The Catalan Combinatorics of the Hereditary Artin Algebras

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Abstract. The Catalan numbers are *one of the most ubiquitous and fascinating sequences of enumerative combinatorics* (Stanley), in particular they count the number of non-crossing partitions of a finite set. In the appendix of these notes we will try to outline in which way the Catalan combinatorics could be seen as the heart of the theory of finite sets, starting with the subsets of cardinality two. If we fix a finite set C of cardinality $n+1$, the subsets of cardinality two may be considered as the positive roots of a root system (in the sense of Lie theory) of Dynkin type $\mathbb{A}_n$, and there are recent proposals to work with generalized non-crossing partitions, starting with any root system (of Dynkin type $\mathbb{A}_n, \mathbb{B}_n, \ldots, \mathbb{G}_2$). The Catalan combinatorics looks for sets of partitions of C which are of relevance and relates them to subsets of the automorphism group $S_{n+1} = \operatorname{Aut}(C)$, this is the Weyl group of type $\mathbb{A}$. The generalized Cartan combinatorics starts directly with a suitable subset of G, where G is any Weyl (or, more generally, any Coxeter) group. It turns out that the representation theory of representation-finite hereditary artin algebras Λ can be used in order to categorify these generalized non-crossing partitions in the Weyl group case. In particular, for the case $\mathbb{A}_n$, one may use the ring Λ_n of all upper triangular $(n \times n)$-matrices with coefficients in a field.

Introduction

A **root system** is a finite set of vectors in a Euclidean vector space satisfying some strong symmetry conditions. Root systems are used as convenient index sets when dealing with semi-simple complex Lie algebras or algebraic groups, but play an important role also in other parts of mathematics. The (crystallographic) root systems have been classified by Killing and Cartan at the end of the 19th century, the different types of irreducible root systems are labeled by the Dynkin diagrams $\mathbb{A}_n, \mathbb{B}_n, ..., \mathbb{G}_2$. As we have mentioned, the definition of the root systems refers to symmetry properties, but it turns out that there are further hidden symmetries which are not at all apparent at first sight. They have been discovered only quite recently and extend the use of root systems considerably.

Always, Λ will be a hereditary artin algebra. If Λ is of finite representation type, it is well-known that the indecomposable Λ-modules correspond bijectively to the positive roots of a root system. The positive roots form in a natural way

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a poset, these posets are called the **root posets.** In the setting of Λ-modules, the ordering is given by looking at subfactors. Root posets play a decisive role in many parts of mathematics: of course in Lie theory, in geometry (hyperplane arrangements) and group theory (reflection groups), but also say in singularity theory, in topology, and even in free probability theory (non-crossing partitions). The aim of this survey will be to report on combinatorial properties of the root posets which have been found in recent years by various mathematicians, often in view of these applications. Several of the results which we will discuss have been generalized to the Kac-Moody root systems. We will focus the attention to the relevance of these properties in the representation theory of hereditary artin algebras and the use of this categorification.

Outline. A root system Φ is a finite subset of a Euclidean space V. If x is a root, we denote by H_x the hyperplane orthogonal to x, and by ρ_x the reflection at H_x. In this way, we attach to Φ (or Φ_+) a finite set $\mathbf{H}(\Phi)$ of hyperplanes in V, such sets are called hyperplane arrangements. The reflections ρ_x generate the corresponding Weyl group W. Using the reflections, one defines on W a partial ordering, the so-called absolute ordering $\leq_a$. Given a Coxeter element c in W, the set $\mathbf{Nc}(W, c)$ of all element $w \in W$ with $w \leq_a c$ is called the lattice of generalized non-crossing partitions. In the case of a root system of type $\mathbb{A}$, one just obtains the usual lattice of non-crossing partitions, as introduced by Kreweras and now used for example in free probability theory.

As we have mentioned, we always will consider a hereditary artin algebra Λ. Let $\operatorname{mod}\Lambda$ be the category of all (left) Λ-modules of finite length. Recall that a full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is said to be *thick* provided it is closed under kernels, cokernels, and extensions, thus it is an abelian exact subcategory, and we say that a thick subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is *exceptional* provided it is categorically equivalent to the module category $\operatorname{mod}\Lambda'$ where Λ' is also a (necessarily hereditary) artin algebra, or, equivalently, provided $\mathcal{C}$ has a generator. We denote by $\mathbf{A}(\operatorname{mod}\Lambda)$ the poset of exceptional subcategories of $\operatorname{mod}\Lambda$, this will be the main object of interest. The central result to be shown asserts that

$$\boxed{\mathbf{A}(\operatorname{mod}\Lambda) \simeq \mathbf{Nc}(W(\Lambda), c(\Lambda)),}$$

where $W(\Lambda)$ and $c(\Lambda)$ are the Weyl group and the Coxeter element, respectively, corresponding to Λ, see Theorem 3.7.4.4.

Chapter 1 is devoted to **numbers** which arise from counting problems dealing with a representation-finite hereditary artin algebra Λ. The numbers we are interested in will depend just on the Dynkin type of Λ (and not on the orientation). Thus, here we deal with what we call Dynkin functions: A Dynkin function $\mathbf{f}$ attaches to any Dynkin diagram Δ an integer, or more generally a real number, sometimes even a set or a sequence of real numbers (for example the sequence of exponents); thus a Dynkin function $\mathbf{f}$ consists of four sequences of numbers, namely $\mathbf{f}(\mathbb{A}_n)$, $\mathbf{f}(\mathbb{B}_n)$, $\mathbf{f}(\mathbb{C}_n)$, $\mathbf{f}(\mathbb{D}_n)$ as well as five additional single values $\mathbf{f}(\mathbb{E}_6)$, $\mathbf{f}(\mathbb{E}_7)$, $\mathbf{f}(\mathbb{E}_8)$, $\mathbf{f}(\mathbb{F}_4)$, $\mathbf{f}(\mathbb{G}_2)$. Typical Dynkin functions are the number of indecomposable modules, the number of tilting modules, the number of complete exceptional sequences. We will analyze some of these Dynkin functions, of particular interest seem to be the prime factorizations of their values.

As we will see, there is a unified, but quite mysterious way to deal with some of these Dynkin functions, namely to invoke the so-called exponents of Δ. Usually, the exponents just fall from heaven: either by looking at the invariant theory of the action of the Weyl group on the ambient space of the root system (Chevalley 1955), or by dealing with the eigenvalues of a Coxeter element (Coxeter, 1951). As Shapiro and Kostant (1959) have shown, there is a third possibility to obtain the exponents, namely looking at the root poset: if r_t is the number of roots of height t, then $(r_1, r_2, \dots)$ is a Young partition and the dual partition is the partition of the exponents. It is of interest that one may determine the exponents inductively, going up step by step in a chain of poset ideals I of Φ_+. A recent result of Sommers-Tymoczko (and Abe-Barakat-Cuntz-Hoge-Terao) based on old investigations of Arnold and Saito (1979) asserts that the set $\mathbf{H} = \mathbf{H}(I)$ of hyperplanes orthogonal to the roots in I is a so-called free hyperplane arrangement. This means that the corresponding module $D(\mathbf{H})$ of $\mathbf{H}$-derivations is free, thus one may consider the degrees of a free generating system of $D(\mathbf{H})$. We obtain in this way an increasing sequence of Young partitions which terminates in the partition of the exponents.

Chapter 2 concerns the classical **tilting theory**, the study of (finitely generated) tilting modules for a hereditary artin algebra. As we will see in this chapter, already the basic setting of tilting theory can be refined, replacing the usually considered torsion pair by a torsion triple or even a torsion quadruple. In this way, tilting theory is put into the realm of the stability theory of King. The study of tilted algebras turns out to be just the study of sincere exceptional subcategories. We also will study perpendicular pairs of exceptional subcategories. Altogether we obtain a wealth of bijections (the Ingalls-Thomas bijections) between sets of modules and subcategories. These bijections explain why we obtain the same Dynkin functions when dealing with quite different counting problems.

Chapter 3 presents the poset $\mathbf{A}(\operatorname{mod}\Lambda)$ of all **exceptional antichains** in $\operatorname{mod}\Lambda$, or, equivalently, of all exceptional subcategories of $\operatorname{mod}\Lambda$. Using the results of Chapter 2, it will be shown that this poset is self-dual (it has a self-duality whose square is essentially the Auslander-Reiten translation). Also, any interval in $\mathbf{A}(\operatorname{mod}\Lambda)$ is again of the form $\mathbf{A}(\operatorname{mod}\Lambda')$ for some hereditary artin algebra Λ', and the maximal chains in $\mathbf{A}(\operatorname{mod}\Lambda)$ correspond bijectively to the complete exceptional sequences of Λ-modules. On the other hand, we will see that $\mathbf{A}(\operatorname{mod}\Lambda)$ can be identified with the poset $\mathbf{Nc}(W(\Lambda), c(\Lambda))$. In this way the theory of generalized non-crossing partitions can be seen as part of the representation theory of hereditary artin algebras. I should stress that the main results outlined in Chapters 2 and 3 are due to Ingalls and Thomas [**57**], and a subsequent paper by Igusa and Schiffler [**56**].

Chapter 4 deals with the special case of the **Dynkin types** $\mathbb{A}$. We denote by Λ_n the path algebra of the linearly oriented quiver of type $\mathbb{A}_n$. We will show that the lattice $\mathbf{A}(\operatorname{mod}\Lambda_n)$ may be identified in a canonical way with the lattice of non-crossing partitions as introduced by Kreweras; this is now an important tool in several parts of mathematics, for example in free probability theory. Thus, one may consider the module categories $\operatorname{mod}\Lambda_n$ as a natural frame for a categorification of the lattices of non-crossing partitions. In particular, here we deal with the Catalan numbers, *one of the most ubiquitous and fascinating sequences of enumerative combinatorics* (Stanley in [**105**]). Also, at the end of Chapter 4, we review some classical problems which are related to the maximal chains in $\mathbf{A}(\operatorname{mod}\Lambda_n)$: namely, to count labeled trees (Sylvester, 1857, Borchardt, 1860, Cayley, 1889), as well as parking functions (Pyke, 1959, Konheim-Weiss, 1966, Stanley, 1997).

The **Appendix.** As an after-thought we will try to discuss the nature of Catalan combinatorics: it seems to us that it should be considered as the heart of the theory of finite sets.

This report concerns the **Catalan combinatorics** and the corresponding Narayana numbers. One may also say that it is about the **cluster complex.** Actually, I will mention the cluster complex only in passing by, but one should be aware that the cluster combinatorics in the Dynkin case is really the combinatorics of the representation-finite hereditary artin algebras as discussed in these lectures. Of course, we deal with the **categorification of combinatorial data,** this is the essence of our considerations. An axiomatic account of this categorification can be found in a recent paper by Hubery and Krause [**53**].

As we have mentioned, Chapters 2 and 3 deal with hereditary artin algebras in general, whereas Chapter 4 and the Appendix restrict the attention to the special case $\mathbb{A}_n$, or better just to $\mathbb{A}_n$ endowed with the linear orientation (the corresponding path algebra will be denoted by Λ_n and will be used as the standard example throughout these lectures). In this way, we present first the general theory and specialize afterwards in order to capture the classical theory of non-crossing partitions in terms of the representation theory of artin algebras. Our account should also allow the interested reader to go the opposite way: to start with Chapter 4 in order to see in which way Λ_n-modules are used for the categorification of partitions (see Section 4.2) and only afterwards to immerge into the general representation theory of artin algebras.

Too late? This report comes late, very late, maybe too late. It concerns objects which have been in the mainstream of representation theory 40 years ago, now they seem to be standard and well understood. The first chapter will focus the attention to a lot of numbers; such numbers had been calculated in the early days of representation theory, but as it seems, never systematically, and only few records are available (by Gabriel-de la Peña and Bretscher-Läser-Riedtmann, as well as by Seidel, a student of Happel). As Assem wrote to me: there should be many student theses at various universities devoted to such calculations, but one did not dare to publish them. The mathematicians working in the representation theory of algebras felt that there would not be an independent interest in these numbers, the only exception may have been Gabriel [**45**]: he pointed out that here the Catalan numbers play a role — but as far as I know never in lectures to a mathematical audience, just in a text written for amateurs and enthusiasts. To repeat: a survey similar to the first parts of these notes may (and should) have been given in the seventies or early eighties of the last century.

Actually, the numbers presented have been discussed, but usually outside of representation theory. We should stress that Chapoton [**28**] presented already in 2002, thus more than 10 years ago, the numbers of clusters, positive clusters and exceptional sequences on his web page, and there is a corresponding survey by Fomin and Reading [**42**] written 2005. Some of the numerology can be traced much further back, namely to considerations concerning singularity theory by Brieskorn and Deligne in the seventies.

Of course, there is an advantage of a late presentation: we are able to present a rather complete picture. But be aware: There are still many open questions. In particular, one misses an interpretation of the numerical data in terms of the exponents (see Chapter 1). Also, given a hereditary artin algebra of Dynkin type Δ, it is not clear how to relate the antichains in the category $\operatorname{mod}\Lambda$ and the antichains in the poset $\Phi_+(\Delta)$, thus to relate non-crossing and non-nesting partitions in a satisfactory way.

Our survey is quite long, but unfortunately it is in no way complete. There are many related topics which we do not touch at all; for example the geometrical realizations of lattices and posets using polyhedra, or important hyperplane arrangements such as the Shi arrangements; even the cluster approach (and the use of cluster categories) is not mentioned explicitly. On the other hand, the topics considered here are restricted to a very narrow setting: a general report should start with hereditary artinian rings, not just artin algebras, in order to cover also non-crystallographic Coxeter groups; it should avoid the restriction to hereditary rings by looking at τ-tilting modules instead of tilting modules; and it should consider generalized (not necessarily finitely generated) tilting modules in order to take into account thick subcategories without covers. Concerning these general settings, many satisfactory results are already known, but a unified theory is still out of reach. Thus, one may say that it really is **too early (not too late)** for a general presentation.

As we have mentioned, the appendix outlines in which way the Catalan combinatorics can be seen as the heart of the theory of finite sets, starting with the subsets of cardinality two, thus with the positive roots of a root system of type $\mathbb{A}$. We do not know which kind of categories could replace the category of finite sets in order to deal with the remaining root systems. Also here, our considerations are open-ended.

The approach. I will try to be as elementary as possible. I will prefer to consider individual modules in contrast to subcategories (thus, instead of dealing with thick subcategories, I usually will work with antichains: a thick subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is an abelian exact subcategory closed under extensions, the corresponding antichain is given by the simple objects of $\mathcal{C}$, and $\mathcal{C}$ is obtained back from the antichain as its extension closure). Given an artin algebra Λ, I will prefer to work with its module category $\operatorname{mod}\Lambda$ and will not touch the corresponding derived category $D^b(\operatorname{mod}\Lambda)$. I know that triangulated categories are now well-known and well-appreciated, but they will not be needed in an essential way.

The survey is based on lectures which were addressed to mathematicians working in the representation theory of finite-dimensional algebras, and they deal with a topic all participants were familiar with, namely the representation theory of hereditary artin algebras: first we consider just representation-finite ones, say corresponding to quivers of finite type, or, more generally, to species of finite type, later then hereditary artin algebras in general. The literature usually restricts to quivers, and avoids species. As I mentioned, I want to be as elementary as possible, but nevertheless we will take into account species. The reason is the following: There is the division between the series $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$, and the exceptional cases. Let me look at the series: always, the case $\mathbb{A}_n$ is considered as the basic case, the three remaining cases $\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ are deviations of $\mathbb{A}_n$ (note that for such a diagram $\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$, a large portion is of type $\mathbb{A}$, there is a difference only at one of the ends). For many counting problems as discussed in the first chapter, the cases $\mathbb{B}_n$ and $\mathbb{C}_n$ yield the same answer, and the formulas which one obtains are really neat, condensed and surprisingly easy to remember, whereas the formulas for $\mathbb{D}_n$ usually look much more complicated, and often they may be considered as variations of the $\mathbb{B}_n$ formulas. Thus, in order to understand the formulas for $\mathbb{D}_n$ properly, it seems to be advisable to look first at the numbers for $\mathbb{A}$ and $\mathbb{B}$ and only afterwards to the case $\mathbb{D}$. This is one of the reasons why we definitely want to include the cases $\mathbb{B}$ (and $\mathbb{C}$), although this requires to work not only with path algebras of quivers, but with hereditary artin algebras in general.

References. This is a survey dealing with contributions by a large number of mathematicians, see also the note N 0.1 at the end of the introduction. I will try to indicate the main sources, but to name all contributors seems to be a nearly hopeless task. The material to be covered is vast and I am not at all an expert in several of the topics, thus sometimes I have to be vague, and provide just some indications. I am grateful to many mathematicians for introducing me to various questions, see the acknowledgments at the end of the introduction.

In Chapter 1, we will deal with a large set of counting problems, and it will turn out that several of these problems yield the same answer. This is of course of great interest and asks for some explanation: to provide natural bijections between the objects in questions. However, this also tends to be a source for priority fights: just think of say 100 equivalent counting problems (see the note N 0.2). Now any problem can be solved individually (so there should be 100 different proofs), or else one can show the equivalence to a similar problem where the answer is known (there are $\binom{100}{2} = 4950$ equivalence proofs). But the situation may be even more complex: in case we deal with a Dynkin function, one may have to consider it case by case, or one can find a unified proof; one may need to rely on computer calculations or find a conceptual proof. And the answer may be given by a magic formula, say in terms of the exponents, and a final proof should explain this!

Acknowledgments. This is the written account for a sequence of four lectures given at the ICRA Workshop August 2014 in Sanya, Hainan, and a related series of lectures September 2014 at SJTU, Shanghai. The author thanks the organizers of the ICRA conference and Zhang Pu from SJTU for the invitations to give these lectures. We have omitted here part of the general considerations on hyperplane arrangements (this was lecture 2), only the final result of Sommers-Tymoczko will be outlined in Section 1.5. The third lecture at Sanya was devoted to the new vision of tilting theory following Ingalls and Thomas; this is now Chapter 2. The material which was presented in lecture 4 has been expanded considerably and now forms Chapters 3 and 4.

A central part of this report is based on a joint project with M. Obaid, K. Nauman, W. Al Shammakh and W. Fakieh from KAU, Jeddah, which was devoted to review and to complement known counting results for support-tilting modules and complete exceptional sequences, see the papers [**71, 72, 73**].

Of course, all my considerations concerning representations of hereditary artin algebras rely on the old collaboration with V. Dlab, those on tilted algebras on the collaboration with D. Happel. I am grateful to H. Krause and L. Hille for stressing the relevance of thick subcategories, and of King's stability theory, respectively. I have learned from G. Röhrle the basic induction principle for hyperplane arrangements; H. Thomas made me aware of the Shapiro-Kostant relation between the exponents and the height partition of the positive roots.

But as the main driving force I have to mention F. Götze, the chairman of the Bielefeld CRC 701. He advised me already several years ago to study non-crossing partitions. He organized joint study groups of the Bielefeld research groups in probability theory and in representation theory in order to raise the mutual interest — for a long time, this seemed to be a hopeless endeavor. One of the topics he always stressed were the parking functions, but I realized only now, when writing up these notes, the direct bijection between the parking functions and complete exceptional sequences for the linearly oriented quiver of type $\mathbb{A}_n$ as outlined at the end of Chapter 4: I had been working on exceptional sequences without being aware of such a relationship (but he seemed to know). Thus, I have to thank the Bielefeld CRC 701 who has supported me in this way (see also the note N 0.3). I should add that the presentation has gained from the Bielefeld workshop on *Non-crossing Partitions in Representation Theory* organized in June 2014 by B. Baumann, A. Hubery, H. Krause, Chr. Stump, and the Bielefeld CRC has to be praised for providing the financial support.

My earlier drawings of the various root posets have been improved by A. Beineke, in addition I have to thank him for his permission to include in the first chapter some of his observations concerning the cubical structure of the root posets.

I am grateful to many mathematicians for answering questions and for helpful comments concerning the presented material, in particular to Th˙ Brüstle, F. Chapoton, X. W. Chen, W. Crawley-Boevey, M. Cuntz, S. Fomin, L. Hille, H. Krause, G. Röhrle, and H. Thomas.

Notes.

N 0.1. We hope that our presentation adds some small improvements to the present knowledge.

This concerns in Chapter 2 the unified treatment of torsion and perpendicular pairs, invoking the stability theory of King; it relies on a systematic use of normalizations of modules (see Section 2.2). Following [**72**], the artin algebras to be used in order to categorify the posets of generalized non-crossing partitions are not assumed to be path algebras of quivers, thus we cover all the symmetrizable Cartan matrices, not just the symmetric ones. There are new drawings of the root posets in Section 1.1.1 which may be helpful for the reader. Our discussion of relevant Dynkin functions and their prime factors reveals the strange prime factor 4759, see Table 2. In Chapter 4 we outline a categorical interpretation of the bijection between non-crossing partitions and binary trees, using perpendicular pairs in $\operatorname{mod}\Lambda_n$, see Theorem 2.6.2.1. And there is an interpretation of Stanley's bijection (between maximal chains of non-crossing partitions and parking functions) in terms of the representation theory of the algebras Λ_n, see Theorem 4.5.3.1. Throughout the survey, we try to stress the relevance of antichains in additive categories.

N 0.2. This is not an exaggeration: there is the famous list by R. Stanley [**105**] on problems which yield the Catalan numbers. There, he exhibits 66 different problems, and many additional ones can be found in his Catalan Addendum [**106**].

N 0.3. The author was project leader at the CRC 701 until June 2013, thus he wants to thank the DFG for the corresponding financial support.

1. Numbers

1.1. The setting. On the one hand, there is the combinatorial setting of the (finite) root systems and the corresponding hyperplane arrangements. On the other hand, there is the representation theoretical setting of dealing with a hereditary artin algebra Λ and its module category $\operatorname{mod}\Lambda$.

1.1.1. **The combinatorial setting.** We consider a (finite) root system $\Phi = \Phi(\Delta)$ in the Euclidean space V, it is the disjoint union of irreducible root systems. A connected root system is of type Δ, where $\Delta = \Delta_n$ is a Dynkin diagram with n vertices, thus one of $\mathbb{A}_n, \dots, \mathbb{G}_2$. Given a root system Φ in a vector space V, we choose a root basis Δ, this is a basis of V which consists of elements of Φ (they are called *simple roots*) such that all elements of Φ are linear combinations of these basis elements with integer coefficients which are either non-negative (the *positive roots*) or non-positive (the *negative roots*). We recall the relevant definitions in note N 1.1 at the end of the chapter.

We denote by Φ_+ the set of positive roots, this is a poset with respect to the ordering $x \le y$ iff $y - x$ is a non-negative linear combination of positive roots (or of simple roots). The objects to be considered in these lectures are the root posets

$$\boxed{\Phi_+ = (\Phi_+, \le)}$$

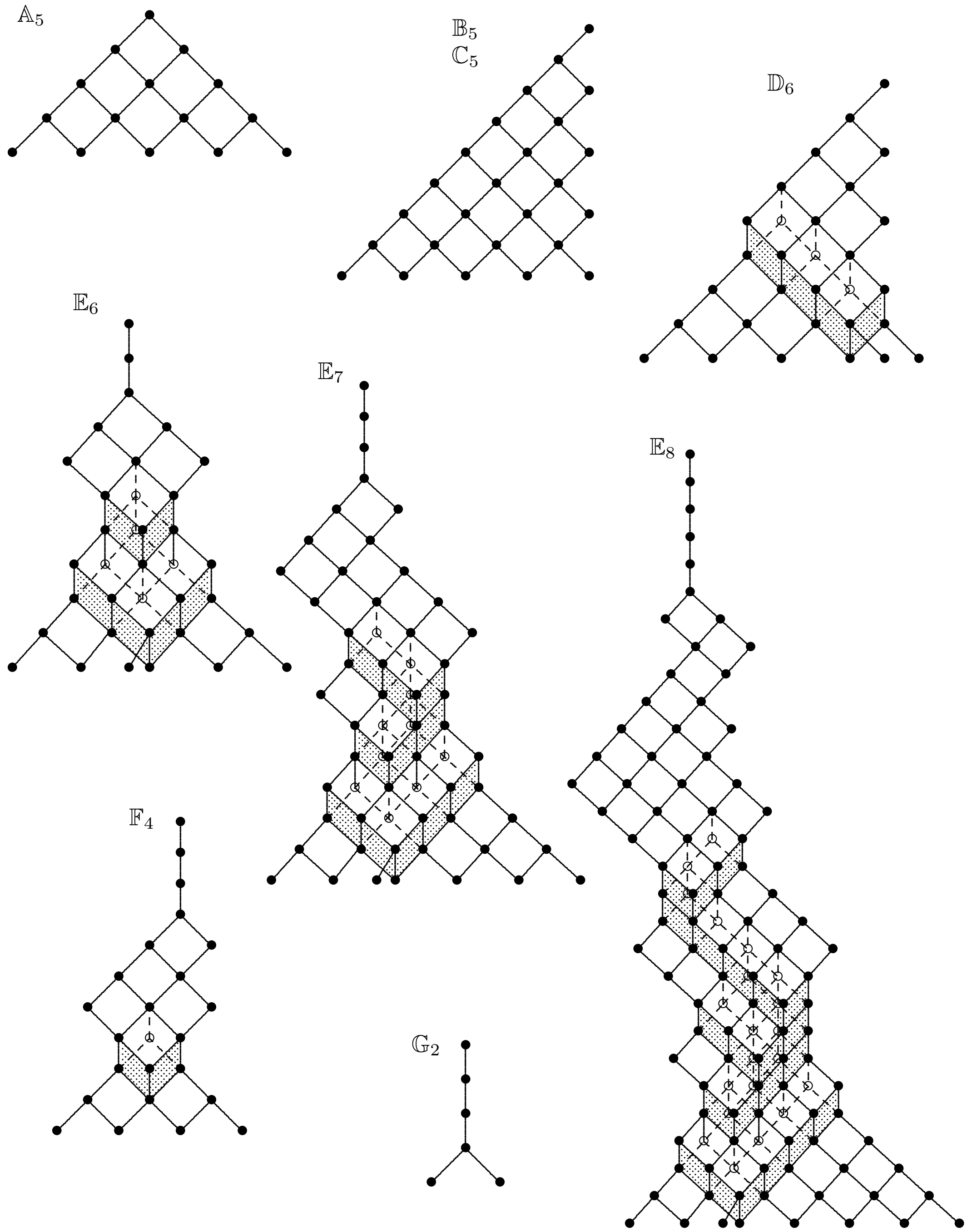
If we write a positive root α as a linear combination of simple roots, the sum of the coefficients is called the *height* of α. For any pair $t \le t'$ of natural numbers, we denote by $\Phi_{t,t'}$ the subposet of Φ_+ given by the positive roots with height s, where $t \le s \le t'$.

The root posets $\Phi_+(\mathbb{A}_n)$ may be identified with the poset of all intervals of the form $[i, j]$ where $0 \le i < j \le n$ are integers, using as ordering the set-theoretical inclusion, see the note N 1.2.

Let us exhibit the Hasse diagrams of some of the root posets, namely the cases $\mathbb{A}_5, \mathbb{B}_5, \mathbb{C}_5, \mathbb{D}_6$, as well as all the exceptional cases. Let us mention some of the properties of a root system which can be seen quite nicely by looking at these Hasse diagrams.

- The lowest row consists of the simple roots, there are precisely n simple roots.
- The row above the lowest row consists of the roots of height 2, there are precisely $n - 1$ roots of height 2, they correspond bijectively to the edges of the Dynkin diagram. In fact, the subposet $\Phi_{1,2}$ given by the two lowest rows may be considered as the incidence graph of the Dynkin diagram.
- There is a unique maximal element in Φ_+: the highest root of Φ.
- The numbers r_t of roots of height t form a decreasing sequence, this will be discussed in detail in Section 1.4.
- The Hasse diagram of Φ_+ may be considered as the 2-dimensional projection of a 3-dimensional object formed by cubes, squares and intervals. This will be discussed in the note N 1.3.

Note that the poset $\Phi_+(\mathbb{B}_n)$ and $\Phi_+(\mathbb{C}_n)$ are isomorphic for any n (this is no longer the case, if we take into account the different root lengths, see the note N 1.4).



1.1.2. **Roots and hyperplanes.** A root system Φ is a subset of a Euclidean space V. If x is a root, we denote by H_x the hyperplane orthogonal to x, and by ρ_x the reflection at H_x. We denote by

$$\boxed{\mathbf{H}(\Phi)}$$

the set of these hyperplanes. These sets will be considered in detail in Section 1.5. Note that for any root x, we have $H_x = H_{-x}$, and we have $H_x \neq H_y$ in case $y \neq \pm x$. This shows that $\mathbf{H}(\Phi)$ can be indexed by the set Φ_+ of positive roots.

We will denote by W the Weyl group, it is the subgroup of $\mathrm{GL}(V)$ generated by the reflections ρ_x with $x \in \Delta$. The Weyl group contains all the reflections ρ_x with $x \in \Phi$, these are just all the reflections in W (a *reflection* in $\mathrm{GL}(V)$ is an element of finite order which fixes pointwise precisely a hyperplane), see the note N 1.5.

If we consider the set $x(1), \ldots, x(n)$ of simple roots with a fixed ordering, the corresponding Weyl group element $c = \rho_{x(n)} \cdots \rho_{x(2)}\rho_{x(1)}$ is called a Coxeter element in W. Actually, instead of fixing such an ordering, we usually work with an orientation Ω of the diagram Δ (but in the more general setting of dealing with arbitrary finite graphs, we only will be interested in orientations without oriented cyclic paths). To choose an *orientation* of a graph means to replace any edge $\{x, y\}$ of the graph by one of the ordered pairs (x, y), (y, x); if it is replaced by (x, y), we will indicate this by drawing an arrow $x \to y$. If there are no oriented cyclic paths, we can order the vertices $x(1), \ldots, x(n)$ in such a way that the existence of an arrow $i \leftarrow j$ implies that $i < j$, and then we attach to it the Coxeter element $c_\Omega = \rho_{x(n)} \cdots \rho_{x(2)}\rho_{x(1)}$; note that c_Ω only depends on the orientation Ω and not on the actual ordering.

Since a Dynkin diagram does not have cycles, we see that *there is a bijection between the orientations Ω of Δ and the Coxeter elements c_Ω in W.* For example, the Dynkin diagram $\mathbb{A}_3$ has four orientations (thus four Coxeter elements):

$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ}$	$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longleftarrow \underset{3}{\circ}$	$\underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ}$	$\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ}$
$\sigma_3\sigma_2\sigma_1$	$\sigma_1\sigma_3\sigma_2 = \sigma_3\sigma_1\sigma_2$	$\sigma_2\sigma_1\sigma_3 = \sigma_2\sigma_3\sigma_1$	$\sigma_1\sigma_2\sigma_3$

We should stress that using Dynkin graphs with orientations, there are two kinds of arrows which should not be confused: namely, besides the arrow heads describing the orientation, the non-simply-laced Dynkin graphs $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4, \mathbb{G}_2$ also use an arrow (usually drawn in the middle of the edge) which indicates the relative length of the corresponding roots. Thus, for example, there are two different Dynkin quivers of type $\mathbb{B}_2$, namely

$\sigma_2\sigma_1$ $\qquad\qquad$ $\sigma_1\sigma_2$

In both cases, the basis root with label 1 is a short root, the root with label 2 a long root (as indicated by the arrow head in the middle of the edge).

1.1.3. **The representation theoretical setting.** We start with a hereditary artin algebra Λ and consider its module category $\mathrm{mod}\,\Lambda$ (modules are left Λ-modules of finite length). The *quiver* $Q(\mathcal{C})$ of an abelian category $\mathcal{C}$ has as vertices the isomorphism classes $[S]$ of the simple objects S, and there is an arrow $[T] \to [S]$ provided $\mathrm{Ext}^1(T, S) \neq 0$. Any arrow carries a valuation which records the dimensions of $\mathrm{Ext}^1(T, S)$ as a module over the endomorphism rings of S and T, respectively. For the category $\mathcal{C} = \mathrm{mod}\,\Lambda$, we just write $Q(\Lambda)$ instead of $Q(\mathrm{mod}\,\Lambda)$. If Λ is representation-finite, the underlying valued graph of $Q(\Lambda)$ turns out to be the disjoint union of Dynkin diagrams and all Dynkin diagram arise in this way.

For further details see the note N 1.6. In Chapter 1, we usually will assume that Λ is representation-finite.

The representation theoretical objects to be considered are the categories $\operatorname{mod}\Lambda$ where Λ is a hereditary artin algebra (in Chapters 1 and 4, they will be assumed to be in addition representation-finite). We denote by $K_0(\Lambda)$ the Grothendieck group of Λ (of all Λ-modules modulo exact sequences); this is the free abelian group with basis the isomorphism classes $[S]$ of the simple modules; if M is a module, $\mathbf{dim}\, M$ is the corresponding element in $K_0(\Lambda)$, thus $\mathbf{dim}\, S = [S]$, and $\mathbf{dim}\, M = \sum [M : S][S]$, where $[M : S]$ is the Jordan-Hölder multiplicity of S in M.

Theorem 1.1.3.1. **(Gabriel 1972, Dlab-Ringel 1973).** *Let Λ be a hereditary artin algebra. Then Λ is representation-finite if and only if the valued quiver $Q(\Lambda)$ is the disjoint union of quivers of Dynkin type. If Λ is of Dynkin type Δ, then the map* **dim** *provides a bijection between the isomorphism classes of the indecomposable Λ-modules and the positive roots of the root system of type Δ.*

We denote by $\operatorname{ind}\Lambda$ a set of indecomposable Λ modules, one from each isomorphism class. If X, Y are in $\operatorname{ind}\Lambda$, we write $X \sqsubseteq Y$ provided X is a subfactor of Y, thus provided there are submodules $Y'' \subseteq Y' \subseteq Y$ with X isomorphic to Y'/Y''.

Theorem 1.1.3.2. **(Dlab-Ringel 1979).** *Let k be a field with at least 3 elements. Let Λ be a hereditary finite-dimensional k-algebra of Dynkin type Δ. Then the map* **dim** *provides an isomorphism of posets*

$$\boxed{\mathbf{dim}\colon (\operatorname{ind}\Lambda,\ \sqsubseteq\ ) \longrightarrow (\Phi_+, \leq)}$$

It should be stressed that the assumption $|k| \geq 3$ is necessary, see N 1.7.

The theorem shows that the root poset $\Phi_+(\Delta)$ can be categorified by the indecomposable Λ-modules, where Λ is a hereditary artin algebra of Dynkin type Δ. We should stress the following: whereas the category $\operatorname{ind}\Lambda$ strongly depends on the orientation of the Dynkin quiver $Q(\Lambda)$, the poset $(\operatorname{ind}\Lambda,\ \sqsubseteq\ )$ does not depend on the orientation (as the poset isomorphism $\mathbf{dim}\colon (\operatorname{ind}\Lambda,\ \sqsubseteq\ ) \ \longrightarrow (\Phi_+, \leq)$ shows).

1.1.4. The aim of this survey is to exhibit combinatorial data which can be derived from the category $\operatorname{ind}\Lambda$, and we are going to emphasize those which do not depend on the orientation of $Q(\Lambda)$ (at least in case $Q(\Lambda)$ is a tree). Of particular importance seems to be the set of all antichains in $\operatorname{mod}\Lambda$: An *antichain* in an additive category $\mathcal{C}$ is a set of pairwise orthogonal bricks (a brick is an object whose endomorphism ring is a division ring); the note N 1.8 will provide an explanation for the terminology. In case $\mathcal{C}$ is abelian, starting with an antichain A, we can consider the full subcategory $\mathcal{E}(A)$ of all objects with a filtration with factors in the antichain: this is a *thick* subcategory (an exact abelian subcategory which is closed under extensions), again let us refer to some comments in the note N 1.9. Conversely, given a thick subcategory of $\mathcal{C}$, the simple objects in this subcategory form an antichain (this is just Schur's Lemma). Thus, there is an obvious bijection between antichains and thick subcategories. An antichain is said to be *exceptional* provided the quiver of $\mathcal{E}(A)$ has no oriented cyclic paths. For hereditary artin algebras Λ, every antichain is exceptional if and only if Λ is representation-finite. Chapter 3 will be devoted to the study of the poset

$$\boxed{\mathbf{A}(\operatorname{mod}\Lambda)}$$

of exceptional subcategories of $\operatorname{mod}\Lambda$.

1.2. Dynkin functions. As the title of part 1 indicates, this part is devoted to numbers, to numbers which arise from counting problems dealing mainly with representation-finite hereditary artin algebras Λ. The numbers we are interested in will depend just on the Dynkin type of Λ, but not on the orientation of $Q(\Lambda)$. Thus, here we deal with what we want to call Dynkin functions.

1.2.1. **Definition.** A *Dynkin function* $\mathbf{f}$ attaches to any Dynkin diagram an integer (or more generally a real number, sometimes even a set or a sequence of real numbers, for example the sequence of exponents); thus we get four sequences of numbers, namely

$$\mathbf{f}(\mathbb{A}_n),\ \mathbf{f}(\mathbb{B}_n),\ \mathbf{f}(\mathbb{C}_n),\ \mathbf{f}(\mathbb{D}_n)$$

as well as five additional single values

$$\mathbf{f}(\mathbb{E}_6),\ \mathbf{f}(\mathbb{E}_7),\ \mathbf{f}(\mathbb{E}_8),\ \mathbf{f}(\mathbb{F}_4),\ \mathbf{f}(\mathbb{G}_2).$$

(Sometimes, such a function is also defined for the remaining Coxeter diagrams $\mathbb{I}_2(t), \mathbb{H}_3, \mathbb{H}_4$.)

Let us draw the attention to Sloane's OEIS [**100**], the *Online Encyclopedia of Integer Sequences.* This is a marvelous tool when dealing with integer sequences, however in our context it would be nice to be able to use a similar data bank, an OEDF (*Online Encyclopedia of Dynkin Functions*), so that the integer sequences which arise say in case $\mathbb{A}_n$ immediately refer to corresponding sequences which arise in the cases $\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$, and also to the numbers which occur for the exceptional cases. If we look at integral sequences $a = (a_1, a_2, a_3, \dots)$, it is usually difficult to predict whether for a finite set of indices n, the values a_n are relevant to determine the whole sequence a. In the case of a Dynkin function $\mathbf{f}$, we are in another realm: here the three numbers $\mathbf{f}(\mathbb{E}_n)$ with $n = 6, 7, 8$ seem always to be exciting numbers.

1.2.2. **Examples.** Here are some Dynkin functions which one may consider starting with hereditary artin algebras Λ of Dynkin type Δ_n. Let us stress that formulations concerning counting of modules are meant as a short form for counting isomorphism classes of modules.

$r(\Delta_n)$ the number of indecomposable modules (thus, the number of positive roots).

$r_t(\Delta_n)$ the number of indecomposable modules of length t (thus, the number of positive roots of height t).

$\operatorname{sinc}(\Delta_n)$ the number of sincere indecomposable modules (thus, the number of sincere positive roots). We recall that the *support* of a module M is the set of simple modules which occur as subfactor of M, and M is called *sincere* provided any simple module belongs to its support.

$\mathbf{d}(\Delta_n) = x_1 \cdots x_n \cdot x_{\text{th}}$, where $x = (x_1, \dots, x_n)$ is the highest root (note that $x = \mathbf{dim}\, M$, where M is the indecomposable module of maximal length) and x_{th} is equal to 1 plus the number of indices i with $x_i = 1$ (it is well-known that x_{th} is just the determinant of the Cartan matrix of type Δ_n). For example, the highest root for $\mathbb{E}_6$ is

$$x = {\scriptstyle 1\;2\;\overset{2}{3}\;2\;1},$$

thus $\mathbf{d}(\mathbb{E}_6) = 72$, namely $1 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot x_{\rm th}$ and $x_{\rm th} = 1 + 2 = 3$. Similarly, the highest root for $\mathbb{A}_n$ is $x = {\scriptstyle 1\;1\;\ldots\;1}$, thus $x_{\rm th} = 1 + n$, therefore $\mathbf{d}(\mathbb{A}_n) = n + 1$.

$\mathbf{c}(\Delta_n)$ the number of complete exceptional sequences. Recall that a sequence $(E_1, \dots, E_t)$ of indecomposable Λ-modules is said to be *exceptional* provided $\operatorname{Ext}^1(E_i, E_j) = 0$ for $i \geq j$ and $\operatorname{Hom}(E_i, E_j) = 0$ for $i > j$ (in case $t = 2$, one calls it an *exceptional pair*). An exceptional sequence $(E_1, \dots, E_t)$ is said to be *complete* provided t is equal to the number of simple Λ-modules.

$\mathbf{a}(\Delta_n)$ the number of antichains in $\operatorname{mod}\Lambda$.

$\mathbf{a}_t(\Delta_n)$ the number of antichains in $\operatorname{mod}\Lambda$ of cardinality t.

$\mathbf{t}(\Delta_n)$ the number of (multiplicity-free) support-tilting modules. We recall that a multiplicity-free module $T = \bigoplus_{i=1}^{t}$ with indecomposable direct summands T_i is said to be a *support-tilting* module, provided $\operatorname{Ext}^1(T, T) = 0$ and t is the cardinality of its support.

$\mathbf{t}_s(\Delta_n)$ the number of (multiplicity free) support-tilting modules which are direct sums of s indecomposable modules (thus with support of cardinality s). In particular:

$\mathbf{t}_n(\Delta_n)$ is the number of (multiplicity free) tilting modules (since a sincere support-tilting module is just a tilting module)

Of course, there are further important Dynkin functions, for example:

$|W|$ the order of the Weyl group W.

$|W_i|$ where W_i is the set of elements $w \in W$ with fixed point space of dimension $n - i$. In particular W_0 consists just of the identity element, W_1 is the set of reflections.

1.2.3. **Verification.** In order to see that we deal with Dynkin functions, it is necessary to check in any case that the numbers in question are independent of the orientation. For example, let us do this for the number $\mathbf{a}_t(\Delta)$ of antichains of cardinality t (or even for the number $\mathbf{a}_t^s(\Delta)$ of antichains of cardinality t with support of cardinality s), since finally these antichains will be our main concern.

LEMMA 1.2.3.1. *The number $\mathbf{a}_r^s$ of antichains of cardinality r with support of cardinality s does not depend on the orientation.*

PROOF. We consider the set $\mathbf{A}_r^s(\operatorname{mod}\Lambda)$ of antichains of cardinality r with support of cardinality s. Let x be a sink and ρ_x the BGP-reflection functor for x, let $\Lambda' = \rho_x \Lambda$. Let S be the simple Λ-module with support x, and S' the simple Λ'-module with support x.

We claim that there is a bijection

$$\eta\colon \mathbf{A}_r(\operatorname{mod}\Lambda) \to \mathbf{A}_r(\operatorname{mod}\Lambda').$$

Thus, let A be an antichain in $\operatorname{mod}\Lambda$ of cardinality r.

Case 1: If S belongs to A, then for the remaining modules A_i in A, we have $(A_i)_x = \operatorname{Hom}(S, A_i) = 0$, thus A_i may be considered as a Λ'-module and $A' = A \setminus \{S\} \cup \{S'\}$ is an antichain in $\operatorname{mod}\Lambda'$ which contains S'.

Case 2. Now assume that S does not belong to A, then $A' = \{\rho_x(A_i) \mid A_i \in A\}$ is an antichain in $\operatorname{mod}\Lambda'$ which does not contain S'.

Now, let us refine this to deal with $\mathbf{A}_r^s$.

Case 1 is as before: If S belongs to the antichain A, then for the remaining modules A_i in A, we have $(A_i)_x = \operatorname{Hom}(S, A_i) = 0$, thus A_i may be considered as a Λ'-module and $A' = A \setminus \{S\} \cup \{S'\}$ is an antichain in $\operatorname{mod}\Lambda'$ which contains S'. Clearly, both A and A' have the same support.

Case 2. Now assume that S does not belong to A, then $A' = \{\rho_x(A_i) \mid A_i \in A\}$ is an antichain in $\operatorname{mod}\Lambda'$ which does not contain S'.

There are four possibilities, whether x belongs to the support of A or to the support of A'. The cardinality of the support changes in case x does not belong to the support of A but to the support of A' or the other way round. Let $\mathbf{B}$ be the antichains A such that x does not belong to the support of A, but to the support of A'. Then these are the antichains in $\operatorname{mod}\Lambda''$ such that at least one of the elements lives on a vertex which is a neighbor of x. But then the antichains in $\mathbf{B}$ (considered as antichains of Λ'-modules) are precisely the antichains $\rho_x(A)$, where A is an antichain of cardinality r with support of cardinality s such that $\rho_x(A)$ has support of cardinality $s-1$. $\square$

In order to obtain A' from A, we have distinguished two cases: we used the functor ρ_x whenever possible, otherwise we see that the support of any element A_i of A is either $\{x\}$ or does not involve x, so that A_i is already a Λ'-module. Thus we can write: $A' = A$. If we look at the corresponding dimension vectors, we see that we deal with **piecewise linear** functions: the dimension vectors of A' are obtained from those of A' partly by using the reflection ρ_x, partly by taking the identity map.

1.2.4. **Table 1.** Here are the values for some of these Dynkin functions.

Δ_n	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
h	$n+1$	$2n$	$2(n-1)$	12 $2^2{\cdot}3$	18 $2{\cdot}3^2$	30 $2{\cdot}3{\cdot}5$	12 $2^2{\cdot}3$	6 $2{\cdot}3$
$\mathbf{d}(\Delta_n)$	$n+1$	2^n	2^{n-1}	72 $2^3{\cdot}3^2$	576 $2^6{\cdot}3^2$	16740 $2^7{\cdot}3^3{\cdot}5$	48 $2^4{\cdot}3$	6 $2{\cdot}3$
$\mathbf{c}(\Delta_n)$	$(n+1)^{n-1}$	n^n	$2(n-1)^n$	41472 $2^9{\cdot}3^4$	1062882 $2{\cdot}3^{12}$	37968750 $2{\cdot}3^5{\cdot}5^7$	432 $2^4{\cdot}3^3$	6 $2{\cdot}3$
$\lvert W\rvert$	$(n+1)!$	$2^n \cdot n!$	$2^{n-1}n!$	51840 $2^7{\cdot}3^4{\cdot}5$	2903040 $2^{10}{\cdot}3^4{\cdot}5{\cdot}7$	696729600 $2^{14}{\cdot}3^5{\cdot}5^2{\cdot}7$	1152 $2^7{\cdot}3^2$	12 $2^2{\cdot}3$
$\lvert\Phi_+\rvert$	$\binom{n+1}{2}$	n^2	$n(n-1)$	36 $2^2{\cdot}3^2$	63 $3^2{\cdot}7$	120 $2^3{\cdot}3{\cdot}5$	24 $2^3{\cdot}3$	6 $2{\cdot}3$

Observation concerning the prime factors which appear: With the exception of $\mathbb{A}_n$, all the prime factors are bounded by n, for $\mathbb{A}_n$ the bound is $n+1$.

1.2.5. **Table 2.** Here are the values for further Dynkin functions.

Δ_n	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$\mathrm{sinc}(\Delta_n)$	1	n	$n-2$	7 7	16 2^4	44 $2^2\cdot 11$	10 $2\cdot 5$	4 2^2
$\mathbf{a}(\Delta_n)=\mathbf{t}(\Delta_n)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$ Catalan numbers	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833 $7^2\cdot 17$	4160 $2^6\cdot 5\cdot 13$	25080 $2^3\cdot 3\cdot 5\cdot 11\cdot 19$	105 $3\cdot 5\cdot 7$	8 2^3
$\mathbf{a}_t(\Delta_n)$	$\frac{1}{n+1}\binom{n+1}{t}\binom{n+1}{t+1}$ Narayana numbers	$\binom{n}{t}^2$	$\mathbf{a}_t(\mathbb{D}_n)$	1 36 204 351 204 36 1	1 63 546 1470 1470 546 63 1	1 120 1540 6120 9518 6120 1540 120 1	1 24 55 24 1	1 6 1
$\mathbf{t}_s(\Delta_n)$	$\frac{n-s+1}{n+1}\binom{n+s}{s}$	$\binom{n+s-1}{s}$	$\mathbf{t}_s(\mathbb{D}_n)$	1 6 20 50 110 228 418	1 7 27 77 187 429 1001 2431	1 8 35 112 299 728 1771 4784 17342	1 4 10 24 66	1 2 5
$\mathbf{t}_n(\Delta_n)$	$\frac{1}{n+1}\binom{2n}{n}$ Catalan numbers	$\binom{2n-1}{n-1}$	$\frac{3n-4}{2n-2}\binom{2n-2}{n-2}$	418 $2\cdot 11\cdot 19$	2431 $11\cdot 13\cdot 17$	17342 $2\cdot 13\cdot 23\cdot 29$	66 $2\cdot 3\cdot 11$	5 5

with

$$\begin{aligned}\mathbf{a}_t(\mathbb{D}_n) &= \binom{n}{t}^2 - \frac{n}{n-1}\binom{n-1}{t-1}\binom{n-1}{t}\\ &= \binom{n}{t}^2 - \binom{n}{t}\binom{n-2}{t-1}\\ &= \binom{n}{t}\binom{n-2}{t-1}\frac{n(n-1)-t(n-t)}{t(n-t)},\end{aligned}$$

for $1 \le t \le n-1$ and

$$\mathbf{t}_s(\mathbb{D}_n) = \frac{n+2s-2}{n+s-2}\binom{n+s-2}{s}$$

for $0 \le s \le n$.

Here are the missing factorizations for $\mathbf{a}_t(\Delta_n)$ and $\mathbf{t}_s(\Delta_n)$:

	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$
$\mathbf{a}_t(\Delta_n)$	$204=2^2\cdot 3\cdot 17$ $351=3^3\cdot 13$	$546=2\cdot 3\cdot 7\cdot 13$ $1470=2\cdot 3\cdot 5\cdot 7^2$	$1540=2^2\cdot 5\cdot 7\cdot 11$ $6120=2^3\cdot 3^2\cdot 5\cdot 17$ $9518=2\cdot 4759$	$55=5\cdot 11$
$\mathbf{t}_s(\Delta_n)$	$20=2^2\cdot 5$ $50=2\cdot 5^5$ $110=2\cdot 5\cdot 11$ $228=2^2\cdot 3\cdot 19$ $418=2\cdot 11\cdot 19$	$27=3^3$ $77=7\cdot 11$ $187=11\cdot 17$ $429=3\cdot 11\cdot 13$ $1001=7\cdot 11\cdot 13$ $2431=11\cdot 13\cdot 17$	$35=5\cdot 7$ $112=2^4\cdot 7$ $299=13\cdot 23$ $728=2^3\cdot 7\cdot 13$ $1771=7\cdot 11\cdot 23$ $4784=2^4\cdot 13\cdot 23$ $17342=2\cdot 13\cdot 23\cdot 29$	$10=2\cdot 5$ $24=2^3\cdot 3$ $66=6\cdot 11$

Observations concerning the prime factors which appear in table 2 as well as in the subsequent material:

The numbers $\mathbf{a}(\Delta)$ and $\mathbf{a}_t(\Delta)$: *Always, at most one prime factor p is greater than h* and with the exception of the central coefficient for $\mathbb{E}_8$, one has $p < n(n-1)$. (The central coefficient for $\mathbb{E}_8$ has the surprising prime factor **4759**.) This is clear for $\mathbb{A}_n$ and $\mathbb{B}_n$. In case $\mathbb{D}_n$, the prime factors $p > h$ must divide $n(n-1) - t(n-t)$, thus $p < n(n-1)$, and only one such prime factor is possible. For the exceptional cases, the factorizations are listed above.

The numbers $\mathbf{t}_s(\Delta_n)$: Here only primes bounded by h play a role.

1.2.6. **Table 3.** Let us draw the attention to the values $f(\Delta)$ which occur for the infinite sequences $\Delta = \mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$. Actually, we prefer to order the columns differently, namely, first $\mathbb{A}_n$, then $\mathbb{D}_n$, and finally the common values for $\mathbb{B}_n, \mathbb{C}_n$.

Δ_n	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{B}_n, \mathbb{C}_n$
h	$n+1$	$2(n-1)$	$2n$
$\mathbf{d}(\Delta_n) = x_1 \cdots x_n x_{\mathrm{th}}$	$n+1$	2^{n-1}	2^n
$\mathbf{c}(\Delta_n)$	$(n+1)^{n-1}$	$2(n-1)^n$	n^n
$\|W\|$	$(n+1)!$	$2^{n-1}n!$	$2^n \cdot n!$
$\|\Phi_+\|$	$\binom{n+1}{2}$	$n(n-1)$	n^2
$\mathrm{sinc}(\Delta_n)$	1	$n-2$	n
$\mathbf{a}(\Delta_n) = \mathbf{t}(\Delta_n)$	$\left]{2n+2 \atop n+1}\right[$ Catalan numbers	$\left[{2n-1 \atop n-1}\right]$	$\binom{2n}{n}$
$\mathbf{a}_t(\Delta_n)$	$\frac{1}{n+1}\binom{n+1}{t}\binom{n+1}{t+1}$ Narayana numbers	$\binom{n}{t}\binom{n-2}{t-1}\frac{n(n-1)-t(n-t)}{t(n-t)}$	$\binom{n}{t}^2$
$\mathbf{t}_s(\Delta_n)$ $0 \le s < n$	$\left]{n+s \atop s}\right[$	$\left[{n+s-2 \atop s}\right]$	$\binom{n+s-1}{s}$
$\mathbf{t}_n(\Delta_n)$	$\left]{2n \atop n}\right[$ Catalan numbers	$\left[{2n-2 \atop n-2}\right]$	$\binom{2n-1}{n-1}$

Here, in three rows, namely the rows for $\mathbf{a}(\Delta_n) = \mathbf{t}(\Delta_n)$ and for all $\mathbf{t}_s(\Delta_n)$, we have used square-bracket notations in order to indicate the parallelity to the binomial coefficients. The binomial coefficients themselves arise in the cases $\mathbb{B}$ and $\mathbb{C}$ (thus in the last column). The notation $\left[{t \atop s}\right] = \frac{s+t}{t}\binom{t}{s}$ (used in the third column) was proposed by Bailey [**11**]: this concerns the cases $\mathbb{D}$. It has been suggested in [**73**] to write similarly $\left]{t \atop s}\right[= \frac{t-2s+1}{t-s+1}\binom{t}{s}$. This is done in the second column and concerns the classical case $\mathbb{A}$.

The reader should observe that for Δ_n equal to $\mathbb{A}_n$ or $\mathbb{B}_n$, the formula given for $\mathbf{t}_s(\Delta_n)$ and $0 \leq s < n$ works also for $s = n$. This is not the case for $\mathbb{D}_n$: Whereas $\binom{2n-2}{n-2} = \binom{2n-2}{n}$, the numbers $\left[{2n-2 \atop n-2}\right]$ and $\left[{2n-2 \atop n}\right]$ are different.

Four observations should be mentioned.

First of all, it seems that often the numbers which arise for $\mathbb{B}$ (and $\mathbb{C}$) are given by very simple expressions, whereas those for $\mathbb{D}$ tend to be similar to those obtained for $\mathbb{B}$, but sometimes much more complicated (see for example the row $\mathbf{a}_t(\Delta_n)$). There has been a tendency in representation theory to avoid working with non-simply-laced Dynkin diagrams, thus restricting the attention to the cases $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$. But to avoid the really nice cases $\mathbb{B}_n$ seems to be a mistake! The slight increase of difficulties when looking at species and not just quivers is definitely honored by the unified numerical picture which one obtains.

Second, the numbers presented in the table are increasing from left to right as soon as we rearrange the columns (as we have done): first the case $\mathbb{A}_n$, then the case $\mathbb{D}_n$ and finally the cases $\mathbb{B}_n$ and $\mathbb{C}_n$. (Actually, there are further reasons to prefer the sequence $\mathbb{A}_n, \mathbb{D}_n, \mathbb{B}_n, \mathbb{C}_n$ over the alphabetical order: This ordering corresponds to the quite natural ordering of geometries, starting with affine geometry, followed by the orthogonal geometry and ending with symplectic geometry — this is the sequence geometries are usually taught, with symplectic geometry as the climax, or as an afterthought which is left to the students as an exercise.)

