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Abstract

In this paper, we show that there is an equivalence between the 2-category of smooth Deligne–Mumford stacks with torus embeddings and actions and the 1-category of stacky fans. To this end, we prove two main results. The first is related to a combinatorial aspect of the 2-category of toric algebraic stacks defined by I. Iwanari [*Logarithmic geometry, minimal free resolutions and toric algebraic stacks*, Preprint (2007)]; we establish an equivalence between the 2-category of toric algebraic stacks and the 1-category of stacky fans. The second result provides a geometric characterization of toric algebraic stacks. Logarithmic geometry in the sense of Fontaine–Illusie plays a central role in obtaining our results.

1. Introduction and main results

The equivalence between the category of toric varieties and the category of fans is a fundamental theorem for toric varieties, and provides a fruitful bridge between the fields of algebraic geometry and combinatorics. It is also useful in various contexts; a typical and most beautiful application, for example, is the toric minimal model program (see [Rei83]). Since simplicial toric varieties have quotient singularities in characteristic zero, a natural problem is to find such an equivalence in the stack-theoretic context. Let k be an algebraically closed base field of characteristic zero. Consider a triple $(\mathcal{X}, \iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}, a : \mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X})$, where $\mathcal{X}$ is a smooth Deligne–Mumford stack of finite type, separated over k, which satisfies the following properties.

(i) The morphism $\iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}$ is an open immersion identifying $\mathbb{G}_m^d$ with a dense open substack of $\mathcal{X}$. (We shall refer to $\mathbb{G}_m^d \hookrightarrow \mathcal{X}$ as a torus embedding.)

(ii) The morphism $a : \mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X}$ is an action of $\mathbb{G}_m^d$ on $\mathcal{X}$ which is an extension of the action $\mathbb{G}_m^d$ on itself. (We shall refer to it as a torus action.)

(iii) The coarse moduli space X for $\mathcal{X}$ is a scheme.

We shall refer to such a triple as a *toric triple*. Note that if $\mathcal{X}$ is a scheme, then $\mathcal{X}$ is a smooth toric variety. A 1-morphism of toric triples

$$(\mathcal{X}, \iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}, a : \mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X}) \to (\mathcal{X}', \iota' : \mathbb{G}_m^{d'} \hookrightarrow \mathcal{X}', a' : \mathcal{X}' \times \mathbb{G}_m^{d'} \to \mathcal{X}')$$

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is a morphism $f:\mathcal{X}\to\mathcal{X}'$ such that the restriction of f to $\mathbb{G}_m^d$ induces a morphism $\mathbb{G}_m^d\to\mathbb{G}_m^{d'}$ of group k-schemes and the diagram

$$\begin{tikzcd} \mathcal{X}\times\mathbb{G}_m^d \arrow[r, "f\times(f|_{\mathbb{G}_m^d})"] \arrow[d, "a"] & \mathcal{X}'\times\mathbb{G}_m^{d'} \arrow[d, "a'"] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{X}' \end{tikzcd}$$

commutes in the 2-categorical sense. A 2-isomorphism $g:f_1\to f_2$ is an isomorphism of 1-morphisms.

Our main goal is to prove the following.

THEOREM 1.1. *There exists an equivalence between the 2-category of toric triples and the 1-category of stacky fans. (See Definition 2.1 for the definition of stacky fans.)*

The non-singular fans form a full subcategory of the category of stacky fans. Thus our equivalence includes the classical equivalence between smooth toric varieties and non-singular fans (see Remark 4.5).

To obtain Theorem 1.1, we need to consider the following two problems.

(i) Construction of toric triples $\mathscr{X}_{(\Sigma,\Sigma^0)}$ associated to a stacky fan (Σ,Σ^0) (which we shall refer to as the associated toric algebraic stack) and the establishment of an equivalence between the groupoid category $\mathrm{Hom}(\mathscr{X}_{(\Sigma_1,\Sigma_1^0)},\mathscr{X}_{(\Sigma_2,\Sigma_2^0)})$ and the discrete category $\mathrm{Hom}((\Sigma_1,\Sigma_1^0),(\Sigma_2,\Sigma_2^0))$ associated to the set of morphisms.

(ii) Geometric characterization of toric algebraic stacks associated to stacky fans.

For the first problem, the construction was given in [Iwa07a] (see also [Iwa06]). In this work, given a stacky fan (Σ,Σ^0), the associated toric algebraic stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ was defined by means of logarithmic geometry. In characteristic zero, it is a smooth Deligne–Mumford stack and has a natural torus embedding and a torus action, i.e. a toric triple. Let us denote by $\mathfrak{Torst}$ the 2-category whose objects are toric 1-morphism of two toric algebraic stacks $f:\mathscr{X}_{(\Sigma,\Sigma^0)}\to\mathscr{X}_{(\Delta,\Delta^0)}$ in $\mathfrak{Torst}$ is a torus-equivariant 1-morphism (cf. Definitions 2.1, 2.3 and 2.6). A 2-morphism $g:f_1\to f_2$ is an isomorphism of 1-morphisms. Then, the following is our answer to the first problem.

THEOREM 1.2. *Working over a field of characteristic zero, there exists an equivalence of 2-categories*

$$\Phi:\mathfrak{Torst}\xrightarrow{\sim}(\text{1-category of stacky fans})$$

which makes the diagram

$$\begin{tikzcd} \mathfrak{Torst} \arrow[r, "\Phi"] \arrow[d, "c"] & (\text{category of stacky fans}) \arrow[d] \\ \mathrm{Simtoric} \arrow[r, "\sim"] & (\text{category of simplicial fans}) \end{tikzcd}$$

commutative. Here Simtoric *is the category of simplicial toric varieties (morphisms in* Simtor *are those that are torus-equivariant), and c is the natural functor which sends the toric algebraic stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ associated to a stacky fan (Σ,Σ^0) to the toric variety X_Σ (cf. Remarks 2.8 and 3.4). The 1-category of stacky fans is regarded as a 2-category.*

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In Theorem 1.2, the difficult issue is to show that the groupoid of torus-equivariant 1-morphisms of toric algebraic stacks is equivalent to the discrete category of the set of the morphisms of stacky fans. For two algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$, the classification of 1-morphisms $\mathcal{X} \to \mathcal{Y}$ and their 2-isomorphisms is a hard problem even when $\mathcal{X}$ and $\mathcal{Y}$ have very explicit groupoid presentations. Groupoid presentations of stacks are ill-suited to dealing with such problems. To overcome this difficulty, our idea is to use the modular interpretation of toric algebraic stacks in terms of *logarithmic geometry* (see § 2). A certain type of resolution of monoids (called (Σ, Σ^0)-free resolutions) plays a role similar to monoid algebras arising from cones in classical toric geometry; by virtue of this notion, we obtain Theorem 1.2 by reducing it to a certain problem concerning log structures on schemes. As a corollary of Theorem 1.2, we show also that every toric algebraic stack admits a smooth torus-equivariant cover by a smooth toric variety (see Corollary 3.10).

The second problem is the geometric characterization. Remembering that we have a *geometric* characterization of toric varieties (see [KKMS73]), we wish to obtain a similar characterization of toric algebraic stacks. Our geometric characterization of toric algebraic stacks is given in the following theorem.

Theorem 1.3. *Assume that the base field k is algebraically closed in characteristic zero. Let $(\mathcal{X}, \iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}, a : \mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X})$ be a toric triple over k. Then there exist a stacky fan (Σ, Σ^0) and an isomorphism of stacks*

$$\Phi : \mathcal{X} \xrightarrow{\sim} \mathscr{X}_{(\Sigma,\Sigma^0)}$$

over k which satisfy the following properties.

