, if we sum over $i = 1, 2, \dots, s$, after some tedious but routine computations (that include rearranging the indices where necessary) we obtain

$$g_s(\sigma) = \sum_{i=1}^s [2C_{s-i}g_{i-1}(\sigma) + 2(s-i+1)C_{s-i}f_{i-1}(\rho) + (s-i+3)C_{s-i}f_{i-1}(\tau) + (i-1)C_{s-i}C_{i-1} - f_{i-1}(\tau)f_{s-i}(\tau)].$$

The theorem now follows by induction on s , if we apply Lemma 2.2 with $f_j = f_j(\rho)$, $g_j = g_j(\sigma)$, and $h_j = f_j(\tau)$. \square

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