Third. We see that the Catalan numbers

$$C_n = \tfrac{1}{n+1}\tbinom{2n}{n} = \left]{2n \atop n}\right[$$

appear as values for the Dynkin diagrams of type $\mathbb{A}$, looking at two different counting problems, namely we have

$$\mathbf{a}(\mathbb{A}_{n-1}) = C_n, \quad \text{as well as} \quad \mathbf{t}_n(\mathbb{A}_n) = C_n$$

(of course, there is an index shift). For the remaining Dynkin diagrams $\mathbb{B}_n, \ldots, \mathbb{G}_2$, these Dynkin functions $\mathbf{a}$ and $\mathbf{t}_n$ take completely different values. Thus, we deal with two different generalizations of the Catalan combinatorics: to look at the set of antichains in $\operatorname{mod}\Lambda$ (this yields the function $\mathbf{a}$) and to look at the tilting modules in $\operatorname{mod}\Lambda$ (this yields the function $\mathbf{t}_n$).

Fourth. The non-zero numbers $\mathbf{t}_s(\Delta_n)$ with $\Delta = \mathbb{A}, \mathbb{B}, \mathbb{D}$ yield three triangles which have similar properties, see [**73**]. The triangle for $\mathbb{A}$ is the Catalan triangle itself (this is Sloane's sequence A009766). The triangle for $\mathbb{B}$ is the triangle A059481, it corresponds to the increasing part of the Pascal triangle (thus it consists of the binomial coefficients $\binom{t}{s}$ with $2s \leq t+1$). The triangle for $\mathbb{D}$ is an expansion of the increasing part of the Lucas triangle A029635: taking the increasing parts of the rows in the Lucas triangle (thus the numbers $\left[{t \atop s}\right]$ with $2s \leq t+1$), one obtains numbers which occur in the triangle $\mathbb{D}$, namely the numbers $\mathbf{t}_s(\mathbb{D}_n)$ with $0 \leq s < n$. The numbers $\mathbf{t}_n(\mathbb{D}_n)$ on the diagonal have to be treated separately: recall that $\mathbf{t}_s(\mathbb{D}_n) = \left[{n+s-2 \atop s}\right]$, whereas $\mathbf{t}_n(\mathbb{D}_n) = \left[{2n-2 \atop n-2}\right]$.

1.2.7. **Why counting?** The result may give an indication about structural similarities. If for two different counting problems we obtain the same numbers, one may ask whether there is a natural bijection between the sets in question. And if we find one, it may turn out such a bijection exists in similar situations looking at sets which are no longer finite.

For example, if the answer to a counting problem turns out to be $(n+1)^{n-1}$, as it is for counting the number of complete exceptional sequences in case $\mathbb{A}_n$, one may try to find a bijection between the complete exceptional sequences and labeled trees. In Section 2.4 we will present a bijection between the support-tilting modules and the normal partial tilting modules. One could have asked for such a bijection as soon as one had observed that in the representation-finite cases the numbers of multiplicity-free modules which are support-tilting or normal coincide.

We should stress that the appearance of the same Dynkin functions in different parts of mathematics has been a great stimulus to look for corresponding relations. As examples, let us mention here questions in singularity theory [**22, 23, 37, 66**] and the study of ideals in Lie theory [**26, 27, 75, 103**].

1.3. The exponents. There is a unified, but quite mysterious way to deal with some of the Dynkin functions, namely to invoke the so-called exponents. If $\Delta = \Delta_n$ is a root system of rank n, there is attached a sequence $\epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_n$ of positive integers, the *exponents.*

1.3.1. The relevance of the exponents is revealed by the classical definitions:

- The **eigenvalues of a Coxeter transformation** c are of the form ζ^{ϵ_i}, where ζ is a primitive h-th root of unity (here, h is the Coxeter number, this is the order of c), and $1 \le i \le n$.
- The degrees of a basic set of **invariants** of W acting on V are $\epsilon_1+1, \dots, \epsilon_n+1$.
- The degrees of a basic set of **H-derivations** (see Section 1.5) are $\epsilon_1, \dots, \epsilon_n$.

Some properties of the sequence of the exponents $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$:

- These are n positive integers.
- If ϵ_i is an exponent, also $h - \epsilon_i$ is an exponent (in particular, $\epsilon_i < h$ for all i).
- The exponents are usually pairwise different, the only exceptions are the cases $\mathbb{D}_{2n}$: here $2n-1$ is twice an exponent.
- Actually, the exponents are easy to remember for the series: for $\mathbb{A}_n$, these are just the first n numbers $1, 2, \dots, n$; for $\mathbb{B}_n$ and $\mathbb{C}_n$, take the first n odd numbers: $1, 3, 5, \dots, 2n-1$, finally for $\mathbb{D}_n$, take the first $n-1$ odd numbers as well as $n-1$ itself, thus $1, 3, 5, \dots, 2n-3, n-1$. (As we have mentioned above, often the cases $\mathbb{D}_n$ turn out to be more complicated than the cases $\mathbb{A}_n$ and $\mathbb{B}_n$).
- *If $p < h$ is a prime number with $(p, h) = 1$, then p is an exponent* (see for example [**55**], 3.20). This result, together with the facts that the number of exponents is n, that 1 is an exponent, and that with ϵ also $h - \epsilon$ is an exponent, determines the exponents uniquely: For $\mathbb{E}_7$, one needs precisely one exponent which is different from 1 and not a prime, this can be only 9; for $\mathbb{E}_6$ one needs precisely two exponents which are different from 1 and not primes, these can be only the numbers 4 and 8 (altogether we see that the exponents for the exceptional cases are prime powers).

1.3.2. **Formulas, using the exponents.**

$$
\begin{aligned}
&(1) & h &= \epsilon_n + 1\\
&(2) & \mathbf{d}(\Delta_n) = x_1 \cdots x_n x_{\rm th} &= \prod \tfrac{\epsilon_i+1}{i}\\
&(3) & \mathbf{c}(\Delta_n) &= \tfrac{1}{|W|} n! h^n = \tfrac{h^n}{x_1\cdots x_n x_{\rm th}} = h^n \prod \tfrac{i}{\epsilon_i+1}\\
&(4) & |W| &= \textstyle\prod(\epsilon_i+1)\\
&(4') & \textstyle\sum_{i=1}^n |W_i| t^i &= \textstyle\prod(1+\epsilon_i t)\\
&(5) & |\Phi_+| &= \tfrac{1}{2} nh = \textstyle\sum \epsilon_i,\\
&(6) & \operatorname{sinc}(\Delta_n) &= nh \textstyle\prod_{i\geq 2} \tfrac{\epsilon_i - 1}{\epsilon_i+1}\\
&(7) & \mathbf{a}(\Delta_n) &= \tfrac{1}{|W|}\textstyle\prod(h+\epsilon_i+1) = \prod \tfrac{h+\epsilon_i+1}{\epsilon_i+1}\\
&(8) & \mathbf{t}_n(\Delta_n)) &= \tfrac{1}{|W|}\textstyle\prod(h+\epsilon_i-1) = \prod \tfrac{h+\epsilon_i-1}{\epsilon_i+1}
\end{aligned}
$$

It is not completely clear to us, who first found these formulas. Formula $(4')$ may have been the first one obtained, it is due to Shephard-Todd [**97**], 1954. Of course, formula (4) is a special case of $(4')$, namely $t = 1$. Some of the other formulas seem to be due to Chapoton.

The formulas for $\mathbf{a}(\Delta_n)$ and $\mathbf{t}_n(\Delta_n)$ are taken from Fomin and Zelevinsky [**43**], 2003. They showed that $\mathbf{a}_n(\Delta)$ is the number of clusters for a cluster algebra of type Δ_n. They write: *we are grateful to Frédéric Chapoton who observed that these expressions, which we obtained on a case by case basis, can be replaced by the unifying formula* $a(\Delta_n) = \prod \frac{h+\epsilon_i+1}{\epsilon_i+1}$. *F. Chapoton brought to our attention that these numbers appear in the study of non-crossing partitions by V. Reiner, C. Athanasiadis and A. Postnikov.* And they add: *The appearance of the exponents in the formula* $a(\Delta_n) = \prod \frac{h+\epsilon_i+1}{\epsilon_i+1}$ *for the number of clusters is a mystery for us at the moment. To add to this mystery, a similar expression can be given for the number of positive clusters,* namely $a'_n(\Delta_n) = \prod \frac{h+\epsilon_i-1}{\epsilon_i+1}$. It seems that this mystery has not been resolved until now. So it is a challenge for the readers.

1.4. The height partition. It seems to be worthwhile to gather as much information as possible on the exponents, in particular about the relationship between the exponents and the positive roots. We saw already:

- the number of exponents is the number of simple roots.
- the sum of the exponents is just the number of positive roots.

But there is a more intense interrelation between the positive roots and the exponents: the exponents may seem to be mysterious, but they are easily obtained from the root poset! Akyildiz and Carrell wrote a paper [**3**] with the title: *Betti numbers of smooth Schubert varieties and the remarkable formula of Kostant, Macdonald, Shapiro, and Steinberg.* The result in question has been presented by Humphreys in his book on Reflection Groups, we will recall it next. The first published proof is by Kostant (1959), with a reference to unpublished investigations of Shapiro, further proofs were given by Macdonald (1972) and Steinberg.

1.4.1. We say that a sequence of numbers $r_1, r_2, \dots, r_t$ is a *Young partition* provided $r_1 \geq r_2 \geq \cdots r_t \geq 0$ (see the note N 1.10). Given a Young partition $r = (r_1, r_2, \dots, r_t)$ its *dual* partition $r' = (r'_1, r'_2, \dots)$ is defined as follows: r'_j is the number of indices i with $r_i \geq j$.

THEOREM 1.4.1.1. **(Shapiro, Kostant).** *Let r_t be the number of roots of height t. Then $r = (r_1, r_2, \dots)$ is a Young partition, called the height partition of Φ_+. The dual partition of the height partition of Φ_+ is the Young partition of the exponents.*

As a consequence, we see

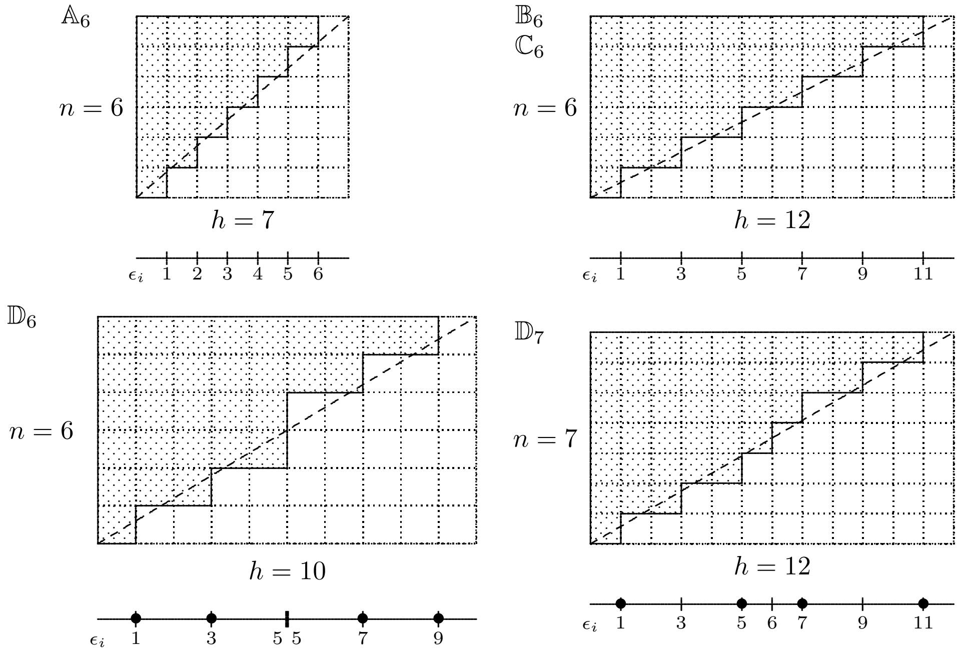
$$\boxed{r_t + r_{h+1-t} = n.}$$

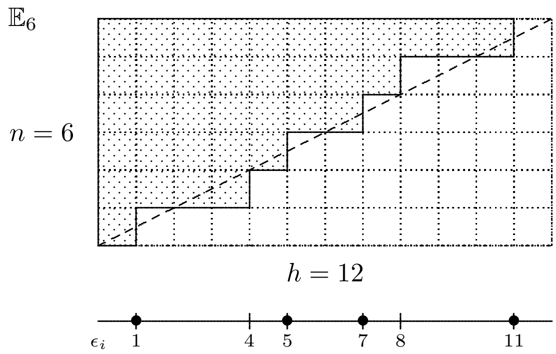
This follows directly from the well-known symmetry condition for the exponents: $\epsilon_t + \epsilon_{n-t+1} = h$.

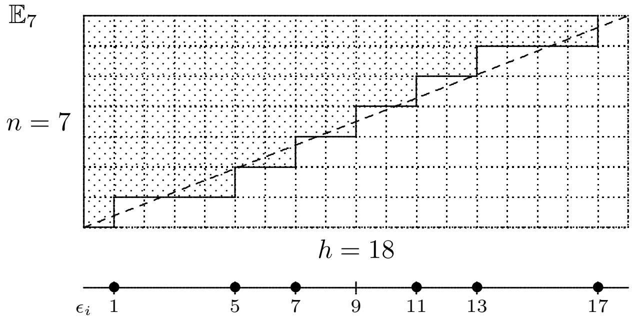
For example: $r_1 = n,\ r_h = 0$. Next, for a connected root system: $r_2 = n - 1,\ r_{h-1} = 1$ (this means that there is a unique highest root).

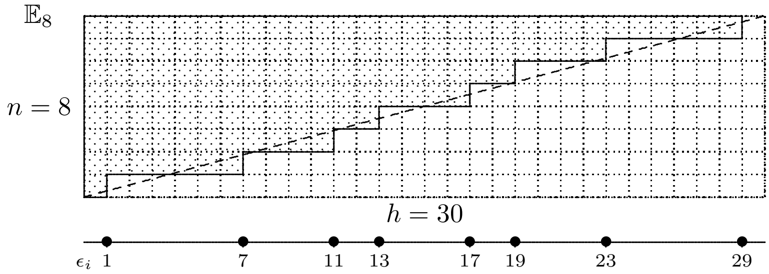
For a similar result concerning the roots of fixed length, see the note N 1.11. .

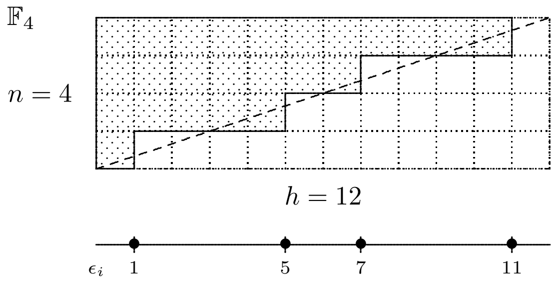
Let us draw the corresponding Young diagrams. Actually, we will draw the Young diagram $Y = Y(\epsilon)$ of the partition $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ of the exponents, using shaded boxes. Thus as the dual partition we see the height partition $(r_1, r_2, \dots)$. We will draw the partition of exponents inside a rectangle R with n rows and h columns (thus, in R, there are altogether nh square boxes), and we may consider the square boxes in $R \setminus Y$ as corresponding bijectively to the negative roots.

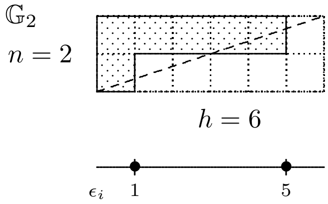



$\mathbb{E}_6$
$n = 6$
$h = 12$
ϵ_i 1 4 5 7 8 11


$\mathbb{E}_7$
$n = 7$
$h = 18$
ϵ_i 1 5 7 9 11 13 17


$\mathbb{E}_8$
$n = 8$
$h = 30$
ϵ_i 1 7 11 13 17 19 23 29


$\mathbb{F}_4$
$n = 4$
$h = 12$
ϵ_i 1 5 7 11

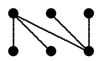

$\mathbb{G}_2$
$n = 2$
$h = 6$
ϵ_i 1 5

1.4.2. **Solid subchains of Φ_+.** Let us have a detailed look at the poset Φ_+. Given a poset P, we call a subposet P' a *solid* subposet provided neighbors in P' are neighbors in P (if $x < y$ are neighbors in the subposet P', this interval cannot be refined in P).

Proposition 1.4.2.1. *Any root poset Φ_+ is the disjoint union of solid subchains which contain a minimal element of Φ_+.*

This strengthens the assertion that the sequence $r_1, r_2, \dots, r_t$ is a Young partition. Namely, write Φ_+ as the disjoint union of solid subchains C_i such that any C_i contains a minimal element of Φ_+. The minimal elements of Φ_+ are just the simple roots, thus the number of subchains C_i is r_1. It follows that r_j is the number of the subchains C_i which have length at least j. It follows that $r_1 \geq r_2 \geq r_3 \geq \cdots$.

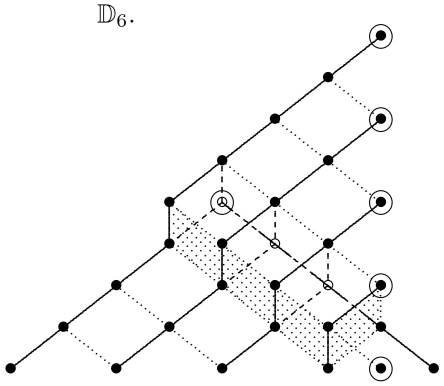
Note that the Young partition property does not imply the solid subchain property. The first example of a connected poset with the Young partition property but without the solid subchain property is as follows:



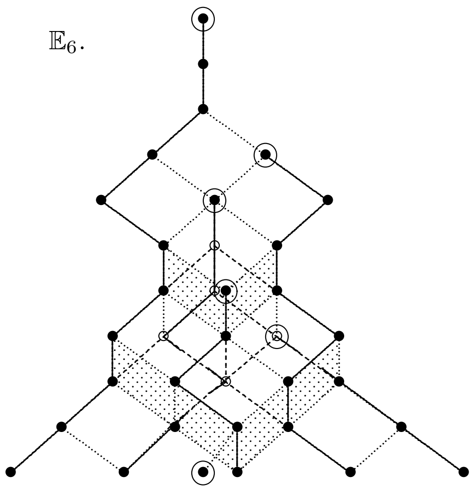
The assertion of the proposition is obvious for the cases $\mathbb{A}_n$ and $\mathbb{B}_n$.

Below we show solid subchains in the cases $\mathbb{D}_6$ and $\mathbb{E}_6$, drawn by solid lines. The largest elements of the solid subchains are always encircled (thus these are suitable roots of height ϵ_i, where $(\epsilon_1, \dots, \epsilon_n)$ is the exponent partition). The case $\mathbb{D}_6$ indicates a general rule how to obtain a solid subchain decomposition in the cases $\mathbb{D}_n$ in general. To find such a decomposition for the cases $\mathbb{E}_7, \mathbb{E}_8$ is more challenging.

$\mathbb{D}_6$.

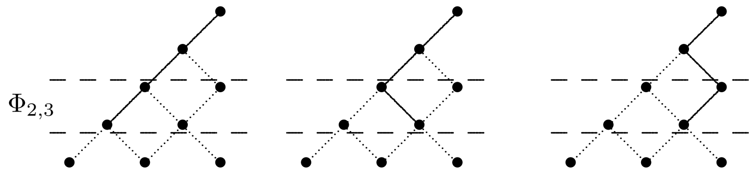


$\mathbb{E}_6$.

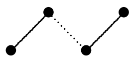


Actually, the existence of a solid subchain decomposition concerns a local property, namely it concerns the bipartite subposets $\Phi_{t-1,t}$ of all roots of height $t-1$ and t: As we know, we have $|\Phi_t| \leq |\Phi_{t-1}|$ (this just means that the height partition is a Young partition). The assertion is that there is a **matching** which involves all the vertices of Φ_t.

In particular, if we want to construct a solid subchain decomposition, we should start at the top of the poset Φ_+ and go down. If the subchains have reached the layer Φ_t, we have to look at $\Phi_{t-1,t}$ and we have to continue the path downwards inside a matching. For example, in case $\mathbb{B}_3$, starting with the maximal element z, the next two choices for a solid chain containing z are arbitrary, but then in $\Phi_{2,3}$ we have to be careful:



The choice in the middle does not work, since $\Phi_{2,3}$ has just one matching namely:



1.5. Inductive determination of the exponents. We have seen that there is a unified, but quite mysterious way to deal with some of the Dynkin functions: to invoke the exponents of Δ. Usually, the exponents seem to fall from heaven: either by looking at the invariant theory of the action of the Weyl group on the ambient space of the root system (Chevalley 1955), or by dealing with the eigenvalues of a Coxeter element (Coxeter, 1951). As we have mentioned, Shapiro and Kostant (1959) and later also Macdonald (1972) have shown that there is a third possibility to obtain the exponents, namely as the conjugate partition of the height partition of the root poset. It is of interest that in this way one may determine the exponents inductively, going up step by step in a chain of poset ideals I of Φ_+. A recent result of Sommers-Tymoczko [**104**] (and Abe-Barakat-Cuntz-Hoge-Terao [**1**]) based on old investigations of Arnold and Saito (1979) asserts that the set $\mathbf{H} = \mathbf{H}(I)$ of hyperplanes orthogonal to the roots in I is a so-called free hyperplane arrangement. This means that the corresponding module $D(\mathbf{H})$ of $\mathbf{H}$-derivations is free, thus one may consider the degrees of a free homogeneous generating system of $D(\mathbf{H})$. We obtain in this way an increasing sequence of Young partitions which terminates in the partition of the exponents.

1.5.1. **Hyperplane arrangements.** We consider a finite-dimensional vector space V over the field of real numbers, say of dimension n. A finite set $\mathbf{H}$ of (pairwise different) subspaces of dimension $n-1$ will be called a *hyperplane n-arrangement,* or just an arrangement. As a general reference for hyperplane arrangements, we refer to the book [**74**] by Orlik and Terao, see also the note N 1.12.

An element $g \in \mathrm{GL}(V)$ is called a *reflection* provided g is an involution (this means $g^2 = 1 \neq g$) and fixes pointwise a hyperplane. A finite subgroup G of $\mathrm{GL}(V)$ generated by reflections is called a (real) *reflection group.*

A hyperplane arrangement in V is called a *Coxeter arrangement* if the hyperplanes are those given by the reflections in a (real) reflection group $G \subseteq \mathrm{GL}(V)$. The *Weyl arrangements* are those given by a Weyl group, thus by a finite root system. Weyl arrangements are of course Coxeter arrangements.

1.5.2. **The module $D(\mathbf{H})$ of H-derivations.** Let $S = \mathbb{R}[V]$ be the ring of regular functions $V \to \mathbb{R}$, this is the symmetric algebra of $V^* = \mathrm{Hom}(V, \mathbb{R})$. If we choose $x_1, \dots, x_n$ in V^*, then S can be identified with the ring $\mathbb{R}[x_1, \dots, x_n]$ of polynomials in the variables $x_1, \dots, x_n$. We always consider S as a $\mathbb{Z}$-graded ring with all variables having degree 1.

Instead of looking at a hyperplane H, we may look at non-zero linear polynomials $\alpha = \alpha_H$ (this is an element of S of degree 1); the corresponding hyperplane is just the kernel of α_H). Note that α_H is determined by H only up to a non-zero scalar (but in the following, this usually will not matter).

Recall the definition of a *derivation* $\theta\colon S \to M$, where M is an S-S-bimodule: it is an $\mathbb{R}$-linear map such that $\theta(fg) = \theta(f)g + f\theta(g)$. The set $\mathrm{Der}(S)$ of derivations $S \to S$ is an S-module, using the following operations: Let θ, θ' belong to $\mathrm{Der}(S)$, and $h \in S$. Then $\theta + \theta'$ is defined by $(\theta + \theta')(s) = \theta(s) + \theta'(s)$; and $h\theta$ is defined by $(h\theta)(f) = h \cdot \theta(f)$. Actually, the following is true (and easy to verify): *The set* $\mathrm{Der}(S)$ *of derivations* $S \to S$ *is a free* S*-module with basis* $D_i = \partial/\partial x_i$ (where we have chosen a basis $e_1, \dots, e_n$ of V). A derivation of the form $\theta = \sum f_i D_i$ with homogeneous polynomials f_i of fixed degree p is said to be *homogeneous of (polynomial) degree* $p = \mathrm{pdeg}\,\theta$. Of course, any element of $\mathrm{Der}(S)$ can be written as a sum of homogeneous polynomials. For $v \in V$, let D_v be the derivation defined by

$$D_v(\alpha) = \alpha(v) \quad \text{for} \quad \alpha \in S_1$$

(and extended to all of S by the derivation rule). In particular, $D_i = D_{e_i}$, where $e_1, \dots, e_n$ is the chosen basis of V. Of course, the map

$$S \otimes_{\mathbb{R}} V \to \mathrm{Der}(S) \quad \text{defined by } s \otimes v \mapsto sD_v$$

for $s \in S$ and $v \in V$ is an isomorphism of graded S-modules.

A typical example of a derivation $S \to S$ is the *Euler derivation* $\theta_E = \sum x_i D_i$; its polynomial degree is 1 and it has the following important property: If $f \in S$ has degree t, then $\theta_E(f) = t \cdot f$, in particular, if α is linear, then $\theta_E(\alpha) = \alpha$.

Let $\mathbf{H}$ be a hyperplane arrangement. The *module* $D(\mathbf{H})$ *of* $\mathbf{H}$*-derivations* is the set of all derivations θ of S such that $\theta(\alpha_H) \subseteq \alpha_H S$ for all $H \in \mathbf{H}$, note that this is an S-submodule of $\mathrm{Der}(S)$. For example, θ_E always belongs to $D(\mathbf{H})$.

1.5.3. **Free arrangements and their exponents.** An arrangement $\mathbf{H}$ is said to be *free* provided the S-module $D(\mathbf{H})$ is free. If $D(\mathbf{H})$ is free, then there is a free set of homogeneous generators and *the weakly increasing sequence of degrees of a minimal set of homogeneous generators is uniquely determined* (see [**74**], 4.18). We call a free set of homogeneous generators of $D(\mathbf{H})$ a *basic set of* $\mathbf{H}$*-derivations* and the weakly increasing sequence of degrees of the set the *exponent partition for* $\mathbf{H}$.

Let $\mathbf{H}$ be a hyperplane arrangement and H an element of $\mathbf{H}$. If both $\mathbf{H}$ and $\mathbf{H}' = \mathbf{H} \setminus \{H\}$ are free arrangements, then there are basic sets for $\mathbf{H}$ and $\mathbf{H}'$ which are nicely related to each other, namely: *There is a basic set* $\{\theta_1, \dots, \theta_l\}$ *of* $\mathbf{H}'$*-derivations and an index* i *such that*

$$\{\theta_1, \dots, \theta_{i-1}, \alpha_H\theta_i, \theta_{i+1} \dots, \theta_l\}$$

is a basic set of $\mathbf{H}$*-derivations* (see [**74**], 4.46).

We call an ordering $\{H_1, \dots, H_m\}$ of the elements of $\mathbf{H}$ a *free ordering* provided all the subarrangements $\{H_1, \dots, H_t\}$ are free for $1 \le t \le m$.

1.5.4. **Ideal subarrangements of Weyl arrangements.** An *ideal subarrangement* is given by the hyperplanes H_x, where x belongs to some fixed ideal I in a root poset Φ_+. If I is an ideal in a root poset Φ_+, let $r_I(t)$ be the number of roots in I of height t. We claim that r_I *is a Young partition.* This follows directly from the fact that the root poset can be covered by n solid subchains, see Section 1.4.

THEOREM 1.5.4.1. **(Sommers-Tymoczko).** *Every ideal of a root poset yields a free hyperplane arrangement whose exponent partition is the dual partition of the height partition* r_I.

The proof of Sommers-Tymoczko [**104**] covered all cases but $\mathbb{F}_4, \mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$; the missing cases were verified by Barakat using a computer. A unified proof was later given by Abe-Barakat-Cuntz-Hoge-Terao [**1**].

COROLLARY 1.5.4.2. *Let* $\mathbf{H}$ *be a Weyl arrangement. Any total ordering which refines the partial ordering of* Φ_+ *is a free ordering.*

1.5.5. **Basic sets of invariants, of differential 1-forms, of derivations.** We assume that G is a (real) reflection group and $\mathbf{H}$ the corresponding hyperplane arrangement. It remains to be seen that the exponents for $\mathbf{H}$ are just the exponents for G (and we will see again that $\mathbf{H}$ is free). We outline here only the main steps. We start with the main theorem of the invariant theory of reflection groups:

THEOREM 1.5.5.1. **(Chevalley, 1955)**. *Let* $G \subseteq \mathrm{GL}(V)$ *be a reflection group, where* V *is an* n*-dimensional vector space. Let* S *be the ring of regular functions on* V *and* $R = S^G$. *Then there are homogeneous polynomials* $f_1, \ldots, f_n$ *which generate* S^G. *And there exists a finite-dimensional* G*-stable graded subspace* U *of* S *such that* $S = R \otimes_{\mathbb{R}} U$ *and the* G*-module* U *is the regular representation.*

The polynomials $f_1, \ldots, f_n$ which generate S^G have to be algebraically independent, thus R is isomorphic to a polynomial ring with free generators $f_1, \ldots, f_n$. We call $f_1, \ldots, f_n$ a *basic set of invariants* and the weakly increasing sequence of degrees of such a set the sequence of *degrees for* G.

PROOF. Bourbaki [**18**] V,5.3. Theorèmes 1, 2. □

The group G operates on $\mathrm{Der}(S)$ as follows

$$(g\theta)(v) = g\theta(g^{-1}v).$$

Let $\mathrm{Der}(S)^G$ be the set of G-invariant derivations of S, thus the set of derivations θ of S with $g\theta = \theta$.

(1) *The graded* S*-modules* $S \otimes_R \mathrm{Der}(S)^G$ *and* $D(\mathbf{H})$ *are isomorphic.*

Proof: It is shown in [**74**], 6.59 that $\mathrm{Der}(S)^G \subseteq D(\mathbf{H})$, this yields a map

$$S \otimes_R \mathrm{Der}(S)^G \to D(\mathbf{H}).$$

One uses the tensor factorization $S = U \otimes R$ in order to show that the map is an embedding. Further calculations in [**74**] (p.237/8) show the surjectivity.

(2) *The* R*-module* $\mathrm{Der}(S)^G$ *is a free* R*-module of rank* n.

For the proof see [**74**], 6.48. We note that the (bijective) map $S \otimes V \to \mathrm{Der}(S)$ given by $s \otimes v \mapsto sD_v$ is G-equivariant, thus $\mathrm{Der}(S)^G \simeq (S \otimes V)^G$, therefore

$$(S \otimes_{\mathbb{R}} V)^G = (R \otimes_{\mathbb{R}} U \otimes_{\mathbb{R}} V)^G = R \otimes_{\mathbb{R}} (U \otimes_{\mathbb{R}} V)^G$$

shows that $(S \otimes_{\mathbb{R}} V)^G$ is a free R-module. Since U is the regular representation of G, the dimension of the space $(U \otimes_{\mathbb{R}} V)^G$ is equal to the dimension of V.

Let $\Omega(S) = \bigoplus S\mathrm{d}x_i$ be the S-module of all differential 1-forms, this is the free S-module with basis $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ (note that $\Omega(S)$ is often denoted by $\Omega^1[V]$; for the general setting, see [**74**], Section 3.5 and Section 4.6, p.123). As usual, given $f \in S$, we write $\mathrm{d}f = \sum D_i(f)\mathrm{d}x_i$, and obtain in this way a derivation $\mathrm{d}\colon S \to \Omega(S)$. The map

$$S \otimes_{\mathbb{R}} V^* \to \Omega(S) \quad \text{defined by } s \otimes \alpha \mapsto s\mathrm{d}\alpha$$

for $s \in S$ and $\alpha \in V^*$ is an isomorphism of graded S-modules. Using this identification, we see that we may consider the elements $\omega \in \Omega(S)$ as maps $\mathrm{Der}(S) \to S$.

The group G operates on $\Omega(S)$ as follows

$$(g\omega)(\theta) = g\omega(g^{-1}\theta),$$

where $\omega \in \Omega(S)$ and $\theta \in \mathrm{Der}(S)$. Let $\Omega(S)^G$ be the set of G-invariant differential 1-forms, thus the set of all ω in $\Omega(S)$ with $g\omega = \omega$.

(3) *The R-modules $\Omega(S)^G$ and $\mathrm{Der}(S)^G$ are isomorphic.*

In order to prove (3), we first note: *For a (real) reflection group $G \subseteq \mathrm{GL}(V)$, the G-modules V and V^* are isomorphic,* therefore also the G-modules $S \otimes V$ and $S \otimes V^*$ are isomorphic. It follows that the R-modules $(S \otimes V)^G$ and $(S \otimes V^*)^G$ are isomorphic.

The map $S \otimes V \to \mathrm{Der}(S)$ defined by $s \otimes v \mapsto s \cdot D_v$ and $S \otimes V^* \to \Omega(S)$ defined by $s \otimes v \mapsto s \cdot \mathrm{d}v$ are bijective and G-equivariant, thus we get isomorphisms of R-modules

$$(S \otimes V)^G \to \mathrm{Der}(S)^G, \quad (S \otimes V^*)^G \to \Omega(S)^G.$$

The assertions (1) and (3) provide isomorphisms of graded S-modules:

$$\boxed{D(\mathbf{H}) \underset{(1)}{\simeq} S \otimes_R \mathrm{Der}(S)^G \underset{(3)}{\simeq} S \otimes_R \Omega(S)^G}$$

and (2) asserts that the module in the middle is free, thus $D(\mathbf{H})$ is a free S-module.

(4) (Solomon 1964). *If $f_1, \dots, f_n$ is a basic set of invariants for G, then $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ is basic set for the R-module $\Omega(S)^G$*, thus a basic set for the S-module $S \otimes_R \Omega(S)^G$.

COROLLARY 1.5.5.2. **(V.I.Arnold, K.Saito, 1979).** *Let G be a reflection group and $\mathbf{H}$ the corresponding reflection arrangement. Then $\mathbf{H}$ is a free arrangement. Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degrees for G, and let $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n$ be the exponents for the corresponding hyperplane arrangement $\mathbf{H}$. Then $\epsilon_i = d_i - 1$ for $1 \leq i \leq n$.*

Remark. Solomon's result explains nicely why the degrees and the exponents for a (real) reflection group differ by 1.

Notes to Chapter 1.

N 1.1. Root systems, root bases, and the Dynkin diagrams. We start with a finite-dimensional Euclidean vector space V, say of dimension n. Given a non-zero element $\alpha \in V$, we may consider the hyperplane H_α orthogonal to α and we denote by $\rho_\alpha \colon V \to V$ the reflection with respect to H_α: it is the identity map on H_α and it sends α to $-\alpha$. Note that if $\alpha \neq 0$ (so that ρ_α is defined), the difference vector $\rho_\alpha(\beta) - \beta$ is a multiple of α, for any $\beta \in V$.

A *root system* Φ in V is a finite set of non-zero elements of V (the *roots*) such that the following conditions are satisfied: The elements of Φ generate V. If α, β belong to V and generate the same real subspace, then $\beta = \pm\alpha$. Finally (and this is the decisive condition): If α, β belong to Φ, then $\rho_\alpha(\beta)$ belongs again to Φ and the difference vector $\rho_\alpha(\beta) - \beta$ is an integral multiple of α.

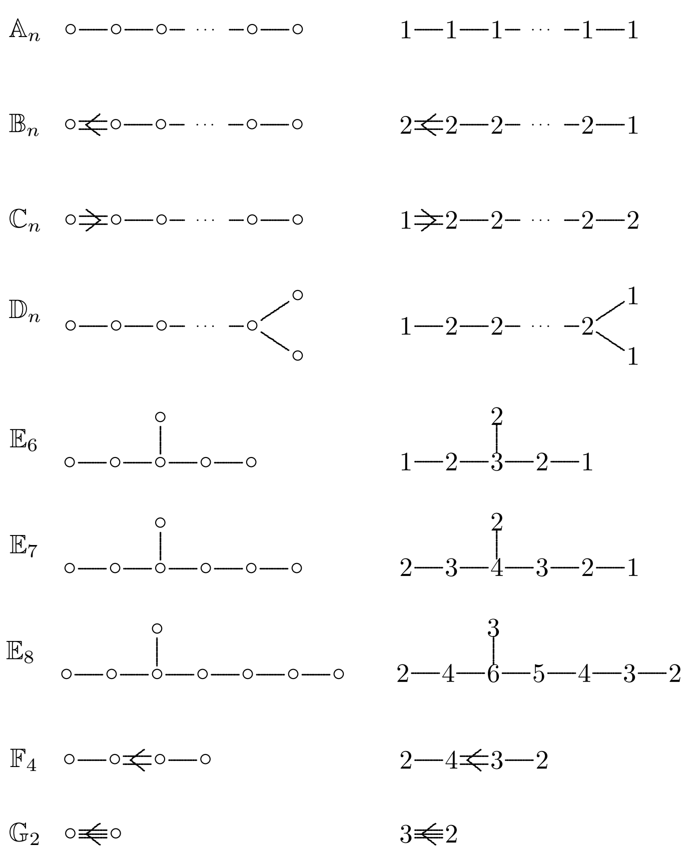
By the very definition, we see that root systems are finite subsets of Euclidean vector spaces with strong symmetry conditions. The subgroup of $\mathrm{GL}(V)$ generated by the reflections ρ_α with $\alpha \in \Phi$ is called the *Weyl group* for Φ. It consists of orthogonal transformations of V which map Φ into itself.

Root system play an important role in many parts of mathematics. In particular, they have been used to obtain a structure theory for finite-dimensional semisimple complex Lie-algebras. Thus, any book which introduces such Lie algebras (for example [**54**]) provides a lot of information about root systems.

Of importance is the following theorem: *Given a root system Φ in V, there is a basis Δ of V which consists of elements on Φ, such that any element of Φ is a linear combination of the elements of Δ using integral coefficients which are either all only non-negative or all non-positive.* Such a basis Δ is called a *root basis* of Φ, it is unique up to an automorphism of V given by an element of the Weyl group W. Given a root basis Δ, the elements of Φ which are linear combinations of the elements of Δ with non-negative coefficients are said to be the *positive* roots, the remaining ones the *negative* roots.

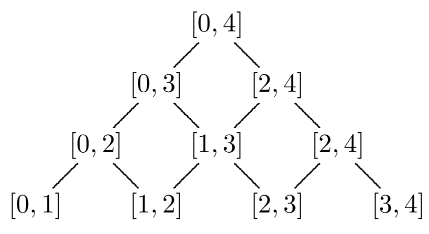
Given a root basis Δ, the possible angles between its elements are very restricted: Either the roots are orthogonal, or else the angle is $120^0, 135^0$ or 150^0. One uses the set Δ as the vertices of a graph, connecting two vertices of Δ by an edge provided they are not orthogonal. If the angle between $\alpha, \beta \in \Delta$ is 135^0 or 150^0, then these vectors must have different length: in case of 135^0 one draws not a single edge, but a double edge $\circ\!\Leftarrow\!\circ$, decorated in the middle by an arrow head pointing to the shorter root. Similarly, in case of 150^0 one draws a triple edge $\circ\!\Lleftarrow\!\circ$, such that the arrow head again points to the shorter root. One obtains in this way the so-called *Dynkin diagram* $\Delta(\Phi)$ of Φ. We exhibit below all the connected Dynkin diagrams which arise in this way.

The connected Dynkin diagrams (left) and the highest roots (right).



Recall that the vertices of a Dynkin diagram $\Delta = \Delta(\Phi)$ are the elements of a root basis of Φ. If we attach to these vertices α numbers c_α (as we do it on the right), we obtain as $\sum_\alpha c_\alpha \alpha$ an element of V. The *highest root* displayed on the right is an element of Φ which is uniquely determined by the property that all the coefficients c_α are maximal.

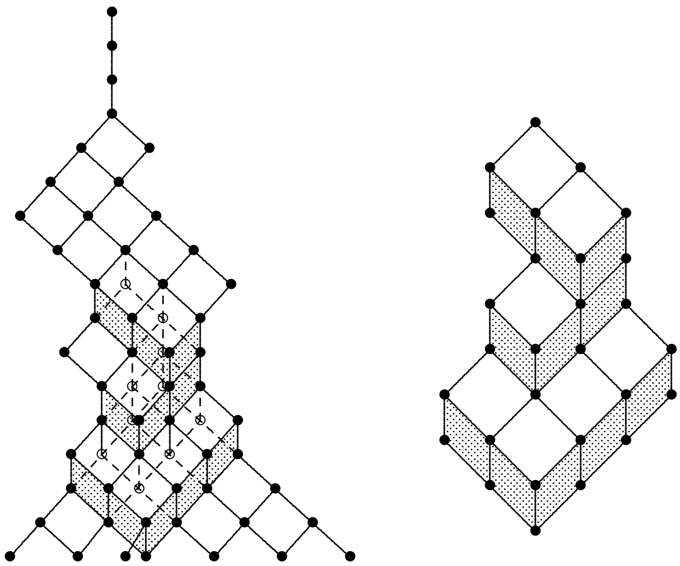
N 1.2. The root system $\mathbb{A}_n$. The root poset $\Phi_+(\mathbb{A}_n)$ may be identified with the set of intervals $[i, j]$ where $0 \le i < j \le n$ are integers, using as partial ordering the (set-theoretical) inclusion of intervals. For example, here is the Hasse diagram of the interval poset for $n = 4$.



A usual realization of the root system of type $\mathbb{A}_n$ (see for example Humphreys [**54**]) is to start with the Euclidean vector space $\mathbb{R}^{n+1}$ with orthonormal basis

$e_0, e_1, \ldots, e_n$. Let V be the subspace of $\mathbb{R}^{n+1}$ orthogonal to the vector $\sum_i e_i$. The vectors in V with integer coefficients and of length $\sqrt{2}$ are just the vectors of the form $e_i - e_j$ with $i \neq j$. The set of these vectors is a root system Φ and the vectors $e_{i-1} - e_i$ with $1 \le i \le n$ form a root basis. One obtains a bijection between the interval poset and Φ_+ by sending the interval $[i, j]$ with $0 \le i < j \le n$ to the vector $e_i - e_j$.

N 1.3. The root posets as 2-dimensional projections of 3-dimensional objects. The root posets may be considered as projections of some 3-dimensional objects formed by cubes, squares and intervals. For example, the root poset $\Phi_+(\mathbb{E}_7)$ shown below on the left may be thought of being obtained from the constellation of ten cubes shown on the right by adding 1- and 2-dimensional pieces (squares and intervals), thus, on the right, we see its 3-dimensional "core":



If P is a finite poset, we may call an interval $[x, y]$ in P a *cuboid* provided it is the product of three proper chains, say $C_1 \times C_2 \times C_3$; its volume is $c_1c_2c_2$, where $c_i + 1$ is the number of vertices of the chain C_i. A *cube* in P is a cuboid of volume 1.

For the Dynkin graphs Δ of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$, Beineke [**14**] has observed that the map

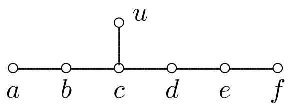
$$[x, y] \mapsto x + y$$

furnishes a bijection between the cuboids in the root poset and the non-thin positive roots, as well as a bijection between the cubes and the non-thin positive roots with precisely 3 odd coefficients (a root is said to be *thin* provided its coefficients are bounded by 1).

We may use the cubes in Φ_+ in oder to describe different levels inside the poset. The root posets of type $\mathbb{A}_n, \mathbb{B}_n, \mathbb{G}_2$ have no cubes, thus there is just one level. Those of type $\mathbb{D}_n$ and $\mathbb{F}_4$ have two levels, the number of levels for $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ is 3, 4 and 6, respectively. It should be mentioned that the set of roots belonging to a fixed level can be described by inequalities concerning the coefficients, and for the higher levels, these sets are always intervals. For example, in case $\mathbb{E}_7$, here is the description of the levels:

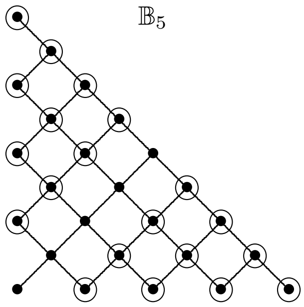
Level	Conditions	Minimal elements	Maximal element	number of roots
1	$u = 0$	6 simple roots	$\begin{smallmatrix} & & 0 & & & \\ 1&1&1&1&1&1 \end{smallmatrix}$	21
2	$u = 1,\ c \leq 1$	$\begin{smallmatrix} & & 1 & & & \\ 0&0&0&0&0&0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & & \\ 1&1&1&1&1&1 \end{smallmatrix}$	13
3	$c = 2,\ d = 1$	$\begin{smallmatrix} & & 1 & & & \\ 0&1&2&1&0&0 \end{smallmatrix}$	$\begin{smallmatrix} & & 1 & & & \\ 1&2&2&1&1&1 \end{smallmatrix}$	9
4	$d \geq 2$	$\begin{smallmatrix} & & 1 & & & \\ 0&1&2&2&1&0 \end{smallmatrix}$	$\begin{smallmatrix} & & 2 & & & \\ 2&3&4&3&2&1 \end{smallmatrix}$	20

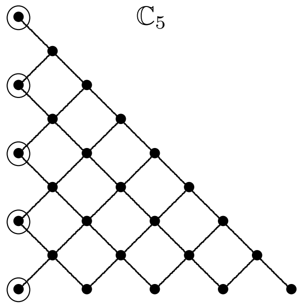
where we denote the coefficients as follows:

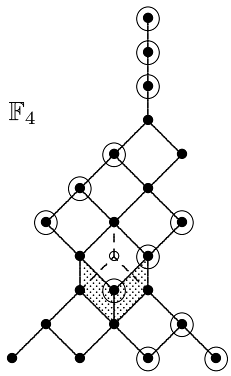


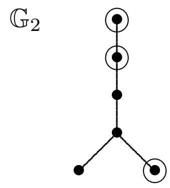
Let us add that the root posets of type $\mathbb{D}_n$ have precisely two levels, each level consists of $\binom{n}{2}$ roots: thus, here we obtain a separation of the positive roots into two classes of equal cardinality.

N 1.4. The root posets with the long roots being marked by circles.









Always, the lowest row consists of the simple roots. Here, the sequence of the simple roots is the same as in the pictures above showing the Dynkin diagrams.

N 1.5. Reflections in the Weyl group W. Here we refer to Humphreys [**55**], 1.2. It asserts: if g is an orthogonal transformation, then gs_xg^{-1} maps gx to $-gx$ and fixes the hyperplane H_{gx} orthogonal to x pointwise. This shows that for

$w \in W$, we have $s_{wx} = ws_xw^{-1}$, thus s_{wx} belongs to W. [**55**] 1.14 asserts that any reflection $w \in W$ is $w = s_x$ for some root x.

N 1.6. The quiver $Q(\Lambda)$ of a hereditary artin algebra. We may assume that Λ is connected, then Λ is a finite-dimensional k-algebra for some field k (namely for k the center of Λ).

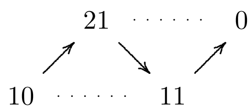
The vertices of $Q(\Lambda)$ are the isomorphism classes $[S]$ of the simple Λ-modules S and there is an arrow $[T] \to [S]$ provided $\operatorname{Ext}^1(T,S) \neq 0$. For S a simple module, let $\Gamma(S) = \operatorname{End}(S)^{\mathrm{op}}$. Let v be the product of $\dim_{\Gamma(T)} \operatorname{Ext}^1(T,S)$ and $\dim \operatorname{Ext}^1(T,S)_{\Gamma(S)}$. Taking into account the function $v\colon Q(\Lambda)_1 \to \mathbb{N}_1$, the quiver $Q(\Lambda)$ is called the *valued quiver* of Λ.

If we assume that Λ is representation-finite, then $v \le 3$ and we draw the edge between S and T as a double edge, in case $v = 2$, and as a triple edge, in case $v = 3$. In the case of a double or triple edge between the vertices $[S_1], [S_2]$, we endow it in the middle with an arrow pointing to $[S_1]$ provided $\dim_k \Gamma(S_1) < \dim_k \Gamma(S_2)$ (let us stress again that these middle arrows should not be confused with the arrows given by the orientation).

It is easy to construct hereditary artin algebras with simply-laced quivers, say of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$: just take the path algebra of a corresponding quiver. For constructing hereditary artin algebras with quiver of type $\mathbb{B}_n, \mathbb{C}_n$ and $\mathbb{F}_4$, we need a field extension $K : k$ of degree 2 (or, more generally, division rings $D_1 \subset D_2$ which are finite-dimensional k-algebras such that $\dim_k D_2 = 2\dim_k D_1$). For example, the matrix algebra $\left[\begin{smallmatrix} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ & \mathbb{C} & \mathbb{C} \\ & & \mathbb{C} \end{smallmatrix}\right]$ is a hereditary artin algebra of type $\mathbb{B}_3$, whereas $\left[\begin{smallmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ & \mathbb{R} & \mathbb{R} \\ & & \mathbb{R} \end{smallmatrix}\right]$ is a hereditary artin algebra of type $\mathbb{C}_3$. In order to realize $\mathbb{G}_2$, let $\Lambda = \left[\begin{smallmatrix} k & K \\ & K \end{smallmatrix}\right]$, where $K : k$ is a field extension of degree 3. In general, for dealing with non-simply-laced quivers, Gabriel has introduced the notion of a “species”: whereas for simply-laced quivers, it is sufficient to work with vertices, arrows and one field k, a species is given by a set of division rings (indexed by the vertices of a quiver) as well as bimodules (indexed by the arrows). For an outline of the representation theory of species, we may refer to several joint papers with Dlab.

N 1.7. Why $|k| > 2$? Here is an example to have in mind. Consider the 3-subspace quiver Q with sink 0, and its k-representation, where k is a field. Let $X = I(0)$ be the indecomposable injective module of length 4 and Y the maximal indecomposable module. Then $\mathbf{dim}\, X < \mathbf{dim}\, Y$. Now $|X|$ and $|Y|$ differ by 1, thus, if X is a subfactor of Y, it is a submodule or a factor module of Y. Since X is injective, it cannot be a submodule of Y. In order to analyze whether X is a factor module of Y, we consider $\operatorname{Hom}(Y,X)$, this is a 2-dimensional k-space. For any field k, there are 3 maps $Y \to X$ which are non-zero and not surjective, all other non-zero maps $Y \to X$ are surjective. Thus, if k is the field with two elements, then there is no surjective map $Y \to X$.

A second example of interest: Consider an artin algebra of type $\mathbb{B}_2$, say with Auslander-Reiten quiver



Let $X = 11$ and $Y = 21$. Then X is a factor module of Y, thus $X \sqsubseteq Y$. But since X is the injective envelope of 10 and the socle of 21 is the direct sum of two copies of 10, we see that Y is a submodule of the direct sum of two copies of 11. This shows that $Y \sqsubseteq X^2$.

N 1.8. Antichains in a poset, antichains in an additive category. Let us motivate the definition. We recall the following: Given a poset P, a chain in P is a subset of pairwise comparable elements, whereas an antichain in P is a subset of pairwise incomparable elements. Now consider the linearization kP of P, were k is a field: this is an additive k-category whose indecomposable objects are the elements of P such that $\mathrm{Hom}_{kP}(x,y) = k$ provided $x \le y$ in P and $\mathrm{Hom}_{kP}(x,y) = 0$ otherwise, such that the composition of maps in kP is given by the multiplication in k, and, finally, such that any object in kP is the direct sum of indecomposable objects. Of course, a subset A of P is an antichain in P if and only if A (considered as a set of objects in kP) consists of pairwise orthogonal bricks (thus, is an antichain in the additive category kp).

N 1.9. Thick subcategories of abelian categories. If $mathcalC$ is an abelian category, we say that a full subcategory $\mathcal{U}$ is a *thick* subcategory, provided it is closed under kernels, cokernels, and extensions. Such a thick subcategory is again an abelian category, and the embedding functor is exact. On the other hand, assume that $\mathcal{U}$ is a full subcategory which is an abelian subcategory. Then $\mathcal{U}$ is a thick subcategory of $\mathcal{C}$ if and only if the embedding functor is exact and $\mathcal{U}$ is closed under extensions. Thick subcategories had been called 'wide" by Hovey [**52**], the denomination "thick" seems to be due to Krause [**63**].

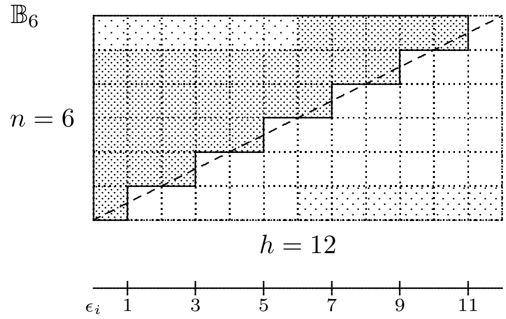
N 1.10. Young partition. In this survey, a sequence of numbers $(r_1, r_2, \dots, r_t)$ with $r_1 \ge r_2 \ge \cdots r_t \ge 0$ is called a *Young partition*. In algebra and number theory, such sequences are usually just called "partitions", but we need also the concept of a partition as used in set theory (see Chapter 4 and the Appendix): a *partition of a set* M is a set of disjoint non-empty subsets M_i of M such that $M = \bigcup M_i$. The non-crossing partitions in type $\mathbb{A}_n$ considered in Chapter 4 are just certain (set-theoretical) partitions of the set $\{1, 2, \dots, n\}$. Note that the word *partition* in the formulation *non-crossing partitions* refers to (set-theoretical) partitions. Of course, Young partitions may be considered as special set-theoretical partitions: namely, the Young partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ with $n = \sum \lambda_i$ may be considered as the partition of the set $M = \{1, 2, \dots, n\}$ with parts $M_j = \{x \in \mathbb{Z} \mid \sum_{i<j} \lambda_i < x \le \sum_{i \le j} \lambda_i\}$ for $1 \le j \le t$.

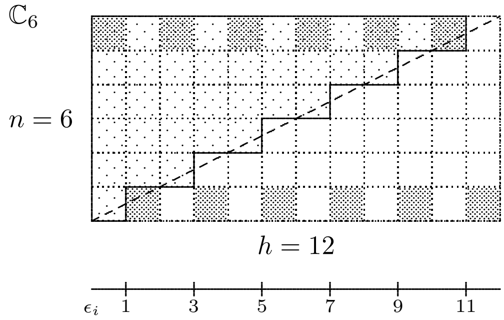
N 1.11. The short roots of fixed height. Let r^s_t be the number of short roots of height t. We should stress that this is usually not a Young partition. But still we have the following symmetry condition:

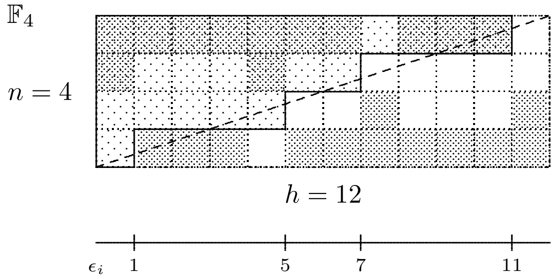
$$\boxed{r^s_t + r^s_{h+1-t} = r^s_1}$$

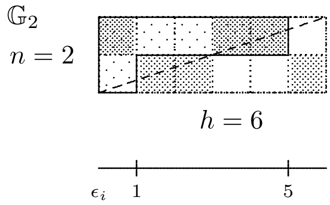
and r^s_1 is equal to $n-1$ in case $\mathbb{B}_n$, to 1 in case $\mathbb{C}_n$ and $\mathbb{G}_2$, and finally to 2 in case $\mathbb{F}_4$.