(i) *The restriction of Φ to $\mathbb{G}_m^d \subset \mathcal{X}$ induces an isomorphism $\Phi_0 : \mathbb{G}_m^d \xrightarrow{\sim} \operatorname{Spec} k[M] \subset \mathscr{X}_{(\Sigma,\Sigma^0)}$ of group k-schemes. Here $N = \mathbb{Z}^d$, $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, Σ is a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$, and $\operatorname{Spec} k[M] \hookrightarrow \mathscr{X}_{(\Sigma,\Sigma^0)}$ is the natural torus embedding (see Definition 2.3).*

(ii) *The diagram*

$$\begin{tikzcd} \mathcal{X} \times \mathbb{G}_m^d \arrow[r, "\Phi\times\Phi_0"] \arrow[d, "m"] & \mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M] \arrow[d, "a_{(\Sigma,\Sigma^0)}"] \\ \mathcal{X} \arrow[r, "\Phi"] & \mathscr{X}_{(\Sigma,\Sigma^0)} \end{tikzcd}$$

commutes in the 2-categorical sense, where $a_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M] \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ is the torus action functor (see § 2.2).

Moreover, such a stacky fan is unique up to isomorphisms.

The essential ingredient in the proof of Theorem 1.3 is a study of (étale) local structures of the coarse moduli map $\mathcal{X} \to X$. We first show that X is a toric variety and then determine the local structure of $\mathcal{X} \to X$ by applying the logarithmic Nagata–Zariski purity theorem which was independently proven by Mochizuki and by Kato.

It is natural and interesting to consider a generalization of our work to positive characteristics. Unfortunately, our proof does not seem to be applicable to the case of positive characteristics; for instance, we impose the assumption of characteristic zero in order to apply the log purity theorem. Furthermore, in positive characteristics, toric algebraic stacks as defined in [Iwa07a] are not necessarily Deligne–Mumford stacks; in fact, they happen to be Artin stacks. Thus, in such a generalized framework, the formulation would need to be modified (see Remark 4.4).

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Informally, Theorem 1.1 implies that the geometry of toric triples could be encoded by the combinatorics of stacky fans. It would be interesting to investigate the geometric invariants of toric triples from the viewpoint of stacky fans. In this direction, we have demonstrated in [Iwa07b] the relationship between integral Chow rings of toric triples and (classical) Stanley–Reisner rings, in which a non-scheme-theoretic phenomenon arises. In another direction, Theorem 1.2 has a nice place in the study of toric minimal model programs from a stack-theoretic and derived categorical perspective.

The paper is organized as follows. In § 2 we recall basic definitions concerning toric algebraic stacks as well as some results that we will use in §§ 3 and 4. In § 3 we present the proof of Theorem 1.2 and its corollaries. In § 4 we give the proof of Theorem 1.3. Finally, in § 5 we discuss the relationships between our work and that of Borisov, Chen and Smith [BCS05] modeling the quotient construction by Cox [Cox95a], as well as the recent papers [FMN07] and [Per08] by Fantechi, Mann, Nironi and Perroni.

We shall systematically use the language of logarithmic geometry, which we assume that readers are familiar with at the level of [Kat88].

1.1 Notations and conventions

(i) We fix a Grothendieck universe $\mathscr{U}$ with $\{0, 1, 2, 3, \ldots\} \in \mathscr{U}$, where $\{0, 1, 2, 3, \ldots\}$ is the set of all finite ordinals. We consider only monoids, groups, rings, schemes and log schemes which belong to $\mathscr{U}$.

(ii) A *variety* is a geometrically integral scheme of finite type which is separated over a field.

(iii) *Toric varieties.* Let $N \cong \mathbb{Z}^d$ be a lattice of rank d and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. If Σ is a fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, we denote the associated toric variety by X_Σ. We usually write $i_\Sigma : T_\Sigma := \operatorname{Spec} k[M] \hookrightarrow X_\Sigma$ for the torus embedding. If k is an algebraically closed field, then by applying Sumihiro's theorem [Sum75, Corollary 3.11], just as in [KKMS73, ch. 1], we obtain the following geometric characterization of toric varieties. Let X be a normal variety which contains an algebraic torus (i.e. $\mathbb{G}_m^d$) as a dense open subset. Suppose that the action of $\mathbb{G}_m^d$ on itself extends to an action of $\mathbb{G}_m^d$ on X. Then there exist a fan Σ and an equivariant isomorphism $X \cong X_\Sigma$.

(iv) *Logarithmic geometry.* All monoids are assumed to be commutative with unit. For a monoid P, we denote by P^{gp} the Grothendieck group of P. A monoid P is said to be *sharp* if whenever $p + p' = 0$ for $p, p' \in P$, we have $p = p' = 0$. For a fine sharp monoid P, an element $p \in P$ is said to be *irreducible* if whenever $p = q + r$ for $q, r \in P$, we have either $q = 0$ or $r = 0$. In this paper, a log structure on a scheme X means a log structure (in the Fontaine–Illusie sense [Kat88]) on the étale site X_{et}. We usually denote simply by $\mathcal{M}$ a log structure $\alpha : \mathcal{M} \to \mathscr{O}_X$ on X, and by $\bar{\mathcal{M}}$ the sheaf $\mathcal{M}/\mathscr{O}_X^*$. Let R be a ring. For a fine monoid P, the canonical log structure on $\operatorname{Spec} R[P]$, denoted by $\mathcal{M}_P$, is the log structure associated to the natural injective map $P \to R[P]$. If there is a homomorphism of monoids $P \to R$ (here we regard R as a monoid under multiplication), we denote by $\operatorname{Spec}(P \to R)$ the log scheme with underlying scheme $\operatorname{Spec} R$ and the log structure associated to $P \to R$. For a toric variety X_Σ, we denote by $\mathcal{M}_\Sigma$ the fine log structure $\mathscr{O}_{X_\Sigma} \cap i_{\Sigma *}\mathscr{O}_{T_\Sigma}^* \hookrightarrow \mathscr{O}_{X_\Sigma}$ on X_Σ; we shall call this log structure the *canonical log structure* on X_Σ. We refer to [Iwa07a] for further properties and notation relating to toric varieties, monoids and log schemes, which are needed in what follows.

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(v) *Algebraic stacks*. We follow the conventions in ([LM00]). For a diagram $X \xrightarrow{a} Y \xleftarrow{b} Z$, we denote by $X \times_{a,Y,b} Z$ the fiber product (a and b are often omitted if no confusion seems likely to arise). Let us review some facts about coarse moduli spaces of algebraic stacks. Let $\mathcal{X}$ be an algebraic stack over a scheme S. A *coarse moduli space* (or *map*) for $\mathcal{X}$ is a morphism $\pi : \mathcal{X} \to X$ to an algebraic space over S such that: (i) π is universal among morphisms from $\mathcal{X}$ to algebraic spaces over S; (ii) for every algebraically closed S-field K the map $[\mathcal{X}(K)] \to X(K)$ is bijective, where $[\mathcal{X}(K)]$ denotes the set of isomorphism classes of objects in the small category $\mathcal{X}(K)$. The fundamental existence theorem for coarse moduli spaces (which we refer to as the Keel–Mori theorem [KM97]) can be stated as follows (this version is sufficient for our purposes). Let k be a field of characteristic zero. Let $\mathcal{X}$ be an algebraic stack of finite type over k with finite diagonal. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$, where X is of finite type and is separated over k, which satisfies the following additional properties: (a) π is proper, quasi-finite and surjective; (b) for any morphism $X' \to X$ of algebraic spaces over k, $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space (cf. [AV02, Lemmas 2.3.3 and 2.2.2]).