Here are the Young diagrams Y for the exponent partitions in the cases $\mathbb{B}_6, \mathbb{C}_6, \mathbb{F}_4, \mathbb{G}_2$, always inserted into a rectangle R with n rows and h columns. The boxes of the Young diagram correspond bijectively to the positive roots, we have shaded these boxes in two different ways: the darker shading marks long roots. Of course, the square boxes in $R \setminus Y$ may be interpreted as corresponding to the negative roots, here we use again a shading, but this time we shade only one kind of the roots: the short ones in case $\mathbb{B}_6$ and the long ones in the cases $\mathbb{C}_6, \mathbb{F}_4, \mathbb{G}_2$.









N 1.12. Hyperplane arrangements. A hyperplane arrangement is a finite set of pairwise different hyperplanes in a fixed vector space V. If V is of dimension n, one calls it an n-arrangement. Such a set is called *real* provided the base field is the field of real numbers and *central* provided all the hyperplanes are subspaces (that means: affine hyperplanes which contain the zero vector). In the lectures, all the hyperplane arrangements considered are real and central.

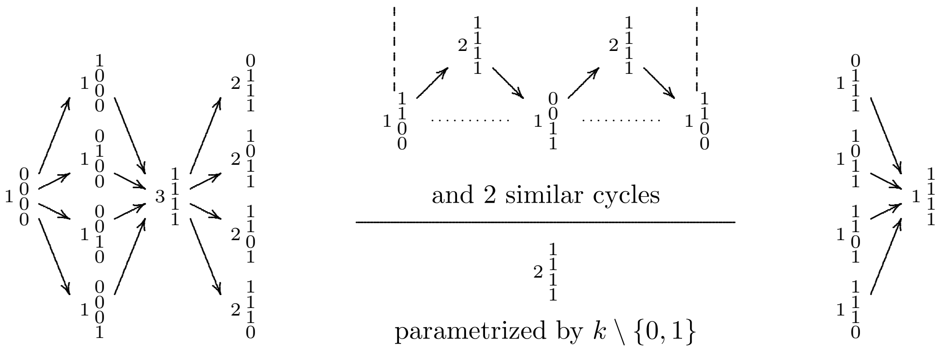
What is an "arrangement"? Just a fancy word for a finite set. Of course, any central hyperplane arrangement $\mathbf{H}$ may be considered as a representation of a quiver, namely of the t-subspace quiver, where t is the cardinality of $\mathbf{H}$, the discussion in Section 1.5 should also be seen as an attempt to propagate a method to deal with such quiver representations, which has been found useful outside of representation theory, namely to look at modules of derivations.

Let us repeat: Hyperplane n-arrangements of cardinality t are representations of the t-subspace quiver with dimension vector $(n; n-1, \dots, n-1)$. Using duality and reflections, they correspond bijectively to the representations with dimension vector $(n; 1, \dots, 1)$, thus to t-tuples of elements of the projective space $\mathbb{P}^{n-1}$. A hyperplane arrangement is called *irreducible* provided the corresponding representation is indecomposable.

Let k be an arbitrary base field. Let Q be the t-subspace quiver (it has $t+1$ vertices and t arrows; one vertex, say with label 0 is a sink, the remaining vertices $1, 2\,dots, t$ are sources; the arrows are of the form $\alpha_i : i \to 0$ with $1 \le i \le n$). A subspace representation of Q is a representation of Q which uses only inclusion maps (or at least monomorphisms). Let M be a subspace representation of Q such that all the vector spaces M_i are one-dimensional, for $1 \leq i \leq t$. Then M is a direct sum of indecomposable subspace representation N of Q such that all the vector spaces N_i with $1 \leq i \leq t$ are at most one-dimensional. Thus, it seems reasonable for us to look for a moment at the full subcategory $\mathcal{U}$ of $\operatorname{mod} kQ$ of direct sums of indecomposable subspace representation N of Q such that all the vector spaces N_i with $1 \leq i \leq t$ are at most one-dimensional. The indecomposable objects in $\mathcal{U}$ are of two different kinds: first of all, there is the simple representation $S(0)$ corresponding to the vertex 0; let us denote by $\emptyset_t$ the direct sum of t copies of $S(0)$ (it is denoted by Φ_t in [**74**]). The remaining indecomposable objects N in $\mathcal{U}$ have the property that the sum of the images of the maps α_i is the total space N_0. It is not difficult to see that *all the indecomposable objects in $\mathcal{U}$ have endomorphism ring k.* This is clear for the simple representation $S(0) = \emptyset_1$. We can assume that we deal with an indecomposable representation X with $\mathbf{dim}\, X = (n; 1, \dots, 1)$. Let $f \neq 0$ be an endomorphism with $f^2 = 0$. Then $\mathbf{dim}\, X = 2\,\mathbf{dim}\operatorname{Ker}(f) + \mathbf{dim}\operatorname{Im}(f)$. But then $\operatorname{Ker}(f)$ is of the form $\emptyset_t$ for some t, impossible. This shows that the endomorphism ring of X is a division ring. The restriction map $f \mapsto f|X_1$ is a ring homomorphism from $\operatorname{End}(X)$ onto $\operatorname{End}_k(X_1) = k$, thus $\operatorname{End}(X) = k$.

Altogether we see: *Any representation in $\mathcal{U}$ is the direct sum of a representation $\emptyset_t$, and pairwise non-isomorphic non-simple representations with endomorphism ring k.*

Example. Already the four subspace quiver shows that the category $\mathcal{U}$ may be quite complicated: in particular, let us stress that there may be non-isomorphic indecomposable representations X, X' in $\mathcal{U}$ with $\operatorname{Hom}(X, X') \neq 0$ and $\operatorname{Hom}(X', X) \neq 0$.



On the left we see the 10 indecomposable preprojective modules which belong to $\mathcal{U}$ (the module farthest left is the simple module $S(0) = \emptyset_1$), on the right the 5 indecomposable preinjective modules which belong to $\mathcal{U}$, the middle part exhibits indecomposable regular modules. Each of the 3 tubes of rank 2 hosts 4 indecomposable modules which belong to $\mathcal{U}$, namely the modules of regular length at most 2, here we see the cycles mentioned above. And there are the remaining indecomposable modules with dimension vector $(2; 1, 1, 1, 1)$, they are modules on the mouth of homogeneous tubes; the number of such modules is $|k| - 2$.

2. Tilting Theory

This chapter concerns the classical tilting theory, the study of (finitely generated) tilting modules for a hereditary artin algebra. In my appendix to the *Handbook of Tilting Theory* (2007) I wrote: *At the time the Handbook was conceived* (= 2002) *there was a common feeling that the tilted algebras (as the core of tilting theory) were understood well and that this part of the theory had reached a sort of final shape. But in the meantime this has turned out to be wrong: the tilted algebras have to be seen as factor algebras of the so called cluster tilted algebras, and it may very well be, that in future the cluster tilted algebras and the cluster categories will topple the tilted algebras* ([**86**], p. 446).

Actually, as we will see in this chapter, already the basic setting of tilting theory should be refined, replacing the usually considered torsion pair $(\mathcal{F}(T), \mathcal{G}(T))$ by a torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$, a torsion triple which is defined by a linear form $\alpha_T : K_0(\Lambda) \to \mathbb{Z}$. Here, $\mathcal{N}(T)$ will be an arbitrary sincere exceptional subcategory, and $\mathcal{N}(T)$ and $(\mathcal{F}(T), \mathcal{G}(T))$ determine each other. In this way, the study of tilted algebras turns out to be just the study of sincere exceptional subcategories. This refinement is due to Ingalls and Thomas (2009), it puts tilting theory into the realm of the stability theory of King. Whereas the impetus for this refinement came from the theory of cluster algebras and cluster tilted algebras, there is now no further need to refer to cluster theory.

In this chapter, we usually will deal with an arbitrary hereditary artin algebra Λ, and do not restrict the attention to the representation-finite ones. The modules which we will consider are left Λ-modules of finite length. The chapter is essentially independent from Chapter 1 and is mainly devoted to advertise findings of Ingalls and Thomas.

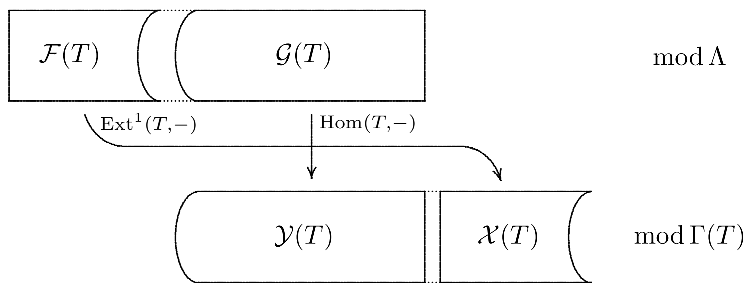
Let us recall the basic definitions of tilting theory. A module T will be called a *partial tilting* module provided it is multiplicity-free (this means that it is the direct sum of pairwise non-isomorphic indecomposable modules) and has no self-extensions (this means that $\mathrm{Ext}^1(T, T) = 0$). A *tilting* module is a partial tilting module which is the direct sum of n indecomposable modules, where n is the number of simple Λ-modules (thus the rank of the Grothendieck group $K_0(\Lambda)$).

A *torsion pair* $(\mathcal{F}, \mathcal{G})$ in $\mathrm{mod}\,\Lambda$ is a pair of full subcategories $\mathcal{F}, \mathcal{G}$ which satisfies the following two properties: first, $\mathrm{Hom}(G, F) = 0$ for all modules $F \in \mathcal{F}$ and $G \in \mathcal{G}$ (we just will write $\mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$ and use a similar convention also elsewhere). Second, any module X with $\mathrm{Hom}(X, \mathcal{F}) = 0$ belongs to $\mathcal{G}$ and every module Y with $\mathrm{Hom}(\mathcal{G}, Y) = 0$ belongs to $\mathcal{F}$. Alternatively, one may replace the second condition by requiring that every module M has a submodule M' which belongs to $\mathcal{G}$ such that M/M' belongs to $\mathcal{F}$, this submodule M' is called the *torsion submodule* of M and is a uniquely determined submodule. The class $\mathcal{G}$ is called a torsion class, the class $\mathcal{F}$ a torsionfree class, the torsion classes are just the full subcategories closed under factor modules and extensions, the torsionfree classes are the full subcategories closed under submodules and extensions. A torsion pair $(\mathcal{F}, \mathcal{G})$ is said to be *split* provided any indecomposable module belongs to $\mathcal{F}$ or to $\mathcal{G}$ (or, equivalently, provided the torsion submodule of any module is a direct summand).

A tilting module T determines a torsion pair $(\mathcal{F}(T), \mathcal{G}(T))$ as follows: $\mathcal{G}(T)$ consists of the modules which are generated by T (thus the factor modules of modules of the form T^m for some natural number m), or equivalently, the modules G satisfying $\mathrm{Ext}^1(T,G)=0$. The modules in $\mathcal{F}(T)$ are the modules F with $\mathrm{Hom}(T,F)=0$.

If T is a tilting module, one is interested in the ring $\Gamma(T)=\mathrm{End}(T)^{\mathrm{op}}$; this is called a *tilted* artin algebra. The indecomposable $\Gamma(T)$-modules are obtained from the indecomposable Λ-modules as follows: The functor $\mathrm{Hom}(T,-)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma(T)$ yields an equivalence between $\mathcal{G}(T)$ and its image category $\mathcal{Y}(T)$, the functor $\mathrm{Ext}^1(T,-)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma(T)$ yields an equivalence between $\mathcal{F}(T)$ and its image category $\mathcal{X}(T)$, and the pair $(\mathcal{Y}(T),\mathcal{X}(T))$ is a torsion pair in $\mathrm{mod}\,\Gamma(T)$ which is split. The functors $\mathrm{Hom}(T,-)$ and $\mathrm{Ext}^1(T,-)$ are called the *tilting functors* given by T.

Let us stress that here we deal with torsion pairs in $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Gamma(T)$, namely with $(\mathcal{F}(T),\mathcal{G}(T))$ and $(\mathcal{Y}(T),\mathcal{X}(T))$, such that the two given subcategories of $\mathrm{mod}\,\Lambda$ are equivalent to the two given subcategories in $\mathrm{mod}\,\Gamma(T)$, but it is a cross-over: the torsion class of $\mathrm{mod}\,\Lambda$ is equivalent to the torsionfree class of $\mathrm{mod}\,\Gamma(T)$, and the torsionfree class of $\mathrm{mod}\,\Lambda$ is equivalent to the torsion class of $\mathrm{mod}\,\Gamma(T)$. One illustrates this as follows:



It should be observed that usually, the torsion pair $(\mathcal{F}(T),\mathcal{G}(T))$ is not split (whereas $(\mathcal{Y}(T),\mathcal{X}(T))$ always splits). Thus, going from $\mathrm{mod}\,\Lambda$ to $\mathrm{mod}\,\Gamma(T)$ via the tilting functors, one "loses" some modules (the indecomposable Λ-modules which are neither torsion, nor torsionfree).

2.1. Linearity of tilting torsion pairs. Following Ingalls and Thomas, we are going to show that the torsion pair $(\mathcal{F}(T),\mathcal{G}(T))$ for any tilting module T is defined by a linear form $\alpha_T\colon K_0(\Lambda)\to\mathbb{Z}$.

2.1.1. **First formulation.**

LEMMA 2.1.1.1. *If* $\alpha\colon K_0(\Lambda)\to\mathbb{R}$ *is a linear form, let*

$$\mathcal{F}(\alpha)\ \ \{M \mid \alpha(M')<0 \quad \textit{for all submodules } 0\neq M' \textit{ of } M\}$$

$$\mathcal{G}(\alpha)\ \ \{M \mid \alpha(M'')\geq 0 \quad \textit{for all factor modules } M'' \textit{ of } M\}$$

Then the pair $(\mathcal{F}(\alpha),\mathcal{G}(\alpha))$ *is a torsion pair.*

Such a torsion pair will be said to be a *linear* torsion pair, or also the *torsion pair defined by* α. The proof of the lemma is not difficult, we will outline it later in a more general setting (see Lemma 2.1.1.5 and the note N 2.1).

Here is a first version of Theorem 2.3.5.1, one of the main results of the chapter.

2.1.1.2. *Let* Λ *be a hereditary artin algebra and* T *a tilting module. Then* $(\mathcal{F}(T),\mathcal{G}(T))$ *is a linear torsion pair, it is defined by a linear form* $\alpha\colon K_0(\Lambda)\to\mathbb{Z}$.

The proof will be given in Section 2.3. It provides a clear recipe for α; in general, as we will see, there will be many different linear forms on $K_0(\Lambda)$ which define the torsion pair $(\mathcal{F}(T), \mathcal{G}(T))$.

The result may look quite innocent, but it has striking consequences. Namely, such a linear form α is just a **stability condition** as considered by King, and it determines not only a torsion **pair,** but even a torsion **triple**. Here is the definition of a torsion triple:

A *torsion triple* $(\mathcal{F}, \mathcal{N}, \mathcal{Q})$ consists of three full subcategories such that we have $\operatorname{Hom}(\mathcal{N}, \mathcal{F}) = \operatorname{Hom}(\mathcal{Q}, \mathcal{F}) = \operatorname{Hom}(\mathcal{Q}, \mathcal{N}) = 0$ and such that any module M has a filtration $0 \subseteq M'' \subseteq M' \subseteq M$ such that M'' belongs to $\mathcal{Q}$, M'/M'' to $\mathcal{N}$ and M/M' to $\mathcal{F}$.

If $\mathcal{A}, \mathcal{B}$ are subcategories of $\operatorname{mod}\Lambda$ (or classes of Λ-modules), we write $\mathcal{A}\int\mathcal{B}$ for the full subcategory of all modules M with a submodule M' in $\mathcal{B}$ such that M/M' is in $\mathcal{A}$.

Lemma 2.1.1.3. *If $(\mathcal{F}, \mathcal{N}, \mathcal{Q})$ is a torsion triple, then $(\mathcal{F}, \mathcal{N}\int\mathcal{Q})$ and $(\mathcal{F}\int\mathcal{N}, \mathcal{Q})$ are torsion pairs and $\mathcal{N} = (\mathcal{F}\int\mathcal{N}) \cap (\mathcal{N}\int\mathcal{Q})$. In particular, all three classes $\mathcal{F}, \mathcal{N}, \mathcal{Q}$ are closed under extensions, $\mathcal{F}$ is closed under submodules and $\mathcal{Q}$ is closed under factor modules.*

Thus, starting with a torsion triple $(\mathcal{F}, \mathcal{N}, \mathcal{Q})$, we obtain two torsion pairs $(\mathcal{F}, \mathcal{G}) = (\mathcal{F}, \mathcal{N}\int\mathcal{Q})$ and $(\mathcal{F}', \mathcal{G}') = (\mathcal{F}\int\mathcal{N}, \mathcal{Q})$ with $\mathcal{F} \subseteq \mathcal{F}'$. *Conversely, let $(\mathcal{F}, \mathcal{G})$ and $(\mathcal{F}', \mathcal{G}')$ be torsion pairs with $\mathcal{F} \subseteq \mathcal{F}'$. Then $(\mathcal{F}, \mathcal{G} \cap \mathcal{F}', \mathcal{G}')$ is a torsion triple.*

Lemma 2.1.1.4. *Let $(\mathcal{F}, \mathcal{N}, \mathcal{Q})$ and $(\mathcal{F}', \mathcal{N}', \mathcal{Q}')$ be torsion triples. If $\mathcal{F} \subseteq \mathcal{F}'$, $\mathcal{N} \subseteq \mathcal{N}'$, $\mathcal{Q} \subseteq \mathcal{Q}'$, then $\mathcal{F} = \mathcal{F}'$, $\mathcal{N} = \mathcal{N}'$, $\mathcal{Q} = \mathcal{Q}'$.*

Let $\alpha\colon K_0(\Lambda) \to \mathbb{R}$ be a linear form. We have already defined $\mathcal{F}(\alpha)$ (as well as $\mathcal{G}(\alpha)$). Let us define in a corresponding way $\mathcal{N}(\alpha)$ and $\mathcal{Q}(\alpha)$.

Let $\mathcal{N}(\alpha)$ be the full subcategory of all modules M with $\alpha(M) = 0$, such that $\alpha(M') \le 0$ for any submodule M' of M (or, equivalently, such that $\alpha(M'') \ge 0$ for any factor module M'' of M). Following King, the modules in $\mathcal{N}(\alpha)$ are usually said to be *α-semistable.*

Let $\mathcal{Q}(\alpha)$ be the full subcategory of all the modules M with $\alpha(M') > 0$ for any non-zero factor module M'' of M (in particular, $\alpha(M) > 0$ provided $M \neq 0$).

Lemma 2.1.1.5. *Let $\alpha\colon K_0(\Lambda) \to \mathbb{R}$ be a linear form. Then $(\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha))$ is a torsion triple such that $\mathcal{N}(\alpha)$ is a thick subcategory.*

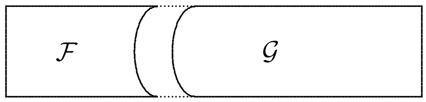
We call $(\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha))$ the *torsion triple defined by α.*

Recall that a subcategory of an abelian category is said to be *thick* provided it is closed under kernels, cokernels and extensions. Any thick subcategory is an abelian category, and the embedding functor is exact.

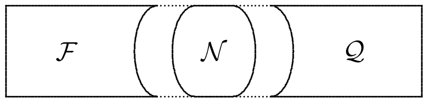
For the proof of 2.1.1.5 (and thus also of Lemma 2.1.1.1) see the note N 2.1 at the end of the chapter. The note N 2.2 draws the attention to some examples of torsion pairs: torsion pairs usually are not linear, and linear torsion pairs are not necessarily tilting (or support-tilting) torsion pairs.

Consequences. Let $(\mathcal{F}, \mathcal{G})$ be the torsion pair defined by some tilting module T. Thus, there is a torsion triple $(\mathcal{F}, \mathcal{N}, \mathcal{Q})$. Whereas the subcategories $\mathcal{F}, \mathcal{G}$ are not symmetric in nature, the categories $\mathcal{F}(\alpha), \mathcal{Q}(\alpha)$ defined by a linear from α are defined in a symmetric way!

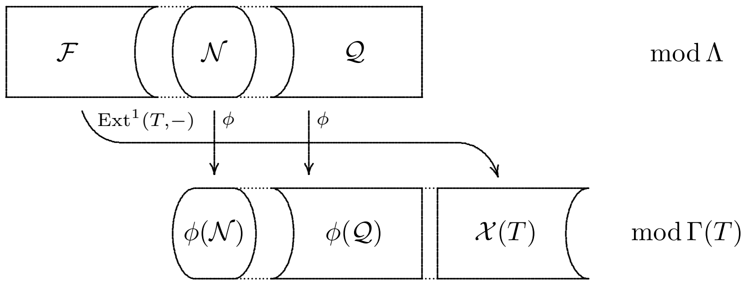
As we have mentioned, the subcategories $\mathcal{F}, \mathcal{G}$ of $\operatorname{mod} \Lambda$ are usually drawn as follows



Now we see that $\mathcal{G} = \mathcal{N} \int \mathcal{Q}$, thus we deal with the following subcategories:



The illustration concerning the tilting functors $\phi = \operatorname{Hom}(T, -)$ and $\operatorname{Ext}^1(T, -)$ has now to be refined as follows:



Remark. The name "tilting" was chosen by Brenner and Butler in view of the tilting of the coordinate axes which one observes when going from $K_0(\Lambda)$ to $K_0(\Gamma(T))$. The linearity assertion supports this observation.

2.1.2. **The support of a module, sincere modules, support tilting modules.** As we have mentioned in Chapter 1, the *support* of a module M consists of all the simple modules S (or better, their isomorphism classes) which occur as composition factors of M. Also, if all simple modules occur as composition factors of M, then M is said to be *sincere*. Given a module M, its *support algebra* Λ_M is the factor algebra Λ/I_M, where I_M is the two-sided ideal generated by all idempotents $e \in \Lambda$ with $eM = 0$. Since I_M annihilates the module M, one may consider M as a Λ_M-module; of course, M considered as a Λ_M-module is sincere. A module T is called a *support-tilting* module, provided T considered as a Λ_T-module is a tilting module.

Support tilting torsion pairs. If T is a support tilting module, then we denote again by $\mathcal{G}(T)$ the full subcategory of all modules generated by T and by $\mathcal{F}(T)$ the class of all modules Y with $\operatorname{Hom}(T, Y) = 0$. Then this is again a torsion pair (note that if $\mathbf{S}$ is the support of T, then all the modules in $\mathcal{G}(T)$ have support in $\mathbf{S}$, whereas $\mathcal{F}(T)$ will contain modules whose support is not contained in $\mathbf{S}$ (provided $\mathbf{S}$ is not the set of all simple modules).

As we will see, 2.1.1.2 above holds true in this more general situation (thus, here we have a second preliminary version of Theorem 2.3.5.1):

2.1.2.1. *Let Λ be a hereditary artin algebra and T a support-tilting module. Then $(\mathcal{F}(T), \mathcal{G}(T))$ is a linear torsion pair, it is defined by a linear form $\alpha : K_0(\Lambda) \to \mathbb{Z}$.*

2.2. Exceptional antichains and normal partial tilting modules. Our main interest will be devoted to the exceptional antichains in $\operatorname{mod}\Lambda$. We will show that the exceptional antichains can be obtained by normalizing a partial tilting module.

2.2.1. **Normality.** Here we go back to the beginning of modern representation theory, namely to Roiter's proof [**90**] of the first Brauer-Thrall conjecture. The first topic which he discusses in his paper is the normalization of a module. Actually, this concept was reinvented several times, for example by Auslander-Smalø when dealing with minimal covers of additive subcategories (a normalization of a module M is a minimal cover of the subcategory $\operatorname{add} M$, where $\operatorname{add} M$ denotes the subcategory of all direct summands of direct sums of copies of M).

We call a module M *normal* provided no proper direct summand of M generates M; this means that $M = M' \oplus M''$ with M' generating M'' (thus even M) implies that $M'' = 0$.

Lemma 2.2.1.1 (**Roiter's Normalization-Lemma**). *Let Λ be an artin algebra. Any module M can be decomposed $M = M' \oplus M''$ such that M' is normal and generates M (or, equivalently, M''). Such a decomposition is unique up to isomorphism.*

For a proof, see the note N 2.3. The module $M' = \nu(M)$ is called a *normalization* of M. If M is multiplicity-free, we write $M = \nu(M) \oplus \nu'(M)$, where $\nu(M)$ is the normalization of M (the assumption on M to be multiplicity-free assures that $\nu(M)$ and $\nu'(M)$ have no indecomposable direct summands which are isomorphic).

2.2.2. **Antichains.** Let us recall from Chapter 1 the definition of an antichain (see also the note N 1.8): An *antichain* A in an additive category $\mathcal{C}$ is a family $A = \{A_i\}_i$ of pairwise orthogonal bricks A_i. Given an antichain in an abelian category $\mathcal{C}$, we may consider its extension closure $\mathcal{E}(A)$, this is the full subcategory of all objects in $\mathcal{C}$ having a filtration with factors in A. It has been shown in [**82**] that $\mathcal{E}(A)$ is a thick subcategory. Conversely, if $\mathcal{C}$ is a length category (an abelian category in which any object has finite length), then any thick subcategory arises in this way: we denote by $S(\mathcal{C})$ the set of simple objects in $\mathcal{C}$; the Schur Lemma asserts that this is an antichain and we have $\mathcal{C} = \mathcal{E}(S(\mathcal{C}))$. Antichains (and thick subcategories) have been considered in several papers. For example, antichains appear in [**46**] under the name Hom-*free sets.*

Starting with an antichain $A = \{A_1, \dots, A_t\}$, its Ext-*quiver* has t vertices, and there is an arrow $j \to i$ provided $\operatorname{Ext}^1(A_j, A_i) \neq 0$. We say that the antichain A is *exceptional* provided its Ext-quiver has no oriented cyclic path (thus, provided we may number the modules A_i in such a way that $\operatorname{Ext}^1(A_i, A_j) = 0$ for all $i \leq j$). Clearly, *A is exceptional if and only if $\mathcal{E}(A)$ is exceptional.*

2.2.3. The following bijections will be the first ones of a long list, see Theorem 2.4.2.1.

2.2.3.1. *There are bijections between:*

(1) *Exceptional antichains.*
(1′) *Exceptional subcategories.*

(1″) *Normal partial tilting modules.*

The bijection between (1) and (1′) has already been mentioned (with reference to [**82**], 1976):

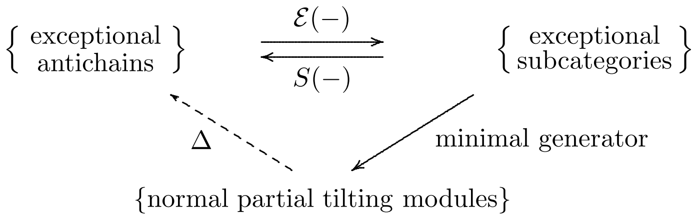
$$\{\text{antichains}\} \quad \underset{S(-)}{\overset{\mathcal{E}(-)}{\rightleftarrows}} \quad \{\text{thick subcategories}\}$$

Given an antichain A, the subcategory $\mathcal{E}(A)$ is abelian and closed under extensions and the simple objects in $\mathcal{E}(A)$ are the elements of A, thus $S(\mathcal{E}(A)) = A$. Conversely, given a thick subcategory $\mathcal{C}$, the set $S(\mathcal{C})$ consists of the simple objects in $\mathcal{C}$. This is an antichain, and $\mathcal{C} = \mathcal{E}(S(\mathcal{C}))$.

Of course, this bijection induces a bijection of the following subsets:

$$\left\{\begin{matrix}\text{exceptional}\\ \text{antichains}\end{matrix}\right\} \quad \underset{S(-)}{\overset{\mathcal{E}(-)}{\rightleftarrows}} \quad \left\{\begin{matrix}\text{exceptional}\\ \text{subcategories}\end{matrix}\right\}$$

But here we can add a third class, namely the normal partial tilting modules:



If $\mathcal{C}$ is an exceptional subcategory, then $\mathcal{C}$ has a minimal generator P (namely, $\mathcal{C}$ is equivalent to the module category of an artin algebra Λ', thus it has a progenerator). Clearly, P, considered as a Λ-module, is a normal partial tilting module.

What about the dashed arrow with the label Δ? This will be discussed now.

2.2.4. We refer to considerations which have been useful in the study of quasi-hereditary algebras (see [**40**]): an exceptional antichain A is a standardizable set, thus there is a quasi-hereditary algebra B such that the subcategory $\mathcal{E}(A)$ is equivalent to the category of Δ-filtered B-modules. Since the standardizable set A consists of pairwise orthogonal modules, the same is true for the Δ-modules of B, and consequently the Δ-modules of B are just the simple B-modules. This shows that the category of Δ-filtered B-modules is the whole category $\operatorname{mod} B$. In this way, we obtain the converse of the dashes arrow: We start with an exceptional antichain A, the paper [**40**] shows how to construct indecomposable objects in $\mathcal{E}(A)$ which are the indecomposable projective objects in $\mathcal{E}(A)$. In this way, we obtain a minimal generator for the abelian category $\mathcal{E}(A)$ (thus a normal partial tilting module).

Now let us discuss the dashed arrow itself. Let $N = \bigoplus_{i=1}^t N_i$ be a normal partial tilting module with indecomposable direct summands N_i. Then non-zero homomorphisms between indecomposables in $\operatorname{add} N$ are monomorphisms (namely, since $\operatorname{Ext}^1(N_j, N_i) = 0$, any non-zero homomorphism $N_i \to N_j$ has to be either injective or surjective). Thus, the indecomposables in $\operatorname{add} N$ form a poset.

Given an indecomposable direct summand N_i of N, the sink map for N_i in $\operatorname{add} N$ cannot be surjective (since N is normal), thus it has to be injective (since

$\operatorname{Ext}^1(N,N)=0$). Let $\Delta(i)$ be its cokernel. Thus, there is an exact sequence

$$(*) \qquad 0 \to N_i' \to N_i \to \Delta(i) \to 0$$

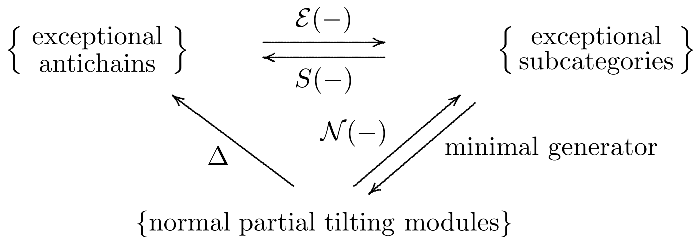
with $N_i' \in \operatorname{add} N$. It is easy to see that these modules $\Delta(i)$ form an antichain and by induction one sees that N_i has a filtration with factors of the form $\Delta(j)$. Of course, this antichain $\Delta(N)=\{\Delta(1),\dots,\Delta(t)\}$ is exceptional.

2.2.5. It is worthwhile to look at the composition of Δ and $\mathcal{E}$. If N is a normal, partial tilting module, let us denote

$$\mathcal{N}(N)=\mathcal{E}(\Delta(N)),$$

then $\mathcal{N}(N)$ *is the smallest thick subcategory containing* N. (Namely, since N has a filtration with factors in $\Delta(N)$, we see that N belongs to $\mathcal{N}(N)$. On the other hand, any object in $\mathcal{N}(N)$ has a filtration with factors in $\Delta(N)$, and the modules in $\Delta(N)$ are cokernels of maps in $\operatorname{add} N$.) Let us add the following obvious assertion: *If* N *is sincere, then also* $\mathcal{N}(T)$ *is sincere.*

The construction $\mathcal{N}(-)$ from the set of normal partial tilting modules to the exceptional subcategories is the inverse of taking a minimal generator.



If T is a support-tilting module, we define

$$\mathcal{N}(T)=\mathcal{N}(\nu(T)).$$

The next section is devoted to a detailed study of the subcategories of the form $\mathcal{N}(T)$.

2.3. The category $\mathcal{N}(T)$. Given a tilting module T, it is the category $\mathcal{N}(T)$ which is of interest. In order to understand $\mathcal{N}(T)$, we need some prerequisites. Let $\Gamma(T)=\operatorname{End}(T)^{\mathrm{op}}$.

2.3.1. **The simple $\Gamma(T)$-modules.** First, let us show that *the decomposition* $T=\nu(T)\oplus\nu'(T)$ *corresponds to the distribution of the simple* $\Gamma(T)$*-modules* $S'(i)$ *into* $\mathcal{Y}(T)$ *and* $\mathcal{X}(T)$.

We note that the indecomposable projective $\Gamma(T)$-modules are the modules $\operatorname{Hom}(T,T_i)$, where T_i is any indecomposable direct summand of T, thus the simple $\Gamma(T)$-modules are the modules $S'(i)=\operatorname{top}\operatorname{Hom}(T,T_i)$.

For any indecomposable direct summand T_i of $T=\bigoplus T_j$, let $T^{(i)}=T/T_i=\bigoplus_{j\neq i}T_j$ and let $g_i\colon T^i\to T_i$ be a minimal right $T^{(i)}$-approximation of T_i.

Proposition 2.3.1.1. *Let T_i be an indecomposable direct summand of the tilting module T. Then there are two possibilities:*

If T_i is a direct summand of $\nu(T)$, then $g_i : T^i \to T_i$ is a monomorphism say with cokernel $\Delta(i)$. In this case, $S'(i) = \operatorname{top}\operatorname{Hom}(T, T_i) = \operatorname{Hom}(T, \Delta(i))$ is a simple $\Gamma(T)$-module which belongs to $\mathcal{Y}(T)$.

If T_i is a direct summand of $\nu'(T)$, then $g_i : T^i \to T_i$ is an epimorphism say with kernel $U(i)$. In this case, $S'(i) = \operatorname{top}\operatorname{Hom}(T, T_i) = \operatorname{Ext}^1(T, U(i))$ is a simple $\Gamma(T)$-module which belongs to $\mathcal{X}(T)$.

Also, let us note:

If S, S' are simple $\Gamma(T)$-modules with an arrow $[S] \to [S']$ in the quiver of $\Gamma(T)$ (thus with $\operatorname{Ext}^1(S, S') \neq 0$), then, if S belongs to $\mathcal{Y}(T)$, also S' belongs to $\mathcal{Y}(T)$.

Proposition 2.3.1.2. *Let T be a tilting module and $\phi = \operatorname{Hom}(T, -)$ the corresponding tilting functor. Then $\mathcal{N}(T)$ is the inverse image under ϕ of the Serre subcategory in $\operatorname{mod}\Gamma(T)$ given by all $\Gamma(T)$-modules whose composition factors belong to $\mathcal{Y}(T)$.*

The proof of the Propositions 2.3.1.1 and 2.3.1.2 will be outlined in N 2.4.

Proposition 2.3.1.3. *T be a support tilting module with normal decomposition $T = \nu(T) \oplus \nu'(T)$. Then*

$$\boxed{\mathcal{N}(T) = \mathcal{G}(T) \cap \mathcal{F}(\nu'(T)).}$$

Proof. Since $\mathcal{N}(T)$ is obtained from $\nu(T)$ by forming cokernels and extensions, we see that $\mathcal{N}(T)$ is contained in $\mathcal{G}(T)$.

Next, we note that $\operatorname{Hom}(\nu'(T), \nu(T)) = 0$. Namely, consider a map $f : T_j \to T_i$ where T_i is an indecomposable direct summand of $\nu(T)$ and T_j is an indecomposable direct summand of $\nu'(T)$. It cannot be surjective, since otherwise with T_j also T_i would be a direct summand of $\nu'(T)$. It also cannot be injective, since there is a non-split epimorphism $T' \to T_j$ with T' in $\operatorname{add} T$ and we have $\operatorname{Ext}^1(T_i, T') = 0$. Thus $f = 0$.

As a consequence, we have $\operatorname{Hom}(\nu'(T), \Delta(i)) = 0$ (since $\operatorname{Ext}^1(T_i, \nu'(T)) = 0$) and therefore $\operatorname{Hom}(\nu'(T), \mathcal{N}(T)) = 0$. Thus $\mathcal{N}(T) \subseteq \mathcal{F}(\nu'(T))$.

Conversely, assume that M belongs to $\mathcal{G}(T) \cap \mathcal{F}(\nu'(T))$. Since M is in $\mathcal{G}(T)$, the minimal right $\operatorname{add} T$-approximation of M provides an exact sequence $0 \to T'' \to T' \to M \to 0$ with $T', T'' \in \operatorname{add}(T)$. Since $\operatorname{Hom}(\nu'(T), M) = 0$, we see that T' belongs to $\operatorname{add}\nu(T)$. Since $\operatorname{Hom}(\nu'(T), \nu(T) = 0)$, we also have $T'' \in \operatorname{add}\nu(T)$. Thus, we see that M is the cokernel of a map in $\operatorname{add}\nu(T)$ and therefore belongs to the thick closure $\mathcal{N}(T)$ of $\nu(T)$. □

2.3.2. **The linearity of the triple** $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$**.** Let T be a tilting module. We use the decomposition $T = \nu(T) \oplus \nu'(T)$ in order to define $\mathcal{N}(T)$ and $\mathcal{Q}(T)$. Let $\mathcal{N}(T)$ be the category of all modules M generated by T such that $\mathrm{Hom}(\nu'(T), M) = 0$) and let $\mathcal{Q}(T)$ be the full subcategory of all modules generated by $\nu'(T)$. Then $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$ *is a torsion triple.*

Namely, let us denote by $\mathcal{F}'$ the full subcategory of all modules M with $\mathrm{Hom}(\nu'(T), M) = 0$. We deal with the two torsion pairs $(\mathcal{F}(T), \mathcal{G}(T))$ and $(\mathcal{F}', \mathcal{Q}(T))$, they satisfy $\mathcal{F}(T) \subseteq \mathcal{F}'$. Since $\mathcal{N}(T) = \mathcal{G}(T) \cap \mathcal{F}'$, we see that $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$ is a torsion triple.

Given a Λ-module M, we may consider the linear form $\langle \mathbf{dim}\, M, - \rangle$ on $K_0(\Lambda)$, we will denote it just by $\langle M, - \rangle$. Here is a third version of 2.1.1.2, the final version will be Theorem 2.3.5.1

2.3.2.1. *Let T be a tilting module. Then $\alpha_T = \langle \nu'(T), - \rangle$ defines the torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$. In particular, $\mathcal{N}(T)$ is a thick subcategory.*

More generally: *If T' is a module which is Morita-equivalent to $\nu'(T)$, then $\langle T', - \rangle$ defines the torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$.*

Let us recall that modules M, M' are said to be *Morita equivalent* provided $\mathrm{add}(M) = \mathrm{add}(M')$.

The final statement of 2.3.2.1 implies the following: *if $\nu'(T)$ is the direct sum of r indecomposable modules, then there are r linearly independent linear forms α_i which define the torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$.* Namely, let $\nu'(T) = \bigoplus_{i=1}^r T_i$ with indecomposable direct summands T_i. Let $T_i' = \nu'(T) \oplus T_i$. Then all the modules T_i' with $1 \le i \le r$ are Morita equivalent to $\nu'(T)$, and the linear forms $\alpha_i = \langle T_i', - \rangle$ with $1 \le i \le r$ are linearly independent.

Proof of 2.3.2.1. Let $\alpha = \langle T', - \rangle$ for some module T' which is Morita equivalent to $\nu'(T)$.

(1) $\mathcal{Q}(T) \subseteq \mathcal{Q}(\alpha)$.

Proof. Let $M \in \mathcal{Q}(T)$ be a non-zero module. Since M is generated by $\nu'(T)$, we must have $\mathrm{Hom}(T', M) \neq 0$. On the other hand, since M is generated by $\nu'(T)$, it belongs to $\mathcal{G}(T)$, thus $\mathrm{Ext}^1(T, M) = 0$, therefore also $\mathrm{Ext}^1(T', M) = 0$. This shows that $\alpha(M) > 0$. Of course, if M'' is a non-zero factor module of M, then also M'' is a non-zero module in $\mathcal{Q}(T)$. Therefore, as we have seen, $\alpha(M'') = 0$.

(2) $\mathcal{N}(T) \subseteq \mathcal{N}(\alpha)$.

Proof. Let M belong to $\mathcal{N}(T)$. Since M belongs to $\mathcal{G}(T)$, we have $\mathrm{Ext}^1(T, M) = 0$, thus $\mathrm{Ext}^1(T', M) = 0$. Since $\mathrm{Hom}(\nu'(T), M)) = 0$, we also have $\mathrm{Hom}(T', M) = 0$. This shows that $\alpha(M) = 0$. If M'' is any factor module of M, then also M'' is generated by T, thus $\mathrm{Ext}^1(T', M) = 0$ and therefore $\alpha(M'') \geq 0$.

(3) $\mathcal{F}(T) \subseteq \mathcal{F}(\alpha)$.

Let M be a non-zero module in $\mathcal{F}(T)$. Since $\mathrm{Hom}(T, M) = 0$, we also have $\mathrm{Hom}(T', M) = 0$. Thus, it remains to show that $\mathrm{Ext}^1(T', M) \neq 0$, or, equivalently, that there exists some indecomposable direct summand T_j of $\nu'(T)$ with $\mathrm{Ext}^1(T_j, M) \neq 0$.

We consider the $\Gamma(T)$-module $\psi(M) = \mathrm{Ext}^1(T, M)$ which belongs to $\mathcal{X}(T)$. The support of its dimension vector is the set of all indices s with $\mathrm{Ext}^1(T_s, M) \neq 0$.

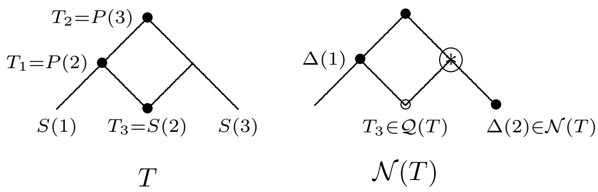
Now assume that $\operatorname{Ext}^1(T_j, M) = 0$ for all indecomposable direct summands of $\nu'(T)$. Then all the composition factors of $\psi(M)$ are of the form $\operatorname{Hom}(T, \Delta(i))$ with T_i an indecomposable direct summand of $\nu(T)$. But these modules $\operatorname{Hom}(T, \Delta(i))$ belong to $\mathcal{Y}(T)$ and $\mathcal{Y}(T)$ is closed under extensions. This implies that $\psi(M)$ belongs to $\mathcal{Y}(T)$, a contradiction. Thus, we see that $\operatorname{Ext}^1(T', M) \neq 0$, therefore $\alpha(M) < 0$.

Of course, if M' is a non-zero submodule of M, then also M' is a non-zero module in $\mathcal{F}(T)$ and therefore $\alpha(M') < 0$. □

Corollary 2.3.2.2. *The direct summand $\nu'(T)$ of T determines T.*

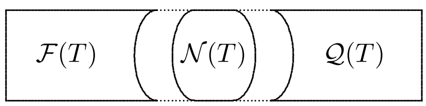
Proof. The module $\nu'(T)$ determines $\alpha_T = \langle \nu'(T), - \rangle$, thus $\mathcal{F}(T)$ and therefore $(\mathcal{F}(T), \mathcal{G}(T))$ and this in turn determines the module T. □

2.3.3. **Remark.** For any tilting module T, we have $\mathcal{G}(T) = \mathcal{N}(T) \int \mathcal{Q}(T)$: any module M in $\mathcal{G}(T)$ has a (uniquely determined) submodule M' which belongs to $\mathcal{Q}(T)$ such that M/M' belongs to $\mathcal{N}(T)$. We should stress that this submodule M' of M usually will not be a direct summand (thus, the torsion pair $(\mathcal{N}(T), \mathcal{Q}(T))$ in the additive category $\mathcal{G}(T)$ may not be split). As an example, consider the linearly oriented quiver of type $\mathbb{A}_3$ with path algebra Λ, with indecomposable projective modules $P(1) \subset P(2) \subset P(3)$ and choose the tilting module $T = \bigoplus T_i$ with $T_1 = P(2)$, $T_2 = P(3)$, $T_3 = S(2)$ (see the left picture below, the bullets mark the modules T_i), here $\nu(T) = T_1 \oplus T_2$ and $\nu'(T) = T_3$.



On the right, the bullets mark the indecomposable modules which belong to $\mathcal{N}(T)$. Note that $S(3) = \Delta(2)$ belongs to $\mathcal{N}(T)$, whereas $T_3 = S(2)$ belongs to $\mathcal{Q}(T)$. The module $P(3)/P(1)$ marked by a big circle is a non-split extension of $T_3 \in \mathcal{Q}(T)$ by $\Delta(2) \in \mathcal{N}(T)$.

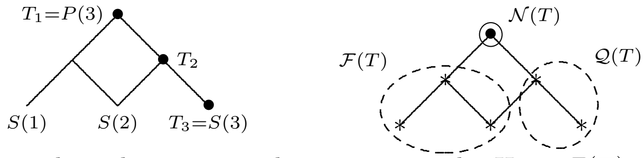
Let us have another look at the torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$ defined by a tilting module T. The following picture



suggests a complete symmetry between the roles of $\mathcal{F}(T)$ and $\mathcal{Q}(T)$, but there is some **asymmetry** involved which we should mention. Whereas the subcategory $\mathcal{Q}(T)$ is generated by $\mathcal{N}(T)$, the subcategory $\mathcal{F}(T)$ is not necessarily cogenerated by $\mathcal{N}(T)$ (but it is always cogenerated by $\mathcal{G}(T)$).

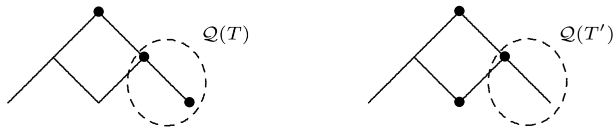
As an example, we consider again the linearly oriented quiver of type $\mathbb{A}_3$, with indecomposable projective modules $P(1) \subset P(2) \subset P(3)$. This time, we take as tilting module T the minimal cogenerator (see the left picture below), here

$\nu(T) = P(3)$, and $\nu'(T) = P(3)/S(1) \oplus S(3)$.



On the right, we show the corresponding torsion triple. Here, $\mathcal{F}(T)$ is not cogenerated by $\mathcal{N}(T)$.

Also, $\mathcal{Q}(T)$ does not determine the torsion triple (in contrast to $\mathcal{F}(T)$). We use the same algebra Λ as before and present two non-isomorphic tilting modules T, T' with $Q(T) = Q(T')$.



T T'

Let us repeat: given a tilting module T, the partial tilting module $\nu'(T)$ determines T uniquely (see the Corollary 2.3.2.2), but the class $\mathcal{Q}(T)$ of modules generated by $\nu'(T)$ does not determine T.

2.3.4. **Remark.** As we see, any tilting torsion pair $(\mathcal{F}(T), \mathcal{G}(T))$ is given by a linear form α on $K_0(\Lambda)$ and in this way, we obtain a sincere exceptional subcategory $\mathcal{N}(\alpha)$ contained in $\mathcal{G}(T)$. But we should remark that there are linear forms α on $K_0(\Lambda)$ with $\mathcal{N}(\alpha)$ being a sincere exceptional subcategory such that $\mathcal{Q}(\alpha)$ is not generated by $\mathcal{N}(\alpha)$ (thus they do not come from a tilting module), and also dually, $\mathcal{F}(\alpha)$ is not cogenerated by $\mathcal{N}(\alpha)$ (thus they do not come from a cotilting module). An example is presented in the note N 2.5.

2.3.5. **The general case of support-tilting modules.** Let T be a support-tilting module. As we have mentioned in Section 2.1.2, $\mathcal{F}(T)$ denotes the full subcategory of all modules M with $\mathrm{Hom}(T, M) = 0$. As for a tilting module, let $\mathcal{N}(T)$ be the thick subcategory generated by $\nu(T)$ and $\mathcal{Q}(T)$ the full subcategory of all modules generated by $\nu'(T)$.

THEOREM 2.3.5.1. **(Ingalls-Thomas).** *Let T be a support-tilting module. Let P be a projective module such that the top of P is the direct sum of all simple modules outside of the support of T. Then the linear form*

$$\alpha = \langle \nu'(T), - \rangle - \langle P, - \rangle$$

defines the torsion triple $(\mathcal{F}(T), \mathcal{N}(T), \mathcal{Q}(T))$.

PROOF. For the proof, see the note N 2.6. □

2.4. The Ingalls-Thomas bijections. We are going to present a large number of sets which correspond bijectively to the set of exceptional antichains.

2.4.1. **Normal partial-tilting modules and support-tilting modules.** We show a bijection between normal partial tilting modules and support-tilting modules. It is sufficient to look at the support of the modules, thus to show a bijection between sincere normal modules and tilting modules.

One direction is given by normalization

$$\{\text{sincere normal modules}\} \;\underset{\nu(-)}{\overset{\dashrightarrow}{\longleftarrow}}\; \{\text{tilting modules}\}$$

The uniqueness of the normalization shows that the map ν going from right to left is well-defined.

The map ν is injective when applied to tilting modules: if T, T' are tilting modules with $\nu(T) = \nu(T')$, then T and T' are isomorphic. Namely, T' is generated by $\nu(T') = \nu(T)$, thus by T. But $T' \in \mathcal{G}(T)$ means that $\operatorname{Ext}^1(T, T') = 0$. Similarly, we see that T belongs to $\mathcal{G}(T')$, thus $\operatorname{Ext}^1(T', T) = 0$. This shows that $T \oplus T'$ is without self-extensions, thus the number of isomorphism classes of indecomposable direct summand of $T \oplus T'$ is equal to the number of simple modules, therefore $\operatorname{add}(T \oplus T') = \operatorname{add} T = \operatorname{add} T'$. Since we assume that T, T' are multiplicity-free, T and T' are isomorphic.

In order to see that ν is surjective, we need to find for any sincere normal partial tilting module N a tilting module T with $\nu(T) = N$, thus we need a module N', such that N' is generated by N, and second, $N \oplus N'$ is a tilting module. The existence (and unicity) of N' is well-known, see the note N 2.7. We call N' a *factor complement* for N.

2.4.2. We need two further definitions: The antichains $A = \{A_1, \ldots, A_t\}$ and $A' = \{A'_1, \ldots, A'_{t'}\}$ are said to be *isomorphic,* provided the modules $\bigoplus_i A_i$ and $\bigoplus_j A'_j$ are isomorphic.

If $\mathcal{C}$ is a subcategory of a module category and $C \in \mathcal{C}$, then C is said to be a *cover* of $\mathcal{C}$ provided any module in $\mathcal{C}$ is generated by C.

THEOREM 2.4.2.1. **(Ingalls-Thomas).** *Let Λ be a hereditary artin algebra. There are bijections between the following data:*

(1) *Exceptional antichains.*
(1′) *Exceptional subcategories.*
(1″) *Normal partial tilting modules.*
(1‴) *Conormal partial tilting modules.*
(2) *Support-tilting modules.*
(3) *Torsion classes with a cover.*
(3′) *Torsionfree classes with a cocover.*

If Λ is in addition representation-finite, then all antichains are exceptional, all thick subcategories have a cover and a cocover, all torsion classes have a cover. all torsionfree classes have a cocover.

Several of these bijections have been mentioned already. We have added here the torsion classes (3), since classical tilting theory asserts the bijection between (2) and (3):

$$\{\text{support tilting modules}\} \;\underset{\text{Ext-projectives}}{\overset{\mathcal{G}(-)}{\rightleftarrows}}\; \{\text{torsion classes with a cover}\}$$

attaching to any support tilting module T the torsion class $\mathcal{G}(T)$ of all the modules generated by T, and to any torsion class $\mathcal{C}$ with a cover the direct sum of all indecomposable Ext^1-projective modules, one from each isomorphism class.

The impressive list of bijections provided by this theorem means of course, that in case Λ is representation finite, say of Dynkin type Δ_n, all the counting problems (as discussed in Chapter 1) have the same answer, namely $\mathbf{a}(\Delta_n)$.

Here are some additional arguments for establishing the asserted bijections:

First, we may show the following: If T is a support-tilting module and $\mathcal{G} = \mathcal{G}(T)$, then $\operatorname{add} T$ is the class of the Ext-projective modules in $\mathcal{G}$. Tilting theory asserts that $\mathcal{G}$ is the class of $\Lambda(T)$-modules M such that $\mathrm{Ext}^1(T, M) = 0$. Let M be in $\mathcal{G}$ and $g\colon T' \to M$ be a right T-approximation of M. Then g is surjective and the kernel M' of g satisfies $\mathrm{Ext}^1(T, M') = 0$, thus belongs to $\mathcal{G}$. If M is Ext-projective, then the exact sequence $0 \to M' \to T' \to M \to 0$ splits, thus M is in $\operatorname{add} T$. This shows that the Ext-projective modules in $\mathcal{G}$ are just the modules in $\operatorname{add} T$.

From (2) to (3); If T is a partial tilting module, let $\mathcal{G}(T)$ be the class of modules generated by T. Then it is well-known (and easy to see) that T is a torsion class. Of course, T is a cover for $\mathcal{G}(T)$.

From (3) to (2): If $\mathcal{C}$ is a torsion class with a cover C, then we attach to it a module T such that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}$. In order to do so, we need to know that the class $\mathcal{E}$ of Ext-projective modules in $\mathcal{C}$ is finite, say $\mathcal{G} = \operatorname{add} T$ for some module T. We also have to show that T is support-tilting.

With C also its normalization $\nu(C)$ is a cover. A normal cover of a torsion class has no self-extension (see Proposition 1 of [**88**]). Let B be a factor complement for $\nu(C)$. As we have seen, $T = \nu(C) \oplus B$ is a support-tilting module. Since B is generated by $\nu(C)$, we have $\mathcal{G}(T) = \mathcal{G}(\nu(C)) = \mathcal{G}(C) = \mathcal{C}$. But we have shown already that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}(T)$.

From (2) to (3) to (2): Let us start with a support-tilting module T and attach to it $\mathcal{G} = \mathcal{G}(T)$. As we have seen, the class of Ext-projectives in $\mathcal{G}$ is $\operatorname{add} T$. We choose T' with $\operatorname{add} T' = \operatorname{add} T$. But this just means that T, T' are Morita equivalent.

From (3) to (2) to (3). We start with a torsion class $\mathcal{C}$ with a cover, we choose a support-tilting module T with $\mathcal{C} = \mathcal{G}(T)$, thus we are back at $\mathcal{C}$.

We have used duality, in order to add some further conditions.

2.4.3. **Remark.** The bijections between the set $(1')$ of thick subcategories $\mathcal{C}$ and the sets (1), $(1'')$ and $(1''')$ can be reformulated as follows: In an abelian category we may look at the semi-simple, the projective and the injective objects: the set of simple objects in $\mathcal{C}$ is an antichain in $\operatorname{mod} \Lambda$, a minimal projective generator in $\mathcal{C}$ is a normal partial tilting module, a minimal injective cogenerator is a conormal partial tilting module.

Conversely, let us start with (1), $(1'')$ or $(1''')$. It has been mentioned already that starting with an antichain A, we take the full subcategory $\mathcal{E}(A)$. Starting with a normal partial tilting module P, the corresponding thick subcategory $\mathcal{C}$ consists of all modules which arise as the cokernel of a map in $\operatorname{add} P$ (in this way, we specify projective presentations of the objects in $\mathcal{C}$). Dually, starting with a conormal partial tilting module I, the corresponding thick subcategory $\mathcal{C}$ consists of all modules which arise as the kernel of a map in add I (in this way, we specify injective presentations of the objects in $\mathcal{C}$).

2.5. Perpendicular pairs and exceptional sequences. If M is a module, the number of isomorphism classes of indecomposable direct summands of M will be called the *rank* of M and denoted by rank M. Of course, the rank of the regular representation ${}_\Lambda\Lambda$ is just the rank of the Grothendieck group $K_0(\Lambda)$. In general, if $\mathcal{A}$ is a length category, we call the rank of the Grothendieck group $K_0(\mathcal{A})$ the rank of $\mathcal{A}$. Thus, if Λ is an artin algebra, the rank of mod Λ is the rank of any projective generator and of any injective cogenerator, and this is the number of simple modules.

We assume in this section that Λ is a hereditary artin algebra of rank n.

2.5.1. Let $\mathcal{U}, \mathcal{V}$ be classes of modules. Let $\mathcal{V}^\perp$ be the full subcategory of all modules U with $\operatorname{Hom}(\mathcal{V}, U) = 0$ and $\operatorname{Ext}^1(\mathcal{V}, U) = 0$. Dually, let ${}^\perp\mathcal{U}$ be the full subcategory of all modules V with $\operatorname{Hom}(V, \mathcal{U}) = 0$ and $\operatorname{Ext}^1(V, \mathcal{U}) = 0$. It is easy to see that subcategories of the form $\mathcal{V}^\perp$ or ${}^\perp\mathcal{U}$ are thick subcategories.

For example, if S is a simple module and $P(S)$ is a projective cover of S, then $P(S)^\perp = \operatorname{mod}\Lambda/\langle e_S\rangle$, where e_S is a primitive idempotent of Λ with $e_S S \neq 0$ and $\langle e_S\rangle$ is the two-sided ideal generated by e_S.

We call $(\mathcal{U}, \mathcal{V})$ a *perpendicular pair* provided $\mathcal{U} = \mathcal{V}^\perp$ and $\mathcal{V} = {}^\perp\mathcal{U}$ (thus, provided $\operatorname{Hom}(V, U) = 0 = \operatorname{Ext}^1(V, U)$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$, and such that the classes $\mathcal{U}$ and $\mathcal{V}$ are maximal with this property); we should point out that writing first $\mathcal{U}$, then $\mathcal{V}$, corresponds to our preference of writing first $\mathcal{F}$, then $\mathcal{G}$ when dealing with a torsion pair $(\mathcal{F}, \mathcal{G})$.

In these lecture, we only will be interested in perpendicular pairs $(\mathcal{U}, \mathcal{V})$ such that $\mathcal{V}$ is exceptional (as we will see, $\mathcal{V}$ is exceptional if and only if $\mathcal{U}$ is exceptional). But we should note that also other perpendicular pairs play a role in the representation theory of artin algebras, see the note N2.8.

LEMMA 2.5.1.1. *Let $\mathcal{V}$ be an exceptional subcategory, let N be a projective generator of $\mathcal{V}$. Then $\mathcal{V}^\perp = N^\perp$.*

PROOF. We only have to show that $N^\perp \subseteq \mathcal{V}^\perp$. Let U belong to $N^\perp$. We have to show that $\operatorname{Hom}(\mathcal{V}, U) = 0$ and $\operatorname{Ext}^1(\mathcal{V}, U) = 0$. Let $V \in \mathcal{V}$. Since V is generated by N, the condition $\operatorname{Hom}(N, U) = 0$ implies that $\operatorname{Hom}(V, U) = 0$. Next, we show that $\operatorname{Ext}^1(V, U) = 0$. Since N is a projective generator of $\mathcal{V}$, there is an exact sequence $0 \to N' \to N'' \to V \to 0$ with N', N'' in add N. It follows that there is an exact sequence $\operatorname{Hom}(N', U) \to \operatorname{Ext}^1(V, U) \to \operatorname{Ext}^1(N'', U)$. Since U belongs to $N^\perp$, it follows that $\operatorname{Ext}^1(V, U) = 0$. □

2.5.2. Let N be a partial tilting module. Given any module X, let us define $\gamma_N(X)$ as follows: Choose an embedding $X \to X'$ such that X'/X is isomorphic to a direct sum of copies of N and such that $\operatorname{Ext}^1(N, X') = 0$. Such an embedding exists: We assume that k is a commutative artinian ring and that Λ is a k-algebra which is module-finite as a k-module. We take a finite generating set $(\epsilon_i)_{i=1}^t$ of $\operatorname{Ext}^1(N, X)$ say as a k-module and form the "universal extension" $0 \to X \to X' \to N^t \to 0$ (so that the t canonical embeddings $N \to N^t$ induce the extensions $\epsilon_1, \dots, \epsilon_t$).

Let $\gamma_N(X) = X'/X''$, where X'' is the N-trace of X' (this is the sum of all images of maps $N \to X'$). Note that this construction is functorial: any map $f\colon X_1 \to X_2$ can be lifted to a map f' $X_1' \to X_2'$, but this lifting is not necessarily unique. Of course, f' maps X_1'' into X_2'', thus it induces a map $\gamma_N(X_1) \to \gamma_N(X_2)$ and this map is uniquely determined by f. Namely, if $f'\colon X_1' \to X_2'$ is a map with zero restriction $f'|X_1$, then f' factors through X_1'/X_1. Since X_1'/X_1 is a direct sum of copies of N, we see that the image of f' is contained in X_2'', thus the induced map $\gamma_N(X_1) \to \gamma_N(X_2)$ is the zero map.

Lemma 2.5.2.1. *Let N be a partial tilting module. Then the functor γ_N maps $\operatorname{mod}\Lambda$ onto $N^\perp$ and its restriction to $N^\perp$ is the identity functor. The module $\gamma_N({}_\Lambda\Lambda)$ is a projective generator for $N^\perp$.*

Proof. Let X be any Λ-module. The epimorphism $X' \to X'/X''$ induces a surjection $\operatorname{Ext}^1(N, X') \to \operatorname{Ext}^1(N, X'/X'')$. Since $\operatorname{Ext}^1(N, X') = 0$, we see that $\operatorname{Ext}^1(N, X'/X'') = 0$. Since N is a partial tilting module, it is easy to see that for any module X' with N-trace X'', the N-trace of X'/X'' is zero. This shows that $\operatorname{Hom}(N, X'/X'') = 0$. Altogether, we see that $\gamma_N(X) = X'/X''$ belongs to $N^\perp$.

If X belongs to $N^\perp$, then $\operatorname{Ext}^1(N, X) = 0$ shows that we can take $X' = X$ with the identity map $X \to X$ as the required embedding. Since $\operatorname{Hom}(N, X) = 0$, the N-trace X'' of $X' = X$ is zero, thus $\gamma_N(X) = X'/X'' = X/0 = X$.

Let $M = \gamma_N({}_\Lambda\Lambda)$ and $X \in N^\perp$. There is an epimorphism $f\colon F \to X$, where F is a free Λ-module, say $F = ({}_\Lambda\Lambda)^s$ for some $s \geq 0$. The map f can be extended to a map $f'\colon F' \to X' = X$. Since the restriction f of f' is surjective, also f' is surjective. f' maps F'' to $X'' = 0$, thus f' induces a map $F'/F'' \to X'/X'' = X$ which again has to be surjective. But this is the map $\gamma_N(f)$, and $F'/F'' = M^s$. This shows that there is a surjective map $M^s \to X$ for some $s \geq 0$, therefore M is a cogenerator for $N^\perp$.

It remains to be seen that M is projective in $N^\perp$. Let X be a module in $N^\perp$. There is an exact sequence $0 \to \Lambda \to \Lambda' \to N^t \to 0$. Since $\operatorname{Ext}^1(\Lambda, X) = 0$ and $\operatorname{Ext}^1(N, X) = 0$, we see that $\operatorname{Ext}^1(\Lambda', X) = 0$. The exact sequence $0 \to \Lambda'' \to \Lambda' \to M \to 0$ yields an exact sequence $\operatorname{Hom}(\Lambda'', X) \to \operatorname{Ext}^1(M, X) \to \operatorname{Ext}^1(\Lambda', X)$. We know already that the last term $\operatorname{Ext}^1(\Lambda', X) = 0$. But we also have $\operatorname{Hom}(\Lambda'', X) = 0$, since Λ'' is generated by N and $\operatorname{Hom}(N, X) = 0$. These two assertions imply that $\operatorname{Ext}^1_\Lambda(M, X) = 0$. Since $N^\perp$ is an exact abelian subcategory of $\operatorname{mod}\Lambda$, the vanishing $\operatorname{Ext}^1_\Lambda(M, X) = 0$ for all X in $N^\perp$ shows that M is projective in $N^\perp$. □

Theorem 2.5.2.2. *Let $(\mathcal{U}, \mathcal{V})$ be a perpendicular pair. Then $\mathcal{U}$ is an exceptional subcategory if and only if $\mathcal{V}$ is an exceptional subcategory.*

Proof. Assume that $\mathcal{V}$ is an exceptional subcategory. Let N be a projective generator of $\mathcal{V}$. As we have seen in Lemma 2.5.1.1 we have $\mathcal{V}^\perp = N^\perp$. Lemma 2.5.2.1 asserts that $\gamma_N({}_\Lambda\Lambda)$ is a projective generator for $N^\perp = \mathcal{V}^\perp$. This shows that $\mathcal{U} = \mathcal{V}^\perp$ is also an exceptional subcategory.

The reverse implication follows by duality. □

2.5.3. **Exceptional sequences.** Recall that a sequence $(E_1, \dots, E_t)$ of indecomposable Λ-modules is said to be *exceptional* provided $\operatorname{Ext}^1(E_i, E_j) = 0$ for $i \geq j$ and $\operatorname{Hom}(E_i, E_j) = 0$ for $i > j$ (in case $t = 2$, one calls it an *exceptional pair*).

PROPOSITION 2.5.3.1. *Let* $(E_1, \ldots, E_t)$ *be an exceptional sequence in* $\operatorname{mod}\Lambda$ *with thick closure* $\mathcal{E}$. *Then* $\mathcal{E}$ *is an exceptional subcategory of rank* t.

Let us stress that for arbitrary exceptional modules E_1, E_2, the thick closure of E_1, E_2 may have arbitrarily large rank, see the note N 2.8.

For the proof of the proposition, we refer to [**84**]. Actually, the main arguments will be outlined in Section 3.5, where we discuss the braid group action on the set of exceptional sequences of fixed length. This braid group action replaces successively an exceptional pair (E, E') by an exceptional pair (E', E'') or (E'', E) with E'' being contained in the thick closure of E, E'. Using the braid group action, one obtains from the given exceptional sequence $(E_1, \ldots, E_t)$ an exceptional sequence $(A_1, \ldots, A_t)$ with the same thick closure, such that the modules $A_1, \ldots, A_t$ are pairwise orthogonal (thus, they form an exceptional antichain). Of course, the thick closure of an exceptional antichain of cardinality t is an exceptional subcategory of rank t.

COROLLARY 2.5.3.2. *Let* $(E_1, \ldots, E_t)$ *be an exceptional sequence in* $\operatorname{mod}\Lambda$ *with thick closure* $\mathcal{E}$. *Then* $t \le n$. *If* $t = n$, *then* $\mathcal{E} = \operatorname{mod}\Lambda$.

PROOF. Let N be a projective generator of $\mathcal{E}$. Then N is a partial tilting module of rank t. This shows that $t \le n$. Now assume that $t = n$. Then N is a tilting module. But the thick closure of a tilting module is $\operatorname{mod}\Lambda$. Namely, there is an exact sequence of the form $0 \to {}_\Lambda\Lambda \to N' \to N'' \to 0$ with $N', N'' \in \operatorname{add}(T)$. This shows that ${}_\Lambda\Lambda$ belongs to the thick closure of N. Of course, any module belongs to the thick closure of ${}_\Lambda\Lambda$. □

COROLLARY 2.5.3.3. *Let* $\mathcal{C}' \subseteq \mathcal{C}$ *be exceptional subcategories of* $\operatorname{mod}\Lambda$. *Let* r *be the rank of* $\mathcal{C}$ *and* r' *the rank of* $\mathcal{C}'$. *Then* $r' \le r$ *and if* $r' = r$, *then* $\mathcal{C}' = \mathcal{C}$.

PROOF. Since $\mathcal{C}$ is an exceptional subcategory, it is equivalent to the module category of a hereditary artin algebra of rank r. The assertions follow directly from Corollary 2.5.3.2. □

THEOREM 2.5.3.4. *If* $\mathcal{V}$ *is an exceptional subcategory of* $\operatorname{mod}\Lambda$ *and has rank* t, *then* $\mathcal{V}^\perp$ *has rank* $n - t$.

PROOF. As we have seen in Theorem 2.5.2.2, the category $\mathcal{U} = \mathcal{V}^\perp$ is again an exceptional subcategory, thus it has finite rank, say rank s. Take an exceptional sequence $(M_1, \ldots, M_s)$ in $\mathcal{C}^\perp$ and an exceptional sequence $(N_1, \ldots, N_t)$ in $\mathcal{C}$. Then $(M_1, \ldots, M_s, N_1, \ldots, N_t)$ is an exceptional sequence in $\operatorname{mod}\Lambda$, thus $s + t \le n$ by Corollary 2.5.3.2. As a consequence, we only have to show that $s \ge n - t$.

We use induction on n. If $n = 1$, then the only exceptional subcategories are 0 and $\operatorname{mod}\Lambda$, and the assertion is clear.

Thus, let us assume that $n \ge 2$. We consider first the case $t = 1$, thus $\mathcal{C} = \operatorname{add} N$ for some exceptional module N. There are two different cases: N may be projective or not. If $N = \Lambda e$ is projective, with e a primitive idempotent of Λ, then $N^\perp = \operatorname{mod}\Lambda/\langle e\rangle$. Of course, $\operatorname{mod}\Lambda/\langle e\rangle$ has rank $n - 1$, as required.

Second, assume that N is not projective. Let $\beta(N)$ be the Bongartz complement for N, see N 2.7. Then $\beta(N)$ belongs to $N^\perp$ and has rank $n - 1$. This shows that $N^\perp$ has rank at least $n - 1$.

Now, let $t \ge 2$, and take an exceptional sequence $(N_1, \ldots, N_t)$ in $\mathcal{C}$. We look at the exceptional subcategory $N_t^\perp$. As we have seen, $N_t^\perp$ has rank $n - 1$. The modules $N_1, \dots, N_{t-1}$ belong to $N_t^\perp$, also the subcategory $\mathcal{C}^\perp$ belongs to $N_t^\perp$. We denote by $\mathcal{C}'$ the thick closure of $N_1, \dots, N_{t-1}$. The subcategory $\mathcal{C}'$ has rank $t-1$ and its perpendicular subcategory inside $N_t^\perp$ is just $\mathcal{C}^\perp$. By induction, we know that the rank of $\mathcal{C}^\perp$ is $(n-1)-(t-1) = n-t$. □

COROLLARY 2.5.3.5. *Let $(E_1, \dots, E_t)$ be an exceptional sequence in* $\operatorname{mod}\Lambda$. *Then there is an exceptional sequence $(M_1, \dots, M_s, E_1, \dots, E_t)$ with $s = n-t$.*

PROOF. Let $\mathcal{E}$ be the thick closure of $E_1, \dots, E_t$ and take for $(M_1, \dots, M_s)$ any exceptional sequence in $\mathcal{E}^\perp$ of cardinality s. Such a sequence exists, since $\mathcal{E}$ is exceptional and has rank s. □

An exceptional sequence $(E_1, \dots, E_t)$ will be said to be *complete* provided $t = \operatorname{rank}(\Lambda)$. The corollary asserts that any exceptional sequence can be completed.

2.6. Torsion pairs and perpendicular pairs. We assume again that Λ is a hereditary artin algebra of rank n. We are going to look at torsion pairs $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{G}$ has a cover and at perpendicular pairs $(\mathcal{U}, \mathcal{V})$ such that $\mathcal{V}$ is an exceptional subcategory. We will see that there is a one-to-one correspondence between these torsion pairs and these perpendicular pairs.

2.6.1. We recall the following. Let $(\mathcal{U}, \mathcal{V})$ be a perpendicular pair. According to Theorem 2.5.2.2, $\mathcal{V}$ is an exceptional subcategory if and only if $\mathcal{U}$ is an exceptional subcategory. If this holds, then Theorem 2.5.3.4 asserts that $\operatorname{rank}\mathcal{U} + \operatorname{rank}\mathcal{V} = n$. Let us now deal with torsion pairs.

LEMMA 2.6.1.1. *Let N be a normal partial tilting module. Then the modules F with* $\operatorname{Hom}(N, F) = 0$ *are the modules cogenerated by $N^\perp$.*

PROOF. First, assume that F is cogenerated by $N^\perp$, thus, there is an embedding $u\colon F \to X$ with $X \in N^\perp$. If $f : N \to F$, then $uf : N \to X$ is zero, thus also $f = 0$.

Conversely, let $\Delta(N) = \{\Delta(1), \dots, \Delta(t)\}$ be the antichain corresponding to N, thus this is an antichain consisting of modules which are cokernels of maps in $\operatorname{add} N$ such that N belongs to $\mathcal{E}(\Delta(N))$. As we know, $\Delta(N)$ is an exceptional antichain, thus we can assume that $\operatorname{Ext}^1(\Delta(j), \Delta(i)) = 0$ for $j \le i$ (note that often one deals with the opposite ordering, but here we deviate from the usual ordering). Since any $\Delta(i)$ is generated by N, any module F' with $\operatorname{Hom}(N, F') = 0$ satisfies $\operatorname{Hom}(\Delta(i), F') = 0$ for all i.

Let $F_0 = F$. Using induction, we construct for $1 \le i \le t$ exact sequences

$$(*) \qquad 0 \to F_{i-1} \to F_i \xrightarrow{p_i} \Delta(i)^{m_i} \to 0$$

such that $\operatorname{Hom}(N, F_i) = 0$ and $\operatorname{Ext}^1(\Delta(j), F_i) = 0$ for $1 \le j \le i$ and such that m_i is chosen minimal. Note that the conditions $\operatorname{Hom}(N, F_i) = 0$ and $\operatorname{Ext}^1(\Delta(j), F_i) = 0$ for $1 \le j \le i$, are satisfied for $i = 0$. Assume that the module F_{i-1} has been constructed already. We take for $(*)$ the universal extension of F_{i-1} from above, using copies of $\Delta(i)$. Thus, by construction, $\operatorname{Ext}^1(\Delta(i), F_i) = 0$. Also, $\operatorname{Hom}(\Delta(i), F_i) = 0$. Namely, a non-zero homomorphism $f : \Delta(i) \to F_i$ cannot map into F_{i-1}, since $\operatorname{Hom}(\Delta(i), F_{i-1}) = 0$, therefore $p_i f \ne 0$. However, $\Delta(i)$ is a brick, thus $p_i f$ has to be a split monomorphism. But this contradicts the minimality of m_i.

We also have $\operatorname{Hom}(\Delta(j), F_i) = 0$ for all $j \neq i$, since $\operatorname{Hom}(\Delta(j), \Delta(i)) = 0$ and, by induction $\operatorname{Hom}(\Delta(j), F_{i-1}) = 0$. Finally, we assert that $\operatorname{Ext}^1(\Delta(j), F_i) = 0$ for all $j < i$. This follows directly from the induction hypothesis that $\operatorname{Ext}^1(\Delta(j), F_{i-1}) = 0$ for all $j < i$ and the fact that $\operatorname{Ext}^1(\Delta(j), \Delta(i)) = 0$ for $j < i$. This completes the inductive construction.

We have obtained in this way an embedding $F \to F_t$ and F_t satisfies both $\operatorname{Hom}(N, F_i) = 0$ and $\operatorname{Ext}^1(\Delta(j), F_t) = 0$ for $1 \leq j \leq t$, therefore also $\operatorname{Ext}^1(N, F_t) = 0$. This shows that F_t belongs to $N^\perp$. □

As a consequence, there is the following result is due to Smalø [**103**].

THEOREM 2.6.1.2. **(Smalø)** *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair. Then $\mathcal{G}$ has a cover if and only if $\mathcal{F}$ has a cocover. If N is a minimal cover of $\mathcal{G}$ and M is a minimal cover of $\mathcal{F}$, then* $\operatorname{rank} M + \operatorname{rank} N = n$.

PROOF. Let N be a minimal generator of $\mathcal{G}$ and $\mathcal{N}$ the thick closure of N. Then N is a normal partial tilting module. A module F belongs to $\mathcal{F}$ if and only if $\operatorname{Hom}(N, F) = 0$, thus, according to Lemma 2.6.1.1 if and only if F is cogenerated by $N^\perp$. Since $\mathcal{M} = N^\perp$ is an exceptional subcategory, it has a minimal cogenerator, say M. Then $M \in N^\perp \subseteq \mathcal{F}$ and any module $F \in \mathcal{F}$ is cogenerated by M. This shows that M is a minimal cocover of $\mathcal{F}$.

We have $\operatorname{rank} N = \operatorname{rank} \mathcal{N}$ and $\operatorname{rank} M = \operatorname{rank} N^\perp$. According to Theorem 2.5.3.4, we have

$$\operatorname{rank} N + \operatorname{rank} M = \operatorname{rank} \mathcal{N} + \operatorname{rank} \mathcal{M} = n.$$

□

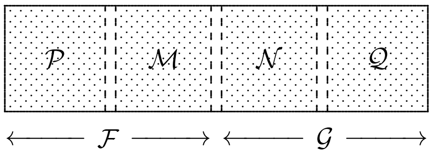
2.6.2. Let us summarize the previous considerations.

THEOREM 2.6.2.1. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair. Assume that N is a minimal cover of $\mathcal{G}$ and M a minimal cocover of $\mathcal{F}$. If $\mathcal{N}$ is the thick closure of N and $\mathcal{M}$ the thick closure of M, then $(\mathcal{M}, \mathcal{N})$ is a perpendicular pair and the subcategories $\mathcal{N}$ and $\mathcal{M}$ are exceptional.*

Conversely, assume that $(\mathcal{U}, \mathcal{V})$ is a perpendicular pair such that $\mathcal{V}$ is exceptional. Let $\mathcal{G}$ be the modules generated by modules in $\mathcal{V}$, let $\mathcal{F}$ be the modules cogenerated by modules in $\mathcal{U}$. Then $(\mathcal{F}, \mathcal{G})$ is a torsion pair such that $\mathcal{G}$ has a cover and $\mathcal{F}$ a cocover.

We obtain in this way a bijection between the torsion pairs $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{G}$ has a cover, and the perpendicular pairs $(\mathcal{U}, \mathcal{V})$ such that $\mathcal{V}$ is exceptional.

Let us assume that $(\mathcal{F}, \mathcal{G})$ is a torsion pair and that N is a minimal cover of $\mathcal{G}$. Then N is a normal partial tilting module. Let $N \oplus N'$ be a support tilting module with N' generated by N. We denote by $\mathcal{Q}$ the full subcategory of all modules generated by N'. Dually, let M be a minimal cocover of $\mathcal{F}$, thus M is a conormal partial tilting module. Let $M \oplus M'$ be a support tilting module with M' cogenerated by M and let $\mathcal{P}$ be the full subcategory of all modules cogenerated by M'. The four subcategories $\mathcal{P}, \mathcal{M}, \mathcal{N}, \mathcal{Q}$ should be seen as being arranged as follows:



with no maps backwards, and such that any module Z has a filtration

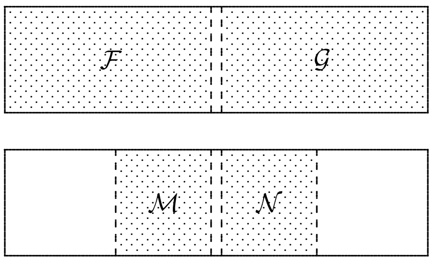
$$Z = Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq Z_4 = 0$$

such that

$$Z_0/Z_1 \in \mathcal{P}, \quad Z_1/Z_2 \in \mathcal{M}, \quad Z_2/Z_3 \in \mathcal{N}, \quad Z_3/Z_4 \in \mathcal{Q},$$

(thus we deal with what one may call a *torsion quadruple* $(\mathcal{P}, \mathcal{M}, \mathcal{N}, \mathcal{Q})$; it refines the given torsion pair $(\mathcal{F}, \mathcal{G})$, since $\mathcal{F} = \mathcal{P}\lceil\mathcal{M}$ and $\mathcal{G} = \mathcal{N}\lceil Q)$.

The bijection between torsion pairs $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{G}$ has a cover and perpendicular pairs $(\mathcal{M}, \mathcal{N})$ of exceptional subcategories may be remembered as follows:

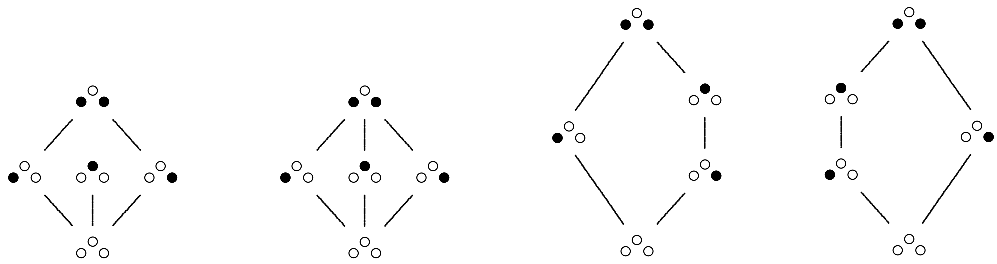


There is given a normal partial tilting module N such that $\mathcal{G}$ are the modules generated by N, whereas $\mathcal{N}$ is the thick closure of N. And, there is given a conormal partial tilting module M such that $\mathcal{F}$ are the modules generated by M, whereas $\mathcal{M}$ is the thick closure of M. It remains to mention that the bijectivity assertion is, of course, part of the Ingalls-Thomas Theorem. For the relationship between a minimal cover N for $\mathcal{G}$ and a minimal cocover M for $\mathcal{F}$, see the note N 2.9.

2.7. Partial orderings on the set of antichains. For any hereditary artin algebra Λ, one may consider the set of antichains in $\operatorname{mod}\Lambda$ and its subset of exceptional antichains. There are several partial orderings on these sets which should not be confused.

- $\leq_s$ the set-inclusion of antichains,
- $\leq$ the set-inclusion of the corresponding thick subcategories, thus $A \leq A'$ means that every element of A has a filtration by objects in in A', thus every element of A belongs to $\mathcal{E}(A')$.
- $\leq_t$ the set-inclusion of the corresponding torsion classes, thus $A \leq_t A'$ means that every element of A is generated by a module in $\mathcal{E}(A')$,
- $\leq_f$ the set-inclusion of the corresponding torsionfree classes, thus $A \leq_f A'$ means that every element of A is cogenerated by a module in in $\mathcal{E}(A')$.

Already the case $\mathbb{A}_2$ shows that all these partial orderings may be different.



$\leq_s$ $\qquad$ $\leq$ $\qquad$ $\leq_t$ $\qquad$ $\leq_f$

We see in this case: the partial ordering $\leq_s$ does not yield a lattice, the three other partial orderings yield lattices (the definition of a lattice will be recalled in N 2.10, And this is a general fact in case Λ is representation-finite:

The partial ordering $\leq$ yields a lattice, namely the lattice of thick subcategories. The partial ordering $\leq_t$ yields a lattice, namely the lattice of torsion subcategories. The partial ordering $\leq_f$ yields a lattice, namely the lattice of torsionfree subcategories.

The partial ordering $\leq$ seems to be the most import one. This is the structure on $A(\operatorname{mod}\Lambda)$ which we will discuss in the next chapter. (But let us add that also the lattice of torsion subcategories, as well as dually, the lattice of torsionfree subcategories are relevant, see the note N 2.11.)

Notes to Chapter 2.

N 2.1. Proof of Lemma 2.1.1.5. Assume that P belongs to $\mathcal{F} = \mathcal{F}(\alpha)$, that R, R' belong to $\mathcal{N} = \mathcal{N}(\alpha)$ and that Q belongs to $\mathcal{Q} = \mathcal{Q}(\alpha)$.

Let $f : R \to P$ be a non-zero homomorphism. The image X is a non-zero submodule of P, thus $\alpha(X) < 0$ and a factor module of R, thus $\alpha(X) \geq 0$, a contradiction.

Let $f : Q \to P$ be a non-zero homomorphism. The image X is a non-zero submodule of P, thus $\alpha(X) < 0$ and a non-zero factor module of Q, thus $\alpha(X) \geq 0$, a contradiction.

Let $f : Q \to R$ be a non-zero homomorphism. The image X is a submodule of R, thus $\alpha(X) \leq 0$ and a non-zero factor module of Q, thus $\alpha(X) \geq 0$, a contradiction.

Let $f : R \to R'$ be a homomorphism. The image X of f is a submodule of R', thus $\alpha(X) \leq 0$, and a factor module of R, thus $\alpha(X) \geq 0$. This shows that $\alpha(X) = 0$. Any submodule X' of X is a submodule of R', thus $\alpha(X') \leq 0$. Thus X belongs to $\mathcal{N}$. Let us show that $\mathcal{N}$ is closed under kernels. Since $\mathcal{N}$ is closed under images, we may assume that $f : R \to R'$ is an epimorphism and consider its kernel Y. We have $\alpha(Y) = \alpha(R) - \alpha(R') = 0$. Also, a submodule Y' of Y is a submodule of R, thus $\alpha(Y') \leq 0$. This shows that Y belongs to $\mathcal{N}$. In order to show that $\mathcal{N}$ is closed under cokernels, it is sufficient to look at the cokernel Z of a monomorphisms $f : R \to R'$. We have $\alpha(Z) = \alpha(R) - \alpha(R') = 0$. Also, any factor module Z' of Z is a factor module of R', thus $\alpha(Z') \geq 0$. This shows that Z belongs to $\mathcal{N}$.

In order to show that $\mathcal{N}$ is closed under extensions, let N be a module with a submodule N' such that both N' and N/N' belong to $\mathcal{N}$. We have $\alpha(N) = \alpha(N') + \alpha(N/N') = 0$. Let U be a submodule of N and $U' = U \cap N'$. Then U' is a submodule of N', thus $\alpha(U') \leq 0$. Since $U/U' = U/(U \cap N') \simeq (U + N')/N'$ is isomorphic to a submodule of N/N', we N belongs to $\mathcal{N}$.

It remains to be shown that every module M has a filtration $M'' \subseteq M' \subseteq M$ with $M'' \in \mathcal{Q}$, $M'/M'' \in \mathcal{N}$, $M/M' \in \mathcal{F}$. As a preparation, we show that $\mathcal{F}$ and $\mathcal{Q}$ are closed under extensions.

Let N be a module with a submodule N' such that both N' and N/N' belong to $\mathcal{F}$. We have to look at non-zero submodules U of N. Consider the submodule $U \cap N'$ of U with factor module $U/(U \cap N') \simeq (U + N')/N'$, thus, we have $\alpha(U) = \alpha(U \cap N') + \alpha((U + N')/N')$. Both summands are non-positive, since we evaluate α at submodules of modules in $\mathcal{F}$. Since U is non-zero, at least one of the modules $U \cap N'$, $(U + N')/N'$ is non-zero and therefore at least one of the values $\alpha(U \cap N')$, $\alpha((U + N')/N')$ is negative, thus $\alpha(U) < 0$.

Similarly, take a module N with a submodule N' such that both N' and N/N' belong to $\mathcal{Q}$. To show that N is in $\mathcal{Q}$, we have to look at non-zero factor modules of N. Thus, let U be a proper submodule of N. The module N/U has the submodule $(U + N')/U \simeq N'/(U \cap N')$, with factor module isomorphic to $N/(U + N')$, thus $\alpha(N/U) = \alpha(N'/(U \cap N')) + \alpha(N/(U + N'))$. Both summands are non-negative, since we deal with factor modules of modules in $\mathcal{Q}$. Since N/U is non-zero, at least one of $N'/(U \cap N')$, $N/(U + N')$ is non-zero, thus at least one $\alpha(N'/(U \cap N'))$, $\alpha(N/(U + N'))$ is positive, therefore $\alpha(N/U) > 0$.

Now we show that every module has a filtration with factors in $\mathcal{F}$, $\mathcal{N}$, $\mathcal{Q}$. First, we show that a module X which has no non-zero factor module in $\mathcal{F}$ and no non-zero submodule in $\mathcal{Q}$ belongs to $\mathcal{N}$. Assume that there is a submodule X' of X with $\alpha(X') > 0$. Choose such a submodule X' which is minimal with this property. We claim that X' belongs to $\mathcal{Q}$. Namely, if X'/U is a non-zero factor module of X' (here U is a submodule of X'), then the minimality of X' implies that $\alpha(U) \leq 0$, thus $\alpha(X'/U) = \alpha(X') - \alpha(U) > 0$. But we assume that X has no non-zero submodule in $\mathcal{Q}$. This contradiction shows that $\alpha(X') \leq 0$ for any submodule X' of X. By duality, we have $\alpha(X'') \leq 0$ for all factor modules X'' of X. In particular, we have both $\alpha(X) \geq 0$ and $\alpha(X) \leq 0$, thus $\alpha(X) = 0$. Altogether we see that X belongs to $\mathcal{N}$.

Second case: Let N be a module without any non-zero submodule in $\mathcal{Q}$. Let N' be a submodule of N such that N/N' belongs to $\mathcal{F}$ and which is minimal with this property. Since $\mathcal{F}$ is closed under extensions, we know that no proper factor module of N' belongs to $\mathcal{F}$. According to the first case, we see that N' belongs to $\mathcal{N}$.

Now we look at the general case. Thus, let M be an arbitrary module. Let M'' be a submodule of M which belongs to $\mathcal{Q}$ and which is maximal with this property. Since we know that $\mathcal{Q}$ is closed under extensions, we know that M/M'' has no non-zero submodule which belongs to $\mathcal{Q}$. By the second case, the module $N = M/M''$ has a submodule N' in $\mathcal{N}$ such that N/N' belongs to $\mathcal{F}$. Let $N' = M'/M''$ for some $M'' \subseteq M' \subseteq M$. Then M'/M'' is in $\mathcal{N}$, whereas $M/M' \simeq N/N'$ belongs to $\mathcal{F}$. This completes the proof.

N 2.2. Examples of torsion pairs. Let us stress that the tilting torsion pairs, even the support-tilting torsion pairs, are quite special torsion pairs. These are linear torsion pairs, but *torsion pairs are usually not linear.*

For example, take the path algebra Λ of the Kronecker quiver

$$\overset{1}{\circ} \leftleftarrows \overset{2}{\circ},$$

take the simple regular representation $R(0) = (k, k; 1, 0)$ and let $\mathcal{G}$ be the torsion class generated by $R(0)$: it consists of one homogeneous tube and all the preinjective modules, the corresponding torsionfree class consists of the remaining tubes and all the preprojective modules. In particular, the simple regular representations and $R(1) = (k, k; 1, 1)$ and $R(\infty) = (k, k; 0, 1)$ both belong to $\mathcal{F}$, but it is not possible to distinguish say $R(0)$ and $R(1)$, using linear forms on $K_0(\Lambda)$.

Also: *Not all linear torsion pairs are tilting torsion pairs or at least support-tilting torsion pairs.* Again, let Λ be the path algebra of the Kronecker quiver and consider the defect function: it takes negative values on the preprojective Kronecker modules, positive values on the preinjective Kronecker modules and vanishes on the regular Kronecker modules.

It is easy to write down all the linear forms of $K_0(\Lambda)$ for the Kronecker algebra Λ, since $K_0(\Lambda) = \mathbb{Z}^2$. We work with the basis $e(1) = (1, 0) = \mathbf{dim}\, S(1)$, $e(2) = (0, 1) = \mathbf{dim}\, S(2)$. In general, given a pair a, b of integers, we define the linear form $\alpha_{(a,b)} \colon K_0(\Lambda) \to \mathbb{Z}$ by $\alpha_{(a,b)}(d_1, d_2) = ad_1 + bd_2$.

Of importance is the linear form $\alpha = \alpha_{(-1,1)}$, thus $\alpha(d_1, d_2) = -d_1 + d_2$. This is the defect function. The classification of the indecomposable Kronecker modules asserts: Any indecomposable Kronecker module belongs to one of the three classes $\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha)$, the modules in $\mathcal{F}(\alpha)$ are the preprojective modules, those in $\mathcal{N}(\alpha)$ the regular modules, those in $\mathcal{Q}(\alpha)$ the preinjective modules.

Let us mention some other special cases in order to outline the variety of possibilities. If $a = b = 0$, then $(\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha)) = (0, \operatorname{mod} \Lambda, 0)$. If both numbers a, b are positive, then we have $(\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha)) = (0, 0, \operatorname{mod} \Lambda)$. If both numbers a, b are negative, then we have $(\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha)) = (\operatorname{mod} \Lambda, 0, 0)$.

For $(a, b) = (-4, 3)$, the category $\mathcal{N}(\alpha_{(-4,3)})$ is $\operatorname{add} P(3)$. The indecomposable modules in $\mathcal{F}(\alpha_{(-4,3)})$ are the Kronecker modules $P(0), P(1), P(2)$, and the modules in $\mathcal{Q}(\alpha_{(-4,3)})$ are the Kronecker modules without a direct summand of the form $P(0), \dots, P(3)$.

For $(a, b) = (-5, 3)$, the category $\mathcal{N}(\alpha_{(-5,3)})$ is the zero category. The indecomposable modules in $\mathcal{F}(\alpha_{(-5,3)})$ are the Kronecker modules $P(0), P(1)$, and the modules in $\mathcal{Q}(\alpha_{(-5,3)})$ are the Kronecker modules without a direct summand of the form $P(0), P(1)$.

Finally, let us remark that *not every linear torsion pair is defined by a linear form with values in $\mathbb{Z}$.* As an example, we can take any wild hereditary algebra. If Λ is the n-Kronecker algebra with $n \geq 3$, say with simple projective module S and simple injective module T, define α by $\alpha(S) = -1$ and $\alpha(T) = \sqrt{2}$. It is not difficult to show that the torsion pair $(\mathcal{F}(\alpha), \mathcal{G}(\alpha))$ cannot be defined by a rational linear form. Other obvious examples are provided by tubular algebras.

N 2.3. Proof of Roiter's Normalization Lemma (following [**85**]). We use the following facts, here X, Y are modules over an artin algebra.

(a) *Let $(f_1, \dots, f_t, g) \colon X \to X^t \oplus Y$ be an injective map for some t, with all the maps f_i in the radical of* $\operatorname{End}(X)$. *Then X is cogenerated by Y.*

(b) *Let $(f_1, \dots, f_t, g) \colon X^t \oplus Y \to X$ be a surjective map for some natural number t, with all the maps f_i in the radical of* $\operatorname{End}(X)$, *then Y generates X.*

Proof. (a) Assume that the radical J of $\operatorname{End}(X)$ satisfies $J^m = 0$. Let W be the set of all compositions w of at most $m-1$ maps of the form f_i with $1 \leq i \leq t$ (including $w = 1_X$). We claim that $(gw)_{w\in W} : X \to Y^{|W|}$ is injective. Take a non-zero element x in X. Then there is $w \in W$ such that $w(x) \neq 0$ and $f_i w(x) = 0$ for $1 \leq i \leq t$. Since $(f_1, \dots, f_t, g)$ in injective and $w(x) \neq 0$, we have $(f_1, \dots, f_t, g)(w(x)) \neq 0$. But $f_i w(x) = 0$ for $1 \leq i \leq t$, thus $g(w(x)) \neq 0$. This completes the proof.

(b) This follows by duality. □

Normalization lemma. *Let $M = M_0 \oplus M_1 = M_0' \oplus M_1'$ be direct decompositions of a module M over an artin algebra and assume that both modules M_0 and M_0' generate M. Then there is a module N which generates M and which is isomorphic to a direct summand of M_0 and M_0'.*

Proof. We may assume that M is multiplicity free. Write $M_0 \simeq N \oplus C$, $M_0' \simeq N \oplus C'$, such that C, C' have no indecomposable direct summand in common. Now, $N \oplus C$ generates $N \oplus C'$ generates $N \oplus C$ generates C. We see that $N \oplus C$ generates C, such that the maps $C \to C$ used belong to the radical of $\operatorname{End}(C)$ (since they factor through $\operatorname{add}(N \oplus C')$ and no indecomposable direct summand of C belongs to $\operatorname{add}(N \oplus C')$). According to (b), N generates C, thus it generates M. □

N 2.4. The simple $\Gamma(T)$-modules. Let T be a tilting module and $\Gamma(T) = \operatorname{End}(T)^{\mathrm{op}}$. A simple $\Gamma(T)$ module belongs either to $\mathcal{Y}(T)$ or to $\mathcal{X}(T)$, since we deal with a torsion pair $(\mathcal{Y}(T), \mathcal{X}(T)$ (actually, this torsion pair is even a split torsion pair, thus any indecomposable module belongs to one of the two subcategories). The presentation of the classical tilting theory in the paper [**49**] used this fact quite prominently, but the actual structure of the simple $\Gamma(T)$-modules has not been discussed there. The new approach to the classical tilting theory stresses the relevance of thick subcategories, and, as it turns out, it is the structure of the simple $\Gamma(T)$-modules which plays a decisive role!

Let $T = \bigoplus_{i=1}^n T_i$ be a tilting module with indecomposable direct summands T_i. The indecomposable projective $\Gamma(T)$-modules are of the form $\operatorname{Hom}(T, T_i)$, let us denote the top of $\operatorname{Hom}(T, T_i)$ by $S'(i)$.

For any i, let $T^{(i)} = T/T_i = \bigoplus_{j\neq i} T_j$ and let $g_i \colon T^i \to T_i$ be a minimal right $T^{(i)}$-approximation of T_i.

First case. *If T_i is a direct summand of $\nu(T)$, then g_i is injective, say with cokernel $\Delta(i)$. Then* $\operatorname{Hom}(T, \Delta(i))$ *is a simple* $\Gamma(T)$*-module, it belongs to* $\mathcal{Y}(T)$ *and its projective cover is* $\operatorname{Hom}(T, T_i)$, *thus* $S'(i) = \operatorname{Hom}(T, \Delta(i))$.

Proof: The map g_i cannot be surjective, since otherwise T_i would belong to $\nu'(T)$. But then g_i has to be injective since $\operatorname{Ext}^1(T_i, T^i) = 0$. Thus, there is an exact sequence of the form

$$0 \to T^i \xrightarrow{g_i} T_i \xrightarrow{p_i} \Delta(i) \to 0$$

(just as the sequence $(*)$). We apply the functor $\operatorname{Hom}(T, -)$ and obtain the exact sequence

$$(**) \qquad 0 \to \operatorname{Hom}(T, T^i) \xrightarrow{\operatorname{Hom}(T,g_i)} \operatorname{Hom}(T, T_i) \xrightarrow{\operatorname{Hom}(T,p_i)} \operatorname{Hom}(T, \Delta(i)) \to 0$$

(the right exactness follows from $\operatorname{Hom}(T,T^i)=0$). Now, $\operatorname{Hom}(T,T_i)$ is indecomposable projective and $Y(i)=\operatorname{Hom}(T,\Delta(i))$ is a factor module. In order to see that $Y(i)$ is simple, we only have to show that any proper submodule U of $\operatorname{Hom}(T,T_i)$ is contained in the image of $\operatorname{Hom}(T,g_i)$. It is sufficient to show this for any proper submodule U which is local. Such a submodule is a factor module of some $\operatorname{Hom}(T,T_j)$. Thus there is given a map $h: T_j \to T_i$ which is not an isomorphism such that $\operatorname{Hom}(T,h)$ maps into U. Since the endomorphism ring of T_i is a division ring, we must have $j\neq i$ and therefore h factors through g_i. But this is what we wanted to show.

Thus we see: The module $Y(i)=\operatorname{Hom}(T,\Delta(i))$ is simple. Obviously, $(**)$ is a minimal projective resolution of $Y(i)$ as a $\Gamma(T)$-module. What is of importance for us is the fact that $Y(i)$ belongs to $\mathcal{Y}(T)$, but this is clear, since $Y(i)=\operatorname{Hom}(T,\Delta(i))$.

Second case. *If T_i is a direct summand of $\nu'(T)$, then g_i is surjective, say with kernel $U(i)$. Then* $\operatorname{Ext}^1(T,U(i))$ *is a simple* $\Gamma(T)$*-module, it belongs to* $\mathcal{X}(T)$ *and its projective cover is* $\operatorname{Hom}(T,T_i)$, *thus* $S'(i)=\operatorname{Ext}^1(T,U(i))$. Proof. To see that g_i is surjective, we only have to note that T_i is generated by $\nu(T)$, thus by $T^{(i)}$. We apply $\operatorname{Hom}(T,-)$ to the exact sequence

$$0\to U(i)\xrightarrow{u_i} T^i \xrightarrow{g_i} T_i \to 0$$

and obtain the exact sequence

$$0\to \operatorname{Hom}(T,U(i)) \xrightarrow{\operatorname{Hom}(T,u_i)} \operatorname{Hom}(T,T^i) \xrightarrow{\operatorname{Hom}(T,g_i)} \operatorname{Hom}(T,T_i) \xrightarrow{d_i} \operatorname{Ext}^1(T,U(i)) \to 0.$$

$(***)$

The map g_i is not a split epimorphism, since T_i does not belong to $\operatorname{add} T^{(i)}$. It follows that $\operatorname{Hom}(T,g_i)$ is not surjective, thus $\operatorname{Ext}^1(T,U(i))\neq 0$. Since $\operatorname{Hom}(T,T_i)$ is indecomposable projective, the map d_i is a projective cover. In particular, $\operatorname{Ext}^1(T,U(i))$ is a local module. with top $S'(i)$.

We look at the dimension vector of $\operatorname{Ext}^1(T,U(i))$. For any index j, the simple module $S'(j)$ occurs as a composition factor of $\operatorname{Ext}^1(T,U(i))$ if and only if $\operatorname{Ext}^1(T_j,U(i))\neq 0$.

Consider an index $j\neq i$ and apply $\operatorname{Hom}(T_j,-)$ to g_i. We obtain the exact sequence

$$\operatorname{Hom}(T_j,T^i) \xrightarrow{\operatorname{Hom}(T_j,g_i)} \operatorname{Hom}(T_j,T_i) \to \operatorname{Ext}^1(T_j,U(i)) \to \operatorname{Ext}^i(T_j,T^i).$$

The last term is zero since $\operatorname{Ext}^1(T,T)=0$. The map $\operatorname{Hom}(T_j,g_i)$ is surjective, since g_i is a $T^{(i)}$-approximation and T_j is a direct summand of $T^{(i)}$. This shows that $\operatorname{Ext}^1(T_j,U(i))=0$.

As a consequence, all composition factors of $\operatorname{Ext}^1(T,U(i))$ are equal to $S'(i)$. But the quiver of $\Gamma(T)$ has no loops, thus $\operatorname{Ext}^1(T,U(i))=S'(i)$. In particular, $S'(i)$ belongs to $\mathcal{X}(T)$.

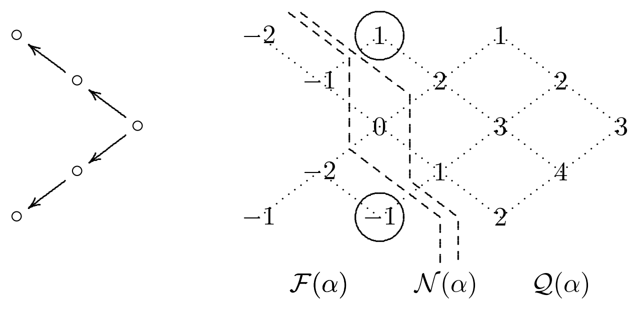
The projective dimension of the simple $\Gamma(T)$-modules $S'(i)$. If T_i is a direct summand of $\nu(T)$, then we have seen that $S'(i)$ belongs to $\mathcal{Y}(T)$. All the modules in $\mathcal{Y}(T)$ have projective dimension at most 1, thus this holds true for $S'(i)$. A minimal projective resolution of $S'(i)$ is given by the sequence $(**)$. On the other hand, in case T_i is a direct summand of $\nu'(T)$, then $S'(i)$ belongs to $\mathcal{X}(T)$, thus the projective dimension of $S'(i)$ may be 2. In case $U(i)$ belongs to $\mathcal{F}(T)$, the exact sequence $(***)$ shows that the projective dimension of $S'(i)$ is equal to 1 (it cannot be zero, since $T^i \neq 0$; of course, we know that the indecomposable projective $\Gamma(T)$-modules belong to $\mathcal{Y}(T)$, thus not to $\mathcal{X}(T)$). On the other hand, if $U(i)$ does not belong to $\mathcal{F}(T)$, then $\operatorname{Hom}(T, U(i) \neq 0$ and the projective dimension of $S'(i)$ is equal to 2.

The subcategory $\phi(\mathcal{N}(T))$. As we know, the functor $\phi = \operatorname{Hom}(T, -)$ yields an equivalence $\mathcal{G}(T) \to \mathcal{Y}(T)$. Since $\operatorname{Ext}^1(T, -)$ vanishes on the category $\mathcal{Y}(T)$, the functor ϕ is exact on exact sequences in $\mathcal{Y}(T)$ (we mean: exact sequences of $\operatorname{mod}\Lambda$ with all terms lying in $\mathcal{Y}(T)$). Now the subcategory $\mathcal{N}(T)$ consists of all Λ-modules with a filtration with factors of the from $\Delta(i)$, thus $\phi(\mathcal{N}(T))$ consists of all the $\Gamma(T)$-modules with a composition series with all factors in $\mathcal{Y}(T)$. This can be reformulated as follows: *$\phi(\mathcal{N}(T))$ is the largest Serre subcategory of* $\operatorname{mod}\Gamma(T)$ *which lies inside $\mathcal{Y}(T)$.*

The set of simple $\Gamma(T)$-modules which belong to $\mathcal{Y}(T)$ is closed under successors in the quiver of $\Gamma(T)$. For the proof, assume that $S'(i), S'(j)$ are simple $\Gamma(T)$-modules with $\operatorname{Ext}^1(S'(i), S'(j)) \neq 0$, and that $S'(i)$ belongs to $\mathcal{Y}(T)$. We have to show that also $S'(j)$ belongs to $\mathcal{Y}(T)$. Since $S'(i)$ belongs to $\mathcal{Y}(T)$, the module T_i is a direct summand of $\nu(T)$. But then also T^i belongs to $\operatorname{add}\nu(T)$. On the other hand, $\operatorname{Ext}^1(S'(i), S'(j)) \neq 0$ means that T_j is a direct summand of T^i, thus in $\operatorname{add}\nu(T)$. It follows that $S'(j)$ belongs to $\mathcal{Y}(T)$.

N 2.5. An example. Here is an example *of a linear form α on $K_0(\Lambda)$ with $\mathcal{N}(\alpha)$ a sincere exceptional subcategory such that $\mathcal{Q}(\alpha)$ is not generated by $\mathcal{N}(\alpha)$ and and such that $\mathcal{F}(\alpha)$ is not cogenerated by $\mathcal{N}(\alpha)$.*

We take as Λ the path algebra of a quiver of type $\mathbb{A}_5$, as shown on the left. We draw the Auslander-Reiten quiver of Λ, but replace the indecomposable modules M by the corresponding values $\alpha(M)$. It is easy to check that these are indeed the values of an additive function on $K_0(\Lambda)$.



The separation between the subcategories $\mathcal{F}(\alpha), \mathcal{N}(\alpha), \mathcal{Q}(\alpha)$ is indicated by dashed lines. In particular, we see that $\mathcal{N}(\alpha) = \operatorname{add}(M)$, where M is the unique sincere indecomposable module. Two modules (or better, their values under α) have been encircled: the upper one belongs to $\mathcal{Q}(\alpha)$, it is not generated by M, the lower one belongs to $\mathcal{F}(\alpha)$, it is not cogenerated by M.

N 2.6. Proof of Theorem 2.3.5.1. It is sufficient to show the inclusions

$$\mathcal{F}(T) \subseteq \mathcal{F}(\alpha),\ \mathcal{N}(T) \subseteq \mathcal{N}(\alpha),\ \mathcal{Q}(T) \subseteq \mathcal{Q}(\alpha).$$

The last two assertions concern modules M with support in $\mathbf{S}$, where $\mathbf{S}$ is the support of T. If the support of M is in $\mathbf{S}$, then $\mathrm{Hom}(P, M) = 0$, thus α and $\langle \nu'(T), - \rangle$ coincide on all submodules and all factor modules of M. Thus, we only have to show the first inclusion. Assume that M belongs to $\mathcal{F}(T)$, let M' be a non-zero submodule of M. If the support of M' is not contained in $\mathbf{S}$, then $\mathrm{Hom}(P, M') \neq 0$ (and $\mathrm{Hom}(\nu'(T).M) = 0$, thus $\alpha(M') < 0$. If the support of M' is contained in $\mathbf{S}$, then M' is a non-zero Λ_T-module with $\mathrm{Hom}(T, M') = 0$, thus we are in the setting of dealing with tilting modules and, as we have seen, $\langle \nu'(T), M' \rangle < 0$.

N 2.7. The Bongartz complement and the dual construction. Let T be a partial tilting module. Choose a universal extension of Λ by copies of T, say $0 \to \Lambda \to \Lambda' \to T^t \to 0$ with $t \in \mathbb{N}$, such that $\mathrm{Ext}^1(T, \Lambda') = 0$, or, equivalently, such that the connecting homomorphism $\mathrm{Hom}(T, T^t) \to \mathrm{Ext}^1(T, \Lambda)$ is surjective. It is easy to see that $\mathrm{Ext}^1(T \oplus \Lambda', T \oplus \Lambda') = 0$. Namely, first of all, we have $\mathrm{Ext}^1(T, T) = 0$ and $\mathrm{Ext}^1(T, \Lambda') = 0$. Next, $\mathrm{Ext}^1(\Lambda, T) = 0$ and $\mathrm{Ext}^1(T, T) = 0$ imply that $\mathrm{Ext}^1(\Lambda', T) = 0$. Finally, $\mathrm{Ext}^1(\Lambda, \Lambda') = 0$ and $\mathrm{Ext}^1(T, \Lambda') = 0$ imply that $\mathrm{Ext}^1(\Lambda', \Lambda') = 0$.

The defining sequence for Λ' shows that $\Lambda' \oplus T$ is Morita equivalent to a tilting module, say to $\beta(T) \oplus T$, where $\beta(T)$ is a direct summand of Λ'. The module $\beta(T)$ is called a *Bongartz complement* $\beta(T)$ for T, it is uniquely determined by T up to isomorphism.

If T is an exceptional module and not projective, then $\mathrm{Hom}(T, \beta(T)) = 0$*, thus $\beta(T)$ belongs to $T^{\perp}$.* Proof. Denote the inclusion map $\Lambda \to \Lambda'$ by u, the projection map $\Lambda' \to T^t$ by q, thus $qu = 0$. We can assume that t is minimal. Let $f\colon T \to \Lambda'$ be a homomorphism. We claim that $qf = 0$. Otherwise, there is a split epimorphism $p : T^t \to T$ such that pqf is non-zero. But a non-zero endomorphism of T is invertible, therefore qf is a split epimorphism. But this implies that t is not minimal, a contradiction. It follows from $qf = 0$ that the image of f is projective, thus a direct summand of T. Since T is indecomposable and not projective, it follows that $f = 0$.

If T is a partial tilting module and sincere, then $\beta(T)$ is cogenerated by T. Proof, following [**85**]. As we know, a sincere partial tilting module is faithful, thus Λ is cogenerated by T. Any embedding $v : \Lambda \to T^s$ gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda' & \xrightarrow{g} & T^t & \longrightarrow & 0 \\ & & \downarrow v & & \downarrow v' & & \| & & \\ 0 & \longrightarrow & T^a & \longrightarrow & M & \longrightarrow & T^t & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Ext}^1(T, T) = 0$, the lower sequence splits, thus Λ' is cogenerated by T. As a consequence, also $\beta(T)$ is cogenerated by T.