2. Preliminaries

In this section, we recall the basic definitions and properties from [Iwa07a] concerning toric algebraic stacks and stacky fans. We fix a base field k of characteristic zero.

2.1 Definitions

In this paper, all fans are assumed to be *finite*, though the theory in [Iwa07a] applies also in the case of *infinite fans*. For a fan Σ, we denote the set of rays by $\Sigma(1)$.

Definition 2.1. Let $N \cong \mathbb{Z}^d$ be a lattice of rank d and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. A stacky fan is a pair (Σ, Σ^0), where Σ is a simplicial fan in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and Σ^0 is a subset of $|\Sigma| \cap N$ called the *free-net of* Σ, which has the following property (♠):

(♠) for any cone σ in Σ, $\sigma \cap \Sigma^0$ is a submonoid of $\sigma \cap N$ which is isomorphic to $\mathbb{N}^{\dim \sigma}$ and is such that for any element $e \in \sigma \cap N$, there exists a positive integer n with $n \cdot e \in \sigma \cap \Sigma^0$.

A morphism $f : (\Sigma \text{ in } N \otimes_{\mathbb{Z}} \mathbb{R}, \Sigma^0) \to (\Delta \text{ in } N' \otimes_{\mathbb{Z}} \mathbb{R}, \Delta^0)$ is a homomorphism of $\mathbb{Z}$-modules $f : N \to N'$ which satisfies the following two properties:

- for any cone σ in Σ, there exists a cone τ in Δ such that $f \otimes_{\mathbb{Z}} \mathbb{R}(\sigma) \subset \tau$;
- $f(\Sigma^0) \subset \Delta^0$.

There exists a natural forgetting functor

$$(\text{category of stacky fans}) \to (\text{category of simplicial fans}), \quad (\Sigma, \Sigma^0) \mapsto \Sigma.$$

It is essentially surjective but *not* fully faithful. Given a stacky fan (Σ, Σ^0) and a ray ρ in $\Sigma(1)$, the initial point P_ρ of $\rho \cap \Sigma^0$ is said to be the *generator* of Σ^0 on ρ. Let Q_ρ be the first point of $\rho \cap N$ and let n_ρ be the natural number such that $n_\rho \cdot Q_\rho = P_\rho$; then the number n_ρ is said to be the *level* of Σ^0 on ρ. Note that Σ^0 is completely determined by the levels of Σ^0 on rays of Σ. Each simplicial fan Σ has the canonical free-net Σ^0_{can} whose level on every ray in Σ is one.

If Σ and Δ are non-singular fans, then a usual morphism of fans $\Sigma \to \Delta$ amounts to a morphism of stacky fans $(\Sigma, \Sigma^0_{\mathrm{can}}) \to (\Delta, \Delta_{\mathrm{can}})$. In other words, the category of non-singular fans is a full subcategory of the category of stacky fans.

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Let us give an example. Let σ be a two-dimensional cone in $(\mathbb{Z}\cdot e_1 \oplus \mathbb{Z}\cdot e_2)\otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^{\oplus 2}$ that is generated by e_1 and e_1+2e_2, and let σ^0 be a free submonoid of $\sigma \cap (\mathbb{Z}\cdot e_1 \oplus \mathbb{Z}\cdot e_2)$ that is generated by $2e_1$ and e_1+2e_2. Note that $\sigma^0 \cong \mathbb{N}^{\oplus 2}$. Then (σ, σ^0) forms a stacky fan. The level of σ^0 on the ray $\mathbb{R}_{\geq 0}\cdot e_1$ (respectively, $\mathbb{R}_{\geq 0}\cdot(e_1+2e_2)$) is 2 (respectively, 1).

Let P be a monoid and let $S \subset P$ be a submonoid. We say that S is *close to* P if for any element e in P there exists a positive integer n such that $n\cdot e$ lies in S. The monoid P is said to be *toric* if P is a fine, saturated and torsion-free monoid.

Let P be a toric sharp monoid and d the rank of P^{gp}. A toric sharp monoid P is said to be *simplicially toric* if there exists a submonoid Q of P generated by d elements such that Q is close to P.

DEFINITION 2.2. Let P be a simplicially toric sharp monoid and d the rank of P^{gp}. The *minimal free resolution* of P is an injective homomorphism of monoids

$$i : P \longrightarrow F,$$

with $F \cong \mathbb{N}^d$, which has the following properties.

(i) The submonoid $i(P)$ is close to F.

(ii) For any injective homomorphism $j : P \to G$ such that $j(P)$ is close to G and $G \cong \mathbb{N}^d$, there exists a unique homomorphism $\phi : F \to G$ such that $j = \phi \circ i$.

We remark that, by [Iwa07a, Proposition 2.4] or Lemma 3.3, there exists a unique minimal free resolution for any simplicially toric sharp monoid. Next, we recall the definition of toric algebraic stacks [Iwa07a]. Immediately after Remark 2.8 we shall give another definition of toric algebraic stacks, which is a more direct presentation in the terms of logarithmic geometry.

DEFINITION 2.3. The *toric algebraic stack* associated to a stacky fan $(\Sigma \text{ in } N\otimes_{\mathbb{Z}}\mathbb{R}, \Sigma^0)$ is a stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ over the category of k-schemes whose objects over a k-scheme X are triples $(\pi : \mathcal{S} \to \mathcal{O}_X, \alpha : \mathcal{M} \to \mathcal{O}_X, \eta : \mathcal{S} \to \mathcal{M})$ such that the following properties hold.

(i) $\mathcal{S}$ is an étale sheaf of submonoids of the constant sheaf M on X determined by $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ such that for every point $x \in X$, we have $\mathcal{S}_x \cong \mathcal{S}_{\bar{x}}$; here $\mathcal{S}_x$ (respectively, $\mathcal{S}_{\bar{x}}$) denotes the Zariski (respectively, étale) stalk.

(ii) $\pi : \mathcal{S} \to \mathcal{O}_X$ is a map of monoids, where $\mathcal{O}_X$ is a monoid under multiplication.

(iii) For $s \in \mathcal{S}$, $\pi(s)$ is invertible if and only if s is invertible.

(iv) For each point $x \in X$, there exists some $\sigma \in \Sigma$ such that $\mathcal{S}_{\bar{x}} = \sigma^{\vee} \cap M$.

(v) $\alpha : \mathcal{M} \to \mathcal{O}_X$ is a fine log structure on X.

(vi) $\eta : \mathcal{S} \to \mathcal{M}$ is a homomorphism of sheaves of monoids such that $\pi = \alpha \circ \eta$ and, for each geometric point $\bar{x}$ on X, $\bar{\eta} : \bar{\mathcal{S}}_{\bar{x}} = (\mathcal{S}/(\text{invertible elements}))_{\bar{x}} \to \bar{\mathcal{M}}_{\bar{x}}$ is isomorphic to the composite

$$\bar{\mathcal{S}}_{\bar{x}} \overset{r}{\hookrightarrow} F \overset{t}{\hookrightarrow} F,$$

where r is the minimal free resolution of $\bar{\mathcal{S}}_{\bar{x}}$ and t is defined as follows.

Each irreducible element of F canonically corresponds to a ray in Σ (see Lemma 2.4 below). Let us denote by e_ρ the irreducible element of F which corresponds to the ray ρ. Then define $t : F \to F$ by $e_\rho \mapsto n_\rho \cdot e_\rho$, where n_ρ is the level of Σ^0 on ρ. We shall refer to $t \circ r : \bar{\mathcal{S}}_{\bar{x}} \to F$ as the (Σ, Σ^0)-*free resolution at* $\bar{x}$ (or the (Σ, Σ^0)-*free resolution of* $\bar{\mathcal{S}}_{\bar{x}} = \mathcal{S}_{\bar{x}}/(\text{invertible elements})$).