The sub complement for a partial tilting module. Let T be a partial tilting module, say with support algebra $\Lambda(T)$. We call the Bongartz complement

$\beta({}_{\Lambda(T)}T)$ for T (considered as a $\Lambda(T)$-module!), the *sub complement* for T. Since ${}_{\Lambda(T)}T$ is a faithful $\Lambda(T)$-module, the sub complement for T is cogenerated by T.

The dual construction. Again, we start with a partial tilting module T. We denote by Z a minimal cogenerator for $\operatorname{mod}\Lambda$. We choose a universal "foundation" of Z by copies of T, that means an exact sequence $0 \to T^t \to Z' \to Z \to 0$ with $t \in \mathbb{N}$ such that $\operatorname{Ext}^1(Z', T) = 0$ (such a foundation is often called a universal extension from below). Then $Z \oplus Z'$ is Morita equivalent to a tilting module, say $T \oplus \beta'(T)$ with $\beta'(T)$ is a direct summand of Z'. The module $\beta'(T)$ is called a *co-Bongartz complement* for T, it is uniquely determined by T up to isomorphism. There are the dual assertions:

If T is an exceptional module and not injective, then $\operatorname{Hom}(\beta'(T), T) = 0$*, thus* $\beta'(T)$ *belongs to* ${}^{\perp}T$.

If T is a partial tilting module and sincere, then $\beta'(T)$ is generated by T.

The factor complement for a partial tilting module. Let T be a partial tilting module, say with support algebra $\Lambda(T)$. We call the co-Bongartz complement $\beta'({}_{\Lambda(T)}T)$ of T (considered as a $\Lambda(T)$-module!) the *factor complement* for T. Since ${}_{\Lambda(T)}T$ is a faithful $\Lambda(T)$-module, we see that the factor complement for T is generated by T.

N 2.8. Thick closures of large rank. *Let E_1, E_2 be exceptional modules and $\mathcal{E}$ the thick closure of E_1, E_2. The rank of $\mathcal{E}$ may be arbitrarily large.*

Example: Let Q be the n-subspace quiver with sink 0. Let $E_1 = S(0)$ and $E_2 = I(0)$. Both are exceptional modules. There is an embedding of E_1 into E_2, its cokernel is the direct sum of the simple modules $S(i)$ with $i \neq 0$. This shows that the thick closure is the category of all representations of Q, this category has rank $n+1$. Note that for $n \geq 2$, neither (E_1, E_2) nor (E_2, E_1) is an exceptional sequence, since $\operatorname{Hom}(E_1, E_2) \neq 0$ and $\operatorname{Ext}^1(E_2, E_1) \neq 0$.

N 2.9. The relationship between a minimal cover N for $\mathcal{G}$ and a minimal cocover M for $\mathcal{F}$, where $(\mathcal{F}, \mathcal{G})$ is a torsion pair.

Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair with minimal cover N for $\mathcal{G}$ and minimal cocover M for $\mathcal{F}$. The corresponding perpendicular pair is $(\mathcal{M}, \mathcal{N})$, where $\mathcal{N}$ is the thick closure of N and $\mathcal{M}$ the thick closure of M. And we know that N is a minimal generator for the abelian category $\mathcal{N}$, whereas M is a minimal cogenerator for the abelian category $\mathcal{M}$. Let N' be the factor complement for N and M' the sub complement for M. There is a direct procedure to obtain $M \oplus M'$ from $N \oplus N'$ and vice versa:

Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair with minimal cover N for $\mathcal{G}$ and minimal cocover M for $\mathcal{F}$. Let N' be the factor complement for N and M' the sub complement for M. Write $N = N^P \oplus N^C$, where N^P is projective and N^C has no indecomposable projective direct summand. Write $M = M^I \oplus M^C$, where M^I is injective and M^C has no indecomposable injective direct summand.

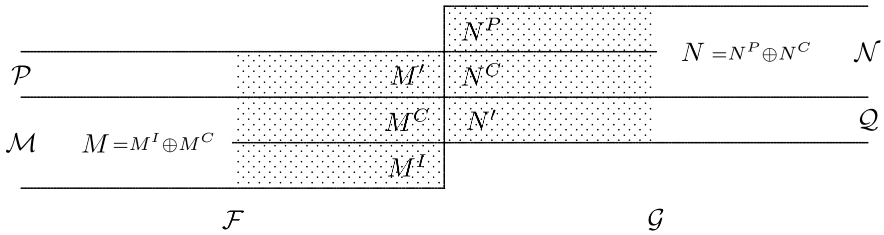
(a) $\tau N^C = M'$ *and* $\tau N' = M^C$.

(b) *The module N^P is the direct sum of all indecomposable projective modules in $\mathcal{G}$, and M^I is the direct sum of all indecomposable injective modules in $\mathcal{F}$.*

(c) *The module $N \oplus N'$ is a tilting $\Lambda_{\mathcal{G}}$-module, the module $M \oplus M'$ is a tilting $\Lambda_{\mathcal{F}}$-module.*

(Here, $\Lambda_{\mathcal{G}}$ is the factor algebra of Λ modulo the annihilator of $\mathcal{G}$, and $\Lambda_{\mathcal{F}}$ is the factor algebra of Λ modulo the annihilator of $\mathcal{F}$.)

Let us sketch the position of the modules involved:



The columns in the middle are related by the Auslander-Reiten translation τ: it shifts the right column to the left one. To be precise: τ sends N^P to zero, N^C to M' and N' to M^C (dually, τ^- sends M^I to zero, M^C to N' and M' to N^C).

For a proof, we refer to Smalø [**103**].

N 2.10. Lattices. Let P be a poset. If a, b are elements of P, and the supremum of a, b exists, then we denote it by $a \vee b$ and call it the *join* of a and b (by definition, $a \vee b$ is an element of P such that $a \leq a \vee b,\ b \leq a \vee b$ and such that for any element c with $a \leq c,\ b \leq c$ we have $a \vee b \leq c$; of course, if $a \vee b$ exists, it is uniquely determined by a and b). Dually, if the infimum of a, b exists, then we denote it by $a \wedge b$ and call it the *meet* of a and b. The poset P is said to be a *lattice* provided meet and join exist for any two elements $a, b \in P$.

N 2.11. The lattice of torsion subcategories. Here are some references: The restriction of the ordering $\leq_t$ to the set of sincere exceptional antichains (or, equivalently, to the set of tilting modules) has been studied in detail by Happel-Unger [**51**], for the use of this ordering, see the references in this paper.

The ordering $\leq_t$ for the exceptional antichains has been investigated for path algebras of quivers by Iyama-Reiten-Thomas-Todorov [**58**], for general hereditary artin algebra Λ see [**85**]. Of course, we obtain in this way just the poset of torsion classes with covers (and these are the functorially finite torsion classes). In case Λ is connected, this poset is a lattice if and only if Λ is representation finite or has just 2 simple modules.

3. The Poset $\mathbf{A}(\operatorname{mod}\Lambda)$ of Exceptional Antichains

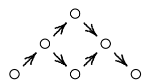
Let Λ be a hereditary artin algebra. We consider **antichains** in the category $\operatorname{mod}\Lambda$. Let us recall that such an antichain A consists of pairwise orthogonal bricks and an antichain A is said to be exceptional provided the quiver of A (with vertices the elements A_i of A and with an arrow from A_i to A_j whenever $\operatorname{Ext}^1(A_i, A_j) \neq 0$) has no oriented cyclic paths. For any hereditary artin algebra, let $\mathbf{A}(\operatorname{mod}\Lambda)$ be the **set of exceptional antichains.** This is a poset with respect to the ordering $A \leq A'$ provided any element of A has a filtration with factors in A' (or, equivalently, provided the extension closure of A is contained in the extension closure of A'). If Λ is representation-finite, then $\mathbf{A}(\operatorname{mod}\Lambda)$ is a lattice (see Theorem 3.2.1.1). If Λ is representation-infinite, then, in general, $\mathbf{A}(\operatorname{mod}\Lambda)$ is not a lattice, but also these posets are of interest! This chapter is devoted to the study of the posets $\mathbf{A}(\operatorname{mod}\Lambda)$ (for the use of the wording “exceptional” see N 3.1).

As an important example, we consider in Chapter 4 the path algebra Λ_n of the linearly oriented quiver of type $\mathbb{A}_n$:

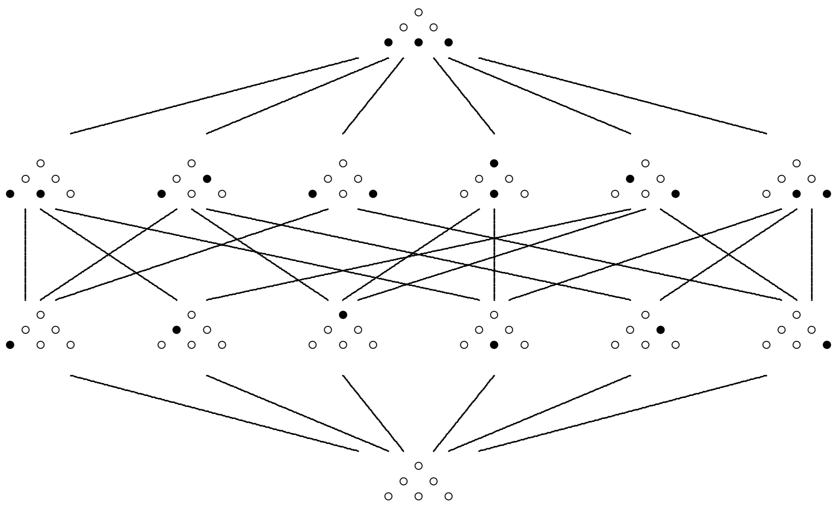
$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n$$

Thus, for $n = 3$, we deal with the quiver Q shown left, its Auslander-Reiten quiver is shown on the right:

Q $\quad 1 \leftarrow 2 \leftarrow 3$



Here is the lattice $\mathbf{A}(\operatorname{mod}\Lambda_3)$. By definition, its elements are antichains, thus sets of (at most 3) indecomposable Λ_3-modules; we use bullets in the Auslander-Reiten quiver to mark such an antichain.



It turns out that the lattices $\mathbf{A}(\operatorname{mod}\Lambda_n)$ can be identified with the lattices of non-crossing partitions (see Theorem 4.2.2.2); these lattices have attracted a lot of interest in recent years and we will provide in Chapter 4 a short survey on the relevance of these lattices. The lattices of non-crossing partitions are related to the Coxeter groups of type $\mathbb{A}$, corresponding lattices have been defined for any finite Coxeter group, and corresponding posets for all Coxeter groups, namely the posets of generalized non-crossing partitions. They will be introduced in Section 3.7.

The representation theory of hereditary artin algebras (or, more generally, of hereditary artinian rings) can be used in order to provide a categorification of the posets of generalized non-crossing partitions and it is the aims of this chapter to provide a direct access to this categorification. Since the lectures just deal with artin algebras (and not artinian rings in general), we restrict to Coxeter groups which arise as Weyl groups for some symmetrizable generalized Cartan matrix. We will show in Section 3.7 that for any hereditary artin algebra Λ of type Δ, the poset $\mathbf{A}(\operatorname{mod}\Lambda)$ is isomorphic to the poset $\mathbf{Nc}(W(\Lambda), c(\Lambda))$, where $W(\Lambda)$ is the Weyl group of type Δ and $c(\Lambda)$ is the Coxeter element in W corresponding to the orientation of Δ given by Λ.

3.1. The poset $\mathbf{A}(\operatorname{mod}\Lambda)$: Definition and first properties. Let Λ be a hereditary artin algebra. Recall that a full subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is called *exceptional* provided it is thick and its quiver has no oriented cyclic paths. We denote by $\mathbf{A}(\operatorname{mod}\Lambda)$ the set of exceptional subcategories of $\operatorname{mod}\Lambda$. This is a poset with respect to set-theoretical inclusion. Equivalently, we may consider $\mathbf{A}(\operatorname{mod}\Lambda)$ as the set of exceptional antichains in $\operatorname{mod}\Lambda$. In terms of antichains, the partial ordering has to be formulated as follows: given two exceptional antichains A, B, we write $A \leq B$ provided any element A_i of the antichain A has a filtration with factors belonging to B.

3.1.1. **The layers of $\mathbf{A}(\operatorname{mod}\Lambda)$.** For any natural number t, let $\mathbf{A}_t(\operatorname{mod}\Lambda)$ be the subset of $\mathbf{A}(\operatorname{mod}\Lambda)$ given by the exceptional subcategories with rank equal to t. Equivalently, we may consider $\mathbf{A}_t(\operatorname{mod}\Lambda)$ as the set of exceptional antichains of cardinality t.

3.1.1.1. The poset $\mathbf{A}(\operatorname{mod}\Lambda)$ has a smallest element, namely the zero subcategory 0, and a greatest element, namely $\operatorname{mod}\Lambda$ itself. Thus $\mathbf{A}_0(\operatorname{mod}\Lambda) = \{0\}$ and $\mathbf{A}_n(\operatorname{mod}\Lambda) = \{\operatorname{mod}\Lambda\}$, where n is the rank of Λ.

3.1.1.2. *The elements of* $\mathbf{A}_1(\operatorname{mod}\Lambda)$ *are the subcategories* $\operatorname{add} X$*, where* X *is an exceptional module.* Namely, here we deal with antichains of cardinality 1, thus with bricks (recall that a brick is a module whose endomorphism ring is a division ring; in particular, a brick is indecomposable), and to say that the quiver of such an antichain has no oriented cyclic paths just means that it consists of a single vertex and no arrow, thus we deal with an indecomposable module without self-extensions.

If P is a poset, then $p \in P$ is said to have *height* t provided any chain $p_0 < p_1 < \cdots < p_s \leq p$ has length $s \leq t$ and there is such a chain with $s = t$.

3.1.1.3. *Let* $\mathcal{A}$ *belong to* $\mathbf{A}(\operatorname{mod}\Lambda)$*. Then* $\mathcal{A}$ *has height* t *in* $\mathbf{A}(\operatorname{mod}\Lambda)$ *if and only if* $\mathcal{A}$ *has rank* t (thus belongs to $\mathbf{A}_t(\operatorname{mod}\Lambda)$).

Proof. Let

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_s \subseteq \mathcal{A}$$

be a chain of exceptional subcategories. Then, according to Corollary 2.5.3, we have

$$\operatorname{rank}\mathcal{A}_0 < \operatorname{rank}\mathcal{A}_1 < \cdots < \operatorname{rank}\mathcal{A}_s \leq \operatorname{rank}\mathcal{A} = t,$$

thus $s \leq t$. On the other hand, let $\mathcal{A} = \mathcal{E}(A_1, \ldots, A_t)$, where $A = \{A_1, \ldots, A_t\}$ is an antichain of cardinality t whose quiver has no oriented cyclic path. Let

$\mathcal{A}(i) = \mathcal{E}(A_1, \dots, A_i)$. Since $\{A_1, \dots, A_i\}$ is an antichain whose quiver has no oriented cyclic path, $\mathcal{A}(i)$ belongs to $\mathbf{A}_i(\operatorname{mod}\Lambda)$ and there is the inclusion chain

$$\mathcal{A}(0) \subset \mathcal{A}(1) \subset \cdots \subset \mathcal{A}(t) = \mathcal{A}.$$

Thus shows that $\mathcal{A}$ has height t as an element of $\mathbf{A}(\operatorname{mod}\Lambda)$. □

3.1.2. **Independence under BGP-reflections.** As we will see in 3.2.4.1, changing the orientation of the quiver of Λ, the corresponding posets $\mathbf{A}(\operatorname{mod}\Lambda)$ may be non-isomorphic. But if we use BGP-reflections, thus changing the orientation of all the arrows at a sink, then we obtain isomorphic posets.

PROPOSITION 3.1.2.1. *Let Λ' be obtained from Λ by changing the orientation at a sink x. Then the posets $\mathbf{A}(\operatorname{mod}\Lambda)$ and $\mathbf{A}(\operatorname{mod}\Lambda')$ are isomorphic.*

PROOF. We have constructed in the proof of Lemma 1.2.3.1 a bijection

$$\eta \colon \mathbf{A}(\operatorname{mod}\Lambda) \to \mathbf{A}(\operatorname{mod}\Lambda')$$

as follows: Let A be an antichain in $\operatorname{mod}\Lambda$. If $S(x)$ belongs to A, then we may consider all elements of A also as Λ'-modules, and we put $\eta(A) = A$. If $S(x)$ does not belong to A, then $\eta(A)$ is obtained from A by applying the BGP-reflection functor ρ_x to all elements of A. We have to show that η preserves and reflects the partial ordering.

Assume that $A \le B$ are antichains in $\operatorname{mod}\Lambda$. If $S(x)$ belongs to A, then also to B (namely, $A \le B$ asserts that $S(x)$ has a filtration with factors in B, but $S(x)$ has only trivial filtrations). Let Λ_0 be obtained from Λ (or Λ') by deleting the vertex x. Let A_i be an element of A different from $S(x)$. Since $A \le B$, the module A_i has a filtration with factors in B, but all these factors are Λ_0-modules, thus A_i considered as a Λ'-module, has a filtration with factors in $\eta(B)$. Therefore $\eta(A) \le \eta(B)$.

Next assume that $S(x)$ belongs neither to A nor to B. If A_i belongs to A, there is a filtration with factors B_{ij} in B. Since σ_x is an exact functor from the category of Λ-modules without direct summands of the form $S_\Lambda(x)$ to the category of Λ'-modules without direct summands of the form $S_{\Lambda'}(x)$, we see that $\sigma_x(A_i)$ has a filtration with factors $\sigma(B_{ij})$.

It remains to consider the case that $S(x)$ does not belong to A, but belongs to B. As a consequence, $\eta(B) = B$. Let A_i be an element of A. There is an exact sequence

$$0 \to S(x)^u \to A_i \to M \to 0$$

such that M is a Λ_0-module, and an exact sequence

$$0 \to M \to \sigma_x(A_i) \to S(x)^v \to 0,$$

with natural numbers u, v. Let $\mathcal{B} = \mathcal{E}(B)$, this is the thick closure of B. Since $A \le B$, we know that A_i belongs to $\mathcal{B}$. Since $S(x)$ belongs to $\mathcal{B}$, also the cokernel M in the first exact sequence belongs to $\mathcal{B}$. The second sequence now shows that $\sigma_x(A_i)$ belongs to $\mathcal{B}$, thus has a filtration with factors in B. Therefore $\eta(A) \le \eta(B)$.

This shows that η preserves the ordering. A similar argument using the BGP-reflection functors at sources shows that η^{-1} also preserves the ordering. □

3.2. The poset $\mathbf{A}(\operatorname{mod}\Lambda)$: Is it a lattice? We will show that in case Λ is representation-finite, $\mathbf{A}(\operatorname{mod}\Lambda)$ is a lattice. If Λ is not representation-finite, then $\mathbf{A}(\operatorname{mod}\Lambda)$ usually is not a lattice.

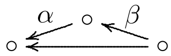
3.2.1. First, we consider the representation-finite case.

THEOREM 3.2.1.1. *If Λ is representation-finite, then $\mathbf{A}(\operatorname{mod}\Lambda)$ is a lattice, and in this case, the meet is given by the set-theoretical intersection.*

PROOF. The set of thick subcategories is closed under (set-theoretical) intersections, thus it is a complete lattice. If Λ is representation-finite, then any thick subcategory is exceptional, thus in this case $\mathbf{A}(\operatorname{mod}\Lambda)$ is just the lattice of thick subcategories. □

3.2.2. **Example.** *The set-theoretical intersection of two elements $\mathcal{X}_1, \mathcal{X}_2$ of $\mathbf{A}(\operatorname{mod}\Lambda)$ may not belong to $\mathbf{A}(\operatorname{mod}\Lambda)$, whereas the meet $\mathcal{X}_1 \wedge \mathcal{X}_2$ in $\mathbf{A}(\operatorname{mod}\Lambda)$ still may exist,* as the following example shows:

Let Λ be the path algebra of the quiver $\widetilde{\mathbb{A}}_{2,1}$

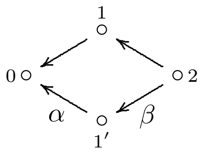


Let P, P', Q, Q' be the indecomposable modules with dimension vectors

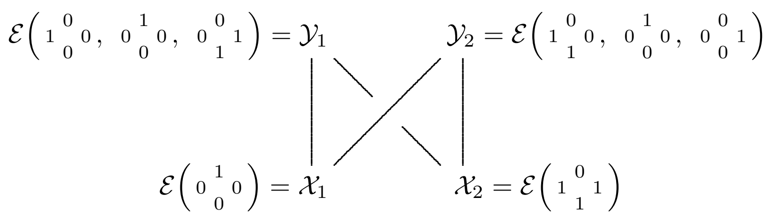
$$\mathbf{dim}\, P = 100,\ \mathbf{dim}\, P' = 110,\ \mathbf{dim}\, Q = 011,\ \mathbf{dim}\, Q' = 001.$$

Then these are exceptional modules and $\mathcal{X} = \mathcal{E}(P, Q)$ and $\mathcal{X}' = \mathcal{E}(P', Q')$ are exceptional subcategories. The category $\mathcal{X}$ consists of all representations M of the quiver such that M_α is bijective, the category $\mathcal{X}'$ consists of all representations M of the quiver such that M_β is bijective. Thus $\mathcal{X} \cap \mathcal{X}'$ consists of all representations M such that both M_α and M_β are bijective, this is a thick subcategory with infinitely many simple objects, all having self-extensions (it is just the full subcategory of $\operatorname{mod}\Lambda$ given by the homogeneous tubes). Since $\mathcal{X} \cap \mathcal{X}'$ contains no exceptional module, we have $\mathcal{X} \wedge \mathcal{X}' = 0$ in $\mathbf{A}(\operatorname{mod}\Lambda)$.

3.2.3. **Example.** *If Λ is the path algebra of a quiver of type $\widetilde{\mathbb{A}}_{2,2}$, then $\mathbf{A}(\operatorname{mod}\Lambda)$ is not a lattice.* We consider the following quiver:



We consider the following elements of $\mathbf{A}(\operatorname{mod}\Lambda)$:

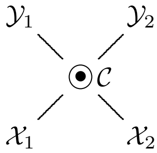


thus, $\mathcal{X}_1 = \operatorname{add} S(1),\ \mathcal{X}_2 = \operatorname{add} \tau S(1)$, and $\mathcal{Y}_2 = S(1')^\perp,\ \mathcal{Y}_1 = (\tau S(1'))^\perp$.

Let us show that $\mathcal{X}_1, \mathcal{X}_2$ have no supremum in $\mathbf{A}(\operatorname{mod}\Lambda)$. Assume, for the contrary, that $\mathcal{C}$ is the supremum, then $\mathcal{X}_1 \subset \mathcal{C} \subset \mathcal{Y}_1$ shows that $\mathcal{C}$ has rank 2. But there is no exceptional subcategory $\mathbb{C}$ which has rank 2 and contains $\mathcal{X}_1$ and $\mathcal{X}_2$. Namely, if C_1, C_2 are the simple objects in $\mathcal{C}$, then the objects in $\mathcal{X}_1$ and in $\mathcal{X}_2$ must have filtrations with factors of the form C_1, C_2, but this implies that C_1, C_2 are the indecomposable modules in $\mathcal{X}_1$ and $\mathcal{X}_2$, and thus the quiver of $\mathcal{C}$ is the oriented cycle with two vertices and two arrows.

It follows that $\mathcal{Y}_1, \mathcal{Y}_2$ have no infimum in $\mathbf{A}(\operatorname{mod}\Lambda)$. Namely, the infimum of $\mathcal{Y}_1, \mathcal{Y}_2$ would be an exceptional subcategory of rank 2 and containing $\mathcal{X}_1, \mathcal{X}_2$, thus the supremum of $\mathcal{X}_1, \mathcal{X}_2$.

One may enlarge $\mathbf{A}(\operatorname{mod}\Lambda)$ by inserting a thick subcategory $\mathcal{C}$ which is both a supremum of $\mathcal{X}_1, \mathcal{X}_2$ and an infimum of $\mathcal{Y}_1, \mathcal{Y}_2$:



There are many possibilities to do so: the supremum of $\mathcal{X}_1, \mathcal{X}_2$ in the lattice of thick subcategories is the extension closure of $\mathcal{X}_1, \mathcal{X}_2$, this is just a single tube of rank 2 in $\operatorname{mod}\Lambda$ (namely the tube containing the simple representation $S(1)$). The infimum of $\mathcal{Y}_1, \mathcal{Y}_2$ in the lattice of thick subcategories consists of the representations M with M_α and M_β both being bijective, these are all the regular modules without indecomposable direct summands in the rank 2 tube containing the simple representation $S(1')$.

Lemma 3.2.3.1. *If $\mathcal{X}$ is an exceptional subcategory of rank at most 2, then the meet $\mathcal{X} \wedge \mathcal{Y}$ exists, for any exceptional subcategory $\mathcal{Y}$.*

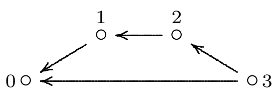
Proof. If $\mathcal{X} \cap \mathcal{Y}$ contains no exceptional modules, then clearly $\mathcal{X} \wedge \mathcal{Y} = 0$. If $\mathcal{X} \cap \mathcal{Y}$ contains precisely one exceptional module M, then $\mathcal{X} \wedge \mathcal{Y} = \operatorname{add} M$. Finally, if If $\mathcal{X} \cap \mathcal{Y}$ contains at least two exceptional modules M_1, M_2, then the condition $\operatorname{rank}\mathcal{X} \leq 2$ implies that $\operatorname{rank}\mathcal{X} = 2$ and that $\mathcal{X}$ has to be the thick closure of M_1, M_2 (see N 3.2), thus $\mathcal{X} \subseteq \mathcal{Y}$ and therefore $\mathcal{X} \wedge \mathcal{Y} = \mathcal{X}$. □

By duality we see: *If Λ has rank n, and $\mathcal{X}$ is an exceptional subcategory of* $\operatorname{mod}\Lambda$ *of rank at least $n-2$, then the join $\mathcal{X} \vee \mathcal{Y}$ exists for any exceptional subcategory $\mathcal{Y}$.*

Corollary 3.2.3.2. *If Λ is a hereditary artin algebra of rank at most 3, then the poset* $\mathbf{A}(\operatorname{mod}\Lambda)$ *always is a lattice.*

The Lemma 3.2.3.1 asserts the existence of meets, by duality, we also have joins.

3.2.4. **Example.** *If Q is the quiver of type $\widetilde{\mathbb{A}}_{3,1}$,*



then $\mathbf{A}(\operatorname{mod} kQ)$ *is a lattice.*

Proof. We want to show that any two elements $\mathcal{X} \neq \mathcal{Y}$ in $\mathbf{A}(\operatorname{mod}\Lambda)$ have a join (the self duality then shows that any two elements also have a meet).

Of course, if $\mathcal{X}$ has rank 3, then any exceptional subcategory containing $\mathcal{X}$ and $\mathcal{Y}$ has to have rank at least 4, thus must be equal to $\operatorname{mod}\Lambda$. Thus we may assume that both $\mathcal{X}, \mathcal{Y}$ have rank at most 2. As we also know, we can assume that neither $\mathcal{X}$ not $\mathcal{Y}$ has rank 2. Thus it remains to consider the case that both subcategories $\mathcal{X}, \mathcal{Y}$ have rank 1, thus we deal with two exceptional modules X, Y. Let us denote by $\mathcal{Z}$ the thick closure of X, Y.

If at least one of the modules X, Y is preprojective, then we show that $\mathcal{Z}$ is exceptional (and therefore the join of $\operatorname{add} X$ and $\operatorname{add} Y$). Using reflection functors, we can assume that one of these modules, say X, is simple projective. Let Y' be the factor module of Y modulo the trace of X in Y. Then Y' is a representation of a quiver of type $\mathbb{A}_3$, thus its thick closure $\mathcal{Z}'$ is exceptional. The simple objects in $\mathcal{Z}'$ together with S are an exceptional antichain and these are the simple objects of $\mathcal{Z}$, thus $\mathcal{Z}$ is exceptional. By duality, we similarly can assume that none of the modules X, Y is preinjective.

Thus, assume that both X, Y are regular modules. In case they form an exceptional pair, then again $\mathcal{Z}$ is exceptional. Thus, at least one of the modules X, Y has regular rank 2. If both have regular rank 2, then $\mathcal{Z}$ contains all three non-homogeneous simple regular modules (the modules with dimension vector $1001, 0100, 0010$), thus $\mathcal{Z}$ is the tube which contains X, Y (of course, this is not an exceptional subcategory). If $\mathcal{C}$ is a thick subcategory which contains $\mathcal{Z}$ properly, then $\mathcal{C}$ has to have rank 4, thus it is just $\mathcal{C} = \operatorname{mod} \Lambda$. This shows that $\operatorname{mod} \Lambda$ is the join of $\operatorname{add} X$ and $\operatorname{add} Y$.

It remains to consider the case that one of the modules X, Y has regular rank 1, the other regular rank 2 and that this is an orthogonal pair. Thus, up to duality, there are two cases: $\{X, Y\} = \{1001, 0110\}$ and $\{X, Y\} = \{0100, 1011\}$. In the first case, there is just one exceptional subcategory $\mathcal{C}$ of rank 3 which contains X, Y, namely the thick closure of $\{1000, 0110, 0001\}$. Similarly, in the second case, there is just one exceptional subcategory $\mathcal{C}$ of rank 3 which contains X, Y, namely the thick closure of $\{1000, 0100, 0011\}$. In both cases the respective subcategory $\mathcal{C}$ is the join of $\operatorname{add} X$ and $\operatorname{add} Y$.

The examples 3.2.3 and 3.2.4 concern quivers with the same underlying graph, they differ only by the choice of the orientation (thus by the choice of a Coxeter element in the Weyl group). This shows:

3.2.4.1. *If Q, Q' are quivers with the same underlying graph, but different orientations, the posets $\mathbf{A}(\operatorname{mod} kQ)$ and $\mathbf{A}(\operatorname{mod} kQ')$ may be non-isomorphic.*

3.3. The poset $\mathbf{A}(\operatorname{mod} \Lambda)$: Intervals. Given elements $a \le b$ in a poset P, we denote by $[a, b] = \{p \in P \mid a \le p \le b\}$ the (closed) interval of elements between a and b. The aim of this section is to show that any interval $[\mathcal{A}, \mathcal{B}]$ in $\mathbf{A}(\operatorname{mod} \Lambda)$ is isomorphic to a poset of the form $\mathbf{A}(\mathcal{C})$, where $\mathcal{C}$ is an element of $\mathbf{A}(\operatorname{mod} \Lambda)$ (thus to a poset which is again of the form $\mathbf{A}(\operatorname{mod} \Lambda')$ for some hereditary artin algebra Λ'), see Theorem 3.3.1.5.

3.3.1. We begin with an analysis of the poset structure of $\mathbf{A}(\operatorname{mod} \Lambda)$.

Lemma 3.3.1.1. *Let $\mathcal{A} \subseteq \mathcal{V}$ be exceptional subcategories of* $\operatorname{mod} \Lambda$, *let a be the rank of $\mathcal{A}$ and v the rank of $\mathcal{V}$. Then both subcategories $\mathcal{V} \cap {}^{\perp}\mathcal{A}$ and $\mathcal{V} \cap \mathcal{A}^{\perp}$ are exceptional subcategories of rank $v - a$, they are the meet $\mathcal{V} \wedge {}^{\perp}\mathcal{A}$, and $\mathcal{V} \wedge \mathcal{A}^{\perp}$ in* $\mathbf{A}(\operatorname{mod} \Lambda)$*, respectively.*

PROOF. We consider $\mathcal{A}^\perp$. In order to show that $\mathcal{V} \cap \mathcal{A}^\perp$ is an exceptional subcategory of rank $v - a$, we can assume that $\mathcal{V} = \operatorname{mod} \Lambda'$ for some hereditary artin algebra Λ'.

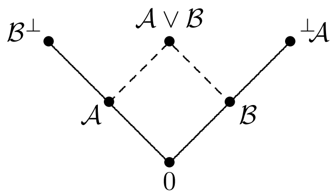
Thus, we deal with an exceptional subcategory $\mathcal{A}$ of $\operatorname{mod} \Lambda'$, and $\mathcal{A}^\perp \cap \mathcal{V}$ is just the right perpendicular category for $\mathcal{A}$ inside $\operatorname{mod} \Lambda'$, thus let us write $\mathcal{V} \cap \mathcal{A}^\perp = \mathcal{A}^{\perp(\operatorname{mod} \Lambda')}$. The claim is that $\mathcal{A}^{\perp(\operatorname{mod} \Lambda')}$ is exceptional and that the rank of $\mathcal{A}^{\perp(\operatorname{mod} \Lambda')}$ is $\operatorname{rank}(\Lambda') - \operatorname{rank}(\mathcal{A}) = v - a$. But this has been shown in Chapter 2.

Since $\mathcal{V} \cap \mathcal{A}^\perp$ is an exceptional subcategory of $\operatorname{mod} \Lambda$, it is the infimum $\mathcal{V} \wedge \mathcal{A}^\perp$ of $\mathcal{V}$ and $\mathcal{A}^\perp$ in $\mathbf{A}(\operatorname{mod} \Lambda)$.

In the same way (or using duality) one deals with ${}^\perp\mathcal{A}$. □

LEMMA 3.3.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be exceptional subcategories of* $\operatorname{mod} \Lambda$, *of rank a and b, respectively. We assume that $\mathcal{A} \subseteq \mathcal{B}^\perp$ or, equivalently, that $\mathcal{B} \subseteq {}^\perp\mathcal{A}$. Then the join $\mathcal{A} \vee \mathcal{B}$ exists in* $\mathbf{A}(\operatorname{mod} \Lambda)$, *it is an exceptional subcategory of rank $a + b$, and we have*

$$\mathcal{A} \vee \mathcal{B} = ({}^\perp\mathcal{A} \cap {}^\perp\mathcal{B})^\perp = {}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp).$$



PROOF. We assume that $\mathcal{A} \subseteq \mathcal{B}^\perp$ (and thus we also have ${}^\perp\mathcal{A} \supseteq {}^\perp(\mathcal{B}^\perp) = \mathcal{B}$).

Note that $\mathcal{B}^\perp$ is exceptional with rank $n - b$. According to Lemma 3.3.1.1, we know that $\mathcal{B}^\perp \cap \mathcal{A}^\perp$ is an exceptional subcategory of rank $n - b - a$. It follows that ${}^\perp(\mathcal{B}^\perp \cap \mathcal{A}^\perp)$ is an exceptional subcategory of rank $n - (n - b - a) = a + b$.

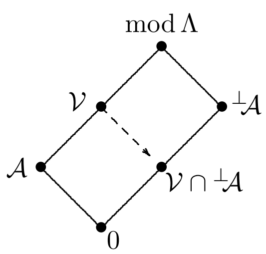
We claim that ${}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp)$ is the join of $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{A}(\operatorname{mod} \Lambda)$. First, we have to show that both $\mathcal{A}$ and $\mathcal{B}$ are contained in ${}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp)$. Since $\mathcal{A}^\perp \supseteq \mathcal{A}^\perp \cap \mathcal{B}^\perp$, we see that $\mathcal{A} = {}^\perp(\mathcal{A}^\perp) \subseteq {}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp)$; similarly, we have $\mathcal{B} \subseteq {}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp)$. Now assume that there is an exceptional subcategory $\mathcal{C}$ with $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{C}$. It follows from $\mathcal{A} \subseteq \mathcal{C}$ that $\mathcal{A}^\perp \supseteq \mathcal{C}^\perp$. Similarly, we have $\mathcal{B}^\perp \supseteq \mathcal{C}^\perp$. Thus $\mathcal{A}^\perp \cap \mathcal{B}^\perp \supseteq \mathcal{C}^\perp$, and therefore ${}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp) \subseteq {}^\perp(\mathcal{C}^\perp) = \mathcal{C}$. This shows that ${}^\perp(\mathcal{A}^\perp \cap \mathcal{B}^\perp) = \mathcal{A} \vee \mathcal{B}$.

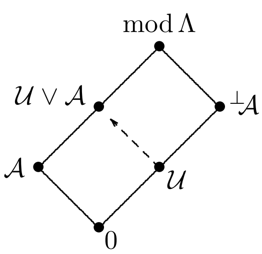
In the same way, one shows that $({}^\perp\mathcal{A} \cap {}^\perp\mathcal{B})^\perp = \mathcal{A} \vee \mathcal{B}$. □

PROPOSITION 3.3.1.3. *Let $\mathcal{A}$ be an exceptional subcategory of* $\operatorname{mod} \Lambda$. *Then the poset* $[\mathcal{A}, \operatorname{mod} \Lambda]$ *is isomorphic to the poset* $\mathbf{A}({}^\perp\mathcal{A})$ *using the maps*

$$\mathcal{V} \mapsto \mathcal{V} \cap {}^\perp\mathcal{A}, \quad \textit{and} \quad \mathcal{U} \mapsto \mathcal{U} \vee \mathcal{A}.$$

for $\mathcal{V}$ in $[\mathcal{A}, \operatorname{mod} \Lambda]$ *and $\mathcal{U}$ in* $\mathbf{A}({}^\perp\mathcal{A})$.





Proof. Lemma 3.3.1.1 asserts that given $\mathcal{V}$ in $[\mathcal{A}, \operatorname{mod}\Lambda]$, the subcategory $\mathcal{V} \cap {}^{\perp}\mathcal{A}$ belongs to $\mathbf{A}({}^{\perp}A)$. Of course, the map $\mathcal{V} \mapsto \mathcal{V} \cap {}^{\perp}\mathcal{A}$ is order preserving. Lemma 3.3.1.2 asserts that given $\mathcal{U}$ in $\mathbf{A}({}^{\perp}A)$, the join $\mathcal{U} \vee \mathcal{A}$ exists, and, of course $\mathcal{U} \vee \mathcal{A}$ belongs to the interval $[\mathcal{A}, \operatorname{mod}\Lambda]$. Also here we see immediately that the map $\mathcal{U} \mapsto \mathcal{U} \vee \mathcal{A}$ is order preserving.

It remains to be shown that the maps defined here are inverse to each other, thus we have to show that

$$ {}^{\perp}((\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \cap \mathcal{A}^{\perp}) = \mathcal{V} \quad \text{and} \quad {}^{\perp}(\mathcal{U}^{\perp} \cap \mathcal{A}^{\perp}) \cap {}^{\perp}\mathcal{A} = \mathcal{U} $$

for $\mathcal{V}$ in $[\mathcal{A}, \operatorname{mod}\Lambda]$ and $\mathcal{U}$ in $\mathbf{A}({}^{\perp}A)$.

First, let us show the inclusion ${}^{\perp}((\mathcal{V}\cap{}^{\perp}\mathcal{A})^{\perp}\cap\mathcal{A}^{\perp}) \subseteq \mathcal{V}$. The inclusion $\mathcal{V}\cap{}^{\perp}\mathcal{A} \subseteq \mathcal{V}$ implies that $(\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \supseteq \mathcal{V}^{\perp}$, thus $(\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \cap \mathcal{A}^{\perp} \supseteq \mathcal{V}^{\perp} \cap \mathcal{A}^{\perp}$ and $\mathcal{V}^{\perp} \cap \mathcal{A}^{\perp} = \mathcal{V}^{\perp}$, since $\mathcal{A} \subseteq \mathcal{V}$ so that $\mathcal{A}^{\perp} \supseteq \mathcal{V}^{\perp}$. But $(\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \cap \mathcal{A}^{\perp} \supseteq \mathcal{V}^{\perp}$ implies that ${}^{\perp}((\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \cap \mathcal{A}^{\perp}) \subseteq {}^{\perp}(\mathcal{V}^{\perp}) = \mathcal{V}$. In addition, we show that the exceptional subcategories ${}^{\perp}((\mathcal{V} \cap {}^{\perp}\mathcal{A})^{\perp} \cap \mathcal{A}^{\perp})$ and $\mathcal{V}$ have the same rank (namely v), as a consequence, they have to be equal.

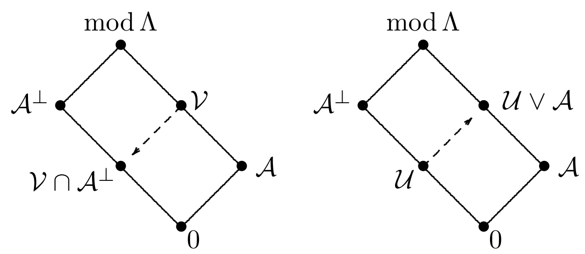
Similarly, we show the inclusion $\mathcal{U} \subseteq {}^{\perp}(\mathcal{U}^{\perp}\cap\mathcal{A}^{\perp})\cap{}^{\perp}\mathcal{A}$. By assumption, $\mathcal{U} \subseteq {}^{\perp}\mathcal{A}$, thus we only have to show that $\mathcal{U} \subseteq {}^{\perp}(\mathcal{U}^{\perp} \cap \mathcal{A}^{\perp})$. But $\mathcal{U}^{\perp} \supseteq \mathcal{U}^{\perp} \cap \mathcal{A}^{\perp}$, therefore $\mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp}) \subseteq {}^{\perp}(\mathcal{U}^{\perp} \cap \mathcal{A}^{\perp})$. And again we show that we deal with exceptional subcategories of the same rank: Since $\mathcal{U} \subseteq {}^{\perp}\mathcal{A}$, Lemma 3.3.1.2 asserts that the rank of $\mathcal{A} \vee \mathcal{U} = {}^{\perp}(\mathcal{U}^{\perp} \cap \mathcal{A}^{\perp})$ is $a + u$. Now $\mathcal{A} \vee \mathcal{U} \supseteq \mathcal{A}$, thus by Lemma 3.3.1.1 the rank of $(\mathcal{A} \vee \mathcal{U}) \cap {}^{\perp}\mathcal{A}$ is $a + u - a = u$. This completes the proof. □

There is the corresponding assertion invoking $\mathcal{A}^{\perp}$:

Proposition 3.3.1.4. *Let $\mathcal{A}$ be an exceptional subcategory of* $\operatorname{mod}\Lambda$. *Then the poset* $[\mathcal{A}, \operatorname{mod}\Lambda]$ *is isomorphic to the poset* $\mathbf{A}(\mathcal{A}^{\perp})$ *using the maps*

$$ \mathcal{V} \mapsto \mathcal{V} \cap \mathcal{A}^{\perp}, \quad \textit{and} \quad \mathcal{U} \mapsto \mathcal{U} \vee \mathcal{A}. $$

for $\mathcal{V}$ in $[\mathcal{A}, \operatorname{mod}\Lambda]$ *and $\mathcal{U}$ in* $\mathbf{A}({}^{\perp}\mathcal{A})$.



Theorem 3.3.1.5. *Let $\mathcal{A} \subseteq \mathcal{B}$ be exceptional subcategories of* $\operatorname{mod}\Lambda$. *Then there is a poset isomorphism*

$$ [\mathcal{A}, \mathcal{B}] \longrightarrow \mathbf{A}(\mathcal{B} \cap \mathcal{A}^{\perp}) $$

defined by $\mathcal{V} \mapsto \mathcal{V} \cap \mathcal{A}^{\perp}$ for $\mathcal{A} \subseteq \mathcal{V} \subseteq \mathcal{B}$, its inverse sends $\mathcal{U}$ to $\mathcal{U} \vee \mathcal{A}$.

Similarly, there is a poset isomorphism

$$ [\mathcal{A}, \mathcal{B}] \longrightarrow \mathbf{A}(\mathcal{B} \cap {}^{\perp}\mathcal{A}) $$

defined by $\mathcal{V} \mapsto \mathcal{V} \cap {}^{\perp}\mathcal{A}$ for $\mathcal{A} \subseteq \mathcal{V} \subseteq \mathcal{B}$, its inverse sends $\mathcal{U}$ to $\mathcal{U} \vee \mathcal{A}$.

This shows that any interval $[\mathcal{A}, \mathcal{B}]$ in $\mathbf{A}(\operatorname{mod}\Lambda)$ is isomorphic to a poset of the form $\mathbf{A}(\mathcal{C})$, where $\mathcal{C}$ is an element of $\mathbf{A}(\operatorname{mod}\Lambda)$, namely $\mathcal{C} = \mathcal{B}\cap\mathcal{A}^{\perp}$ or $\mathcal{C} = \mathcal{B}\cap {}^{\perp}\!\mathcal{A}$.

Let us discuss two special cases.

3.3.2. **Neighbors.** Recall that a pair p, p' of elements of a poset P are called *neighbors* provided $p < p'$ and such that the interval $[p, p']$ contains no other element of P. A direct consequence of the proposition is the following assertion:

PROPOSITION 3.3.2.1. *Let $\mathcal{A} \subseteq \mathcal{B}$ be exceptional subcategories of* $\operatorname{mod}\Lambda$. *Then these are neighbors in the poset* $\mathbf{A}(\operatorname{mod}\Lambda)$ *if and only if the rank of $\mathcal{A}$ and $\mathcal{B}$ differs by* 1.

If $\mathcal{A} \subset \mathcal{B}$ are neighbors in $\mathbf{A}(\operatorname{mod}\Lambda)$, *then there are (uniquely determined) exceptional modules $M_{\mathcal{A}}^{\mathcal{B}}$ and $N_{\mathcal{A}}^{\mathcal{B}}$ such that*

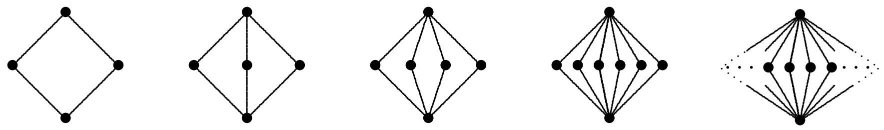
$$\operatorname{add} M_{\mathcal{A}}^{\mathcal{B}} = \mathcal{B}\cap\mathcal{A}^{\perp} \quad \textit{and} \quad \operatorname{add} N_{\mathcal{A}}^{\mathcal{B}} = \mathcal{B}\cap {}^{\perp}\!\mathcal{A}.$$

PROOF. If $\mathcal{A} \subset \mathcal{B}$ are neighbors, any of the subcategories $\mathcal{B}\cap\mathcal{A}^{\perp}$ and $\mathcal{B}\cap {}^{\perp}\!\mathcal{A}$ is exceptional and of rank 1, thus the additive category generated by an exceptional module. □

Section 3.5 will be devoted to the study of maximal chains in $\mathbf{A}(\operatorname{mod}\Lambda)$. Given a maximal chain, we will deal with the corresponding sequence of modules $M_{\mathcal{A}}^{\mathcal{B}}, N_{\mathcal{A}}^{\mathcal{B}}$ where $\mathcal{A} \subset \mathcal{B}$ belong to the chain. Also, as an example, we will exhibit all the modules $M_{\mathcal{A}}^{\mathcal{B}}, N_{\mathcal{A}}^{\mathcal{B}}$ for the case of $\Lambda = \Lambda_3$.

3.3.3. **Intervals of height 2.** Such an interval is isomorphic to $\mathbf{A}(\operatorname{mod}\Lambda')$, where Λ' is a hereditary artin algebra with precisely two simple modules. In case Λ' is representation-finite, it is of type $\mathbb{A}_1 \times \mathbb{A}_1$, $\mathbb{A}_2, \mathbb{B}_2$, or $\mathbb{G}_2$. In case Λ' is representation-infinite, it has an infinite preprojective component and an infinite preinjective component, and the exceptional modules are just the indecomposable modules which are preprojective or preinjective.

Here are these possibilities.



$\mathbb{A}_1{\times}\mathbb{A}_1$ $\mathbb{A}_2$ $\mathbb{B}_2$ $\mathbb{G}_2$ infinite

In particular, an interval of height 2 is never a chain.

Remark. Let us stress already here that the posets $\mathbf{A}(\operatorname{mod}\Lambda)$ with Λ of rank 2 (and thus all intervals of height 2) are not only lattices, but come equipped with a cyclic rotation, in case Λ is representation-finite, and with a totally ordering of $\mathbf{A}_1(\operatorname{mod}\Lambda)$ in case Λ is representation-infinite. The neighbors in $\mathbf{A}_1(\operatorname{mod}\Lambda)$ are just the exceptional pairs of Λ-modules. We will use this in the proof of Theorem 3.6.1.1.

A direct way to see the cyclic or total ordering of $\mathbf{A}_1(\operatorname{mod}\Lambda)$ is to look at the Auslander-Reiten quiver of Λ. If Λ is not connected (the case $\mathbb{A}_1 \times \mathbb{A}_1$), the indecomposable modules are the simple modules, say S and T and both (S, T) and (T, S) are exceptional pairs. If Λ is representation-finite and connected, say with m

indecomposable modules (here $m = 3, 4$, or 6), we may order them as $(X_1, \dots, X_m)$ such that $\mathrm{Hom}(X_i, X_{i+1}) \neq 0$ for $1 \leq i < m$. Then the pairs (X_i, X_{i+1}) with $1 \leq i < m$ as well as the pair (X_m, X_1) are the exceptional pairs. Finally, if Λ is representation-infinite, the exceptional modules are just the indecomposable modules which are preprojective or preinjective. We label the indecomposable preprojective modules as $X_0, X_1, \dots, X_i, \dots$, the indecomposable preinjective modules as $\dots, X_i, \dots, X_{-2}, X_{-1}$ such that $\mathrm{Hom}(X_i, X_{i+1}) \neq 0$ for $i \leq -2$ and for $i \geq 0$. Then the pairs (X_i, X_{i+1}) with $i \in \mathbb{Z}$ are the exceptional pairs.

3.4. The poset $\mathbf{A}(\operatorname{mod}\Lambda)$: Automorphisms and anti-automorphisms. As we will see, the poset $\mathbf{A}(\operatorname{mod}\Lambda)$ is self-dual and has, in general, many automorphisms.

3.4.1. If $\mathcal{A}$ is an exceptional subcategory of $\operatorname{mod}\Lambda$, let $\delta(\mathcal{A}) = \mathcal{A}^{\perp}$.

THEOREM 3.4.1.1. *The map δ is a poset anti-automorphism of $\mathbf{A}(\operatorname{mod}\Lambda)$.*

PROOF. Of course, we always have $\mathcal{A} \subseteq {}^{\perp}(\mathcal{A}^{\perp})$. In case $\mathcal{A}$ is an exceptional subcategory, we have the equality $\mathcal{A} = {}^{\perp}(\mathcal{A}^{\perp})$ (otherwise usually not, see the note N 3.3). Namely, we have shown in Chapter 2 that in this case $(\mathcal{A}^{\perp}, \mathcal{A})$ is a perpendicular pair, but this just means that $\mathcal{A} = {}^{\perp}(\mathcal{A}^{\perp})$. Similarly, for $\mathcal{B}$ an exceptional subcategory, we have $\mathcal{B} = ({}^{\perp}\mathcal{B})^{\perp}$, thus δ is a bijective map from $\mathbf{A}(\operatorname{mod}\Lambda)$ to itself with inverse $\delta^{-1}(\mathcal{A}) = {}^{\perp}\mathcal{A}$.

Of course, δ reverses the partial ordering: $\mathcal{A} \subseteq \mathcal{B}$ implies that $\mathcal{A}^{\perp} \supseteq \mathcal{B}^{\perp}$. Similarly, δ^{-1} reverses the partial ordering. Thus δ is a poset anti-automorphism. □

Remark. In the case $\Lambda = \Lambda_n$, the anti-automorphism δ is just the Kreweras complement (as introduced by Kreweras in 1972), see Theorem 4.3.2.1.

COROLLARY 3.4.1.2. *The map δ provides a bijection between $\mathbf{A}_t(\operatorname{mod}\Lambda)$ and $\mathbf{A}_{n-t}(\operatorname{mod}\Lambda)$, for $0 \leq t \leq n$.*

The special case $t = 1$ should be mentioned explicitly: *The map δ provides a bijection between the elements in $\mathbf{A}_{n-1}(\operatorname{mod}\Lambda)$ and the exceptional modules.*

Note that the restriction of δ to $\mathbf{A}_{n-1}(\operatorname{mod}\Lambda)$ can be written as follows: if $\mathcal{A}$ belongs to $\mathbf{A}_{n-1}(\operatorname{mod}\Lambda)$, then $\delta(\mathcal{A}) = \operatorname{add} N_{\mathcal{A}}^{\operatorname{mod}\Lambda}$.

We define a permutation $\overline{\tau}$ on the set of isomorphism classes of indecomposable Λ-modules as follows: if X is indecomposable, let

$$\overline{\tau}X = \begin{cases} \tau X & \text{if } X \text{ is not projective,} \\ I(S) & \text{if } X = P(S), \end{cases}$$

where $\tau = D\,\mathrm{Tr}$ is the Auslander-Reiten translation, $P(S)$ the projective cover of the simple module S and $I(S)$ the injective envelop of S. We call $\overline{\tau}$ the *extended Auslander-Reiten translation* for $\operatorname{mod}\Lambda$.

THEOREM 3.4.1.3. *The map δ^2 is a poset automorphism of $\mathbf{A}(\operatorname{mod}\Lambda)$ and we have*

$$\delta^2(\mathcal{A}) = \overline{\tau}(\mathcal{A})$$

for any $\mathcal{A} \in \mathbf{A}(\operatorname{mod}\Lambda)$.

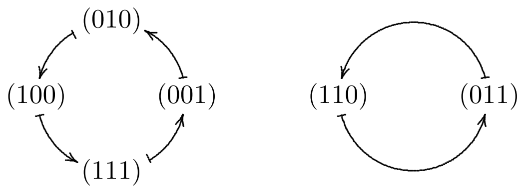
PROOF. The first assertion follows directly from Theorem 3.4.1.1. In order to show the second assertion, it is sufficient to consider the case when $\mathcal{A}$ has rank one. Thus, let $\mathcal{A} = \operatorname{add} X$ where X is an exceptional module. Let us show that $\overline{\tau}(X)$ belongs to $\delta^2(\mathcal{A}) = \mathcal{A}^{\perp\perp}$. Thus, given $Y \in \mathcal{A}^{\perp}$, we claim that $\operatorname{Hom}(Y, \overline{\tau}X) = 0$ and $\operatorname{Ext}^1(Y, \overline{\tau}X) = 0$.

First, assume that $X = P(S)$ for some simple module S, thus $\overline{\tau}(X) = I(S)$. Then $P(S)^{\perp}$ consists of the Λ-modules Y which do not have S as a composition factor, thus $\operatorname{Hom}(Y, I(S)) = 0$. Of course, also $\operatorname{Ext}^1(Y, I(S)) = 0$, since $I(S)$ is injective.

There is the second case that X is not projective. Let $Y \in \mathcal{A}^{\perp}$, thus $\operatorname{Hom}(X, Y) = 0$ and $\operatorname{Ext}^1(X, Y) = 0$. Since X is indecomposable and not projective, $\operatorname{Hom}(Y, \tau X) = D\operatorname{Ext}^1(X, Y) = 0$ and $\operatorname{Ext}^1(Y, \tau X) = D\operatorname{Hom}(\tau X, \tau Y) = D\operatorname{Hom}(X, Y) = 0$.

Now $\overline{\tau}(X)$ is non-zero and belongs to $\delta^2(\mathcal{A})$; since $\delta^2(\mathcal{A})$ has rank 1, it follows that $\delta^2(\mathcal{A}) = \operatorname{add} \overline{\tau}(X) = \overline{\tau}(\mathcal{A})$. □

For example, let Λ_3 be the path algebra Λ of a linearly ordered quiver Q of type $\mathbb{A}_3$. Then the permutation $\overline{\tau}$ of $A_1(\operatorname{mod} \Lambda_3)$ (or of the indecomposable Λ_3-modules) has two orbits:

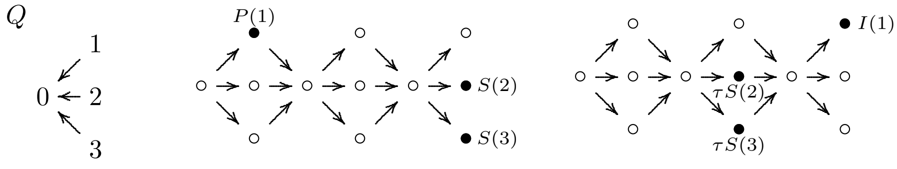


Looking at the lattice $\mathbf{A}(\operatorname{mod} \Lambda_3)$, the elements of rank 1 behave differently: four of the six elements have 3 upper neighbors, these are the elements in the first $\overline{\tau}$-orbit, the remaining two form the second $\overline{\tau}$-orbit, each of them has only 2 upper neighbors.

COROLLARY 3.4.1.4. *The automorphism* δ^2 *of* $\mathbf{A}(\operatorname{mod} \Lambda)$ *is the identity if and only if* $\operatorname{mod} \Lambda$ *is a semismiple algebra.*

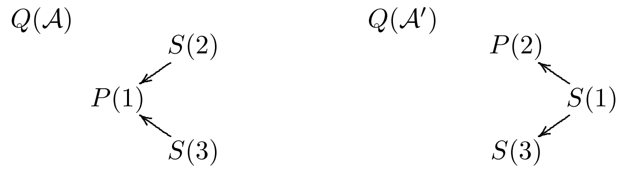
3.4.2. **Remark.** *If* $\mathcal{A}$ *is a thick subcategory of* $\operatorname{mod} \Lambda$*, then* $\delta^2(\mathcal{A})$ *has the same rank as* $\mathcal{A}$*, but may not be equivalent (as a category) to* $\mathcal{A}$*.*

Here is an example. Let Λ be the path algebra of the 3-subspace quiver Q as shown below on the left. To the right, we show twice the Auslander-Reiten quiver of Λ; first we mark the antichain $P(1), S(2), S(3)$ by bullets, on the right we mark in the same way the modules $\overline{\tau}P(1) = I(1), \overline{\tau}S(2) = \tau S(2), \overline{\tau}S(3) = \tau S(3)$



Let $\mathcal{A}$ be the thick subcategory generated by the antichain $P(1), S(2), S(3)$, thus $\mathcal{A}' = \delta^2(\mathcal{A})$ is the thick subcategory generated by the modules $I(1) = S(1), \tau S(2)$,

$\tau S(3)$, thus by the antichain $P(2), P(3), S(1)$. Here are the corresponding quivers:



As we will see in Proposition 4.3.3.1, the algebra $\Lambda = \Lambda_n$: has the property that for any thick subcategory $\mathcal{A}$, the subcategory $\delta^2(\mathcal{A})$ is equivalent to $\mathcal{A}$.

3.5. The poset $\mathbf{A}(\operatorname{mod}\Lambda)$: Maximal chains, complete exceptional sequences. Given a pair $\mathcal{A} \subset \mathcal{B}$ of neighbors in $\mathbf{A}(\operatorname{mod}\Lambda)$, we have denoted in Section 3.3.2 by $M_{\mathcal{A}}^{\mathcal{B}}$ the unique indecomposable module in $\mathcal{B} \cap \mathcal{A}^{\perp}$, and by $N_{\mathcal{A}}^{\mathcal{B}}$ the unique indecomposable module in $\mathcal{B} \cap {}^{\perp}\mathcal{A}$.

3.5.1. Recall that a sequence $(M_1, \ldots, M_t)$ of Λ-modules is called an *exceptional sequence* provided all the modules M_i are exceptional and $\operatorname{Hom}(M_j, M_i) = 0 = \operatorname{Ext}(M_j, M_i)$ for all $i < j$. There is the following equivalent inductive definition: The empty sequence is exceptional, and for $t \geq 1$, the sequence $(M_1, \ldots, M_t)$ is exceptional if and only $(M_1, \ldots, M_{t-1})$ is exceptional and M_t is an exceptional module in ${}^{\perp}(\operatorname{add}\{M_1, \ldots, M_{t-1}\})$, or, equivalently, if and only if $(M_2, \ldots, M_t)$ is exceptional and M_1 is an exceptional module in $(\operatorname{add}\{M_2, \ldots, M_t\})^{\perp}$.

Recall that an exceptional sequence $(M_1, \ldots, M_t)$ is said to be complete provided $t = \operatorname{rank}(\Lambda)$. Two exceptional sequences $(M_1, \ldots, M_t)$ and $(M_1', \ldots, M_{t'}')$ are called *isomorphic* provided $t = t'$ and M_i is isomorphic to M_i', for $1 \leq i \leq t$. We denote by $\mathbf{E}(\operatorname{mod}\Lambda)$ the set of (isomorphism classes of) complete exceptional sequences in $\operatorname{mod}\Lambda$.

There are some obvious complete exceptional sequences: Assume that the quiver of Λ has the vertex set $\{1, 2, \ldots, n\}$ and that for any arrow $i \leftarrow j$ we have $i < j$. Given a vertex i, we denote as usual the corresponding simple module by $S(i)$, the projective cover of $S(i)$ by $P(i)$, the injective envelope of $S(i)$ by $I(i)$. Then the following sequences

$$\big(P(1), P(2), \ldots, P(n)\big), \quad \big(S(n), S(n-1), \ldots, S(1)\big), \quad \big(I(1), I(2), \ldots, I(n)\big)$$

are complete exceptional sequences.

Theorem 3.5.1.1. *The isomorphism classes of complete exceptional sequences of Λ-modules correspond bijectively to the maximal chains in the poset $\mathbf{A}(\operatorname{mod}\Lambda)$.*

A bijection

$$\boxed{N \colon \mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda)) \longrightarrow \mathbf{E}(\operatorname{mod}\Lambda))}$$

is given as follows: Let

$$(*) \qquad 0 = \mathcal{A}(0) \subset \mathcal{A}(1) \subset \cdots \subset \mathcal{A}(n) = \operatorname{mod}\Lambda$$

be a maximal chain in $\mathbf{A}(\operatorname{mod}\Lambda)$, then

$$N(\mathcal{A}(0), \ldots, \mathcal{A}(n)) = \big(N_{\mathcal{A}(0)}^{\mathcal{A}(1)}, N_{\mathcal{A}(1)}^{\mathcal{A}(2)}, \cdots, N_{\mathcal{A}(n-1)}^{\mathcal{A}(n)}\big).$$

Conversely, if $(N_1, N_2, \dots, N_n)$ *is a complete exceptional sequence, let* $\mathcal{A}(i)$ *be the thick closure of* $N_1, \dots, N_i$*. then we obtain a maximal chain* $(*)$ *in* $\mathbf{A}(\operatorname{mod}\Lambda)$.

Addendum 3.5.1.2. *There is a second bijection*

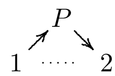
$$\boxed{M : \mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda)) \longrightarrow \mathbf{E}(\operatorname{mod}\Lambda))}$$

given as follows: Starting with a maximal chain $(*)$*, let*

$$M(\mathcal{A}(0), \dots, \mathcal{A}(n)) = \big(M^{\mathcal{A}(n)}_{\mathcal{A}(n-1)}, \cdots, , M^{\mathcal{A}(2)}_{\mathcal{A}(1)}, M^{\mathcal{A}(1)}_{\mathcal{A}(0)}\big).$$

Conversely, if $(M_1, M_2, \dots, M_n)$ *is a complete exceptional sequence, let* $\mathcal{A}(i)$ *be the thick closure of* $M_{n-i+1}, \dots, M_n$*. then we obtain a maximal chain* $(*)$ *in* $\mathbf{A}(\operatorname{mod}\Lambda)$.

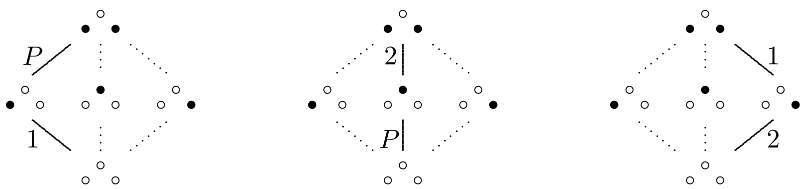
3.5.2. **Example: The algebra Λ_2.** For the algebra Λ_2 with Auslander-Reiten quiver



there are precisely three exceptional sequence, namely the sequence $(1, P)$ of the indecomposable projectives, the sequence $(P, 2)$ of the indecomposable injectives, and the sequence $(2, 1)$ of the simple modules.

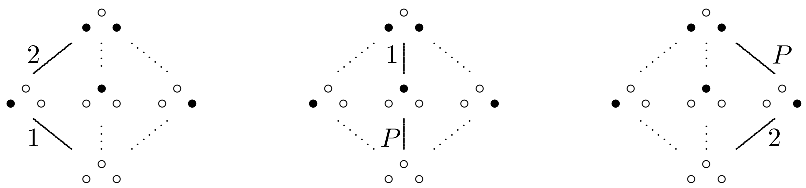
Similarly, the lattice $\mathbf{A}(\operatorname{mod}\Lambda_2)$ has precisely three maximal chains. The map N attaches to these chains in $\mathbf{A}(\operatorname{mod}\Lambda_2)$ the exceptional sequences as shown below (they have to be read going upwards):

N



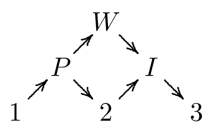
The map M attaches to the maximal chains in $\mathbf{A}(\operatorname{mod}\Lambda_2)$ the exceptional sequences as shown next (now they have to be read going downwards):

M

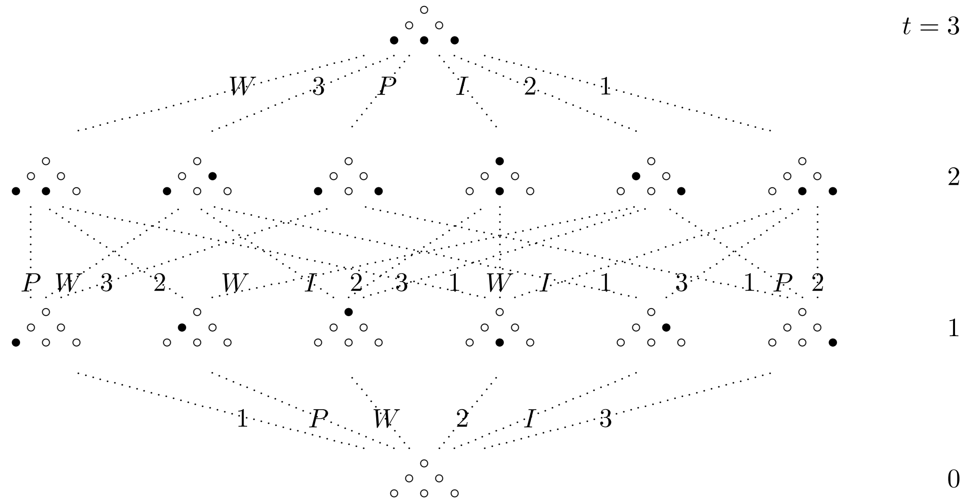


3.5.3. **Example: The algebra Λ_3.** As a second example, we consider the algebra Λ_3 (with the following quiver Q); we denote the simple module $S(i)$ just by i, and put $P = P(2),\ W = P(3) = I(1),\ I = I(2)$.

$$Q \qquad 1 \leftarrow 2 \leftarrow 3$$

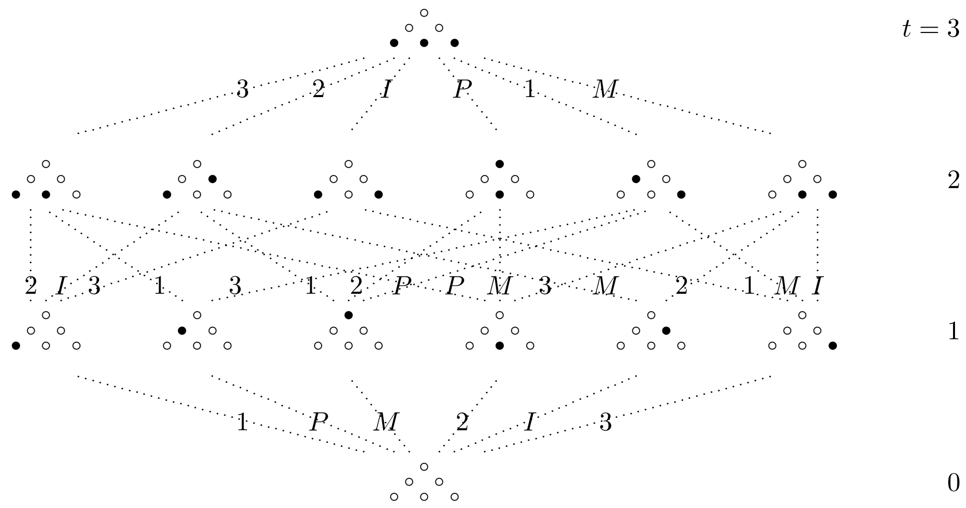


Here is the lattice $\mathbf{A}(\operatorname{mod}\Lambda_3)$. We have added the modules $N^{\mathcal{B}}_{\mathcal{A}}$ for every pair $\mathcal{A} < \mathcal{B}$ of neighbors to the (dotted) line connecting $\mathcal{A}$ and $\mathcal{B}$:



We obtain all the complete exceptional sequences **going up** in the lattice and using the modules $N_{\mathcal{A}}^{\mathcal{B}}$. For example, on the left hand side, we obtain in this way the sequence $(1, P, W)$ of indecomposable projective modules, on the right hand side one sees the sequence $(3, 2, 1)$ of the simple modules, whereas the exceptional sequence $(W, I, 3)$ of the indecomposable injective modules can be traced somewhere in the middle).

Here is again the lattice $\mathbf{A}(\operatorname{mod}\Lambda_3)$, now we show the modules $M_{\mathcal{A}}^{\mathcal{B}}$ for every pair $\mathcal{A} < \mathcal{B}$ of neighbors:



We obtain all the complete exceptional sequences **going down** in the lattice, using the modules $M_{\mathcal{A}}^{\mathcal{B}}$. (For example, on the left hand side, we obtain in this way the sequence $(3, 2, 1)$ of simple modules, on the right hand side there is the sequence $(M, I, 3)$ of the indecomposable injective modules.)

For an inductive formula for determining the number of complete exceptional sequences we refer to [**71**].

3.6. The braid group operation on $\mathbf{E}(\operatorname{mod}\Lambda)$ and on $\mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda))$. Recall that the braid group B_n on n strands is the group with generators $\sigma_1, \dots \sigma_{n-1}$ and the relations

$$\boxed{\begin{array}{lll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for} & 1 \le i \le n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \ge 2 \end{array}.}$$

The braid group B_n operates on $\mathbf{E} = \mathbf{E}(\operatorname{mod}\Lambda)$ as follows: Let $E = (E_1, \dots, E_n)$ be a complete exceptional sequence in $\operatorname{mod}\Lambda$ (and we may assume that $n \ge 2$). Let $\mathcal{E}(i)$ be the thick subcategory generated by E_i and E_{i+1} and $\overline{\tau}_i$ the extended Auslander-Reiten translation for $\mathcal{E}(i)$. Then we define

$$\sigma_i(E_1, \dots, E_n) = (E_1, \dots, E_{i-1}, \overline{\tau}_i E_{i+1}, E_i, E_{i+2}, \dots, E_n).$$

Note that $\sigma_i(E_1, \dots, E_n)$ is the unique complete exceptional sequence which is of the form $(E_1, \dots, E_{i-1}, ?, E_i, E_{i+2}, \dots, E_n)$. It is easy to see that the braid group relations are satisfied (see the note N 3.4).

3.6.1. We can use the map

$$M : \mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda)) \to \mathbf{E}(\operatorname{mod}\Lambda)$$

defined in Theorem 3.5.1.1 (or also the map N defined in the Addendum 3.5.1.2) in order to provide a corresponding braid group operation on the set $\mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda))$.

THEOREM 3.6.1.1. *The braid group acts transitively on the set* $\mathbf{E}$ *of complete exceptional sequences.*

In the case of the path algebra of a quiver, this result is due to Crawley-Boevey [**34**], for the general case, we refer to [**85**].

PROOF. Let $n = \operatorname{rank}\Lambda$. Let $E = (E_1, \dots, E_n)$ be an exceptional sequence. We show that using the braid group operation, we obtain from E an exceptional antichain. We show this by induction on the sum $|E|$ of the length of the modules E_i.

The proof is based on the following observation: For any $1 \le i \le n-1$, there is some $t = t(i)$ such that

$$\sigma_i^t(E) = (E_1, \dots, E_{i-1}, T, S, E_{i+2}, \dots, E_n)$$

with $\operatorname{Hom}(T, S) = 0$, and either we have already $\operatorname{Hom}(E_i, E_{i+1}) = 0$ or else $|\sigma_i^t(E)| < |E|$.

Thus, assume that $\operatorname{Hom}(E_i, E_j) \ne 0$ for some $i \ne j$. We show that there is a braid group element γ such that $|\gamma(E)| < |E|$.

Since E is exceptional, we must have $i < j$ and we can assume that $j - i$ is minimal. Let $f : E_i \to E_j$ be a non-zero map. Such a map is either injective or surjective. Up to duality, we can assume that f is injective. We define $E' = \sigma_{j-2} \cdots \sigma_i(E)$ and show that $|E'| = |E|$ and $|\sigma_{j-1}^t(E')| = |E'|$, for some t, thus $|\gamma(E)| < |E|$ for $\gamma = \sigma_{j-1}^t \sigma_{j-2} \cdots \sigma_i$.

Thus, first let us consider any s with $i < s < j$. By the minimality of $j - i$, we have $\mathrm{Hom}(E_i, E_s) = 0$. We also have $\mathrm{Ext}^1(E_i, E_s) = 0$ (namely, the map $f\colon E_i \to E_j$ is injective, thus

$$\mathrm{Ext}^1(f, E_s)\colon \mathrm{Ext}^1(E_j, E_s) \to \mathrm{Ext}^1(E_i, E_s)$$

is surjective, but on the other hand $\mathrm{Ext}^1(E_j, E_s) = 0$, since $j > s$). Thus, applying the braid group element $\sigma_{j-2} \cdots \sigma_i$ to E, we deal just with a permutation: the entry E_i is shifted from the position i to the position $j - 1$, we obtain

$$E' = \sigma_{j-2} \cdots \sigma_i(E) = (E_1, \dots, E_{i-1}, E_{i+1}, \cdots, E_{j-1}, E_i, E_j, \cdots, E_n).$$

and $|E'| = |E|$.

Second, we apply powers of σ_{j-1} to E'. We have

$$\sigma_{j-1}^t(E') = (E'_1, \dots, E'_{j-2}, T, S, E'_{j+1}, \dots, E'_n)$$

for some t with $\mathrm{Hom}(T, S) = 0$. Now E_i, E_j both belong to $\mathcal{E}(S, T)$ and at most one of these modules is simple in $\mathcal{E}(S, T)$, it follows that $|\sigma^t(E')| < |E'| = |E|$.

Finally, we note that an element $E = (E_1, \dots, E_n)$ in $\mathbf{E}$ which is an antichain consists of the simple modules. Namely, $\mathcal{E}$ is an exceptional subcategory of $\mathrm{mod}\,\Lambda$ of rank n, thus according to Corollary 2.5.3.2 we have $\mathrm{mod}\,\Lambda = \mathcal{E}$. Thus, every simple Λ-module has a filtration with factors in E and therefore belongs to E.

Of course, the antichains in $\mathbf{E}$ are obtained from each other by a sequence of transpositions of simple modules which do not extend each other, thus by elements of the braid group. □

3.7. Generalized non-crossing partitions. We come now to the central concept.

3.7.1. **The absolute length.** We start with a Coxeter group $W = (W, S)$, thus W is a group, S is a finite set of generators of W such that $s^2 = 1$ and $(st)^{m(s,t)} = 1$ where $m(s,t) = m(t,s) \geq 2$ are natural numbers or the symbol ∞, and these are the defining relations. The cardinality of S is called the *rank* of (W, S). If (W, S) is a Coxeter group of rank n, the elements of the form $s_1 s_2 \cdots s_n$, where $s_1, \dots, s_n$ are the elements of S (in some order) are called the *Coxeter elements* of (W, S).

The elements which are conjugate to an element in S are called *reflections*, we denote by T the set of reflections. Any element $w \in W$ is a product of elements from S, thus a product of reflections, and we write $|w|_a$ for the least number t such that w can be written as a product of t reflections. If $v, w \in W$, write $v \leq_a w$ provided $|v|_a + |v^{-1}w|_a = |w|_a$; this is a partial ordering on W called the *absolute ordering*. If $v \leq_a w$, we say that v is an *absolute factor* of w.

If w is of absolute length n, say $w = t_1 t_2 \cdots t_n$ with reflections $t_1, \dots, t_n$, then obviously any prefix word $t_1 t_2 \cdots t_r$ with $r \leq n$ is an absolute factor of w of absolute length r, but actually any word obtained from $t_1 t_2 \cdots t_n$ by deleting $n - r$ letters is an absolute factor of w of absolute length r. Namely, consider the case that we delete the letter t_i, thus, we consider the word uv where $u = t_1 \cdots t_{i-1}$ and $v = t_{i+1} \cdots t_n$. Then $(uv)^{-1}w = v^{-1}u^{-1}ut_iv = v^{-1}t_iv$ is in T, thus of absolute length 1. Of course, we see directly that $uv = t_i \cdot t_{i-1}t_{i+1} \cdots t_n$ is of absolute length at most $n - 1$.

3.7.2. **The definition.** The poset

$$\mathbf{Nc}(W,c) = \{w \in W \mid w \leq_a c\}$$

(using $\leq_a$ as ordering) is called the poset of *generalized non-crossing partitions of type* (W,c). Of course, if c, c' are conjugate Coxeter elements in W, then the posets $\mathbf{Nc}(W,c)$ and $\mathbf{Nc}(W,c')$ are isomorphic. In case the graph $\Delta(W,S)$ is a tree, in particular, if W is finite, then all Coxeter elements are conjugate. But in general, there are Coxeter elements c, c' which are not conjugate.

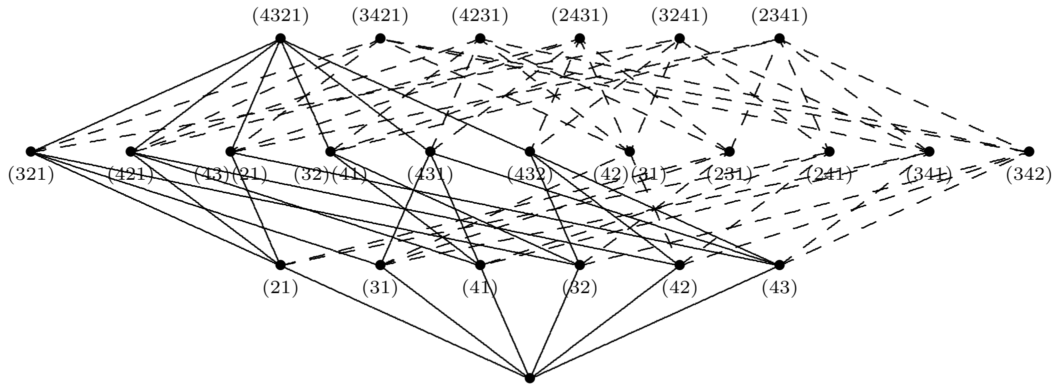
We have seen above that the posets $\mathbf{A}(\operatorname{mod}\Lambda)$, where Λ is the path algebra of the quiver $\widetilde{\mathbb{A}}_{2,2}$ or $\widetilde{\mathbb{A}}_{3,1}$ behave differently, see the examples 3.2.3 and 3.2.4. Note that the quivers $\widetilde{\mathbb{A}}_{2,2}$, $\widetilde{\mathbb{A}}_{3,1}$ have the same underlying graph, but they are distinguished by the orientation, thus by the choice of a Coxeter element. As we have seen, the posets $\mathbf{A}(\operatorname{mod}\Lambda)$ are not isomorphic (one is a lattice, the other one not). As we will see in Theorem 3.7.4.4, this means that the corresponding posets $\mathbf{Nc}(W(\Lambda), c(\Lambda))$ are not isomorphic, thus the Coxeter elements $c(\Lambda)$ cannot be conjugate.

For any natural number r, let $\mathbf{Nc}_r(W,c)$ be the subset of $\mathbf{Nc}(W,c)$ of all elements of absolute length r. Using the sets $\mathbf{Nc}_r(W,c)$ with $0 \leq r \leq n$, the poset $\mathbf{Nc}(W,c)$ is layered: If $v <_a w$ are neighbors in $\mathbf{Nc}_r(W,c)$ and w belongs to $\mathbf{Nc}_r(W,c)$, then v belongs to $\mathbf{Nc}_{r-1}(W,c)$. The set $\mathbf{Nc}_0(W,c)$ consists just of the identity element of W, the set $\mathbf{Nc}_n(W,c)$ of the single element c.

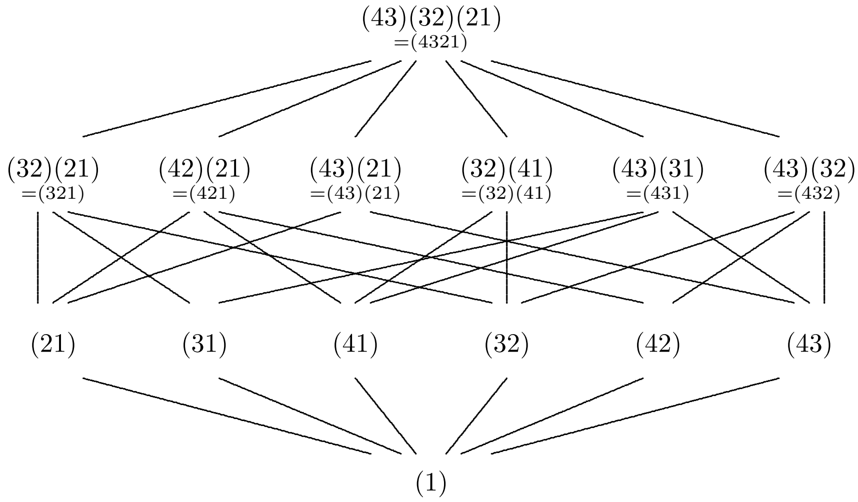
The absolute length was introduced Brady-Watt and Bessis in 2002 and 2003 in analogy to the usual length function $|-|$ which has been considered in Lie theory much earlier, see the note N 3.5.

3.7.3. **Example. The case $\mathbb{A}_n$.** The corresponding Weyl group is $W = S_{n+1}$, the symmetric group on $n+1$ letters $1, 2, \dots, n+1$. Usually we will consider the Coxeter element $c_{n+1} = (n, n+1)\cdots(23)(12) = (n+1, n, \cdots, 2, 1)$.

For $n = 3$, the poset $(W, \leq_a)$ looks as follows:



The solid edges describe the subset $\mathbf{Nc}(S_4, c_4)$, the dashed edges are the remaining ones. The following picture shows just $\mathbf{Nc}(S_4, c_4)$:



Here the permutations π are written as products of reflections as well as in cycle notation, but we note the following inconsistency: The cycle $(a_1, a_2, \dots, a_t)$ is the permutation which sends a_i to a_{i+1}, for $1 \le i < t$ and a_t to a_1, thus one uses the shift to the right. On the other hand, the product $\pi\pi'$ of two permutations π, π' means that we apply first π', then π. In particular, the product (32)(21)is the permutation $(32)(21) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$, and this is the cycle $\psi = (321) = (3, \psi(3), \psi^2(3))$.

We will see in Chapter 4 that the lattice $\mathbf{Nc}(S_n, c_n)$ can be identified in a natural way with the classical lattice $\mathbf{Nc}(n)$ of the non-crossing partitions as introduced by Kreweras in [**62**], see Theorem 4.2.5.1.

3.7.4. **The map** $\mathrm{cox}: \mathbf{A}(\mathrm{mod}\,\Lambda) \to \mathbf{Nc}(W(\Lambda), c(\Lambda))$**.** Given an exceptional module E, we denote by ρ_E the reflection with respect to $\mathbf{dim}\,E$, this is a reflection in the Coxeter group $W(\Lambda), c(\Lambda))$.

We define a map

$$\mathrm{cox}\colon \mathbf{A}(\mathrm{mod}\,\Lambda) \to W(\Lambda)$$

as follows: Let $\mathcal{A}$ be an exceptional subcategory of $\mathrm{mod}\,\Lambda$ of rank t, then

$$\mathrm{cox}(\mathcal{A}) = \rho_{A_1} \cdots \rho_{A_t},$$

where $\{A_1, \dots, A_t\}$ is an exceptional antichain with $\mathcal{A} = \mathcal{E}(A_1, \dots, A_t)$. Note that the composition $\rho_{A_1} \cdots \rho_{A_t}$ is independent from the choice of $\{A_1, \dots, A_t\}$: namely a different choice $\{A'_1, \dots, A'_t\}$ with $\mathcal{A} = \mathcal{E}(A'_1, \dots, A'_t)$ is obtained from $\{A_1, \dots, A_t\}$ by just permuting modules A_i, A_j with $\rho_{A_i}\rho_{A_j} = \rho_{A_j}\rho_{A_i}$.

In particular, we have

$$\boxed{\mathrm{cox}(\mathrm{mod}\,\Lambda) = c(\Lambda).}$$

The main observation is the following:

LEMMA 3.7.4.1. *Let* $(E_1,\dots,E_n)$ *and* $(E'_1,\dots,E'_n)$ *be complete exceptional sequences in* $\operatorname{mod}\Lambda$. *Then*

$$\rho_{E_1}\cdots\rho_{E_n}=\rho_{E'_1}\cdots\rho_{E'_n}.$$

PROOF. Since the braid group operates transitively on the set of exceptional sequences, it is sufficient to assume that $(E'_1,\dots,E'_n)=\sigma_i(E_1,\dots,E_n)$, thus we can assume that $n=2$ and that we deal with exceptional pairs of the form (E,E') and (E'',E). But in this case $\mathbf{dim}\,E''=\pm\rho_E(\mathbf{dim}\,E')$ and therefore $\rho_{E''}=\rho_E\rho_{E'}(\rho_E)^{-1}$, thus $\rho_E\rho_{E'}=\rho_{E''}\rho_E$, thus $\rho_{E''}\rho_E=\rho_E\rho_{E'}$. □

COROLLARY 3.7.4.2. *If* $(E_1,\dots,E_n)$ *is any complete exceptional sequence in* $\operatorname{mod}\Lambda$*, then*

$$\boxed{\rho_{E_1}\cdots\rho_{E_n}=c(\Lambda).}$$

PROPOSITION 3.7.4.3. *Let* $\operatorname{mod}\Lambda$ *be of rank* n *and let* $\mathcal{A}$ *be an exceptional subcategory of rank* m*. Then there is the following product formula in* $W(\Lambda)$*:*

$$\boxed{\operatorname{cox}(\mathcal{A})\cdot\operatorname{cox}({}^{\perp}\mathcal{A})=\operatorname{cox}(\operatorname{mod}\Lambda)).}$$

PROOF. Let $(E_1,\dots,E_m)$ be a complete exceptional sequence of $\mathcal{A}$ and let $(E_{m+1},\dots,E_n)$ be a complete exceptional sequence of ${}^{\perp}\mathcal{A}$. Then $(E_1,\dots,E_n)$ is a complete exceptional sequence of $\operatorname{mod}\Lambda$. We have $\operatorname{cox}(\mathcal{A})=\sigma_{E_1}\cdots\sigma_{E_m}$, $\operatorname{cox}(\mathcal{A}^{\perp})=\sigma_{E_{m+1}}\cdots\sigma_{E_m}$, and $\operatorname{cox}(\operatorname{mod}\Lambda)=\sigma_{E_1}\cdots\sigma_{E_n}$, thus

$$\operatorname{cox}(\mathcal{A})\cdot\operatorname{cox}({}^{\perp}\mathcal{A})=\sigma_{E_1}\cdots\sigma_{E_m}\cdot\sigma_{E_{m+1}}\cdots\sigma_{E_m}=\sigma_{E_1}\cdots\sigma_{E_n}=\operatorname{cox}(\operatorname{mod}\Lambda)).$$

□

In particular, since

$$|\operatorname{cox}(\mathcal{A})|_a=m,\quad |\operatorname{cox}({}^{\perp}\mathcal{A})|_a=n-m,$$

we see in this way that for any exceptional subcategory $\mathcal{A}$, we have

$$\operatorname{cox}(\mathcal{A})\le_a\operatorname{cox}(\operatorname{mod}\Lambda)=c(\Lambda),$$

thus $\operatorname{cox}(\mathcal{A})$ belongs to $\mathbf{Nc}(W(\Lambda),c(\Lambda))$.

THEOREM 3.7.4.4. **(Igusa-Schiffler-Thomas).** *The map*

$$\boxed{\operatorname{cox}\colon \mathbf{A}(\operatorname{mod}\Lambda)\longrightarrow\mathbf{Nc}(W(\Lambda),c(\Lambda)).}$$

is an isomorphism of posets.

In case Λ is of finite or tame representation type, the result is due to Ingalls and Thomas [**57**], for the general case, see [**56**] and also [**64**]. The proof will be outlined in the next section.

3.8. The braid group operation on $\mathbf{F}(W,c)$. Given any group G, the braid group B_n operates on the set G^n of n-tuples of elements of G as follows:

$$\sigma_i(g_1,\dots,g_n) = (g_1,\dots,g_{i-1},g_ig_{i+1}g_i^{-1},g_i,g_{i+1},\dots,g_n)$$

we shift the i-th entry one place to the right and insert on the i-place the conjugate $g_ig_{i+1}g_i^{-1}$ of g_{i+1}; this operation is sometimes called the *Hurwitz action*, see for example [**13**]. Note that for $\sigma_i(g_1,\dots,g_n) = (g'_1,\dots,g'_n)$, the products coincide:

$$g_1g_2\cdots g_n = g'_1g'_2\cdots g'_n.$$

3.8.1. Given a Coxeter group (W,S) of rank n with set of reflections T, one may consider the set T^n of n-tuples of reflections. Since T is closed under conjugation, the Hurwitz action sends T^n into itself. If c is an element of W (of interest for us is the case that c is Coxeter element), we denote by $\mathbf{F}(W,c)$ the set of n-tuples $(t_1,\dots,t_n)$ of elements of T with product $t_1t_2\cdots t_n = c$. There is the following theorem parallel to Theorem 3.6.1.1 (for the proof, we refer to [**56**]).

Theorem 3.8.1.1. **(Igusa-Schiffler).** *Let (W,S) be a Coxeter group of rank n and c a Coxeter element. The braid group B_n operates transitively on the set $\mathbf{F}(W,c)$.*

Given an exceptional sequence $E = (E_1,\dots,E_n)$ we define

$$\rho(E) = (\rho_{E_1},\dots,\rho_{E_n}),$$

this is an n-tuple in T and according to Corollary 3.7.4.2, the product is just the Coxeter element c, thus $\rho(E)$ belongs to $\mathbf{F}(W(\Lambda),c(\Lambda)$, this shows that we obtain a map

$$\rho\colon \mathbf{E}(\operatorname{mod}\Lambda) \to \mathbf{F}(W(\Lambda),c(\Lambda)).$$

Corollary 3.8.1.2. (1) *If $E = (E_1,\dots,E_n)$ is a sequence of exceptional modules and $\rho_{E_1}\cdots\rho_{E_n} = c(\Lambda)$, then E is an exceptional sequence.*

(2) *The map $\rho\colon \mathbf{E}(\operatorname{mod}\Lambda) \to \mathbf{F}(W(\Lambda),c(\Lambda))$ is a B_n-equivariant bijection.*

Proof. The map which attaches to an exceptional module X the reflection ρ_X is an injective map from the set of isomorphism classes of exceptional modules to the set T, thus the map $\rho\colon \mathbf{E}(\operatorname{mod}\Lambda) \to T^n$ is injective. As we know, the image of this map ρ lies in $\mathbf{F}(W(\Lambda),c(\Lambda))$. If $1 \le i < n$, and E is an exceptional sequence, then we know that $\rho(\sigma_i(E)) = \sigma_i\rho(E)$. It follows that the map $\rho : \mathbf{E}(\operatorname{mod}\Lambda) \to \mathbf{F}(W(\Lambda),c(\Lambda))$ is B_n-equivariant. In order to show that this map is surjective, consider an element $(t_1,\dots,t_n)$ in $\mathbf{F}(W(\Lambda),c(\Lambda))$.

Let $S_1,\dots,S_n$ be the simple Λ-modules ordered in such a way that $(S_1,\dots,S_n)$ is an exceptional sequence. Let $\rho_{S_i} = s_i$, thus , thus $\rho(S_1,\dots,S_n) = s_1\cdots s_n = \operatorname{cox}(\Lambda)$ and therefore $(s_1,\dots,s_n)$ belongs to $\mathbf{F}(W(\Lambda),c(\Lambda))$. Since B_n operates transitively on $\mathbf{F}(W(\Lambda),c(\Lambda))$, there is an element $g\in B_n$ such that $(t_1,\dots,t_n) = g(s_1,\dots,s_n)$. If we apply g to $(S_1,\dots,S_n)$, we obtain an exceptional sequence $(E_1,\dots,E_n)$, and

$$\rho(E_1,\dots,E_n) = \rho(g(S_1,\dots,S_n)) = g\rho(S_1,\dots,S_n) = g(s_1,\dots,s_n) = (t_1,\dots,t_n).$$

$\square$

For $0 \le r \le n$ let us denote by $\mathbf{F}_r(W,c)$ the set of r-tuples $(t_1,\dots,t_r)$ in T which can be completed to an n-tuple $(t_1,\dots,t_n) \in \mathbf{F}(W,c)$. And we denote by $\mathbf{E}_r(\operatorname{mod}\Lambda)$ the set of exceptional sequences in $\operatorname{mod}\Lambda$ of length r.

Proposition 3.8.1.3. *The map*

$$\rho\colon \mathbf{E}_r(\operatorname{mod}\Lambda) \to \mathbf{F}_r(W(\Lambda), c(\Lambda))$$

defined by $\rho(E_1,\dots,E_r) = (\rho_{E_1},\dots,\rho_{E_r})$ *is a bijection.*

Proof. Since any exceptional sequence $(E_1,\dots,E_r)$ can be completed to a complete exceptional sequence $(E_1,\dots,E_n)$, we see that our map ρ sends $\mathbf{E}_r(\operatorname{mod}\Lambda)$ into $\mathbf{F}_r(W(\Lambda), c(\Lambda))$.

On the other hand, given an element $(t_1,\dots,t_r)$ in $\mathbf{F}_r(W(\Lambda), c(\Lambda))$, it can be completed to an n-tuple $(t_1,\dots,t_n) \in \mathbf{F}(W,c)$. As we have seen, $(t_1,\dots,t_n) = \rho(E_1,\dots,E_n)$ for some complete exceptional sequence $(E_1,\dots,E_n)$ and therefore $(t_1,\dots,t_r) = \rho(E_1,\dots,E_r)$. □

Let us stress that the injectivity assertion implies the following:

If $E_1,\dots,E_r$ are exceptional modules such that $\rho_{E_1}\cdots\rho_{E_r} = \operatorname{cox}(\mathcal{A})$, where $\mathcal{A}$ is exceptional of rank r, then all the modules $E_1,\dots,E_r$ belong to $\mathcal{A}$.

Theorem 3.8.1.4. *The map* $\operatorname{cox}\colon \mathbf{A}_r(\operatorname{mod}\Lambda) \to \mathbf{Nc}_r(W(\Lambda), c(\Lambda))$ *is bijective, and there is the following diagram*

$$\begin{array}{ccc} \mathbf{E}_r(\operatorname{mod}\Lambda) & \xrightarrow{\ \rho\ } & \mathbf{F}_r(W(\Lambda), c(\Lambda)) \\ \downarrow\gamma & & \downarrow\mu \\ \mathbf{A}_r(\operatorname{mod}\Lambda) & \xrightarrow{\ \operatorname{cox}\ } & \mathbf{Nc}_r(W(\Lambda), c(\Lambda)) \end{array}$$

where γ sends any exceptional sequence E to the exceptional subcategory generated by E and μ is the multiplication map which sends $(t_1,\dots,t_r)$ to $\mu(t_1,\dots,t_r) = t_1\cdots t_r$.

Proof. It is clear that the diagram commutes.

By the definition of $\mathbf{Nc}_r(W(\Lambda), c(\Lambda))$, the map μ is surjective. Since ρ is bijective and μ is surjective, we see that cox is surjective.

Let us show that cox is injective. Let $\mathcal{A}, \mathcal{A}'$ be exceptional subcategories of rank r of $\operatorname{mod}\Lambda$ such that $\operatorname{cox}(\mathcal{A}) = \operatorname{cox}(\mathcal{A}')$. Let $E = (E_1,\dots,E_r)$ be a complete exceptional sequence in $\mathcal{A}$ and $(E'_1,\dots,E'_r)$ a complete exceptional sequences in $\mathcal{A}'$. Let $t_i = \rho_{E_i}$ and $t'_i = \rho_{E'_i}$, then $\operatorname{cox}(E) = t_1\cdots t_r$ and $\operatorname{cox}(E') = t'_1\cdots t'_r$. Now $(t_1,\dots,t_r)$ belongs to $\mathbf{F}_r(W(\Lambda), c(\Lambda))$, thus $(t_1,\dots,t_r)$ can be completed to an element $(t_1,\dots,t_n)$ in $\mathbf{F}(W(\Lambda), c(\Lambda))$. But also the element $(t'_1,\dots,t'_r,t_{r+1},\dots,t_n)$ belongs to $\mathbf{F}(W(\Lambda), c(\Lambda))$, since it is an n-tuple of elements of T with product equal to $c(\Lambda)$. For $r+1 \le i \le n$, write $t_i = \rho_{E_i}$ for some exceptional module E_i.

It follows from part (1) of Corollary 3.8.1.2 that both $(E_1,\dots,E_r,E_{r+1},\dots,E_n)$ and $(E'_1,\dots,E'_r,E_{r+1},\dots,E_n)$ are (complete) exceptional sequences of $\operatorname{mod}\Lambda$. Let $\mathcal{B}$ be the thick subcategory generated by $E_{r+1},\dots,E_n$. Since both $E_1,\dots,E_r$ as well as $E'_1,\dots,E'_r$ are complete exceptional sequences in $\mathcal{B}^\perp$, we see that $\mathcal{A} = \gamma(E_1,\dots,E_r) = \mathcal{B}^\perp = \gamma(E'_1,\dots,E'_r) = \mathcal{A}'$. □

3.8.2. **Proof of Theorem 3.7.4.4.** We have seen already that

$$\mathrm{cox}\colon \mathbf{A}(\mathrm{mod}\,\Lambda) \to \mathbf{Nc}(W(\Lambda), c(\Lambda))$$

is bijective. It remains to be shown: Given exceptional subcategories $\mathcal{A}, \mathcal{A}'$ of $\mathrm{mod}(\Lambda)$, then $\mathcal{A} \subseteq \mathcal{A}'$ if and only if $\mathrm{cox}\,\mathcal{A} \leq_a \mathrm{cox}\,\mathcal{A}'$.

Of course, if $\mathcal{A} \subseteq \mathcal{A}'$, then take a complete exceptional sequence $(E_1, \dots, E_r)$ of $\mathcal{A}$, extend it to a complete exceptional sequence $(E_1, \dots, E_s)$ of $\mathcal{A}'$. It follows that

$$\mathrm{cox}(\mathcal{A}) = \rho_{E_1} \cdots \rho_{E_r} \leq_a \rho_{E_1} \cdots \rho_{E_s} = \mathrm{cox}(\mathcal{A}').$$

Conversely, assume that $\mathrm{cox}(\mathcal{A}) \leq_a \mathrm{cox}(\mathcal{A}')$. Again, let $(E_1, \dots, E_r)$ be a complete exceptional of $\mathcal{A}$, thus $\mathrm{cox}(\mathcal{A}) = \rho_{E_1} \cdots \rho_{E_r}$. Since $\mathrm{cox}(\mathcal{A}) \leq_a \mathrm{cox}(\mathcal{A}')$, we find reflections $t_{r+1}, \dots, t_s$ such that $\mathrm{cox}(\mathcal{A}') = \rho_{E_1} \cdots \rho_{E_r} t_{r+1} \cdots t_s$, where s is the rank of $\mathcal{A}'$. According to Proposition 3.8.1.3, it is possible to write $t_i = \rho_{E_i}$ for $r+1 \leq i \leq s$, where the E_i are exceptional modules. Then $(E_1, \dots, E_s)$ is a sequence of s exceptional modules with product $\mathrm{cox}(\mathcal{A}')$, and $\mathcal{A}'$ has rank s, thus all the modules $E_1, \dots, E_s$ belong to $\mathcal{A}'$. In particular, $\mathcal{A} \subseteq \mathcal{A}'$.

Notes to Chapter 3.

N 3.1. Exceptionality. We follow here a long and well-established tradition to call an indecomposable module without self-extensions an exceptional module, as started by Crawley-Boevey [**34**] in analogy to the exceptional vector bundles discussed by Rudakov [**91**] and his collaborateurs. But the reader should be aware that this terminology seems to be misleading in our context. What is called exceptional has to be considered, in a non-commutative setting, not as an extraordinary feature, but as a quite typical behaviour. Of course, dealing with commutative rings, only few modules will be exceptional. Thus, from a commutative perspective, the existence of large numbers of exceptional vector bundles was rated as an exceptional behaviour (after all, algebraic geometry is often considered as part of commutative algebra).

N 3.2. Artin algebras of rank 2. *If* $\mathrm{rank}\,\Lambda = 2$*, and* $X_1 \neq X_2$ *are non-isomorphic exceptional* Λ*-modules, then the abelian closure of the set* $\{X_1, X_2\}$ (the closure under kernels, cokernels) is $\mathrm{mod}\,\Lambda$.

Proof. The assertion is clear if X_1, X_2 are orthogonal (namely, in this case X_1, X_2 are just the simple Λ-modules. Thus we can assume that $\mathrm{Hom}(X_1, X_2) \neq 0$. Up to duality, we can assume that X_1, is preprojective and we use induction on the number of predecessors of X_1. If X_1 is simple, then we obtain the second simple module as a direct summand of the cokernel of a map $X_1^t \to X_2$. If X_1 is not simple, then X_1 generates X_2, say there is a surjective map $f : X_1^t \to X_2$. If X_1' is an indecomposable direct summand of X_1, then X_1' has less predecessors than X_1. □

N 3.3. Thick subcategories $\mathcal{X}$ with proper inclusion $\mathcal{X} \subset {}^{\perp}(\mathcal{X}^{\perp})$. Consider, for example, the Kronecker algebra Λ with sink 0 and source 1. Let $\mathcal{N}$ be the subcategory of regular modules (these are the direct sums of indecomposable modules M with $\dim M_0 = \dim M_1$). This is a thick subcategory and $\mathcal{N}^{\perp} = 0$, thus ${}^{\perp}(\mathcal{N}^{\perp}) = \mathrm{mod}\,\Lambda$.

N 3.4. The braid group relations. The relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ are obviously satisfied. In order to show that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, we can assume that $i = 1$ and $n = 3$. Thus, let $E = (E_1, E_2, E_3)$ be a complete exceptional sequence and let $\overline{\tau_*}$ be the extended Auslander-Reiten translation for the thick subcategory generated by E_1 and E_3. One sees immediately that both $\sigma_1\sigma_2\sigma_1 E$ and $\sigma_2\sigma_1\sigma_2 E$ are of the form $(?, \overline{\tau_*}E_3, E_1)$. But there is just one complete exceptional sequence of this form.

N 3.5. The length function on a Coxeter group. Given a Coxeter group (W, S) and an element $w \in W$, one denotes by $|w|$ the least number t such that w can be written as a product of t elements of S. If $v, w \in W$, then one sets $v \leq w$ provided $|v| + |v^{-1}w| = |w|$. For the properties of this ordering, we refer to [**18**], and [**54, 55**].

4. The Hereditary Artin Algebra Λ_n

4.1. The lattice $\mathbf{Nc}(n)$ of non-crossing partitions of an n-element set.

4.1.1. A *partition* of a set S is a set of pairwise disjoint non-empty subsets S_i of S such that $S = \bigcup_i S_i$; the subsets S_i are called the *parts.* We will denote the set of all partitions of S by $\Pi(S)$, and we write $\Pi(n)$ instead of $\Pi(\{1, 2, \dots, n\})$.

In order to visualize such a partition, one sometimes uses arcs, as in the following example:  depicts the partition $\{\{1, 2, 4\}, \{3\}\}$ of the set $\{1, 2, 3, 4\}$; here, the bullets are the integers $1, 2, 3, 4$ (in this order), the arcs indicate that the corresponding integers belong to the same part, the parts are just the transitive closure. To be more precise, let us assume that $S = \{1, 2, \dots, n\}$. An *arc* of the partition P of $S = \{1, 2, \dots, n\}$ is a pair $[a, b]$ of integers $1 \le a < b \le n + 1$ such that a, b belong to the same part of P whereas no element c with $a < c < b$ belongs to this part. We denote by $A(P)$ the set of arcs of P. We call a set of arcs a *partition arc set* provided for any pair of arcs $[a, b], [a', b']$ in the set, we have $a = a'$ iff $b = b'$.

Lemma 4.1.1.1. *Let $S = \{1, 2, \dots, n\}$. The map A provides a bijection between the partitions of S and the partition arc sets in S.*

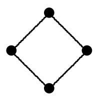
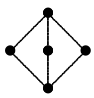
Proof. See the note N 4.1. □

A partition is said to be *non-crossing* provided given elements $i < i' < j < j'$ with i, j in the same part and i', j' in the same part, then all elements belong to the same part (this means that arcs do not intersect properly.) For $n = 4$, all partitions but one are non-crossing, the exception is the partition $\{\{1, 3\}, \{2, 4\}\}$ with picture  . There is the following (quite obvious) characterization of the set of arcs of a non-crossing partition.

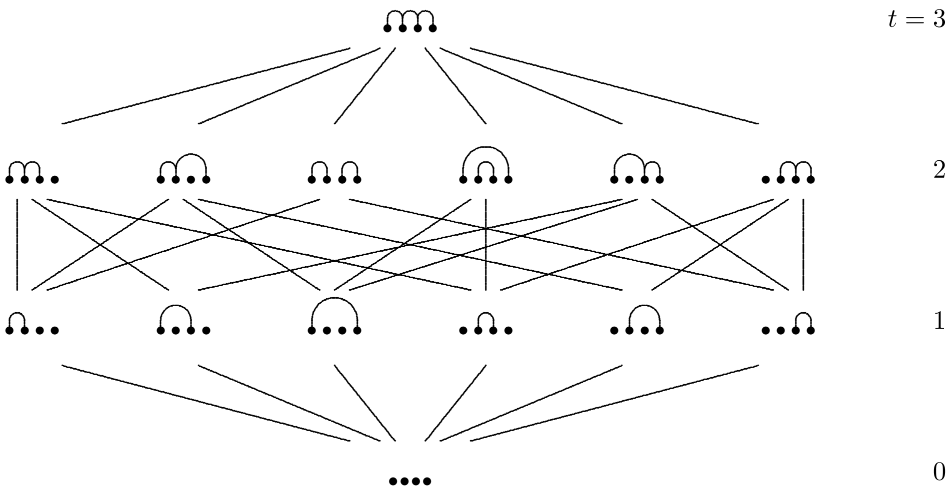
Lemma 4.1.1.2. *A set A of pairs $[a, b]$ with $1 \le a < b \le n$ is the set of arcs of a non-crossing partition $P \in \mathbf{Nc}(n)$ if and only if the following condition is satisfied: If $[a, b], [a', b'] \in A$ and $a \le a' < b \le b'$, then $a = a'$ and $b = b'$.*

We denote by $\mathbf{Nc}(n)$ the poset of non-crossing partitions of the set $\{1, 2, \dots, n\}$; the order which is used is the (opposite of the) *refinement* order: to write parts as disjoint union of subsets. If P, P' are partitions of the same set, we write $P \le P'$ provided any part of P is contained in a part of P', or, equivalently, provided any part of P' is the union of parts of P. The unique minimal element of $\mathbf{Nc}(n)$ is the partition with n parts, the unique maximal element is the partition with just one part. The *layers* are formed by the partitions with a fixed number of parts. The poset $\mathbf{Nc}(n)$ has n different layers $\mathbf{Nc}_t(n)$, with $0 \le t \le n-1$, where $\mathbf{Nc}_t(n)$ denotes the set of non-crossing partitions with $n - t$ parts.

All the posets $\mathbf{Nc}(n)$ are lattices (this is not quite obvious, but it will be shown later), they are self-dual, and have further interesting properties. For example, there are only two possibilities for the intervals of height 2, namely

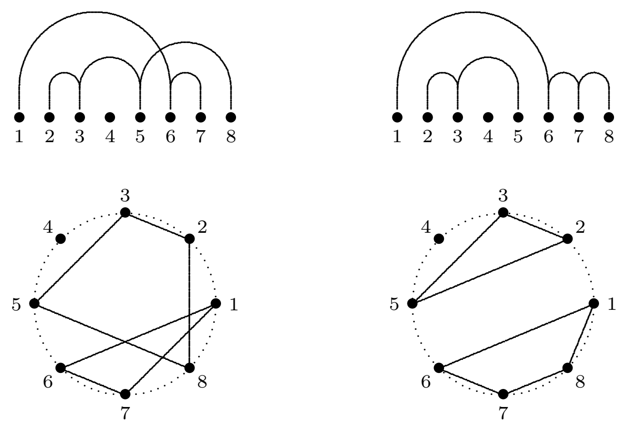
Here is the lattice $\mathbf{Nc}(4)$ of non-crossing partitions of a set with 4 elements.



The lattices $\mathbf{Nc}(n)$ are typical examples of the lattices studied in Chapter 3: as we will see, they can be identified with the lattices $\mathbf{A}(\mathrm{mod}\,\Lambda)$, where Λ is hereditary of type $\mathbb{A}$ (see Theorem 4.2.2.2). But let us stress already here the index shift: the lattice $\mathbf{Nc}(n)$ is considered to be of type $\mathbb{A}_{n-1}$).

Another way to visualize partitions of a set S is to arrange the vertices of S consecutively on a circle and to consider for every part $\{a_1 < a_2 < \cdots < a_m\}$ the convex polygon with vertices $a_1, a_2, \ldots, a_m$ (this just means to draw for $m \geq 3$ another arc, namely joining a_m and a_1). Here are two examples:

$\{\{1,6,7\},\{2,3,5,8\},\{4\}\}$ $\qquad$ $\{\{1,6,7,8\},\{2,3,5\},\{4\}\}$



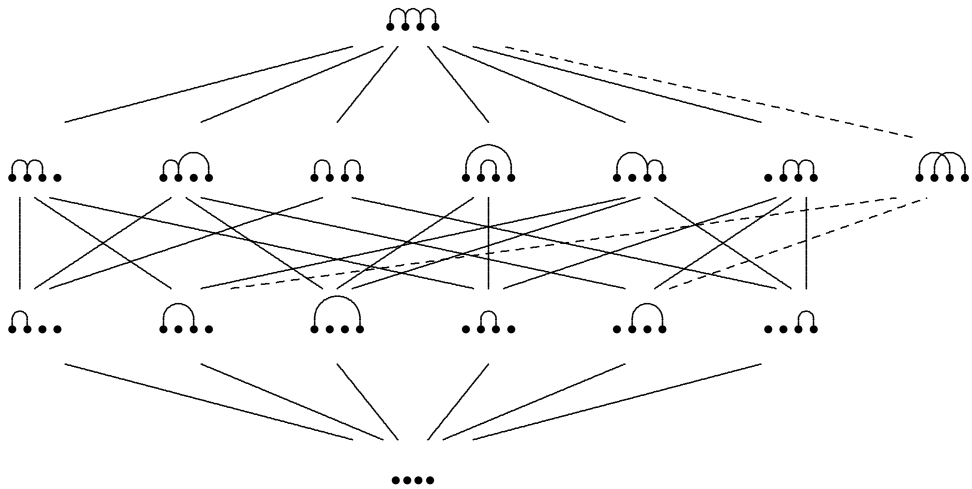
The partition on the left is crossing, that on the right is non-crossing.