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A set of morphisms from $(\pi : \mathcal{S} \to \mathcal{O}_X, \alpha : \mathcal{M} \to \mathcal{O}_X, \eta : \mathcal{S} \to \mathcal{M})$ to $(\pi' : \mathcal{S}' \to \mathcal{O}_X, \alpha' : \mathcal{M}' \to \mathcal{O}_X, \eta' : \mathcal{S}' \to \mathcal{M}')$ over X is, in the case where $(\mathcal{S}, \pi) = (\mathcal{S}', \pi')$, the set of isomorphisms of log structures $\phi : \mathcal{M} \to \mathcal{M}'$ such that $\phi \circ \eta = \eta' : \mathcal{S} = \mathcal{S}' \to \mathcal{M}'$, and is an empty set if $(\mathcal{S}, \pi) \neq (\mathcal{S}', \pi')$. With the natural notion of pull-backs, $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is a fibered category.

By [AMRT75, theorem on p. 10], $\mathrm{Hom}_{k\text{-schemes}}(X, X_\Sigma) \cong \{$all pairs $(\mathcal{S}, \pi)$ on X satisfying (i), (ii), (iii) and (iv)$\}$. Hence there exists a natural functor $\pi_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \longrightarrow X_\Sigma$ which simply forgets the data $\alpha : \mathcal{M} \to \mathcal{O}_X$ and $\eta : \mathcal{S} \to \mathcal{M}$. Moreover, $\alpha : \mathcal{M} \to \mathcal{O}_X$ and $\eta : \mathcal{S} \to \mathcal{M}$ are morphisms of the étale sheaves, and thus $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is a stack with respect to the étale topology. Objects of the form $(\pi : M \to \mathscr{O}_X, \mathscr{O}_X^* \hookrightarrow \mathscr{O}_X, \pi : M \to \mathscr{O}_X^*)$ determine a full subcategory of $\mathscr{X}_{(\Sigma,\Sigma^0)}$, i.e. the natural inclusion $i_{(\Sigma,\Sigma^0)} : T_\Sigma = \mathrm{Spec}\, k[M] \hookrightarrow \mathscr{X}_{(\Sigma,\Sigma^0)}$. This commutes with the torus embedding $i_\Sigma : T_\Sigma \hookrightarrow X_\Sigma$.

Lemma 2.4. *With notation as in Definition 2.3, let e be an irreducible element in F and let n be a positive integer such that $n \cdot e \in r(\bar{\mathcal{S}}_{\bar{x}})$. Let $m \in \mathcal{S}_{\bar{x}}$ be a lifting of $n \cdot e$. Suppose that $\mathcal{S}_{\bar{x}} = \sigma^\vee \cap M \subset M$. Then there exists a unique ray $\rho \in \sigma(1)$ such that $\langle m, \zeta_\rho \rangle > 0$, where ζ_ρ is the first lattice point of ρ and $\langle \bullet, \bullet \rangle$ is the dual pairing, which does not depend on the choice of liftings. Moreover, this correspondence defines a natural injective map*

$$\{\text{irreducible elements of } F\} \to \Sigma(1).$$

Proof. This is fairly elementary (and follows from [Iwa07a]), but we shall give the proof of the completeness. Since the kernel of $\mathcal{S}_{\bar{x}} \to \bar{\mathcal{S}}_{\bar{x}}$ is $\sigma^\perp \cap M$, $\langle m, v_\rho \rangle$ does not depend upon the choice of liftings m. Taking a splitting $N \cong N' \oplus N''$ such that $\sigma \cong \sigma' \oplus \{0\} \subset N'_\mathbb{R} \oplus N''_\mathbb{R}$, where σ' is a full-dimensional cone in $N'_\mathbb{R}$, we may (and will) assume that σ is a full-dimensional cone, i.e. that $\sigma^\vee \cap M$ is sharp. Let $\iota : \sigma^\vee \cap M \hookrightarrow \sigma^\vee$ be the natural inclusion and $r : \sigma^\vee \cap M \hookrightarrow F$ the minimal free resolution. Then there exists a unique injective homomorphism $i : F \to \sigma^\vee$ such that $i \circ r = \iota$. By this embedding, we can regard F as a submonoid of $\sigma^\vee$. Since $r : \sigma^\vee \cap M \hookrightarrow F \subset \sigma^\vee$ is the minimal free resolution and $\sigma^\vee$ is a simplicial cone, for each ray $\rho \in \sigma^\vee(1)$ the initial point of $\rho \cap F$ is an irreducible element of F. Since $\mathrm{rk}\, F^{\mathrm{gp}} = \mathrm{rk}\, (\sigma^\vee \cap M)^{\mathrm{gp}} = \dim \sigma^\vee = \dim \sigma$, each irreducible element of F lies on one of rays of $\sigma^\vee$. This gives rise to a natural bijective map from the set of irreducible elements of F to $\sigma^\vee(1)$. Since σ and $\sigma^\vee$ are simplicial, we have a natural bijective map $\sigma^\vee(1) \to \sigma(1)$, $\rho \mapsto \rho^\star$, where $\rho^\star$ is the unique ray which does not lie in $\rho^\perp$. Therefore, the composite map from the set of irreducible elements of F to $\sigma(1)$ is a bijective map, and our claim follows. $\square$

Remark 2.5.

(i) The above definition works over arbitrary base schemes.

(ii) If Σ is a non-singular fan, then $\mathscr{X}_{(\Sigma,\Sigma^0_{\mathrm{can}})}$ is the toric variety X_Σ.

2.2 Torus actions

The torus action functor

$$a_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \times \mathrm{Spec}\, k[M] \longrightarrow \mathscr{X}_{(\Sigma,\Sigma^0)}$$

is defined as follows. Let $\xi = (\pi : \mathcal{S} \to \mathcal{O}_X, \alpha : \mathcal{M} \to \mathcal{O}_X, \eta : \mathcal{S} \to \mathcal{M})$ be an object in $\mathscr{X}_{(\Sigma,\Sigma^0)}$. Let $\phi : M \to \mathcal{O}_X$ be a map of monoids from a constant sheaf M on X to $\mathcal{O}_X$, i.e. an X-valued point of $\mathrm{Spec}\, k[M]$. Here $\mathcal{O}_X$ is regarded as a sheaf of monoids under multiplication. We define $a_{(\Sigma,\Sigma^0)}(\xi, \phi)$ to be $(\phi \cdot \pi : \mathcal{S} \to \mathcal{O}_X, \alpha : \mathcal{M} \to \mathcal{O}_X, \phi \cdot \eta : \mathcal{S} \to \mathcal{M})$, where $\phi \cdot \pi(s) := \phi(s) \cdot \pi(s)$ and

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$\phi \cdot \eta(s) := \phi(s) \cdot \eta(s)$. Let $h : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism in $\mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M]$ from (ξ_1, ϕ) to (ξ_2, ϕ), where $\xi_i = (\pi : \mathcal{S} \to \mathcal{O}_X, \alpha : \mathcal{M}_i \to \mathcal{O}_X, \eta_i : \mathcal{S} \to \mathcal{M}_i)$ for $i = 1, 2$ and $\phi : M \to \mathcal{O}_X$ is an X-valued point of $\operatorname{Spec} k[M]$. We define $a_{(\Sigma,\Sigma^0)}(h)$ to be h. We remark that this action commutes with the torus action of $\operatorname{Spec} k[M]$ on X_Σ.

Definition 2.6. For $i = 1, 2$, let (Σ_i in $N_{i,\mathbb{R}}, \Sigma_i^0$) be a stacky fan and $\mathscr{X}_{(\Sigma_i,\Sigma_i^0)}$ the associated toric algebraic stack. Put $M_i = \operatorname{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z})$. Let us denote by $a_{(\Sigma_i,\Sigma_i^0)} : \mathscr{X}_{(\Sigma_i,\Sigma_i^0)} \times \operatorname{Spec} k[M_i] \to \mathscr{X}_{(\Sigma_i,\Sigma_i^0)}$ the torus action. A 1-morphism $f : \mathscr{X}_{(\Sigma_1,\Sigma_1^0)} \to \mathscr{X}_{(\Sigma_2,\Sigma_2^0)}$ is *torus-equivariant* if the restriction f_0 of f to $\operatorname{Spec} k[M_1] \subset \mathscr{X}_{(\Sigma_1,\Sigma_1^0)}$ defines a homomorphism of group k-schemes $f_0 : \operatorname{Spec} k[M_1] \to \operatorname{Spec} k[M_2] \subset \mathscr{X}_{(\Sigma_2,\Sigma_2^0)}$ and the diagram

$$\begin{tikzcd} \mathscr{X}_{(\Sigma_1,\Sigma_1^0)} \times \operatorname{Spec} k[M_1] \arrow[r, "f \times f_0"] \arrow[d, "a_{(\Sigma_1,\Sigma_1^0)}"] & \mathscr{X}_{(\Sigma_2,\Sigma_2^0)} \times \operatorname{Spec} k[M_2] \arrow[d, "a_{(\Sigma_2,\Sigma_2^0)}"] \\ \mathscr{X}_{(\Sigma_1,\Sigma_1^0)} \arrow[r, "f"] & \mathscr{X}_{(\Sigma_2,\Sigma_2^0)} \end{tikzcd}$$

commutes in the 2-categorical sense. Similarly, we define the torus-equivariant (1-)morphisms from a toric algebraic stack (or toric variety) to a toric algebraic stack (or toric variety). We remark that $\pi_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \to X_\Sigma$ is torus-equivariant.

2.3 Algebraicity

Now, let us recall some results which will be needed later.

Theorem 2.7 (see [Iwa07a, Theorem 4.5]). *The stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is a smooth Deligne–Mumford stack of finite type that is separated over k, and the functor $\pi_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \to X_\Sigma$ is a coarse moduli map.*

Remark 2.8. By [Iwa07a], we can define the toric algebraic stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ over $\mathbb{Z}$. The stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is an (not necessarily Deligne–Mumford) Artin stack over $\mathbb{Z}$. In characteristic zero, toric algebraic stacks are always Deligne–Mumford.

Let $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ be a functor (not necessarily torus-equivariant). Then, by the universality of coarse moduli spaces, there exists a unique morphism $f_c : X_\Sigma \to X_\Delta$ such that $f_c \circ \pi_{(\Sigma,\Sigma^0)} = \pi_{(\Delta,\Delta^0)} \circ f$.

Here we shall give another presentation of $\mathscr{X}_{(\Sigma,\Sigma^0)}$, which is more directly represented in terms of logarithmic geometry; it is important for later proofs. Let $(\mathscr{U}, \pi_{\mathscr{U}})$ be the universal pair on X_Σ satisfying (i)–(iv) in Definition 2.3, corresponding to $\operatorname{Id}_{X_\Sigma} \in \operatorname{Hom}(X_\Sigma, X_\Sigma)$. It follows from the construction in [AMRT75] that the log structure associated to $\pi_{\mathscr{U}} : \mathscr{U} \to \mathscr{O}_{X_\Sigma}$ is the canonical log structure $\mathcal{M}_\Sigma$ on X_Σ. By [Iwa07a, Proposition 4.4], the stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is naturally isomorphic to the stack $\mathscr{X}_\Sigma(\Sigma^0)$ over the toric variety X_Σ, which is defined as follows. For any morphism $f : Y \to X_\Sigma$, objects in $\mathscr{X}_\Sigma(\Sigma^0)$ over $f : Y \to X_\Sigma$ are morphisms of fine log schemes $(f, \phi) : (Y, \mathcal{N}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ such that for every geometric point $\bar{y} \to Y$, $\bar{\phi} : f^{-1}\bar{\mathscr{U}}_{\bar{y}} = f^{-1}\bar{\mathcal{M}}_{\Sigma,\bar{y}} \to \bar{\mathcal{N}}_{\bar{y}}$ is a (Σ, Σ^0)-free resolution. (We shall call such a morphism $(Y, \mathcal{N}) \to (X, \mathcal{M}_\Sigma)$ a *Σ^0-FR morphism*.) A morphism $(Y, \mathcal{N})_{/(X_\Sigma,\mathcal{M}_\Sigma)} \to (Y', \mathcal{N}')_{/(X_\Sigma,\mathcal{M}_\Sigma)}$ in $\mathscr{X}_\Sigma(\Sigma^0)$ is a $(X_\Sigma, \mathcal{M}_\Sigma)$-morphism $(\alpha, \phi) : (Y, \mathcal{N}) \to (Y', \mathcal{N}')$ such that $\phi : \alpha^*\mathcal{N}' \to \mathcal{N}$ is an isomorphism.

Remark 2.9. Let $(f, h) : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of log schemes. If $h : f^*\mathcal{N} \to \mathcal{M}$ is an isomorphism, we say that (f, h) is *strict*.

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We refer to $\mathscr{X}_{(\Sigma,\Sigma^0)} \cong \mathscr{X}_\Sigma(\Sigma^0)$ as the toric algebraic stacks (or toric stacks) associated to (Σ, Σ^0).

Let us collect some technical results, Lemmas 2.10, 2.11 and 2.12 (cf. [Iwa07a, Propositions 2.17, 2.18 and 3.5]), which we will apply to the proofs of Theorems 1.2 and 1.3. Let (Σ, Σ^0) be a stacky fan. Assume that Σ is a cone σ such that $\dim \sigma = \operatorname{rk} N$, i.e. it is full-dimensional. Set $P = \sigma^\vee \cap M$ ($M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$). The monoid P is a simplicially toric sharp monoid, and there is a natural isomorphism $X_\Sigma \cong \operatorname{Spec} k[P]$.

LEMMA 2.10. *Let $r : P \to \mathbb{N}^d$ be the minimal free resolution. Let us denote by e_ρ the irreducible element in $\mathbb{N}^d$ which corresponds to a ray ρ in σ, and let $t : \mathbb{N}^d \to \mathbb{N}^d$ be the map defined by $e_\rho \mapsto n_\rho \cdot e_\rho$ where n_ρ is the level of $\sigma^0 = \Sigma^0$ on ρ. Let $(\operatorname{Spec} k[P], \mathcal{M}_P)$ and $(\operatorname{Spec} k[\mathbb{N}^d], \mathcal{M}_{\mathbb{N}^d})$ be toric varieties with canonical log structures, and let $(\pi, \eta) : (\operatorname{Spec} k[\mathbb{N}^d], \mathcal{M}_{\mathbb{N}^d}) \to (\operatorname{Spec} k[P], \mathcal{M}_P)$ be the morphism of fine log schemes induced by $l := t \circ r : P \to \mathbb{N}^d \to \mathbb{N}^d$. Then (π, η) is a Σ^0-FR morphism.*