The lattice of non-crossing partitions was introduced and studied by **Kreweras** (1972). Actually, Becker was considering non-negative partitions already in 1948, he showed that the number of elements of $\mathbf{Nc}(n)$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

For a survey concerning the set of non-crossing partitions we may refer to Simion (2000).

The lattice of non-crossing partitions plays a decisive role in **free probability theory.** The free probability theory was initiated by Voiculescu (1985), the main ingredient is the use of non-commuting variables (in contrast to the commuting ones in ordinary probability theory). Many calculations are rather similar to the classical case, but surprisingly, some turn out to become easier. Namely, it is common in classical probability theory that formulas involve the summation over all partitions of some finite set. Looking at the corresponding sums in free probability theory it happens quite frequently that several of the summands are zero, namely those indexed by crossing partitions, thus, in this case it is sufficient to sum over the non-crossing partitions. For example, this happens for the so-called central limit theorem. Speicher (1990) gave a proof of this theorem using systematically non-crossing partitions. One should be aware that for n large, the number of non-crossing partitions of an n-element set is much smaller than the number of all partitions. As we saw above, for $n = 4$, there are 15 partitions and all but one are non-crossing. But for $n = 20$, there are 51 724 158 235 372 partitions, and only 6 564 120 420 are non-crossing. Also, as soon as one deals just with non-crossing partitions, one may rely on properties of the lattice of non-crossing partitions — and, as we have already mentioned, the lattice of non-crossing partitions has some quite surprising properties. The book by Nica and Speicher (2006) is presently the main reference for the use of non-crossing partitions in free probability theory.

4.1.2. Let us compare for a moment the lattice of all partitions with the lattice of non-crossing partitions, for some fixed n. The set of non-crossing partitions is of course a part of the lattice of all partitions (it is not a sublattice, but at least a meet sublattice). Here again the case $n = 4$:

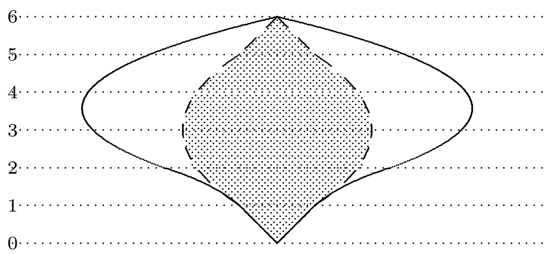


The number of partitions of an n-element set into exactly k non-empty parts is the *Stirling number of the second kind* $\left\{ {n \atop t} \right\}$ (we use the so-called Karamata notation, as advertised by Knuth [**60**], a competing notation would be $S(n,k)$). The sequence of the Stirling numbers of the second kind form the triangle A048993 in Sloane's OEIS [**100**].

The partitions with $n-1$ parts are just given by a two-element subset (the remaining parts are singletons), the number is $\binom{n}{2} = |\Phi_+(\mathbb{A}_{n-1})|$. The partitions with $n-1$ parts are of course non-crossing.

If we look at partitions with 2 parts, the number is $2^{n-1}-1$. (There are 2^n ordered pairs of complementary subsets S and S'. In one case, S is empty, and in another S' is empty, so 2^n-2 ordered pairs of non-empty subsets remain. Finally, since we want unordered pairs rather than ordered pairs we have to divide this last number by 2). On the other hand, the number of non-crossing partitions with 2 parts is again $\binom{n}{2}$: as we have mentioned, the lattice of non-crossing partitions is selfdual, a very remarkable property.

Here is a sketch for $n=7$, the shaded part indicates the set of non-crossing partitions inside the set of all partitions. On the right, we have written the corresponding numbers of partitions with $7-t$ parts, first the number $|\,\mathbf{Nc}_t(7)|$ of non-crossing partitions with $7-t$ parts, then the number $\left\{ {7 \atop 7-t} \right\}$ of all partitions with $7-t$ parts; always $0 \le t \le 6$.



t	$\lvert\,\mathbf{Nc}_t(7)\rvert$	$\left\{ {7 \atop 7-t} \right\}$
6	1	1
5	21	63
4	105	301
3	175	350
2	105	140
1	21	21
0	1	1

The partition lattices have the following layers

1	1	1	1	1	1	1	1	...
	1	3	7	15	31	63	127	
		1	6	25	90	301	966	
			1	10	65	350	1701	
				1	15	140	1050	
					1	21	266	
						1	28	
							1	

whereas the lattices of non-crossing partitions have the layers

1	1	1	1	1	1	1	1	...
	1	3	6	10	15	21	28	
		1	6	20	50	105	196	
			1	10	50	175	490	
				1	15	105	490	
					1	21	196	
						1	28	
							1	

The number of all partitions of a set of cardinality n is called the *Bell numbers* B_n (thus $B_n = |\Pi(n)|$), whereas the number of non-crossing partition in $\Pi(n)$ is the Catalan number C_n. Here is a comparison of the Bell and Catalan numbers for $n \le 10$:

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796
B_n	1	1	2	5	15	52	203	877	4140	21147	115975

It is well-known (and easy to see) that $\lim_{n\to\infty} C_n/B_n = 0$.

4.2. The categorification of $\mathbf{Nc}(n)$. Let us return to the path algebra Λ_n of the linearly oriented quiver of type $\mathbb{A}_n$:

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow n$$

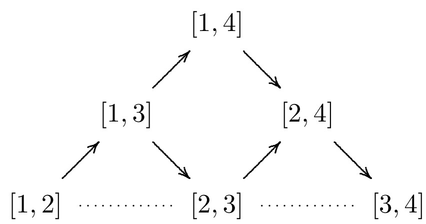
and recall that for any hereditary artin algebra Λ, we denote by $\mathbf{A}(\operatorname{mod}\Lambda)$ the set of (isomorphism classes of) exceptional antichains in $\operatorname{mod}\Lambda$. As we know, the exceptional antichains correspond bijectively to the exceptional subcategories, let us denote here by $\underline{\mathcal{A}}(\operatorname{mod}\Lambda)$ the poset of exceptional subcategories of $\operatorname{mod}\Lambda$ (with ordering being given by the set-theoretical inclusion). Usually, we will identify the two sets $\mathbf{A}(\operatorname{mod}\Lambda)$ and $\underline{\mathcal{A}}(\operatorname{mod}\Lambda)$, but for the purpose of the following dictionary, we should make a distinction. We denote by $\mathcal{E}$ and S the canonical poset isomorphisms

$$\boxed{\mathbf{A}(\operatorname{mod}\Lambda_n) \;\underset{S}{\overset{\mathcal{E}}{\rightleftarrows}}\; \underline{\mathcal{A}}(\operatorname{mod}\Lambda_n)}$$

Here, $\mathcal{E}$ maps the antichain A to the full subcategory $\mathcal{E}(A)$ of all modules with a filtration with factors in A, whereas the inverse bijection S sends the exceptional subcategory $\mathcal{A}$ to the set $S(\mathcal{A})$ of simple objects in $\mathcal{A}$.

4.2.1. **The indecomposable Λ_n-modules.** We deviate from the usual notation for the indecomposable representation of Λ_n. We will denote the indecomposable projective module of length j by $[1, j+1]$ and the factor module $[1, j+1]/[1, i]$ with $1 \le i \le j$ by $[i, j+1]$. In particular, the simple Λ_n-modules are denoted by $S(i) = [i, i+1]$, and the indecomposable module with socle $S[i]$ and length t by $[i, i+t]$. (The usual labeling of the indecomposable module of length t uses its composition factors, thus a sequence of t consecutive numbers.) Let us repeat: In this chapter, the indecomposable Λ_n-modules are given by the intervals $[a, b]$ with $1 \le a < b \le n+1$ and the support of the module $[a, b]$ are the simple modules $S(x) = [x, x+1]$ with $a \le x \le b-1$, in particular, the length of $[a, b]$ is $b - a$.

Using this notation, the Auslander-Reiten quiver of Λ_3 looks as follows:



As we have mentioned, the simple Λ_n-modules are the modules of the form $S(i) = [i, i+1]$ with $1 \le i \le n$. Note that $\operatorname{Ext}^1(S(j), S(i)) \neq 0$ if and only if $j = i+1$, thus $\operatorname{Ext}^1([a, a+1], [c, c+1]) \neq 0$ if and only if $c = a+1$. Also, we should mention that all the indecomposable Λ_n-modules are bricks (after all, Λ_n is representation directed).

The essential (and trivial) observation which will be used in this chapter is already suggested by our notation: *there is a bijection between the arcs in* $\{1, 2, \ldots, n+1\}$ *and the indecomposable* Λ_n*-modules,* both the arcs and the indecomposable modules are denoted by the intervals $[a, b]$ with $1 \le a < b \le n+1$.

4.2.2. **The map $A\colon \mathbf{Nc}(n+1) \to \mathbf{A}(\operatorname{mod}\Lambda_n)$.** We have defined $A(P)$ for any partition P of $\{1, 2, \ldots, n+1\}$ as the set of arcs $[a, b]$ for P, see Lemma 4.1.1.1. Of course, we may interpret such an arc $[a, b]$ also as an indecomposable Λ_n-module. Thus, given a partition $P \in \Pi(n+1)$, let $A(P)$ be the set of all the arcs of P, or as the set of the corresponding Λ_n-modules.

PROPOSITION 4.2.2.1. *Let P be a partition of $\{1, 2, \ldots, n+1\}$. Then P is non-crossing if and only if $A(P)$ is an antichain in* $\operatorname{mod}\Lambda_n$.

For further equivalent properties, see the note N 4.2.

PROOF. First, assume that P is crossing, thus there are two arcs $[a, b], [c, d]$ which are crossing. We may assume that $a < c$, thus $a < c < b < d$. But then $[c, b]$ is a non-zero factor module of $[a, c]$ and also a submodule of $[c, d]$. It follows that $\operatorname{Hom}([a, b], [c, d]) \neq 0$, thus the modules $[a, b], [c, d]$ are non-isomorphic, but not orthogonal, and therefore $A(P)$ is not an antichain in $\operatorname{mod}\Lambda_n$. Conversely, assume that $A(P)$ is not an antichain. Then there are different arcs $[a, b], [c, d]$ such that $\operatorname{Hom}([a, b], [c, d]) \neq 0$. This implies that $a \leq c < b \leq d$. If $a = c$, then c, b, d belong to the same part, thus also $b = d$, since $[c, d]$ is an arc. This is a contradiction, since we assume that $[a, b]$ and $[c, d]$ are different. Therefore $a < c$. Similarly, we see that $b < d$. This shows that the arcs $[a, b]$ and $[c, d]$ are crossing. □

It follows that the restriction of the map A defined on $\Pi(n+1)$ is a bijection $A\colon \mathbf{Nc}(n+1) \to \mathbf{A}(\operatorname{mod}\Lambda_n)$.

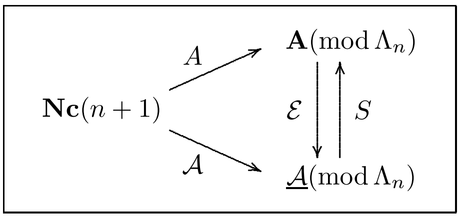
THEOREM 4.2.2.2. *The map $A\colon \mathbf{Nc}(n+1) \to \mathbf{A}(\operatorname{mod}\Lambda_n)$ is a poset isomorphism.*

PROOF. It remains to show: If P, P' are non-crossing partitions, then $P \leq P'$ if and only if $A(P) \leq A(P')$.

First, assume that $P \leq P'$. Let $[a, b] \in A(P)$. Then $a < b$ and a, b belong to the same part of P. Since $P \leq P'$, the elements a, b belong to the same part of P', thus, there are arcs $[a_i, a_{i+1}] \in A(P')$ with $1 \leq i < t$ such that $a = a_1$ and $a_t = b$. But this means that the Λ_n-module $[a, b]$ has a filtration with factors $[a_i, a_{i+1}]$ and $1 \leq i < t$. These modules $[a_i, a_{i+1}]$ belong to $A(P')$. Thus, we see that $[a, b]$ belongs to $A(P')$.

Conversely, assume that $A(P) \leq A(P')$. We want to that $P \leq P'$. Let $\{a_1 < a_a < \cdots < a_t\}$ be a part of P. The pairs $[a_i, a_{i+1}]$ with $1 \leq i < t$ are arcs for P. Since $A(P) \leq A(P')$, we see that any interval $[a_i, a_{i+1}]$ can be refined, say $a_i = a_{i1} < a_{i2} < \cdots < a_{is_i} = a_{i+1}$ such that the intervals $[a_{ij}, a_{i,j+1}]$ with $1 \leq j < s_i$ belong to $A(P')$. It follows that the elements $a_i = a_{i1}, a_{i2}, \cdots, a_{is_i} = a_{i+1}$ belong to the same part of P', thus all the a_i with $1 \leq i < t$ belong to the same part of P'. This shows that $P \leq P'$. □

ADDENDUM 4.2.2.3. *If we define $\mathcal{A} = \mathcal{E}(A)$, there is the following commutative diagram of poset isomorphisms:*

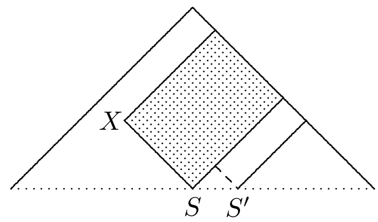


The map $\mathcal{A}$ provides a bijection between the non-crossing partitions of the set $S = \{1, 2, \ldots, n+1\}$ and the thick subcategories of $\operatorname{mod}\Lambda_n$. The structure of the thick subcategories $\mathcal{A}$ in $\operatorname{mod}\Lambda_n$ is very restricted. The essential observation is the following:

PROPOSITION 4.2.2.4. *Any connected thick subcategory* $\mathcal{A}$ *of* $\operatorname{mod}\Lambda_n$ *of rank* t *is equivalent (as a category) to* $\operatorname{mod}\Lambda_t$.

PROOF. It is easy to see that we only have to deal with $\mathcal{A}$ being of rank 3. Thus, let $\mathcal{A}$ be a connected thick subcategory of $\operatorname{mod}\Lambda_n$ of rank 3. We show that the quiver of $\mathcal{A}$ cannot have two sources.

Let X be a sink in $Q(\mathcal{A})$, let S be its top and $S' = \tau^{-1}S$.



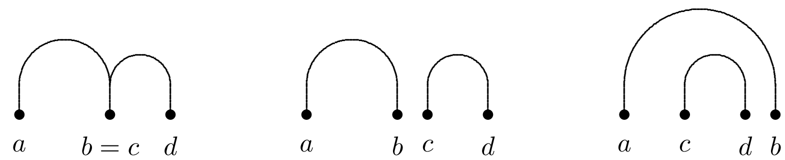
The shaded rectangle are the modules Z with $\operatorname{Hom}(X, Z) \neq 0$. The indecomposable modules Y such that X, Y is an orthogonal pair and $\operatorname{Ext}^1(Y, X) \neq 0$ are the modules Y with $\operatorname{soc} Y = S'$, these are the modules in the ray starting at S'.

There is no pair of orthogonal modules Y with $\operatorname{soc} Y = S'$. This shows that the quiver of $\mathcal{A}$ cannot have two sources. By duality, it also cannot have two sinks. Thus $\mathcal{A}$ has to be equivalent to $\operatorname{mod}\Lambda_3$. □

The aim of our further considerations is to provide a dictionary between the combinatorial assertions concerning $\mathbf{Nc}(n+1)$ and the representation theoretical assertions concerning $\mathbf{A}(\operatorname{mod}\Lambda_n)$ and $\underline{\mathcal{A}}(\operatorname{mod}\Lambda_n)$.

4.2.3. **Pairs of arcs.** For a non-crossing partition P, there are three types of pairs $\{[a, b], [c, d]\}$ of different arcs. We may assume that $a \leq c$. As we have seen above, this implies that $a < c$ and similarly, we have $b \neq d$.

Case 1: $b = c$ Case 2: $b < c$ Case 3: $c < b$



In case 3, the support of the module $[c, d]$ is a proper subset of the support of $[a, b]$; actually, the module $M = [a, b]$ has a filtration $0 \subset M' \subset M'' \subset M$ with non-zero modules M' and M/M'', such that M''/M' is isomorphic to $[c, d]$. In the cases 1 and 2, the modules $[a, b], [c, d]$ have disjoint support. In case 1, we have $\operatorname{Ext}^1([c, d], [a, b]) \neq 0$, namely, there is a non-split exact sequence

$$0 \to [a, b] \to [a, d] \to [c, d] \to 0,$$

whereas $\operatorname{Ext}^1([c, d], [a, b]) = 0$ in case 2 (and also in case 3).

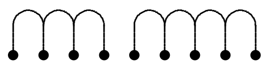
Let us repeat that we define $\mathcal{A} = \mathcal{F}(A)$. Equivalently, we may specify directly the indecomposable objects of $\mathcal{A}(P)$, these are the Λ_n-modules of the form $[a, b]$

where $a < b$ are elements in the same block of P. The parts of P of cardinality $t \geq 2$ correspond bijectively to the blocks of $\mathcal{A}$ of rank $t-1$. Here is the recipe: Given a part $\{a_1 < \cdots < a_t\}$ of P, one sends it under A to the antichain $[a_1, a_2], [a_2, a_3], \ldots, [a_{t-1}, a_t]$ in $\operatorname{mod}\Lambda_n$, and $\mathcal{A}(P)$ is just the thick closure of this antichain. The simple objects $[a_1, a_2], [a_2, a_3], \ldots, [a_{t-1}, a_t]$ in $\mathcal{A}(P)$ have a quiver of the form $1 \leftarrow 2 \leftarrow \cdots \leftarrow t-1$ and the indecomposables in this block are the Λ_n-modules of the form $[a_i, a_j]$ with $1 \leq i < j \leq t$.

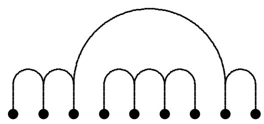
The main point to have in mind is the following: the arcs of P are identified with the simple objects in $\mathcal{A}(P)$. In particular, the number of arcs of A is the rank of $\mathcal{A}(A)$. For example, the partition P with just one part (and therefore with arcs between i and $i+1$, for all $1 \leq i \leq n$) is mapped under $\mathcal{A}$ to the full module category $\operatorname{mod}\Lambda_n$. The subcategory $\mathcal{A}(P)$ is sincere if and only if 1 and $n+1$ belong to the same part of P.

4.2.4. **Maximal elements in $\mathbf{Nc}(n+1)$.** Let P be a maximal element in $\mathbf{Nc}(n+1)$, thus P consists of precisely two parts P_1 and P_2 and we may assume that 1 belongs to P_1. There are two different cases:

Case 1: The elements 1 and $n+1$ do not belong to the same part, thus $n+1$ belongs to P_2. Let x be the maximal element of P_1. Then the arcs are just the pairs $i, i+1$ with $1 \leq i \leq n$ and $i \neq x$. Here is an example:



Case 2: The elements 1 and $n+1$ are in the same part, thus now also $n+1$ belongs to P_1. Let x be the largest number with $x < P_2$ and y the largest element of P_2. Then the arcs are the pairs $i, i+1$ with $1 \leq i \leq n$ and $i \notin \{x, y\}$ as well as $x, y+1$. Here is an example:



(there is also the special case of $y = x+1$; in this case P_2 is a singleton).

Here is the reformulation in terms of $\mathcal{A}(P)$. In the first case, $\mathcal{A}(P)$ is a Serre subcategory of $\operatorname{mod}\Lambda_n$; it consists of all Λ_n-modules such that a fixed simple module (in the example, it is $S(x)$) is not a composition factor (in particular, $\mathcal{A}(P)$ is not sincere).

In the second case, $\mathcal{A}$ is sincere. The simple objects in $\mathcal{A}$ are the simple modules $S(i)$ with $i \notin \{x, y\}$ as well as the module $M = [x, y+1]$ corresponding to the arc from x to $y+1$. Note that M, considered as a Λ_n-module, has a filtration $M' \subseteq M'' \subset M$ with M' and M/M'' being simple Λ_n-modules (namely $M' = S(x), M/M'' = S(y)$), such that M''/M' belongs to $\mathcal{A}$ (it may be zero, otherwise it is indecomposable).

4.2.5. **The poset isomorphism $\iota_n\colon \mathbf{Nc}(n+1) \to \mathbf{Nc}(S_n, c_n)$.** There are the poset isomorphisms

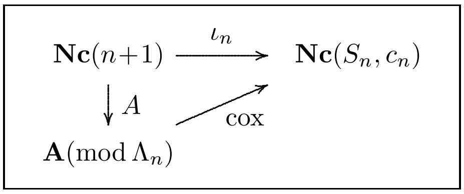
$$A\colon \mathbf{Nc}(n+1) \to \mathbf{A}(\operatorname{mod}\Lambda_n) \quad \text{and} \quad \operatorname{cox}\colon \mathbf{A}(\operatorname{mod}\Lambda_n) \to \mathbf{Nc}(W(\Lambda_n), c(\Lambda_n))$$

as established in Theorem 4.2.2.2 and Theorem 3.7.4.4, respectively. Of course, $W(\Lambda_n) = S_n$ and $c(\Lambda_n) = c_n = (n, n-1, \ldots, 2, 1)$. The composition yields an isomorphism

$$\iota_n = \operatorname{cox} A \colon \mathbf{Nc}(n{+}1) \to \mathbf{Nc}(S_n, c_n).$$

Thus, there is the following theorem:

THEOREM 4.2.5.1. *The posets* $\mathbf{Nc}(n{+}1)$ *are* $\mathbf{Nc}(S_n, c_n)$ *isomorphic. There is a commutative diagram*



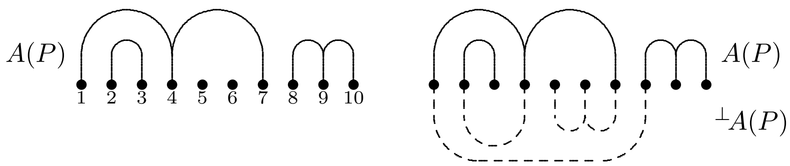
of poset isomorphisms.

Remark. Here, in order to identify $\mathbf{Nc}(n{+}1)$ and $\mathbf{Nc}(S_n, c_n)$ we have used as intermediate step the categorification $\mathbf{A}(\operatorname{mod} \Lambda_n)$. This seems to be a proper procedure in lectures devoted to mathematicians working in representation theory, but it hides the purely combinatorial nature of the identification ι_n. Actually, the identification ι_n has to be considered as the starting point for the theory of generalized non-crossing partitions.

4.3. Perpendicular pairs and the Kreweras complement. We have defined in section 3.4.1 the anti-automorphism δ of $\underline{\mathcal{A}}(\operatorname{mod} \Lambda_n)$ by $\delta(\mathcal{A}) = \mathcal{A}^{\perp}$. We are going to show that in terms of non-crossing partitions, this is precisely the Kreweras complement as introduced by Kreweras [**62**].

4.3.1. **Perpendicular pairs.** Let P be a non-crossing partition of $n+1$ elements. Besides the arcs (which as before we will draw as solid lines) we will consider also coarcs and draw them using dashes lines. By definition, the *coarcs* of P are the pairs $[a, z+1]$, where $a \le z \le n$ and a is the minimal element of a part of P, whereas z is its maximal element. Thus, *if p is the number of parts of P, then the number of coarcs is $p-1$.* Namely, the coarcs correspond bijectively to the parts which do not contain the element $n+1$). As a consequence, the number of arcs and coarcs together is n (since the number of arcs is $n+1-p$ and no coarc is an arc). We denote the set of coarcs by ${}^{\perp}A(P)$.

As an example, consider the partition $P = \big\{\{1,4,7,\}, \{2,3\}, \{5\}, \{6\}, \{8,9,10\}\big\}$. On the left, we show the usual arc diagram for P, on the right, we have added the coarcs: the part $\{1,4,7\}$ gives the coarc $[1,8]$, the part $\{2,3\}$ gives the coarc $[2,4]$, the singletons $\{5\}$ and $\{6\}$ give the coarcs $[5,6]$ and $[6,7]$, respectively (there is no coarc corresponding to the part $\{8,9,10\}$, since this is the part which contains the element $n+1$):



One should observe that the dashed lines (the coarcs) are the arcs of a new partition, namely of the partition $\{\{1,8\},\{2,4\},\{3\},\{5,6,7\},\{9\},\{10\}\}$. Here is the general argument:

Proposition 4.3.1.1. *The set* ${}^{\perp}A(P)$ *is the set of simple objects in* ${}^{\perp}\mathcal{A}(P)$*, in particular it is an antichain in* $\operatorname{mod}\Lambda_n$*, thus the arc set of a non-crossing partition.*

Proof. First, let us use Lemma 4.1.1.2 in order to show that the coarcs are the arcs of a non-crossing partition. Thus, assume that $[a, z+1]$ and $[a', z'+1]$ are coarcs and $a \le a' < z+1 \le z'+1$. It follows that $a \le a' \le z \le z'$. If $a = a'$, or $a' = z$ of $z = z'$, then the coarcs are given by the same part of P, thus we have $a = a'$ and $z = z'$ (therefore also $z+1 = z'+1$). Thus it remains to consider the case that $a < a' < z < z'$. But this is impossible, since we assume that P is non-crossing.

As a consequence, the set of coarcs is the set of simple objects in some thick subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda_n$. We claim that $\mathcal{X} \subseteq {}^{\perp}\mathcal{A}(P)$. It is sufficient to show the following: if $[x, y]$ is an arc and $[a, z+1]$ is a coarc, then we have $\operatorname{Hom}([a, z+1],[x, y]) = 0 = \operatorname{Ext}^1([a, z+1],[x, y])$.

First, assume for the contrary that $\operatorname{Hom}([a, z+1],[x, y]) \neq 0$. Since the Λ_n-top of $[a, z+1]$ is $S(z)$, we see that $S(z)$ has to be a composition factor of $[x, y]$, thus $x \le z < y$. We must have $x < z$, since otherwise z and y belong to the same part, but then z is not maximal in this part. Since P is non-crossing, we must have $x \le a$. Using again that z and y belong to different parts, we see that $x < a$. The Λ_n-socle of $[x, y]$ is $S(x)$. Since we assume that $\operatorname{Hom}([a, z+1],[x, y]) \neq 0$, the simple module $S(x)$ has to be a composition factor of $[a, z+1]$. But this implies that $a \le x$, a contradiction. This shows that $\operatorname{Hom}([a, z+1],[x, y]) = 0$.

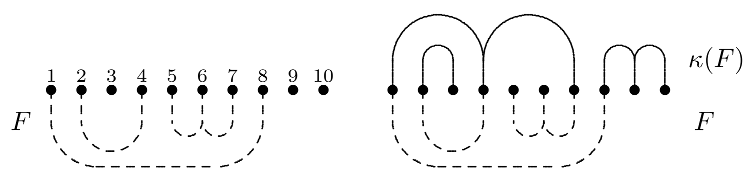
A similar argument shows that for $a \ge 2$ we have $\operatorname{Hom}([x, y],[a-1, z]) \neq 0$. The Auslander-Reiten formula implies that in this case $\operatorname{Ext}^1([a, z+1],[x, y]) = 0$. Of course, if $a = 1$, then $[a, z+1]$ is projective, thus also in this case $\operatorname{Ext}^1([a, z+1],[x, y]) = 0$.

We have shown that $\mathcal{X} \subseteq {}^{\perp}\mathcal{A}(P)$. Now the subcategories $\mathcal{X}$ and ${}^{\perp}\mathcal{A}(P)$ have the same rank, namely $p-1$, where p is the number of parts of P. Thus, $\mathcal{X} = {}^{\perp}\mathcal{A}(P)$. □

4.3.2. **The Kreweras complement.** It follows from Proposition 4.3.1.1 that the coarcs determine uniquely the arcs, the rule to obtain the arcs from the coarcs is just a dual procedure:

Let F be a partition with arc set $A(F)$. For any part of F with minimal element $a > 1$ and maximal element b, take the arc $[a-1, b]$. Denote by $F^{\perp}$ the set of these arcs. The partition generated by $F^{\perp}$ is called the *Kreweras complement* $\kappa(F)$ of F.

For example, starting with $F = \{\{1,8\},\{2,4\},\{3\},\{5,6,7\},\{9\},\{10\}\}$ we obtain the partition $\kappa(F) = \{\{1,4,7,\},\{2,3\},\{5\},\{6\},\{8,9,10\}\}$. On the left, we show the arc diagram for F using dotted arcs, on the right, we have added the arcs of $\kappa(F)$ as solid arcs. The arcs of F are drawn as dotted lines, since they are the coarcs for $\kappa(F)$.



The part $\{2,4\}$ of F yields the arc $[1,4]$ in $\kappa(F)$, the part $\{3\}$ in F yields the arc $[2,3]$ in $\kappa(F)$, and so on.

Remark. The usual definition of the Kreweras complement of a non-crossing partition F of $S = \{1,2,\dots,n\}$ is as follows: One considers the totally ordered set

$$1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$$

and takes as $\kappa(F) \in \mathbf{Nc}(\overline{1},\overline{2},\dots,\overline{n}) \simeq \mathbf{Nc}(1,2,\dots,n)$ the largest partition such that $F \sqcup \kappa(F)$ is a non-crossing partition of $\{1,\overline{1},\dots,n,\overline{n}\}$. It is not difficult to verify that the two definitions of κ coincide.

Theorem 4.3.2.1. *The following diagram commutes*

$$\begin{array}{ccc} \mathbf{Nc}(n{+}1) & \xrightarrow{\ \kappa\ } & \mathbf{Nc}(n{+}1) \\ \big\downarrow A & & \big\downarrow A \\ \mathbf{A}(\operatorname{mod}\Lambda_n) & \xrightarrow{\ \delta\ } & \mathbf{A}(\operatorname{mod}\Lambda_n) \end{array}$$

Thus, if we use A in order to identify $\mathbf{Nc}(n{+}1)$ and $\mathbf{A}(\operatorname{mod}\Lambda_n)$, then δ and κ coincide.

4.3.3. **The automorphism δ^2 of $\mathbf{A}(\operatorname{mod}\Lambda_n)$.**

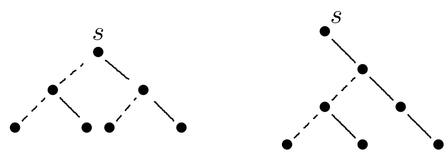
Proposition 4.3.3.1. *If $\mathcal{A}$ is a thick subcategory of* $\operatorname{mod}\Lambda_n$*, then $\delta^2(\mathcal{A})$ is equivalent, as a category, to $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be the product of the connected subcategories $\mathcal{A}_1,\dots,\mathcal{A}_s$. Theorem 3.4.1.3 shows that $\delta^2(\mathcal{A}_i) = \overline{\tau}(\mathcal{A}_i)$. In particular, $\delta^2(\mathcal{A}_i)$ is a connected thick subcategory of $\operatorname{mod}\Lambda_n$ with the same rank as $\mathcal{A}_i$, thus. according to Proposition 4.2.2.4, the categories $\delta^2(\mathcal{A}_i)$ and $\mathcal{A}_i$ are equivalent. □

4.4. Non-crossing partitions and binary trees. There is an interesting relationship between non-crossing partitions and binary trees.

4.4.1. **Binary trees.** Binary trees (sometimes called rooted binary trees) are defined inductively. The empty set is a binary tree, a non-empty binary tree is a triple (L,s,R), where L, R are binary trees and s is a singleton, called the root. We draw binary trees as graphs with two kinds of edges, solid ones and dashed ones. The definition is again by induction: Let $B = (L,s,R)$ be a binary tree. We take the graphs corresponding to L and R and an additional vertex with label s; if L is non-empty, we connect s with the root of L by a dashed edge; if R is non-empty, we connect s with the root of R by a solid edge. We call the roots of L and R the *successors* of s. In this way, we see that any vertex of B has at most two successors.

Here are two examples of binary trees B of cardinality 7, we usually draw the root (contrary to its name) at the top of the picture, in both cases we have marked it by s.

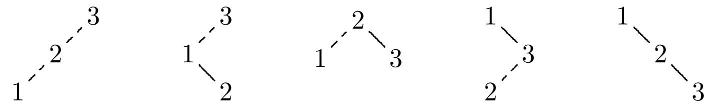


For the binary tree $B = (L, s, R)$ on the left, both binary trees L, R have cardinality 3. On the right, L is empty and the cardinality of R is 6.

There is an *intrinsic numbering* ν_B of the vertices of a binary tree B. We use again induction. Of course, if B is empty, nothing has to be done. Now assume $B = (L, s, R)$ and let L be of cardinality t. Let x be a vertex of B. Then we define

$$\nu_B(x) = \begin{cases} \nu_L(x) & \text{if} \quad x \in L, \\ t+1 & \phantom{\text{if}} \quad x = s, \\ \nu_R(x) + t + 1 & \phantom{\text{if}} \quad x \in R. \end{cases}$$

Here is the intrinsic numbering of the binary trees with 3 vertices:



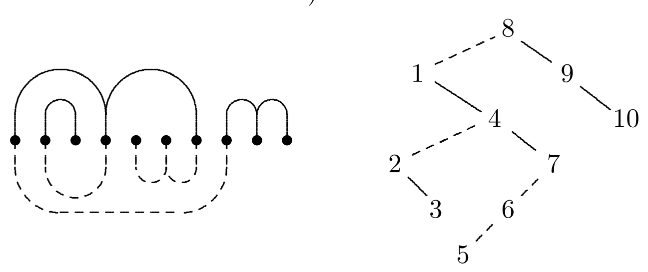
Let P be a non-crossing partition of $n+1$ elements. We define the graph $B(P) = A(P) \cup {}^{\perp}A(P)$ as follows: its vertices are the numbers $\{1, 2, \ldots, n+1\}$ and we use the arcs and the coarcs of P as edges. It is a labeled graph: any edge is either solid (if it is an arc) or dashed (if it is a coarc).

Proposition 4.4.1.1. *The labeled graph $B(P) = A(P) \cup {}^{\perp}A(P)$ is a binary tree.*

Proof. As we know, the number of edges is n. Let us show for $x < n+1$ that there is a path in $B(P)$ starting at x and ending at a vertex x' with $x < x'$ (it follows then by induction that there is a path from x to $n+1$). Let a be the minimal element and z the maximal element of the part of P which contains x. Then x and a are connected by a sequence of arcs, and $[a, z+1]$ is a coarc. Thus x is connected in $B(P)$ by a path from x to $x' = z+1$ and $x \le z < z+1$.

Since $B(P)$ is a connected graph with n edges and $n+1$ vertices, we see that $B(P)$ is a tree. If x is a vertex, there is at most one vertex y such that $[x, y]$ is an arc, and at most one vertex a such that $[a, x]$ is a coarc. This shows that $B(P)$ is a binary tree using the arcs a solid edges and the coarcs as dashed edges. □

4.4.2. Let us return to the example $P = \{\{1, 4, 7, \}, \{2, 3\}, \{5\}, \{6\}, \{8, 9, 10\}\}$. On the left, we reproduce the picture of the binary tree $B(P)$ as shown above, on the right we rearrange the vertices in order to see better the binary tree structure (in particular, now the edges are straight lines, the root 8 is the top vertex, and the successors of a vertex x are below x).



We note the following: *The root of $B(P)$ is the minimal element of the part of P which contains $n+1$ and the maximal element of the part of ${}^{\perp}P$ which contains* 1. In terms of $\operatorname{mod} \Lambda_n$, the root of $B(P)$ corresponds to cutting the indecomposable sincere Λ-module $I = [1, n+1]$ into a submodule M which belongs to ${}^{\perp}\mathcal{A}(P)$ and its factor module I/M which belongs to $\mathcal{A}(P)$. Namely, there is the following lemma:

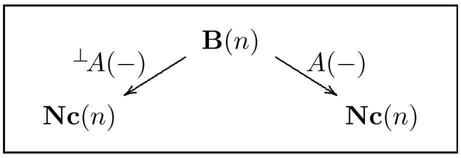
Lemma 4.4.2.1. *Let I/M be the largest factor module of I which belongs to $\mathcal{A}(P)$. Then M belongs to ${}^{\perp}\mathcal{A}(P)$.*

Proof. Assume that X belongs to $A(P)$ and $\operatorname{Hom}(M, X) \neq 0$. In particular, we have $M \neq 0$, thus I/M is not projective and $\tau(I/M) = \operatorname{rad} I/ \operatorname{rad} M$. Now X cannot be injective, since it has top M as a composition factor (it would be a factor module of I which belongs to $\mathcal{A}(P)$, and of larger length than I/M). □

These considerations indicate an interesting interpretation of the base set of $\mathbf{Nc}(n+1)$, these are the numbers $1, 2, \ldots, n+1$: we may call these elements the possible *cuts*, since they serve to describe the cuts of the module $[1, n+1]$, or also the cuts of the quiver of Λ_n. The Lemma yields the cut s, where $S(s)$ is the socle of I/M provided $I/M \neq 0$, and $s = n+1$ in case $I/M = 0$.

4.4.3. Let us denote by $\mathbf{B}(n)$ the set of binary trees with n vertices. The previous considerations may be summarized as follows:

Theorem 4.4.3.1. *Let B be a binary tree with n vertices and use the intrinsic numbering. The set $A(B)$ of solid edges of B is a non-crossing partition, also the set of dashed edges of B is a non-crossing partitions, it is just ${}^{\perp}A(B)$. The maps*



are bijective maps.

The composition of the maps from left to right

$$\mathbf{Nc}(n) \xrightarrow{({}^{\perp}A(-))^{-1}} \mathbf{B}(n) \xrightarrow{A(-)} \mathbf{Nc}(n)$$

is just the Kreweras complement κ defined by $\kappa(A) = A^{\perp}$ for $A \in \mathbf{Nc}(n)$, see Theorem 4.3.2.1.

4.5. The n^{n-2}-problems: Maximal chains in $\mathbf{Nc}(n)$, parking functions, labeled trees. We have mentioned in Chapter 1 that $\mathbf{c}(\mathbb{A}_n) = (n+1)^{n-1}$: the number of complete exceptional sequences for a quiver of type $\mathbb{A}_n$ is given by $(n+1)^{n-1}$. There is a wealth of mathematical counting problems with the answer $(n+1)^{n-1}$ or n^{n-2} and one tries to find canonical bijections.

4.5.1. **Here are some of the n^{n-2}-problems:**

(a) Maximal chains in the lattice $\mathbf{Nc}(n)$ of non-crossing partitions.

Let us denote by $\mathbf{M}(\mathbf{Nc}(n))$ the set of maximal chains in the lattice $\mathbf{Nc}(n)$. Then

$$|\,\mathbf{M}(\mathbf{Nc}(n))| = (n+1)^{n-1}.$$

We hope that the further discussions in this section will provide a better understanding of this equality.

(b) The set $[1,n]^{n-2}$. Of course, n^{n-2} counts the number of (set-theoretical) functions

$$[1,n-2] \to [1,n];$$

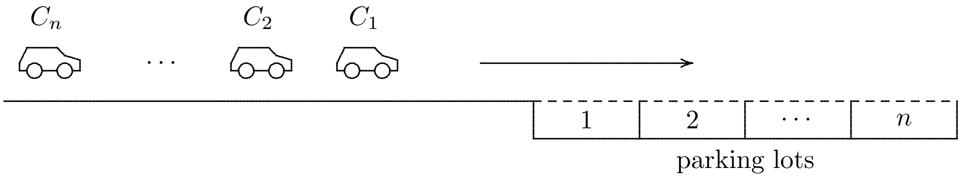
let us denote the set of these functions by $[1,n]^{n-2}$, we may consider this set as the set of sequences $c_1,\dots,c_{n-2}$ of integers c_i with $1 \le i \le n$.

(c) Parking functions. We denote by $\mathbf{P}(n)$ the set of *parking functions* for n cars, these are the endofunctions f of the set $[1,n] = \{1,2,\dots,n\}$ such that $\{x \mid f(x) \le i\}$ has cardinality at least i for all i, or, equivalently, provided the non-decreasing rearrangement $b_1,\dots,b_n$ of the value sequence $f(1),\dots,f(n)$ satisfies $b_i \le i$ for all i. Instead of looking at the function $f:[1,n]\to[1,n]$, we also may call the corresponding n-tuple $(f(1),\dots,f(n))$ a parking function for n cars (the coefficients of such an n-tuple are positive integers and it follows directly from the definition of a parking function that they are bounded by n). For example, the parking functions for $n=3$ are obtained from

$$(1,1,1),\ (1,1,2),\ (1,1,3),\ (1,2,2),\ (1,2,3)$$

by taking all rearrangements; thus there are precisely $16 = 1+3+3+3+6$ parking functions for $n=3$.

Parking functions were introduced in 1966 by Konheim and Weiss (and by Pyke in 1959). Why are these functions called parking functions? Consider n cars $C_1,\dots,C_n$ and n parking lots labeled $1,2,\dots,n$. The cars arrive in arbitrary order at the parking lot, and the car C_i is required to park in the lot $f(i)$ provided it is free, otherwise in the next free parking lot. *The function f is a parking function if and only if any car finds a parking lot if the cars arrive in some fixed order, and in this case any car finds a parking lot if the cars arrive in any order.*



We have:

$$\boxed{|\mathbf{P}(n)| = (n+1)^{n-1}.}$$

The proof follows immediately from the following lemma:

Lemma 4.5.1.1. *For any sequence $a=(a_1,\dots,a_n)\in[1,n+1]^n$, there are $n+1$ elements $a'\in[1,n+1]^n$ such that $a'-a \mod n+1$ is constant and precisely one of these elements a' is a parking function for n cars.*

Proof, using again a parking lot simulation. This time, $n+1$ parking lots are arranged in a circle, labeled $1,2,\dots,n+1$ in consecutive order. There are n cars C_i, where $1\le i\le n$, they come in the order $C_1,\dots,C_n$. The car C_i starts at the parking lot a_i, takes it, if it is free, otherwise it takes the next free lot (driving around the circle). Since there are only n cars, but $n+1$ lots, any car will find a free lot and at the end, precisely one of the lots, say $l(a)$ will remain free (we have fixed the order of the cars, so that we don't have to show that $l(a)$ is well-defined). Clearly, a is a parking function for n cars if and only if $l(a)=n+1$. On the other hand, if we fix some c with $1 \leq c \leq n+1$ and define $a' = (a'_1, \ldots, a'_n)$ by $a'_i = a_i + c \mod n+1$, for all i, then $l(a') \equiv l(a) + c \mod n+1$. □

The formula $|\,\mathbf{P}(n)| = (n+1)^{n-1}$ follows, since the cardinality of $[1, n+1]^n$ is $(n+1)^n$, thus $|\,\mathbf{P}(n)| = \frac{1}{n+1}(n+1)^n = (n+1)^{n-1}$. Here is a reformulation in terms of bijections:

Corollary 4.5.1.2. *For any sequence $a = (a_1, \ldots, a_{n-1}) \in [1, n+1]^{n-1}$, let $u(a) = (a_1, \ldots, a_{n-1}, 1) \in [1, n+1]^n$. If $b \in [1, n+1]^n$, let $p(b)$ be the parking function for n cars with $p(b) - b \mod n+1$ being constant. Then the map*

$$\boxed{pu\colon [1, n+1]^{n-1} \longrightarrow \mathbf{P}(n)}$$

is a bijection.

4.5.2. **From maximal chains in $\mathbf{M}(\mathbf{Nc}(n))$ to parking functions.** Starting with a maximal chain $A^{(0)}, \ldots, A^{(n)}$ in $\mathbf{Nc}(n)$, Stanley has defined a sequence $\lambda(A^{(0)}, \ldots, A^{(n)}) \in \mathbb{N}^n$ as follows: Assume that $P < P'$ are neighbors in $\mathbf{Nc}(n)$, thus P, P' are two non-crossing partitions of $\{1, 2, \ldots, n\}$ and there are two different parts P_i, P_j of P such that P' is obtained from P by replacing the parts P_i, P_j by the disjoint union $P_i \cup P_j$; we assume that the minimal element of P_i is smaller than the minimal element of P_j and define $\lambda'(P, P')$ as the maximal element belonging to P_i and being smaller than all elements of P_j (in particular, always we have $\lambda'(P, P') \leq n-1$). Then there is the definition

$$\lambda(P^{(0)}, \ldots, P^{(n)}) = \big(\lambda'(P^{(0)}, P^{(1)}), \lambda'(P^{(1)}, P^{(2)}), \ldots, \lambda'(P^{(n-1)}, P^{(1)})\big).$$

For example, if we consider the maximal chain

$$1|2|3|4|5 < 1|25|3|4 < 125|3|4 < 125|34 < 12345$$

in $\mathbf{Nc}(5)$, we have

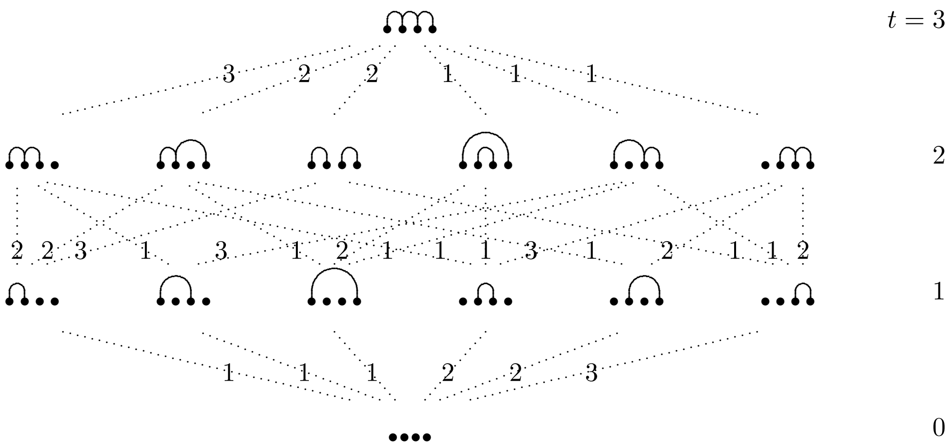
$$\lambda\big(1|2|3|4|5, 1|25|3|4, 125|3|4, 125|34, 12345\big) = (2, 1, 3, 2).$$

Theorem 4.5.2.1. **(Stanley, 1997).** *The map*

$$\boxed{\lambda\colon \mathbf{M}(\mathbf{Nc}(n+1)) \to \mathbf{P}(n).}$$

is a bijection.

Here is the case $n = 3$. For any pair $P < P'$ of neighbors in $\mathbf{Nc}(4)$ we have added the number $\lambda'(P, P')$ to the (dotted) line connecting P and P':



Let us reformulate the map λ in terms of $\mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda_n))$, where $\Lambda = \Lambda_n$ is the path algebra of the linearly directed quiver of type $\mathbb{A}_n$ (so that $\mathbf{A}(\operatorname{mod}\Lambda_n)$ can be identified with $\mathbf{A}(\mathbf{Nc}(n+1))$). It is sufficient to define the numbers $\lambda'(A, B)$, where $A < B$ are antichains in $\operatorname{mod}\Lambda_n$ of cardinality $t-1$ and t, respectively, for some t. In case A is a subset of B, say $B = A \cup \{B_t\}$, then $\lambda'(A, B) = i$ where $S(i) = \operatorname{soc}_\Lambda B_t$. Otherwise, there is some element A_i in A such that A_i has a proper filtration with factors in B. In this case, $\lambda'(A, B) = i$ where $S(i) = \operatorname{soc}_\Lambda A_i$.

We have exhibited in Theorem 3.5.1.1 and its Addendum 3.5.1.2 bijections between the set $\mathbf{M}(\mathbf{A}(\operatorname{mod}\Lambda_n))$ of all maximal chains in the poset $\mathbf{A}(\operatorname{mod}\Lambda_n)$ and the set $\mathbf{E}(\operatorname{mod}\Lambda_n)$ of all complete exceptional sequences in the category $\operatorname{mod}\Lambda_n$. We are going to describe Stanley's bijection in terms of complete exceptional sequences.

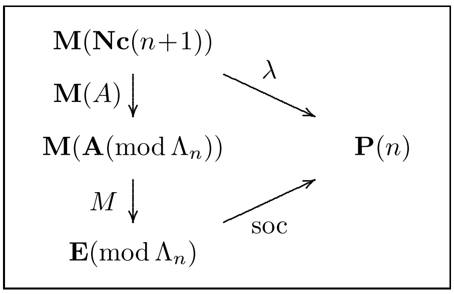
4.5.3. If $(X_1, \dots, X_t)$ is a sequence of indecomposable Λ_n-modules, we write $\operatorname{soc}(X_1, \dots, X_t) = (s_1, \dots, s_t)$, where $\operatorname{soc}_\Lambda X_i = S(s_i)$, for $1 \le i \le t$. In this way, $\operatorname{soc}(X_1, \dots, X_t)$ is an element of $[1, n]^t$.

THEOREM 4.5.3.1. *If E is complete exceptional sequence in* $\operatorname{mod}\Lambda_n$*, then* $\operatorname{soc} E$ *belongs to* $\mathbf{P}(n)$ *and the map*

$$\boxed{\operatorname{soc}\colon \mathbf{E}(\operatorname{mod}\Lambda_n) \to \mathbf{P}(n)}$$

is a bijection.

There is the following commutative diagram



In order to prove this, it is sufficient to show: *If* $A < B$ *are neighbors in* $\mathbf{Nc}(n+1)$ *and* $\mathcal{A}, \mathcal{B}$ *are the corresponding thick subcategories of* $\operatorname{mod}\Lambda_n$, *then*

$$\boxed{\operatorname{soc}_\Lambda M_{\mathcal{A}}^{\mathcal{B}} = S(\lambda'(A,B))\ .}$$

Proof. Since $A < B$ are neighbors, there are two different parts A_i, A_j of A such that B is obtained from A by replacing the parts A_i, A_j by the disjoint union $A_i \cup A_j$; we assume that the minimal element a_i of A_i is smaller than the minimal element a_j of A_j. Then, by definition, $x = \lambda'(A,B)$ is the maximal element which belongs to A_i and is smaller than all elements of A_j. We denote by y the maximal element of A_j. Since $x < a_j \le y$, the module $M = [x,y]$ is indecomposable and its Λ-socle is $S(x)$. Since x, y belong to the same part of B, we know that M belongs to $\mathcal{B}$.

Let U be a simple object in $\mathcal{A}$, say $U = [u,v]$ for some numbers $1 \le u < v \le n+1$. We want to show that both $\operatorname{Hom}(U,M) = 0 = \operatorname{Ext}^1(U,M)$ thus $M \in \mathcal{A}^\perp$.

Let us assume that $\operatorname{Hom}(U,M) \ne 0$. Since $S(x)$ is the socle of M, we see that $S(x)$ is a composition factor of U, thus $u \le x < v$. Let us show that we even have $y < v$. Now U belongs to some block of $\mathcal{A}$. First, assume that U belongs to $\mathcal{A}_i$. Then $v < a_j$ is impossible by the maximality of x. Also, $v = a_j$ and $v = y$ are impossible, since $A_i \cap A_j = \emptyset$. Finally, $a_j < v < y$ is impossible, since A is non-crossing. It follows that $y < v$. Note that U cannot belong to $\mathcal{A}_j$, since $[x, x+1]$ is a composition factor of U, but $x < a_j$ and a_j is the minimal element A_j. It follows that U belongs to a block A_t with $t \notin \{i,j\}$. This block $\mathcal{A}_t$ is a block of $\mathcal{B}$, thus, since B is non-crossing and $u < x < v$ it follows that ($u < a_i$ and) $y < v$. But the condition $y < v$ implies that $\operatorname{Hom}([u,v],[x,y]) = 0$, a contradiction. Thus, we have $\operatorname{Hom}(U,M) = 0$.

The assertion $\operatorname{Ext}^1(U,M) = 0$ is clear in case $y = n+1$, since then M is injective in $\operatorname{mod}\Lambda$. Let us assume that $\operatorname{Ext}^1(U,M) \ne 0$, thus $y < n+1$ and the Auslander-Reiten formula shows that $\operatorname{Hom}([x+1,y+1],U) \ne 0$ (since $\tau_\Lambda^{-1} = [x+1,y+1]$). It follows that U has $[y,y+1]$ as Λ-composition factor, thus $u \le y < v$. We claim that we even have $u \le x$.

Now U cannot belong to $\mathcal{A}_j$, since the indecomposable modules in $\mathcal{A}_j$ are of the form $[z,z']$ with $z' \le y$. Assume that U belongs to $\mathcal{A}_i$. Then we must have $u = x$ (namely, in this case x cannot be maximal in A_i, thus there is $x' \in A_i$ with $x < x'$ and no other element of A_i in-between; then $[x,x']$ is the only simple object of $\mathcal{A}_i$ with Λ-composition factor $[y,y+1]$, therefore $U = [x,x']$). Finally, if U belongs to $\mathcal{A}_t$ with $t \notin \{i,j\}$, then A_t is a part of B. Since B is non-crossing and $U = [u,v]$ has the Λ-composition factor $[y,y+1]$, it follows that $u < a_i$ (and $y < v$), therefore $u < x$. Since $u \le x$, it follows that $\operatorname{Hom}([x+1,y+1],[u,v]) = 0$, a contradiction. Thus $\operatorname{Ext}^1(U,M) = 0$.

Altogether, we see that $M \in \mathcal{B} \cap \mathcal{A}^\perp$, therefore $M = M_{\mathcal{A}}^{\mathcal{B}}$. This completes the proof. □

Remark. It is easy to see directly: *If* $M = (M_1, \dots, M_n)$ *is a complete exceptional sequence of* Λ_n*-modules, then* $\operatorname{soc} M$ *is a parking function* (see also N 4.3).

Proof. We have to show that for any integer i with $1 \le i \le n$, the number of modules M_j such that $\operatorname{soc} M_j = S(i')$ for some $i' \le i$ is at most i. Let us delete these modules M_j from the sequence M. The remaining modules form an exceptional sequence M' of Λ'-modules, where Λ' is the path algebra of the quiver $i+1 \leftarrow \cdots \leftarrow n$, this quiver has $n-i$ vertices, thus any exceptional sequence of Λ'-modules consists of at most $n-i$ modules. This shows that we have deleted from M at least i modules. □

The Stanley bijection shows, in particular, that *for $\Lambda = \Lambda_n$, the map $M \mapsto \operatorname{soc} M$ defined on the set of complete exceptional sequences is injective.* It seems to be of interest to find a direct proof. But one should be aware that for here we deal with a special feature of the Λ_n-modules which does not hold for an arbitrary artin algebra (see the note N 4.4). which looks at the quiver $\mathbb{A}_3$ with 2 sources. The note N 4.6 presents an algorithm how to recover a tilting module T from $\operatorname{top} T$.

4.5.4. Using the k-duality $M \mapsto M^*$ from the category $\operatorname{mod} \Lambda_n$ to the category $\operatorname{mod} \Lambda_n^{\mathrm{op}}$, we see that there is also the following bijection: If $(X_1, \dots, X_t)$ is an exceptional sequence in $\operatorname{mod} \Lambda_n$, then $(X_t^*, \dots, X_1^*)$ is an exceptional sequence for $\Lambda' = \Lambda_n^{\mathrm{op}}$. Let $S_{\Lambda'}(i) = S_\Lambda(\pi i)$, where π is the permutation of $\{1, 2, \dots, n\}$ which reverses the order.

Theorem 4.5.4.1. *If $(X_1, \dots, X_t)$ is an exceptional sequence in $\operatorname{mod} \Lambda_n$, let $\operatorname{top}(X_1, \dots, X_t) = (t_n, \dots, t_1)$, where $\operatorname{top}_\Lambda X_i = S(\pi t_i)$, for $1 \le i \le t$. Then*

$$\boxed{\operatorname{top} \colon \mathbf{E}(\operatorname{mod} \Lambda_n) \to \mathbf{P}(n)}$$

is a bijection.