LEMMA 2.11. *Let $(q, \gamma) : (S, \mathcal{N}) \to (\operatorname{Spec} k[P], \mathcal{M}_P)$ be a morphism of fine log schemes, let $c : P \to \mathcal{M}_P$ be a chart, and let $\bar{s}$ be a geometric point on S. Suppose that there exists a morphism $\xi : \mathbb{N}^d \to \bar{\mathcal{N}}_{\bar{s}}$ such that the composite $\xi \circ l : P \to \bar{\mathcal{N}}_{\bar{s}}$ is equal to $\bar{\gamma}_{\bar{s}} \circ \bar{c}_{\bar{s}} : P \to q^{-1}\bar{\mathcal{M}}_{P,\bar{s}} \to \bar{\mathcal{N}}_{\bar{s}}$ (with notation as in Lemma 2.10). Assume that $\xi : \mathbb{N}^d \to \bar{\mathcal{N}}_{\bar{s}}$ étale locally lifts to a chart. Then there exists an étale neighborhood U of $\bar{s}$ in which we have a chart $\varepsilon : \mathbb{N}^d \to \mathcal{N}$ such that the diagram*

$$\begin{tikzcd} P \arrow[r, "l"] \arrow[d, "c"'] & \mathbb{N}^d \arrow[d, "\varepsilon"] \\ q^*\mathcal{M}_P \arrow[r, "\gamma"] & \mathcal{N} \end{tikzcd}$$

commutes and the composite $\mathbb{N}^d \xrightarrow{\varepsilon} \mathcal{N} \to \bar{\mathcal{N}}_{\bar{s}}$ is equal to ξ.

Let $l : P \to \mathbb{N}^d$ be the homomorphism in Lemma 2.10. Let us denote by $G := ((\mathbb{N}^d)^{\mathrm{gp}}/l^{\mathrm{gp}}(P^{\mathrm{gp}}))^D$ the Cartier dual of the finite group $(\mathbb{N}^d)^{\mathrm{gp}}/l^{\mathrm{gp}}(P^{\mathrm{gp}})$. The finite group scheme G naturally acts on $\operatorname{Spec} k[\mathbb{N}^d]$ as follows. For a k-ring A, an A-valued point $a : (\mathbb{N}^d)^{\mathrm{gp}}/l^{\mathrm{gp}}(P^{\mathrm{gp}}) \to A^*$ of G sends an A-valued point $x : \mathbb{N}^d \to A$ (a map of monoids) of $\operatorname{Spec} k[\mathbb{N}^d]$ to $a \cdot x : \mathbb{N}^d \to A$, $n \mapsto a(n) \cdot x(n)$. Since G is étale over k ($\mathrm{ch}(k) = 0$), the quotient stack $[\operatorname{Spec} k[\mathbb{N}^d]/G]$ is a smooth Deligne–Mumford stack [LM00, Proposition (10.13.1)] whose coarse moduli space is $\operatorname{Spec} k[\mathbb{N}^d]^G = \operatorname{Spec} k[P]$, where $k[\mathbb{N}^d]^G \subset k[\mathbb{N}^d]$ is the subring of functions invariant under the action of G. The quotient $[\operatorname{Spec} k[\mathbb{Z}^d]/G]$ is an open representable substack of $[\operatorname{Spec} k[\mathbb{N}^d]/G]$, which defines a torus embedding.

PROPOSITION 2.12. *There exists an isomorphism $[\operatorname{Spec} k[\mathbb{N}^d]/G] \to \mathscr{X}_\Sigma(\Sigma^0)$ of stacks over $\operatorname{Spec} k[P]$ which sends the torus in $[\operatorname{Spec} k[\mathbb{N}^d]/G]$ onto that of $\mathscr{X}_\Sigma(\Sigma^0)$. Moreover, the natural composite $\operatorname{Spec} k[\mathbb{N}^d] \to [\operatorname{Spec} k[\mathbb{N}^d]/G] \to \mathscr{X}_\Sigma(\Sigma^0)$ corresponds to $(\operatorname{Spec} k[\mathbb{N}^d], \mathcal{M}_{\mathbb{N}^d}) \to (\operatorname{Spec} k[P], \mathcal{M}_P)$.*

2.4 Log structures on toric algebraic stacks

Let $i_{(\Sigma,\Sigma^0)} : T_\Sigma = \operatorname{Spec} k[M] \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ denote the torus embedding. The complement $\mathscr{D}_{(\Sigma,\Sigma^0)} := \mathscr{X}_{(\Sigma,\Sigma^0)} - T_\Sigma$ with reduced closed substack structure is a normal crossing divisor (see [Iwa07a, Theorem 4.17] or Proposition 2.12). The stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$ has the log structure $\mathcal{M}_{(\Sigma,\Sigma^0)}$ arising

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from $\mathscr{D}_{(\Sigma,\Sigma^0)}$ on the étale site $\mathscr{X}_{(\Sigma,\Sigma^0),\mathrm{et}}$. Moreover, we have

$$\mathcal{M}_{(\Sigma,\Sigma^0)} = \mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}} \cap i_{(\Sigma,\Sigma^0)*}\mathscr{O}^*_{T_\Sigma} \subset \mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}},$$

where $\mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}} \cap i_{(\Sigma,\Sigma^0)*}\mathscr{O}^*_{T_\Sigma}$ denotes the subsheaf of $\mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}}$ consisting of regular functions on $\mathscr{X}_{(\Sigma,\Sigma^0)}$ whose restriction to T_Σ is invertible. The coarse moduli map $\pi_{(\Sigma,\Sigma^0)} : \mathscr{X}_{(\Sigma,\Sigma^0)} \to X_\Sigma$ induces a morphism of log stacks, $(\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)}) : (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)}) \to (X_\Sigma, \mathcal{M}_\Sigma)$. Here $h_{(\Sigma,\Sigma^0)} : \pi^*_{(\Sigma,\Sigma^0)}\mathcal{M}_\Sigma \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ arises from the following natural diagram.

$$\begin{tikzcd}
\pi^{-1}_{(\Sigma,\Sigma^0)}\mathcal{M}_\Sigma \arrow[r] \arrow[d] & \mathcal{M}_{(\Sigma,\Sigma^0)} \arrow[d] \\
\pi^{-1}_{(\Sigma,\Sigma^0)}\mathscr{O}_{X_\Sigma} \arrow[r] & \mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}}
\end{tikzcd}$$

Similarly, a functor $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ such that $f(T_\Sigma) \subset T_\Delta \subset \mathscr{X}_{(\Delta,\Delta^0)}$ naturally induces the canonical homomorphism $h_f : f^*\mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ which is induced by $f^{-1}\mathcal{M}_{(\Delta,\Delta^0)} \to f^{-1}\mathscr{O}_{\mathscr{X}_{(\Delta,\Delta^0)}} \to \mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}}$. We shall refer to this homomorphism h_f as the *homomorphism induced by* f.