Using the bijections soc and top from $\mathbf{E}(\operatorname{mod} \Lambda_n)$ to $\mathbf{P}(n)$, we obtain a non-trivial involution $\xi = \operatorname{top} \circ (\operatorname{soc})^{-1}$ of $\mathbf{P}(n)$; for example, we have $\xi(1,1,1) = (1,2,3)$, and $\xi(3,2,1) = (3,2,1)$, see the note N 4.5.

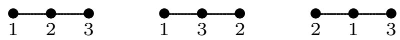
4.5.5. A parking function is said to be *primitive* provided it is weakly increasing. Given a tilting module T with direct summands T_i, we may index the direct summands in such a way that $E(T) = (T_1, \dots, T_n)$ is exceptional and the socle sequence is non decreasing. Then *E furnishes a bijection from the set of tilting modules to the set of primitive parking functions.* See also N 4.6.

4.5.6. We return to our attempt to provide an overview on some of the most relevant n^{n-2}-problems.

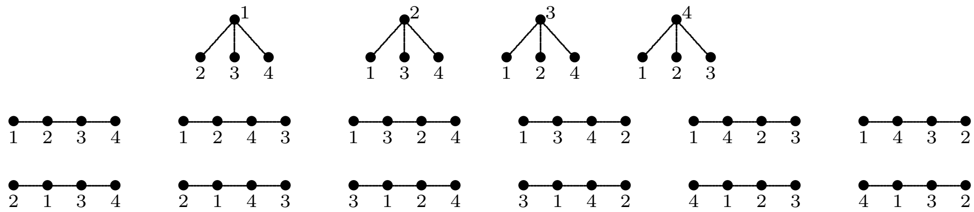
(d) Labeled trees. The most famous n^{n-2}-problem seems to be the problem of counting labeled trees. For a solution of this problem one often refers to Cayley (1889), however, already Sylvester (1857) and Borchardt (1860) discussed the problem.

A *labeled tree* T with n vertices is a tree with vertex set $\{1, 2, \dots, n\}$. We should stress that counting labeled trees means to count the actual number of such trees, not just the number of isomorphism classes (usually, throughout of these lectures, counting meant to count isomorphism classes) — but actually, we also could phrase it as follows: looking at labeled trees T, T', to say that they are isomorphic (as labeled trees) means that $T' = T$, and the only automorphism of a labeled tree is the identity map.

We denote by $\mathbf{LT}(n)$ the set of labeled trees with n vertices. Here are the pictures for 3 vertices (there are 3^1 possibilities):



And here are the pictures for 4 vertices (there are $4^2 = 16$ possibilities):



It was shown by Borchardt (1860), Sylvester (1857) and Cayley (1889) that

$$|\,\mathbf{LT}(n)| = n^{n-2}.$$

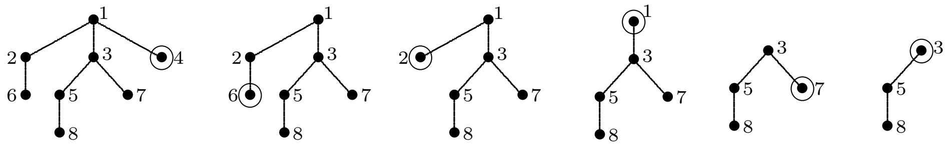
A constructive proof has been given by Prüfer (1918), by attaching to any labeled tree T a sequence $c(T)$ of numbers, now called the *Prüfer code* of T. It is defined as follows:

If T has $n \geq 3$ vertices, let x_1 be the leaf with the lowest label and c_1 its (unique) neighbor. Now remove the vertex x_1, we get a tree T_2 with labels $\{1, \dots, n\} \setminus \{x_1\}$ and if $n \geq 4$ continue: Let x_2 be the leaf with the lowest label and c_2 its (unique) neighbor. Altogether we get two sequences

$$x_1, \quad x_2, \quad x_3, \quad ..., \quad x_{n-2},$$
$$c_1, \quad c_2, \quad c_3, \quad ..., \quad c_{n-2}.$$

The Prüfer code of T is the sequence $c(T) = (c_1, c_2, \dots, c_{n-2})$. This is a sequence of $n-2$ integers c_i with $1 \leq c_i \leq n$. The decisive fact is that one can recover T from $c(T)$.

Example for calculating $c(T)$**.** In any step $i = 1, 2, \dots, 6$, the leaf with lowest label is encircled; the label c_i of its neighbor is exhibited below the tree. In this way, the lower row shows the code $c = (c_1, \dots, c_6)$.



$c_i =$ 1 2 1 3 3 5

Theorem 4.5.6.1. **(Prüfer, 1918).** *The Prüfer code* $c(T)$ *provides a bijection*

$$\mathbf{LT}(n) \to [1, n]^{n-2}.$$

A proof is given in many books dealing with graph theory. For a general discussion of the formula $|\,\mathbf{LT}(n)| = n^{n-2}$, see also Chapter 24 of [**2**].

Notes to Chapter 4.

N 4.1. The bijection between the partitions and partition arc sets. We assume that $S = \{1, 2, \dots, n\}$. Arcs in S are pairs $[a, b]$ with $1 \le a < b \le n$. A set U of arcs in S is said to be a *partition arc set* provided for $[a, b], [a', b']$ in U, we have $a = a'$ iff $b = b'$.

Given a partition P of S, we have defined $A(P)$ as the set of arcs for P, where an arc for P is a pair $[a, b]$ with $1 \le a < b \le n+1$ such that a, b belong to the same part of P whereas no element c with $a < c < b$ belongs to this part. Claim: *$A(P)$ is a partition arc set.* Namely, assume that $[a, b], [a, b']$ are arcs for P. We may assume that $b \le b'$. Now a, b, b' belong to the same part of P. Since $[a, b']$ is an arc for P, it follows that $b = b'$. On the other hand, assume that $[a, b], [a', b]$ are arcs for P. We may assume that $a \le a'$. Thus a, b, b' belong to the same part of P. Since $[a, b']$ is an arc for P, it follows that $a = a'$.

On the other hand, assume that U is a partition arc set in S. Let P be the smallest equivalence relation defined by the arcs which belong to U. We claim that $U = A(P)$. Namely, any part of P is of the form $\{a_1 < a_2 < \cdots < a_t\}$ such that the pairs $[a_i, a_{i+1}]$ with $1 \le i < t$ are arcs in U.

N 4.2. Characterization of non-crossing partitions. The Proposition 4.2.2.1 asserts that a partition is non-crossing if and only if $A(P)$ is an antichain. There are further ways to characterize non-crossing partitions using properties of $A(P)$:

Lemma. *Let P be a partition. Then the following assertions are equivalent:*

(i) *P is non-crossing.*
(ii) *$A(P) \cup \{0\}$ is closed under kernels.*
(iii) *$A(P) \cup \{0\}$ is closed under images.*
(iv) *$A(P) \cup \{0\}$ is closed under cokernels.*

The proof is left as an exercise.

N 4.3. More parking functions. *If $M = (M_1, \dots, M_n)$ is a complete exceptional sequence of Λ_n-modules, then also the length sequence $|M| = (|M_1, |, \dots, |M_n|)$ is a parking function for n cars.*

Proof. Let $1 \le i < n$ be an integer. Let M' be obtained from $M = (M_1, \dots, M_n)$ by deleting all modules M_j with $|M_j| \le i$. We have to show that at most $n - i$ modules remain. We consider the factor category $\mathcal{F} = \operatorname{mod}\Lambda/\mathcal{X}$ of $\operatorname{mod}\Lambda$, where $\Lambda = \Lambda_n$ and $\mathcal{X}$ is the ideal of all maps in $\operatorname{mod}\Lambda$ which factor through a direct sum of modules of length at most i. The Auslander-Reiten quiver of Λ_n shows that $\operatorname{mod}\Lambda_n/\mathcal{X}$ is equivalent to $\operatorname{mod}\Lambda'$, with $\Lambda' = \Lambda_{n-i}$. We claim that M' is an exceptional sequence in $\operatorname{mod}\Lambda_{n-i}$ (therefore it contains at most $n-i$ elements). Namely, consider two modules M_s, M_t in M of length at least $i+1$ such that $s > t$. We have to show that $\operatorname{Hom}_{\mathcal{F}}(M_s, M_t) = 0 = \operatorname{Ext}^1_{\mathcal{F}}(M_s, M_t)$. Of course, we have $\operatorname{Hom}_\Lambda(M_s, M_t) = 0$ and also $\operatorname{Ext}^1_\Lambda(M_s, M_t) = 0$. The first equality implies immediately that also $\operatorname{Hom}_{\mathcal{F}}(M_s, M_t) = 0$. If M_s is a projective Λ-module, then M_s is also projective in $\mathcal{F}$, thus $\operatorname{Ext}^1_{\mathcal{F}}(M_s, M_t) = 0$. If M_s is not projective as a Λ-module, then $\operatorname{Ext}^1_\Lambda(M_s, M_t) = 0$ implies that $\operatorname{Hom}_\Lambda(M_t, \tau_\Lambda M_s) = 0$. and therefore $\operatorname{Hom}_{\mathcal{F}}(M_t, \tau_\Lambda M_s) = 0$. But the Auslander-Reiten quivers show that $\tau_\Lambda M_s = \tau_{\mathcal{F}} M_s$, therefore we see that $\operatorname{Ext}^1_{\mathcal{F}}(M_s, M_t) = 0$. $\square$

However, the map $M \mapsto |M|$ from the set of complete exceptional sequences to the set of parking functions is not bijective! If $n = 3$, the exceptional sequences $(1, 3, P)$ and $(3, 1, P)$ both are sent to the parking function $(1, 1, 2)$. On the other hand, there is no exceptional sequence M with $|M| = (1, 2, 2)$.

N 4.4. The socle of a complete exceptional sequence. For $\Lambda = \Lambda_n$, the Stanley bijection shows that the map $M \mapsto \operatorname{soc} M$ defined on the set of complete exceptional sequences is injective. For a general hereditary artin algebra Λ, this assertion is not true, already the quiver $\mathbb{A}_3$ with two sources provides counterexamples: Let 1 be the sink and $2, 2'$ the sources. Then

$$\big(S(1), S(2), S(2')\big),\quad \big(S(1), S(2'), S(2)\big),\quad \big(S(2), S(2'), I(1)\big),\quad \big(S(2'), S(2), I(1)\big)$$

are complete exceptional sequences (M_1, M_2, M_3) with $\operatorname{soc}(M_1, M_2.M_3) = (1, 1, 1)$.

N 4.5. The functions soc **and** top **on the set of complete exceptional sequences.** Here we show explicitly the bijections between $\mathbf{M}(\mathbf{Nc}(\operatorname{mod}\Lambda_n))$ and $\mathbf{P}(n)$ given by the functions soc and top:

$\mathbf{P}(n)$	$\mathbf{M}(\mathbf{Nc}(\operatorname{mod}\Lambda_n)$	$\mathbf{P}(n)$
$\operatorname{soc}(M_1, M_2, M_3)$	(M_1, M_2, M_3)	$\operatorname{top}(M_1, M_2, M_3)$
111	$1\ P\ M$	123
112	$P\ M\ 2$	212
121	$P\ 2\ M$	122
211	$2\ 1\ M$	132
113	$1\ M\ 3$	113
131	$1\ 3\ P$	213
311	$3\ 1\ P$	231
122	$M\ 2\ I$	121
212	$2\ M\ I$	112
221	$2\ I\ 1$	312
123	$M\ I\ 3$	111
132	$M\ 3\ 2$	211
213	$I\ 1\ 3$	131
231	$I\ 3\ 1$	311
312	$3\ P\ 2$	221
321	$3\ 2\ 1$	321

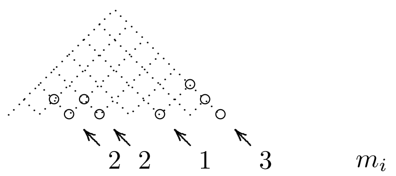
N 4.6. Algorithm to recover a tilting module T from its top. Assume that $T = \bigoplus_r T_r$ is a (multiplicity-free) tilting module. For any i, let m_i be the number of summands of T with top i. We claim that we can recover T from the numbers m_i with $1 \le i \le n$.

By induction on i, the position of the modules with top i can be determined as follows: Let I be the set of integers $1 \le i < n$ such that $m_i > 0$. In case $i \in I$, we denote by $T(i)$ the direct summand of T with top i which has maximal length. Let us denote by R_i the rectangle which is the support of the hammock starting at $\tau^- T(i)$.

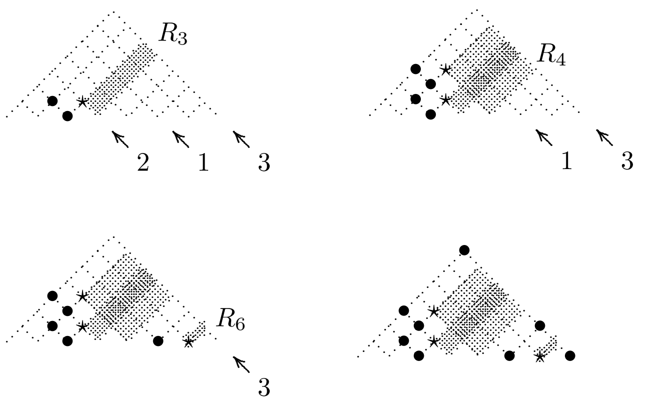
Assume we have determined the position of the modules with top $i' < i$, thus we know the rectangles $R_{i'}$ with $i' \in I$ and $i' < i$. Let $\mathcal{Z}(i)$ be the modules with top i which belong to at least one of the rectangles $R_{i'}$ with $i' \in I$ and $i' < i$. Then the direct summands of T with top i are the m_i modules of smallest possible length which belong to the coray $\mathcal{C}_i$ and not to $\mathcal{Z}(i)$ (the coray $\mathcal{C}_i$ consists of the modules with top $S(i)$, this is just the southeast diagonal ending in $S(i)$). (Namely, a direct summand T' of T with top i cannot belong to a rectangle $R_{i'}$, since otherwise $\mathrm{Ext}^1(T', T(i)) \neq 0$. On the other hand, by assumption, there are precisely m_i direct summands of T with top i; in particular, there is one such module. Recall that $T(i)$ is the direct summand of T with top i of largest length. Since $\mathrm{Ext}^1(T, T(i)) = 0$, also $\mathrm{Ext}^1(T, Y) = 0$ for any factor module Y of $T(i)$. Let Y be a factor module of $T(i)$ which does not belong to $\mathcal{Z}(i)$. Then clearly $T \oplus Y$ has no self-extensions, thus Y has to be a direct summand of T. This shows that all the non-zero factor modules of $T(i)$ which do not belong to $\mathcal{Z}(i)$ are direct summands of T.)

Let us look at the start of the induction, thus let i be minimal with $m_i > 0$. Then $\mathcal{Z}(i) = \emptyset$, thus the direct summands of T with top i are just the modules with top i and length at most m_i.

Example. Let T be a tilting module with top sequence $(3, 3, 4, 4, 6, 8, 8, 8)$. Recall that m_i denotes the number of indecomposable direct summands of T with top i, thus $(m_1, \ldots, m_8) = (0, 0, 2, 2, 0, 1, 0, 3)$. The m_i direct summands of T with top i lie in the coray $\mathcal{C}_i$. We have marked by a circle the m_i modules with top i of smallest possible length, an arrow (and the number m_i) points to the coray $\mathcal{C}_i$. The aim is to shift the vertices inside the coray $\mathcal{C}_i$ so that we obtain a tilting module.



First, we consider the coray $\mathcal{C}_3$, since 3 is the smallest index i with $m_i > 0$. As we know, the direct summands of T in this coray have to be those of smallest possible length, thus of length 1 and 2. In particular, $T(3)$ has length 2. We shade the rectangle R_3 starting at $\tau^- T(3)$, see the first of the following pictures. Always, the vertices marked by a bullet are the direct summands of T which are already determined (and arrows point to the corays which have to be considered later). The vertices marked by a star are the modules $\tau^- T(i)$ with $i \in I$, the corresponding rectangles R_i starting at such vertices are shaded.



In the second step, we have determined the direct summands of T with top 4. The modules in $N(4)$ are the modules with top 4 and length 1 and 2. There are 2 remaining modules on the coray $\mathcal{C}_4$. Since $m_4 = 2$, we see that the remaining modules on the coray $\mathcal{C}_4$ all are direct summands of T. It follows that $T(4)$ has length 4. We have shaded the rectangle R_4 (starting at $\tau^- T(4)$).

The third step concerns the direct summands of T with top 6. Here, $N(6)$ consists of the modules with top 6 and length between 2 and 5. It follows that $T(6)$ has length 1 and we shade the rectangle R_6 (note that it consists just of two modules).

The final step deals with the modules with top 8. Now $N(8)$ consists of the modules with top 8 and length $2, 4, 5, 6, 7$; the remaining three modules with top 8 have to be the remaining summands of T; they have length $1, 3, 8$.

Note: we have considered a tilting module with top sequence $(3, 3, 4, 4, 6, 8, 8, 8)$. If we renumber the simple modules using the permutation π, we see that we deal with the non-increasing parking function $(6, 6, 5, 5, 3, 1, 1, 1)$.

How to find the factor complement for a normal partial tilting module? Let N be a normal partial tilting module. We can assume in addition that N is sincere, thus that $P(n)$ is a direct summand (namely: a sincere partial tilting module is faithful. and any faithful module has $P(n)$ as direct summand).

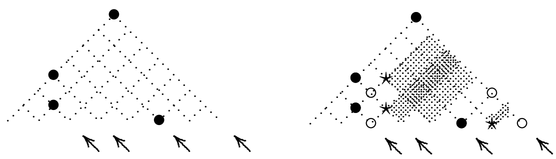
To say that N is normal means that N has at most one direct summand, say $N(i)$, from the coray $\mathcal{C}_i$. Let I be the set of indices $1 \le i < n$ such that i occurs in the top of N, thus

$$N = P(n) \oplus \bigoplus_{i \in I} N(i).$$

For $i \in I$, we define R_i as the rectangle starting in $\tau^- N(i)$, and we denote by R the union of the sets R_i.

The factor complement for N consists of all the proper non-zero factor modules of the modules $N(i)$ which do not belong to R.

Example. Consider Λ_8. Let N be the direct sum of the modules $[2]3$, $[4]4$, 6, $[8]8$. Here is the Auslander-Reiten quiver. On the left, we mark the modules $N(i)$ by bullets. The arrows below point to the corresponding corays.



On the right, we have marked the modules $\tau^- N(i)$ with $i \in I$ by a star $*$ and we have shaded the corresponding rectangles R_i. By definition, R is the union of these rectangles.

The circles show the position of the direct summands of the factor complement: these are the proper non-zero factor modules of the modules $N(i)$ which do not belong to R. Note that the bullets and the circles together provide a tilting module (the unique tilting module with normalization N).

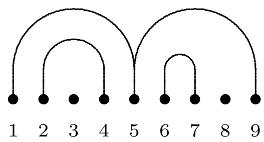
Let us draw the corresponding non-crossing partition. We start with the module

$$N = [2]3 \oplus [4]4 \oplus 6 \oplus [8]8,$$

the corresponding antichain is

$$\begin{aligned} A &= \{\ \Delta(3),\ \Delta(4),\ \Delta(6),\ \Delta(8)\ \} = \{\ [2]3,\ [4]4,\ 6,\ [4]8\ \} = \left\{\ \begin{smallmatrix}3\\2\end{smallmatrix},\ \begin{smallmatrix}4\\3\\2\\1\end{smallmatrix},\ 6,\ \begin{smallmatrix}8\\7\\6\\5\end{smallmatrix}\ \right\} \\ &= \{\ [2,4],\ [1,5],\ [6,7],\ [5,9]\ \} \end{aligned}$$

thus, the non-crossing partition is as follows:



N 4.7. Antichains in Φ_+, where Φ is a (finite) root system say of type Δ. In this last chapter we should have considered also the set $\mathbf{A}(\Phi_+)$ of **antichains** in the root poset (the set of "non-nesting partitions").

Let Λ be a hereditary artin algebra of type Δ. Then

$$\boxed{\ |\,\mathbf{A}(\Phi_+)| = |\,\mathbf{A}(\operatorname{mod}\Lambda)|,\ }$$

thus, there are bijections between $|\,\mathbf{A}(\Phi_+)|$ and $|\,\mathbf{A}(\operatorname{mod}\Lambda)|$, but it is still an open problem to find a natural one, which takes into account that we even have:

$$\boxed{\ |\,\mathbf{A}_t(\Phi_+)| = |\,\mathbf{A}_t(\operatorname{mod}\Lambda)|\ }$$

The relationship between non-nesting and non-crossing is still a mystery.

N 4.8. The equality $\mathbf{t}(\mathbb{A}_n) = \mathbf{t}_{n+1}(\mathbb{A}_{n+1})$. *There is a natural bijection between the tilting Λ_{n+1}-modules T and the support-tilting Λ_n-modules M.* It is defined as follows:

We denote by τ_{n+1} the Auslander-Reiten translation for $\operatorname{mod}\Lambda_{n+1}$. Given M, let $I = \bigoplus_{i\in V} I(i)$, where V is the set of vertices $1 \le i \le n+1$ which do not belong to the support of $\tau_{n+1}^{-}M$ (and $I(i)$ is the injective module corresponding to the vertex i). The Auslander-Reiten formula for $\operatorname{mod}\Lambda_{n+1}$ asserts that $\operatorname{Ext}^1(I(i), M) = D\operatorname{Hom}(\tau_{n+1}^{-}M, I(S)) = 0$ for $i \in V$. Since I is injective, we see that $\operatorname{Ext}^1(T \oplus I, T \oplus I) = 0$. Let s be the cardinality of the support of M. Then V has cardinality $n+1-s$, thus $M \oplus I$ is the direct sum t pairwise non-isomorphic indecomposable modules, thus $M \oplus I$ is a tilting Λ_{n+1}-module.

Conversely, assume that T is tilting Λ_{n+1}-module, let $T_1, \dots, T_s$ be indecomposable non-injective direct summands of T. Then $M = \bigoplus_{i=1}^{s}$ is a support-tilting Λ_n-module.

5. Appendix

In the appendix we will try to outline in which way the classical Catalan combinatorics could be seen as the heart of the theory of finite sets, starting with the subsets of cardinality two. Given a finite set C, we are going to find relations between subsets, quotient sets and automorphisms of C. We follow considerations of Knuth [**60**], Biane [**16**] and Armstrong [**5**]. We start with an overview over some typical Catalan problems as discussed by Stanley [**105, 106**].

5.1. What is Catalan combinatorics? A first answer.

5.1. What is Catalan combinatorics? A first answer. Catalan combinatorics is usually considered just as collecting a wealth of counting problems which are defined for all natural numbers n such that the answer $f(n)$ is the Catalan number $f(n) = \frac{1}{n+1}\binom{2n}{n}$, or, equivalently, such that $f(0) = 1$ and such that for $n \geq 1$ the recursion formula

$$f(n+1) = \sum_{t=0}^{n} f(t)f(n-t)$$

is satisfied. These problems concern finite sets with some decoration, with some additional structures. There are the famous collections by Stanley [**105, 106**] which ask for direct enumerations and for providing bijections. The relationship between any two of these problems is of varied nature: often only small changes are needed to transform one problem into the other, but sometimes it takes some effort to provide reasonable bijections. The literature is full of such examples, but no systematic treatment of the Catalan problems seems to be available. The most important examples of Catalan problems are the non-crossing and the non-nesting partitions.

5.1.1. **Partitions of a finite set.** Given a partition P of the set $\{1, 2, \dots, n\}$, we consider its set $A(P)$ of arcs: An *arc* of P is a pair (i, j) with $i < j$ which belong to the same part such that no element x with $a < x < b$ belongs to this part (alternatively, we could use the cyclic order on the set $\{1, 2, \dots, n\}$ and convex polygons). We may consider the arcs as elements of the positive root set $\Phi_+(\mathbb{A}_{n-1})$, thus the function A provides an embedding of the set $\mathrm{Quot}(n)$ of all partitions of $\{1, 2, \dots, n\}$ into the set $\mathrm{Sub}(\Phi_+(\mathbb{A}_{n-1}))$:

$$\boxed{\mathrm{Quot}(n) \;\subseteq\; \mathrm{Sub}(\Phi_+(\mathbb{A}_{n-1})).}$$

5.1.2. **Non-crossing partitions.** Let us start with the non-crossing partitions and with counting problems which are directly related. Recall that a partition of the set $\{1, 2, \dots, n\}$ is said to be *non-crossing,* provided given elements $i < i' < j < j'$ with i, j in the same part and i', j' in the same part, then all elements belong to the same part. As we have mentioned in Chapter 4, we visualize a partition by drawing the vertices of the set $\{1, 2, \dots, n\}$ in the natural order, as well as all the arcs (alternatively, we could use the cyclic order on the set $\{1, 2, \dots, n\}$ and convex polygons).

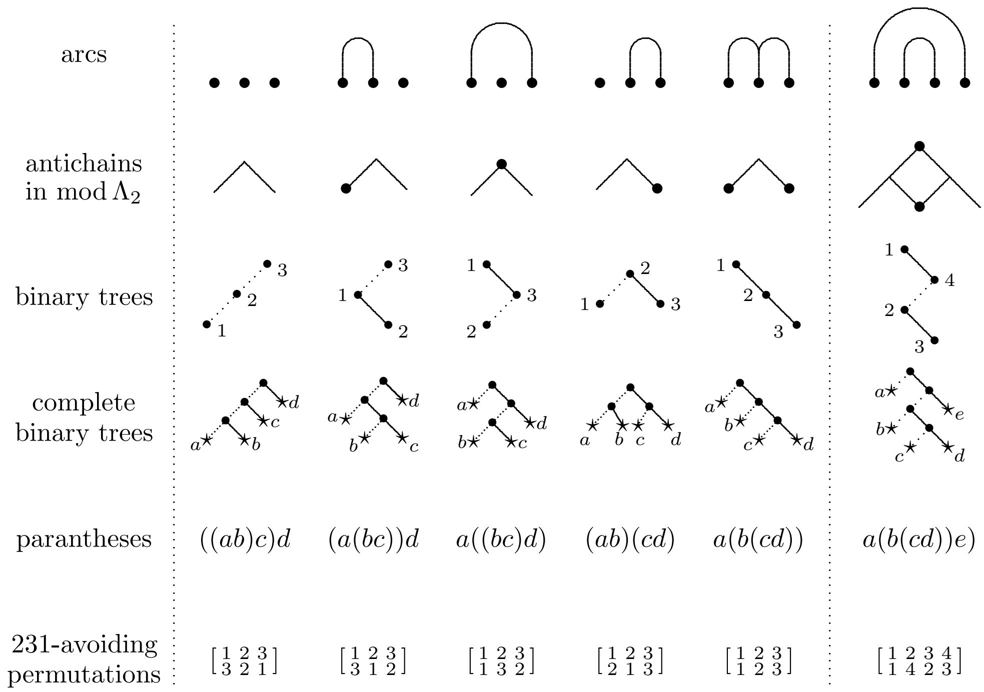
As we have mentioned, the set $A(P)$ of arcs of a partition P can be considered as a subset of Φ_+, thus, we may interpret the arcs as indecomposable Λ_{n-1}. We saw already in Proposition 4.2.2.1, that a partition P is non-crossing if and only if $A(P)$ is an **antichains in** $\mathrm{mod}\,\Lambda_{n-1}$.

In Proposition 4.4.1.1 we have outlined how to attach to a non-crossing partition P a **binary tree** $B(P)$. A binary tree is called a **complete binary trees,** provided any vertex has either two or none successors. There is an obvious bijection between the binary trees B with $n \geq 1$ vertices and the complete binary trees $\widetilde{B}$ with $2n+1$ vertices: if B is a binary tree with $n \geq 1$ vertices, then add to every vertex x with $t \leq 1$ successors $2-t$ successors, in order to obtain $\widetilde{B}$. Conversely, if $\widetilde{B}$ is a complete binary tree with more than one vertices, delete all its leaves, in order to recover B.

Starting with a complete binary tree, label the leaves by pairwise different letters $a, b, c, \dots$, going from left to right. The binary tree structure provides **parentheses** for the word $abc\cdots$.

A permutation π in S_n is said to be a 231-**avoiding permutation** provided there are no numbers $a < b < c$ in $\{1, 2, \dots, n\}$ with $\pi(b) < \pi(c) < \pi(a)$. These permutations are also said to be **stack-sortable,** since such permutations can be achieved by a computer using a single stack; thus, here, we are in the realm of computer science! The problem of sorting an input sequence using a single stack was first posed by Knuth (1968). Given a binary tree B with n vertices, the corresponding stack-sortable permutation π_B is defined inductively as follows: endow B with its intrinsic numbering and if $B = (L, s, R)$, then $\pi_B(1) = s$, the values of $2, \dots, s$ under π_B are $\pi_L(1), \dots, \pi_L(s-1)$, in this order, and the values of $s+1, \dots, n$ under π_B are $\pi_R(1)+s, \dots, \pi_R(n-s)+s$, in this order.

Here are various incarnations of the **non-crossing partitions,** starting with the arc diagram, the antichains in the Auslander-Reiten quiver of mod Λ_{n-1}, as well as the corresponding binary tree. We show the case $n = 3$, as well as the only nesting partition for $n = 4$. The leaves of the complete binary trees are marked as stars $\star$.

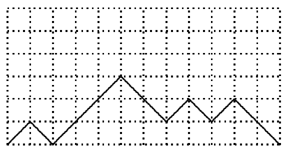


The recursion formula $f(n+1) = \sum_{t=0}^{n} f(t)f(n-t)$ is most easily seen when looking at the set of binary trees $B = (L, s, R)$. If B is of cardinality $n+1$, then L has to be of cardinality t with $0 \le t \le n$, this is the summation index. If we fix t, then there are $f(t)$ possiblities for L and $f(n-t)$ possibilities for R.

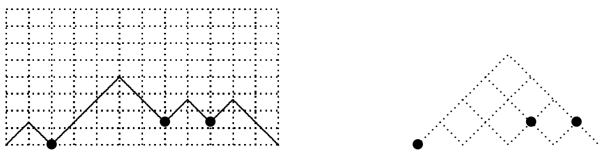
5.1.3. **Non-nesting partitions.** A partition of the set $\{1, 2, \dots, n\}$ is said to be *non-nesting,* provided given elements $i < i' < j < j'$ with i, j' in the same part and i', j in the same part, then all elements belong to the same part.

Given a non-nesting partition, we consider again its arc diagram. We recall that the set $A(P)$ of arcs of a partition P can be considered as a subset of Φ_+. Clearly, a partition P is non-nesting if and only if $A(P)$ is an **antichains in** Φ_+.

There is another model for the set of non-nesting partitions of $\{1, 2, \dots, n\}$, namely the set of **Dyck paths** of length $2n$. These are paths in the integral lattice $\mathbb{Z}^2$ from $(0,0)$ to $(2n, 0)$, inside the positive cone $\mathbb{N}^2$, using just northeast and southeast arrows. Here is an example for $n = 6$.



We get a bijection between the Dyck paths of length $2n$ and the set of elements of $\mathbf{A}(\Phi_+(\mathbb{A}_{n-1}))$ as follows: Attach to a Dyck path the set of its valleys: this is an antichain in the root poset $\Phi_+(\mathbb{A}_{n-1})$. For example, the valleys of the Dyck path shown above are the three vertices marked by bullets:



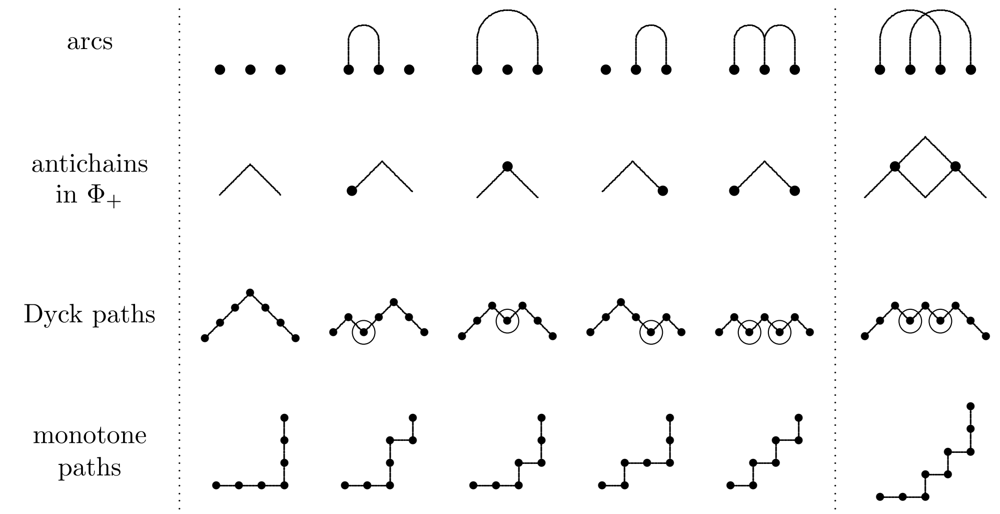
Clearly, *the set of valleys of a Dyck path is an antichain in* Φ_+ *and any antichain is obtained in this way.*

Dyck paths often are drawn as **monotonic paths:** as lattice paths from $(0,0)$ to (n,n) using only east arrows and north arrows of length 1 and staying inside the triangle with vertices $(0,0), (n,0), (n,n)$. The monotonic paths are obtained from the Dyck paths using a rotation by $225°$ (or, alternatively, by a rotation by -$45°$ degrees followed by a vertical reflection). Monotonic paths can be encoded as words using the letters X (for the east arrows) and Y (for the north arrows). Thus, we deal with words using n letters X, and n letters Y, such that for any initial subword the number of letters X is greater or equal to the number of letters Y. These words are called **Dyck words** of length $2n$.

Replacing in a Dyck word X by an opening bracket (and Y by a closing bracket), we obtain n pairs of **parentheses** which are correctly matched. Alternatively, we may use **(**± 1**)-sequences**, these are sequences with n entries equal to 1 and n entries equal to -1. such that all (initial) partial sums are non-negative.

A permutation π in S_n is said to be a **321-avoiding permutation** provided there are no numbers $a < b < c$ in $\{1, 2, \dots, n\}$ with $\pi(c) < \pi(b) < \pi(a)$. There is the following bijection between monotone paths and 321-avoiding permutations (see for example [**33, 90**]): Let $\pi(1), \ldots, \pi(n)$ be the value sequence of the permutation π. Define $y_1 = 0$. For $i \geq 2$, let $y_i = \pi(i)$ in case $i \geq 2$ and $p(i) < p(j)$ for some $j < i$; otherwise let $y_i = y_{i-1}$. The edges with endpoints $(i-1, y_i)$ and (i, y_i) are the X-steps of the monotone path (the corresponding Y-steps are determined uniquely by knowing the X-steps). Conversely, given a monotone path, the left-most X-steps in any row, say from $(i-1, y_i)$ and (i, y_i), show the value $p(i) = y_i$. The remaining values $\pi(j)$ are determined going downwards as the maximum of the numbers not yet used.

Let us exhibit various incarnations of the **non-nesting partitions,** for $n = 3$, as well as the only crossing partition for $n = 4$. For the Dyck paths, we have encircled the valleys (note that they correspond to the elements of the antichain).



Dyck words	$XXXYYY$	$XXYYXY$	$XXYXYY$	$XYXXYY$	$XYXYXY$	$XXYXYXYY$
parentheses	((()))	(())()	(()())	()(())	()()()	(()()())
(± 1)-sequences	(+++−−−)	(++−−+−)	(++−+−−)	(+−++−−)	(+−+−+−)	(++−+−+−−)
321-avoiding permutations	$\begin{bmatrix} 1&2&3 \\ 1&2&3 \end{bmatrix}$	$\begin{bmatrix} 1&2&3 \\ 1&3&2 \end{bmatrix}$	$\begin{bmatrix} 1&2&3 \\ 2&3&1 \end{bmatrix}$	$\begin{bmatrix} 1&2&3 \\ 2&1&3 \end{bmatrix}$	$\begin{bmatrix} 1&2&3 \\ 3&1&2 \end{bmatrix}$	$\begin{bmatrix} 1&2&3&4 \\ 3&4&1&2 \end{bmatrix}$

In order to verify again the recursion formula $f(n+1) = \sum_{t=0}^{n} f(t)f(n-t)$, consider for example the Dyck words. Any Dyck word w of positive length can be written in a unique way in the form Xw_1Yw_2 where w_1, w_2 are (possibly empty) Dyck words. For the Dyck words of length $2(n+1)$ use the summation index t, take as w_1 any Dyck word of length $2t$ and for w_2 any Dyck word of length $2(n-t)$.

5.2. What is Catalan combinatorics? A second answer. Let us try to outline a general setting behind these Catalan problems.

5.2.1. The typical examples of Catalan problems concern finite sets with some additional structure, but it seems to us that Catalan combinatorics should be considered as a kind of heart of finite set theory in itself.

It is the identification

$$\boxed{\mathbf{Nc}(n) \simeq \mathbf{Nc}(S_n, c_n)}$$

which provides the essential hint (and as we have seen, it also is the starting point for the general theory when dealing with arbitrary Dynkin diagrams): it shows that the set $\mathbf{Nc}(n)$ of non-crossing partitions (as the typical Catalan object) is an intrinsic invariant of the automorphism group S_n of the finite set $C = \{1, 2, \dots, n\}$. Of course, we need to specify not only the set C but in addition also a total (or, at least, a cyclic) ordering of the set C in order to obtain the Coxeter element c_n, but actually, all the Coxeter elements c are conjugate, and conjugations yield isomorphisms of the corresponding subsets $\mathbf{Nc}(S_n, c)$. Thus, we want to stress that the identification $\mathbf{Nc}(n) \simeq \mathbf{Nc}(S_n, c_n)$ shows that this typical Catalan object can be realized both as a subset of the set of all the partitions of C (or, equivalently, of the set $\mathrm{Quot}(C)$ of all quotient sets of C), as well as as a subset of the automorphism group $\mathrm{Aut}(C) = S_n$ of C. Let us expand these considerations by looking in more detail at the category of finite sets, taking into account for any finite set C also the set $\mathrm{Sub}(C)$ of all subsets of C.

Modern mathematics considers mathematical structures as objects of a corresponding category. As many nice categories, the category of sets has the following two important properties: first, any morphism can be factored as an epimorphism followed by a monomorphism, and second, a morphism which is both a monomorphism and an epimorphism is an isomorphism. In such a category $\mathcal{C}$, the basic information about any object is given by

- the automorphism group $\mathrm{Aut}(C)$ of C,
- the poset $\mathrm{Sub}(C)$ of all subobjects of C,
- the poset $\mathrm{Quot}(C)$ of all quotient objects of C.

Combining these data, one obtains further specifications about a given object C, as well as about its relation to other objects C', for example about the set of morphisms $\mathcal{C}(C, C')$. In particular, we recover in this way also the set $\mathrm{End}(C) = \mathcal{C}(C, C)$ of endomorphisms of C (which of course could be considered as one of the initial data), let us add this to the previous list:

- the semigroup $\mathrm{End}(C)$ of all endomorphisms of C.

Note that this set $\mathrm{End}(C)$ may be thought of as combining the data mentioned before: the group $\mathrm{Aut}(C)$ is a subgroup of $\mathrm{End}(C)$, whereas any quotient object as well as any non-empty subset of C can be realized as the image of an endomorphism. Thus, let us concentrate on these four kinds of data in the special case of the category of finite sets.

In many categories $\mathcal{C}$ there are obvious relations between some of these data. For example, if $\mathcal{C}$ is the module category of a ring, the subobjects of any C correspond bijectively to the quotient objects.

Our concern is the category of finite sets. For any finite set C, all the data mentioned are again finite sets, but usually these sets are quite different. Here is the table of the corresponding cardinalities for the sets C of cardinality n with $n \le 10$, this concerns the columns 2 to 5. The last three columns present the numbers which we want to discuss. Here, $\mathrm{Sub}(C)'$ denotes the set of subsets of C of cardinality at least 2.

$\|C\|$	$\|\mathrm{End}(C)\|$	$\|\mathrm{Aut}(C)\|$	$\|\mathrm{Sub}(C)\|$	$\|\mathrm{Quot}(C)\|$	$\|\Phi_+(\mathbb{A}_{n-1})\|$	$\|\mathrm{Sub}(C)'\|$	$\|\mathbf{Nc}(n)\|$
n	n^n	$n!$	2^n	Bell	$\binom{n}{2}$	Euler	Catalan
0	1	1	1	1	0	0	1
1	1	1	2	1	0	0	1
2	4	2	4	2	1	1	2
3	27	6	8	5	3	4	5
4	256	24	16	15	6	11	14
5	3125	120	32	52	10	26	42
6	46656	720	64	203	15	57	132

Before we proceed, let us insert in which way $|\mathrm{End}(C)|$ can be obtained from numbers of subobjects, of quotient object and of automorphisms. where C is a set of cardinality n. We denote by $\mathrm{Sub}(t, C)$ the set of subsets of C of cardinality t, of course $|\mathrm{Sub}(t, C)| = \binom{n}{t}$ is just a binomial coefficient. We denote by $\mathrm{Quot}(C, t)$ the set of partitions of C with t parts, thus $|\mathrm{Quot}(C, t)| = \left\{ {n \atop t} \right\}$ is a Stirling numbers of the second kind. Here is the decisive formula:

$$\begin{aligned} |\mathrm{End}(C)| &= \sum_{t=0}^{n} |\mathrm{Quot}(C,t)| \cdot |\mathrm{Aut}(t)| \cdot |\mathrm{Sub}(t,C)| \\ &= \sum_{t=0}^{n} \left\{ {n \atop t} \right\} \cdot t! \cdot \binom{n}{t} \end{aligned}$$

Altogether, we obtain the following triangle (Sloane A090657), on the left side, we exhibit the factorization $\left\{ {n \atop t} \right\} \cdot t! \cdot \binom{n}{t}$, on the right side, the corresponding product.

n			sum
0	1·1·1	1	1
1	0·1·1 1·1·1	0 1	1
2	0·1·1 1·1·2 1·2·1	0 2 2	4
3	0·1·1 1·1·3 3·2·3 1·6·1	0 3 18 6	37
4	0·1·1 1·1·4 7·2·6 6·6·4 1·24·1	0 4 84 144 24	256
5	0·1·1 1·1·5 15·2·10 25·6·10 10·24·5 1·120·1	0 5 300 1500 1200 120	3125

5.2.2. Let us draw the attention to the role of the quotient sets $\mathrm{Quot}(C)$ in finite set theory. If C is a set of cardinality n, the cardinality of $\mathrm{Quot}(C)$ is given by the Bell number B_n. There is an inductive way to determine the Bell numbers, but contrary to the other cardinalities discussed here, there is no explicit formula. The binomial coefficients which count $\mathrm{Sub}(C)$ are ubiquitous, they are taught already in school and are used in all parts of mathematics, whereas the Bell numbers which count $\mathrm{Quot}(C)$ really stand in the shadow.

Algebraists usually appreciate not only subobjects, but also quotient objects — one of the reason for the triumphal rise of the theory of abelian categories seems to be that it has put quotient objects on an equal footing with subjects. Of course, there are categories such as the category of groups where the set of subobjects provide much more information than the set of quotient objects. In contrast, one has to be aware that dealing with sets without any additional structure, the wealth of quotient sets (thus the number of partitions) exceeds by far that of the subsets.

Looking at the wealth of quotient objects $\mathrm{Quot}(C)$ of a given finite set, one is tempted to look for proper subsets of $\mathrm{Quot}(C)$ which may be of relevance. This is the realm of the Catalan combinatorics: to single out important subsets of $\mathrm{Quot}(C)$ and to relate them to subsets of $\mathrm{Aut}(C)$.

Some interesting subsets of $\mathrm{Quot}(C)$:

- Non-crossing partitions.
- Non-nesting partitions.

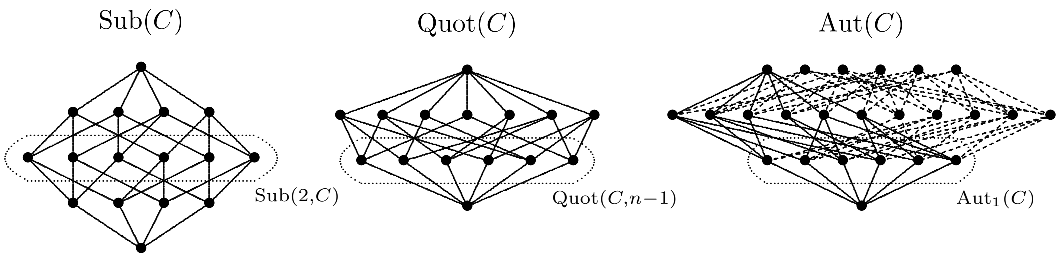
Some interesting subsets of $\mathrm{Aut}(C)$:

- The set of w with $w \leq_a c$ for some Coxeter element c.
- Pattern avoiding permutations (for example avoiding the pattern 321 or 132).

5.2.3. We should stress that the three relevant sets $\mathrm{Sub}(C), \mathrm{Quot}(C), \mathrm{Aut}(C)$ can be (and should be) considered as posets with a natural level structure (here, C is a set of cardinality n):

- $\mathrm{Sub}(C)$ is a poset (even a lattice) with respect to inclusion of subsets, and $\mathrm{Sub}(t, C)$ are the subsets of cardinality t, with $0 \leq t \leq n$. Thus there are $n+1$ levels.
- $\mathrm{Quot}(C)$ is a poset (again even a lattice) with respect to the inverse refinement order, the levels are given by the sets and $\mathrm{Quot}(C, n-t)$ of the partitions with $n-t$ parts, for $1 \leq t \leq n$. Thus, there are n levels.
- $\mathrm{Aut}(C) = S_n$ is a poset with respect to the absolute order $\leq_a$ (see Chapter 3), and $\mathrm{Aut}_t(C)$ is the set of permutation with $n-t$ cycles, for $1 \leq t \leq n$. Again, there are n levels.

Here are these posets for $n = 4$.



The set $\Phi_+(\mathbb{A}_{n-1})$. We have encircled by a dotted line the subsets

$$\mathrm{Sub}(2, C),\ \mathrm{Quot}(C, n-1),\ \mathrm{Aut}_1(C).$$

For any n, we claim that there are canonical bijections between these sets. Note that the set $\mathrm{Sub}(C)_2$ may be considered as the set of Φ_+ of positive roots for the Dynkin diagram $\mathbb{A}_{n-1}$, thus we claim that

$$\boxed{\Phi_+(\mathbb{A}_{n-1}) = \mathrm{Sub}(2, C) \simeq \mathrm{Quot}(C, n-1) \simeq \mathrm{Aut}_1(C).}$$

The bijections are given as follows: A positive root in $\Phi_+(\mathbb{A}_{n-1})$ may be considered as a pair (x, y) of natural numbers $1 \le x < y \le n$, or, equivalently as the set $\{x, y\}$ in $\mathrm{Sub}(2, C)$. The corresponding partition of $\{1, 2, \ldots, n\}$ has $n-1$ parts, namely the part $\{x, y\}$ as well as the remaining singletons; these are the elements of $\mathrm{Quot}(C, n-1)$. The corresponding permutation exchanges x and y and fixes the remaining elements, these are the elements of $\mathrm{Aut}_1(C)$.

5.2.4. **The canonical map $P\colon \mathrm{Sub}(C) \to \mathrm{Quot}(C)$.** There is a canonical map

$$\boxed{P\colon \mathrm{Sub}(C) \to \mathrm{Quot}(C)}$$

defined as follows: If U is a non-empty subset of C, then $P(U)$ is the partition with U as one part, the remaining parts being singletons; the image $P(\emptyset)$ of the empty set is defined as the discrete partition (all parts are singletons).

Let us denote by $\mathrm{Sub}(C)'$ the set of subsets of C of cardinality at least 2. If C has cardinality n, then $\mathrm{Sub}(C)'$ has cardinality $2^n - (n+1)$, these numbers are called the *Euler numbers* (see [**100**], A000295). Also, we denote by $\mathrm{Quot}(C)_c$ the set of all partitions of C with precisely one part of cardinality greater than 1 (the index c stands for *connected*).

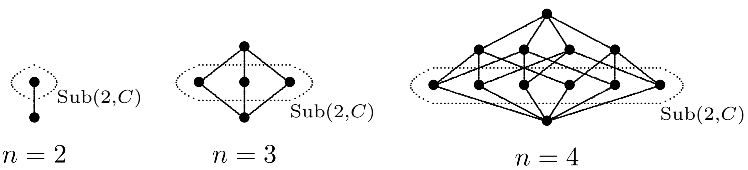
There is the canonical map

$$U\colon \mathrm{Quot}_c(C) \to \mathrm{Sub}(C)',$$

which sends a partition in $\mathrm{Quot}(C)_c$ to the only part of cardinality greater than 1. The maps U, P provide inverse poset isomorphisms

$$\boxed{\mathrm{Quot}_c(C) \simeq \mathrm{Sub}(C)'.}$$

Note that the poset $\mathrm{Sub}(C)'$ becomes a lattice if we add the empty set as zero element. Here are these lattices for $n = 2, 3, 4$, and we encircle the set $\mathrm{Sub}(2, C)$ of minimal elements of $\mathrm{Sub}(C)'$ by a dotted line:



PROPOSITION 5.2.4.1. *The partitions in* $\mathrm{Quot}_c(C)$ *are both non-crossing and non-nesting.*

PROOF. Consider a partition P in $\mathrm{Quot}_c(C)$. Assume that there are elements $a < b < c < d$ with a, c belonging to the same part, and b, d belonging to the same part. Then we deal with parts of cardinality at least 2. Since we assume that P belongs to $\mathrm{Quot}(C)_c$, there is precisely one part of cardinality at least 2, thus all the elements a, b, c, d belong to the same part. This shows that an P is non-crossing.

Similarly, one shows that P is non-nesting. $\square$

5.2.5. **The categorification of** $\mathrm{Sub}(C)' = \mathrm{Quot}_c(C)$**.** The identification of $\mathrm{Quot}_c(C)$ and $\mathrm{Sub}(C)'$ using the maps U, P can be reformulated in terms of the categorification by $\mathrm{mod}\,\Lambda_{n-1}$ as follows: Let $\mathbf{A}_c(\mathrm{mod}\,\Lambda_n)$ be the set of (non-empty) antichains in $\mathrm{mod}\,\Lambda_n$ with connected quiver (or, equivalently, the set of all exceptional subcategories of $\mathrm{mod}\,\Lambda_n$ which are connected).

Let $2 \le t \le n$. There is a bijection between $\mathbf{A}_c(\mathrm{mod}\,\Lambda_{n-1})_{t-1}$ and $\mathrm{Sub}(t, C)$ as follows: A subset in $\mathrm{Sub}(t, C)$ is of the form $\{a_1, \dots, a_t\}$ with $1 \le a_1 < \cdots < a_t \le n$, it is sent to the antichain $\{(a_1, a_2), \dots, (a_{t-1}, a_t)\}$, and this antichain belongs to $\mathbf{A}_c(\mathrm{mod}\,\Lambda_{n-1})$. Thus, we see: The map $A : \mathbf{Nc}(n) \to \mathbf{A}(\mathrm{mod}\,\Lambda_{n-1})$ provides a bijection

$$\boxed{\mathrm{Quot}_c(C) \to \mathbf{A}_c(\mathrm{mod}\,\Lambda_{n-1}),}$$

where C is a set of cardinality n.

5.2.6. **The canonical map** $O\colon \mathrm{End}(C) \to \mathrm{Quot}(C)$**.** There is a canonical map

$$O\colon \mathrm{End}(C) \to \mathrm{Quot}(C),$$

it is defined as follows: If f is an endomorphism of C, the parts of $O(f)$ are the orbits of f in C. The orbit of $c \in C$ under f is the smallest subset of C which contains c and which contains with every element x also $f(x)$ and $f^{-1}(x)$ (of course, if C is a finite set and f is invertible, the orbit of c under f consists of the elements $f^t(c)$ with $t \ge 0$).

The map O is surjective, even the restriction of O to $\mathrm{Aut}(C)$ is surjective: Given a partition P of C, we obtain a permutation f of C with $O(f) = P$ as follows: For any part P_i of P, let $f|P_i$ by a cyclic permutation of P_i.

For any total ordering L of C, there is a map $\pi_L : \mathrm{Quot}(C) \to \mathrm{Aut}(C)$ such that the composition

$$\boxed{\mathrm{Quot}(C) \xrightarrow{\pi_L} \mathrm{Aut}(C) \xrightarrow{O} \mathrm{Quot}(C)}$$

is the identity map. The map π_L is defined as follows: Let P be a partition of C, with parts $P_1, \dots, P_t$. If P_i consists of the elements $a_{i,1}, a_{i,2}, \dots, a_{i,t(i)}$ we can assume that $a_{i,1} < a_{i,2} < \cdots < a_{i,t(i)}$ with respect to the ordering L, and we may consider the corresponding cycle $c_i = (a_{i,1}, a_{i,2}, \dots, a_{i,t(i)})$. Then $\pi_L(P) = c_1 \cdots c_t$. Of course, this is a permutation of C and $O(\pi_L(P)) = P$.

Note: *If P is non-crossing with respect to L, then $\pi_L(P) \in W(S_n, c_L)$.* In general, one may ask which "Catalan subsets" of $\mathrm{Aut}(C)$ and of $\mathrm{Quot}(C)$ correspond to each other under the map O.

We have seen in Chapter 4 that the lattice $\mathbf{Nc}(n)$ can be identified with $\mathbf{Nc}(S_n, c_n)$.

$$\begin{array}{ccc}
\mathbf{Nc}(n) & \xrightarrow{\iota_n} & \mathbf{Nc}(S_n, c_n) \\
\downarrow \simeq & \nearrow_{\mathrm{cox}} & \\
\mathbf{A}(\mathrm{mod}\,\Lambda_{n-1}) & &
\end{array}$$

with $c_n = (n, n-1, \ldots, 2, 1)$. This map ι_n is part of the following larger commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Quot}(n) & \xrightarrow{\pi_n} & S_n & \xrightarrow{O} & \mathrm{Quot}(n) \\
\uparrow{\scriptstyle \iota_n} & & \uparrow & & \\
\mathbf{Nc}(n) & \xrightarrow{\iota_n} & \mathbf{Nc}(S_n, c_n) & & \\
\downarrow{\scriptstyle \simeq} & \nearrow{\scriptstyle \mathrm{cox}} & & & \\
\mathbf{A}(\mathrm{mod}\,\Lambda_{n-1}) & & & &
\end{array}$$

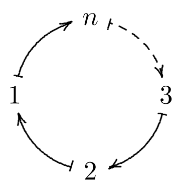
Here, $\pi_n = \pi_L$ with the natural ordering $L = (1 < 2 < \cdots < n)$.

5.2.7. The interesting numbers are provided by the following chain of inclusion maps. Here, we denote by $\iota_L : \mathbf{Nc}(n) \to \mathrm{Quot}(C)$ the canonical inclusion after we have identified (C, L) with $\{1, 2, \ldots, n\}$, where L is a total ordering of the set C of cardinality n.

C	$\mathrm{Sub}(2, C) = \Phi_+(\mathbb{A}_{n-1})$ $\subseteq$	$\mathrm{Sub}(C)'$ $\xrightarrow{P}$	$\mathbf{Nc}(n)$ $\xrightarrow{\iota_L}$	$\mathrm{Quot}(C)$ $\xrightarrow{\pi_L}$	$\mathrm{Aut}(C)$
$\lvert C\rvert = n$	$\binom{n}{2}$	Euler	Catalan	Bell	$n!$
0	0	0	1	1	1
1	0	0	1	1	1
2	1	1	2	2	2
3	3	4	5	5	6
4	6	11	14	15	24
5	10	26	42	52	120
6	15	57	132	203	720

All but the last two maps shown in the upper row are canonical, the last two ι_L and π_L depend on the choice of L, say the choice of a linear ordering such as $L = \{1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n\}$.

But note that π_L actually only depends on the corresponding cyclic ordering



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