The log structure $\mathcal{M}_{(\Sigma,\Sigma^0)}$ on $\mathscr{X}_\Sigma(\Sigma^0) = \mathscr{X}_{(\Sigma,\Sigma^0)}$ has the following modular interpretation. Let $f : Y \to \mathscr{X}_\Sigma(\Sigma^0) = \mathscr{X}_{(\Sigma,\Sigma^0)}$ be a morphism from a k-scheme Y, corresponding to a Σ^0-FR morphism $(Y, \mathcal{M}_Y) \to (X_\Sigma, \mathcal{M}_\Sigma)$. We attach the log structure $\mathcal{M}_Y$ to $f : Y \to \mathscr{X}_\Sigma(\Sigma^0)$, and this gives rise to the log structure $\mathcal{M}'_{(\Sigma,\Sigma^0)}$ on $\mathscr{X}_\Sigma(\Sigma^0) = \mathscr{X}_{(\Sigma,\Sigma^0)}$. We claim that $\mathcal{M}_{(\Sigma,\Sigma^0)} = \mathcal{M}'_{(\Sigma,\Sigma^0)}$. To see this, first note the following observation. Let $U \to \mathscr{X}_\Sigma(\Sigma^0)$ be an étale cover by a scheme U and let $\mathrm{pr}_1, \mathrm{pr}_2 : U \times_{\mathscr{X}_\Sigma(\Sigma^0)} U \rightrightarrows U$ be the étale groupoid. A log structure on $\mathscr{X}_\Sigma(\Sigma^0)$ amounts to a descent data $(\mathcal{M}_U, \mathrm{pr}_1^*\mathcal{M}_U \cong \mathrm{pr}_2^*\mathcal{M}_U)$ where $\mathcal{M}_U$ is a fine log structure on U. Given a data $(\mathcal{M}_U, \mathrm{pr}_1^*\mathcal{M}_U \cong \mathrm{pr}_2^*\mathcal{M}_U)$, if $\mathcal{M}_U$ arises from a normal crossing divisor on U, then $\mathcal{M}_U \subset \mathscr{O}_U$ and $\mathrm{pr}_1^*\mathcal{M}_U = \mathrm{pr}_2^*\mathcal{M}_U \subset \mathscr{O}_{U \times_{\mathscr{X}_\Sigma(\Sigma^0)} U}$. By Proposition 2.12, there is an étale cover $f : U \to \mathscr{X}_\Sigma(\Sigma^0)$ such that $f^*\mathcal{M}'_{(\Sigma,\Sigma^0)}$ arises from the divisor $f^{-1}(\mathscr{D}_{(\Sigma,\Sigma^0)})$. Then, from the above observation and the equality $f^*\mathcal{M}_{(\Sigma,\Sigma^0)} = f^*\mathcal{M}'_{(\Sigma,\Sigma^0)} \subset \mathscr{O}_U$, we conclude that $\mathcal{M}'_{(\Sigma,\Sigma^0)}$ is isomorphic to $\mathcal{M}_{(\Sigma,\Sigma^0)}$ up to a unique isomorphism. For generalities concerning log structures on stacks, we refer to [Ols03, § 5].

Remark 2.13. The notion of stacky fans was introduced in [BCS05, § 3]. We should remark that in [BCS05], given a stacky fan (Σ, Σ^0) whose rays in Σ span the vector space $N_\mathbb{R}$, Borisov, Chen and Smith constructed a smooth Deligne–Mumford stack over $\mathbb{C}$ whose coarse moduli space is the toric variety X_Σ, called the toric Deligne–Mumford stack. Their approach is a generalization of the global quotient constructions of toric varieties due to D. Cox. However, it seems quite difficult, using their machinery, to show that the 2-category (or the associated 1-category) of toric Deligne–Mumford stacks in the sense of [BCS05] is equivalent to the category of stacky fans. In § 5, we explain the relationship of our work with [BCS05].

3. The proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. As in § 2, we continue to work over the fixed base field k of characteristic zero. The proof proceeds in several steps.

Lemma 3.1. *Let $\mathscr{X}_{(\Sigma,\Sigma^0)}$ and $\mathscr{X}_{(\Delta,\Delta^0)}$ be toric algebraic stacks arising from stacky fans $(\Sigma$ in $N_{1,\mathbb{R}}, \Sigma^0)$ and $(\Delta$ in $N_{2,\mathbb{R}}, \Delta^0)$, respectively. Let $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ be a functor*

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such that $f(T_\Sigma) \subset T_\Delta \subset \mathscr{X}_{(\Delta,\Delta^0)}$, and let $f_c : X_\Sigma \to X_\Delta$ be the morphism induced by f (see Remark 2.8). Then there exists a natural commutative diagram of log stacks, as follows.

$$\begin{tikzcd}
(\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)}) \arrow[r, "(f,h_f)"] \arrow[d, "V(\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)})"'] & (\mathscr{X}_{(\Delta,\Delta^0)}, \mathcal{M}_{(\Delta,\Delta^0)}) \arrow[d, "(\pi_{(\Delta,\Delta^0)}, h_{(\Delta,\Delta^0)})"] \\
(X_\Sigma, \mathcal{M}_\Sigma) \arrow[r, "(f_c, h_{f_c})"] & (X_\Delta, \mathcal{M}_\Delta)
\end{tikzcd}$$

Proof. We use the same notation as in § 2.4. Note that f_c commutes with torus embeddings. We define $h_{f_c} : f_c^*\mathcal{M}_\Delta \to \mathcal{M}_\Sigma$ to be the homomorphism induced by $f_c^{-1}\mathcal{M}_\Delta \to f_c^{-1}\mathscr{O}_{X_\Delta} \to \mathscr{O}_{X_\Sigma}$. Since h_f, h_{f_c}, $h_{(\Sigma,\Sigma^0)}$ and $h_{(\Delta,\Delta^0)}$ are induced by the homomorphisms of structure sheaves (see § 2.4), $h_{(\Sigma,\Sigma^0)} \circ \pi^*_{(\Sigma,\Sigma^0)} h_{f_c} : (f_c \circ \pi_{(\Sigma,\Sigma^0)})^*\mathcal{M}_\Delta \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ is equal to $h_f \circ f^* h_{(\Delta,\Delta^0)} : (\pi_{(\Delta,\Delta^0)} \circ f)^*\mathcal{M}_\Delta \to \mathcal{M}_{(\Sigma,\Sigma^0)}$. Thus we have the desired diagram. □

Proposition 3.2. *With notation as in Lemma 3.1, if f is torus-equivariant, then the morphism f_c is torus-equivariant. Moreover, the morphism f_c corresponds to the map of fans $L : \Sigma \to \Delta$ such that $L(\Sigma^0) \subset \Delta^0$.*

Proof. Clearly, the restriction of f_c to T_Σ induces a homomorphism of group k-schemes $T_\Sigma \to T_\Delta \subset X_\Delta$. Note that $\mathscr{X}_{(\Sigma,\Sigma^0)} \times T_\Sigma \xrightarrow{f \times (f|_{T_\Sigma})} \mathscr{X}_{(\Delta,\Delta^0)} \times T_\Delta \xrightarrow{a_{(\Delta,\Delta^0)}} \mathscr{X}_{(\Delta,\Delta^0)}$ is isomorphic to $\mathscr{X}_{(\Sigma,\Sigma^0)} \times T_\Sigma \xrightarrow{a_{(\Sigma,\Sigma^0)}} \mathscr{X}_{(\Sigma,\Sigma^0)} \xrightarrow{f} \mathscr{X}_{(\Delta,\Delta^0)}$. Since $X_{(\Sigma,\Sigma^0)} \times T_\Sigma$ (respectively, $X_{(\Delta,\Delta^0)} \times T_\Delta$) is a coarse moduli space for $\mathscr{X}_{(\Sigma,\Sigma^0)} \times T_\Sigma$ (respectively, $\mathscr{X}_{(\Delta,\Delta^0)} \times T_\Delta$), f_c is torus-equivariant. Set $M_i = \mathrm{Hom}_{\mathbb{Z}}(N_i, \mathbb{Z})$ for $i = 1, 2$. Let $L^\vee : M_2 \to M_1$ be the homomorphism of abelian groups that is induced by the homomorphism of group k-schemes $f_c|_{T_\Sigma} : T_\Sigma \to T_\Delta$. The dual map $L : N_1 = \mathrm{Hom}_{\mathbb{Z}}(M_1, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M_2, \mathbb{Z}) = N_2$ yields the map of fans $L_{\mathbb{R}} : \Sigma$ in $N_{1,\mathbb{R}} \to \Delta$ in $N_{2,\mathbb{R}}$, which corresponds to the morphism f_c. To complete the proof of this proposition, it suffices to show that $L(\Sigma^0) \subset \Delta^0$. To do this, we may assume that $(\Sigma, \Sigma^0) = (\sigma, \sigma^0)$ and $(\Delta, \Delta^0) = (\delta, \delta^0)$, where σ and δ are cones. We need the following lemma.

Lemma 3.3. *If σ is a full-dimensional cone, then the (σ, σ^0)-free resolution (see Definition 2.3) of $\sigma^\vee \cap M_1$ is given by*

$$\sigma^\vee \cap M_1 \to \{m \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle m, n\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } n \in \sigma^0\}.$$

Proof. Let $P := \sigma^\vee \cap M_1$. We first give the proof for the case of $\sigma^0 = \sigma^0_{\mathrm{can}}$. Assume that $\sigma^0 = \sigma^0_{\mathrm{can}}$. Let d be the rank of M_1. Here $M_1 = P^{\mathrm{gp}}$. Let S be a submonoid in $\sigma \cap N_1$ which is generated by the first lattice points of rays in σ, that is, $S = \sigma^0_{\mathrm{can}}$ and $S \cong \mathbb{N}^d$. Put $F := \{h \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle h, s\rangle \in \mathbb{Z}_{\geq 0}$ for any $s \in S\}$. It is clear that F is isomorphic to $\mathbb{N}^d$. Since $\sigma^\vee \cap M_1 = \{h \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle h, s\rangle \in \mathbb{Z}_{\geq 0}$ for any $s \in \sigma \cap N_1\}$, we have $P \subset F \subset M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. We will show that the natural injective map $i : P \to F$ is the minimal free resolution (see Definition 2.2). Since the monoid $\sigma \cap N_1$ is a fine sharp monoid, it has only finitely many irreducible elements and is generated by these irreducible elements [Ols03, Lemma 3.9]. Let $\{\zeta_1, \ldots, \zeta_r\}$ (respectively, $\{e_1, \ldots, e_d\}$) be irreducible elements of $\sigma \cap N_1$ (respectively, F). For each irreducible element e_i in F, we put $\langle e_i, \zeta_j\rangle = a_{ij}/b_{ij} \in \mathbb{Q}$ with some $a_{ij} \in \mathbb{Z}_{\geq 0}$ and $b_{ij} \in \mathbb{N}$. Then we have

$$(\Pi_{0 \leq j \leq r} b_{ij}) \cdot e_i(\sigma \cap N_1) \subset \mathbb{Z}_{\geq 0},$$

and thus $i : P \to F$ satisfies property (i) of Definition 2.2. To show our claim, it suffices to prove that $i : P \to F$ satisfies property (ii) of Definition 2.2. Let $j : P \to G$ be an injective

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homomorphism of monoids such that $j(P)$ is close to G and $G \cong \mathbb{N}^d$. The monoid P has the natural injection $l : P \to M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. On the other hand, for any element e in G there exists a positive integer n such that $n \cdot e \in j(P)$. Therefore we have a unique homomorphism $\alpha : G \to M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ which extends $l : P \to M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ to G. We claim that there exists a sequence of inclusions

$$P \subset F \subset \alpha(G) \subset M_1 \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If we put $\alpha(G)^\vee := \{f \in N_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle p, f\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } p \in \alpha(G)\} \cong \mathbb{N}^d$ and $F^\vee := \{f \in N_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle p, f\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } p \in F\} = S \cong \mathbb{N}^d$, then our claim is equivalent to saying that $\alpha(G)^\vee \subset F^\vee$. However, this latter statement is clearly true. Indeed, the sublattice S is the maximal sublattice of $\sigma \cap N_1$ which is free and close to $\sigma \cap N_1$. The sublattice $\alpha(G)^\vee$ is also close to $\sigma \cap N_1$, and thus each irreducible generator of $\alpha(G)^\vee$ lies on a ray of σ.

Finally, we consider the general case. Put $F_{\mathrm{can}} := \{h \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle h, s\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } s \in \sigma^0_{\mathrm{can}}\}$ and $F := \{h \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle h, s\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } s \in \sigma^0\}$. (Note that the notation has changed.) Then we have a natural injective map $F_{\mathrm{can}} \to F$, because $\sigma^0 \subset \sigma^0_{\mathrm{can}}$. Given a ray $\rho \in \sigma(1)$, the corresponding irreducible element of F_{can} (respectively, F) (see Lemma 2.4) is $m_\rho \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ (respectively, $m'_\rho \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q}$) such that $\langle m_\rho, \zeta_\rho\rangle = 1$ (respectively, $\langle m'_\rho, n_\rho\zeta_\rho\rangle = 1$) and $\langle m_\rho, \zeta_\xi\rangle = 0$ (respectively, $\langle m'_\rho, \zeta_\xi\rangle = 0$) for any ray ξ with $\xi \neq \rho$. Here, for each ray α, ζ_α denotes the first lattice point of α and n_α denotes the level of σ^0 on α. The natural injection $F_{\mathrm{can}} \to F$ identifies m_ρ with $n_\rho m'_\rho$. Thus $\sigma^\vee \cap M_1 \to F$ is a (σ, σ^0)-free resolution of $\sigma^\vee \cap M_1$. □

We now continue the proof of Proposition 3.2. We shall assume that $L(\Sigma^0) \nsubseteq \Delta^0$ and show that such an assumption gives rise to a contradiction. First, we show a contradiction for the case where σ and δ are full-dimensional cones. Set $P := \sigma^\vee \cap M_1$ and $Q := \delta^\vee \cap M_2$. Note that since σ and δ are full-dimensional, P and Q are sharp (i.e. unit-free). Let us denote by o (respectively, o') the origin of $\operatorname{Spec} k[P]$ (respectively, of $\operatorname{Spec} k[Q]$), which corresponds to the ideal (P). Then f_c sends o to o'. Consider the composite $\alpha : \operatorname{Spec} \mathscr{O}_{\operatorname{Spec} k[\mathbb{N}^d], \bar{s}} \to \operatorname{Spec} k[\mathbb{N}^d] \to [\operatorname{Spec} k[\mathbb{N}^d]/G] \cong \mathscr{X}_{(\Sigma,\Sigma^0)}$ of natural morphisms (see Proposition 2.12), where s is the origin of $\operatorname{Spec} k[\mathbb{N}^d]$. Then, by Lemma 3.1, there exists the following commutative diagram.

$$\begin{tikzcd}
M_2 \arrow[d, "L^\vee"'] & Q = \alpha^{-1}\pi^{-1}_{(\Sigma,\Sigma^0)} f_c^{-1}\bar{\mathcal{M}}_\Delta \arrow[l] \arrow[r] \arrow[d] & \alpha^{-1} f^{-1}\bar{\mathcal{M}}_{(\Delta,\Delta^0)} \arrow[d] \\
M_1 & P = \alpha^{-1}\pi^{-1}_{(\Sigma,\Sigma^0)}\bar{\mathcal{M}}_\Sigma \arrow[l] \arrow[r] & \alpha^{-1}\bar{\mathcal{M}}_{(\Sigma,\Sigma^0)}
\end{tikzcd}$$

On the other hand, set $F := \{m \in M_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle m, n\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } n \in \sigma^0\}$ and $F' := \{m \in M_2 \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle m, n\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } n \in \delta^0\}$. Then, by the above lemma, the (σ, σ^0)-free resolution $\alpha^{-1}\pi^{-1}_{(\Sigma,\Sigma^0)}\bar{\mathcal{M}}_\Sigma \to \alpha^{-1}\bar{\mathcal{M}}_{(\Sigma,\Sigma^0)}$ can be identified with the natural inclusion $P := \sigma^\vee \cap M_1 \hookrightarrow F$ (the monoid $\alpha^{-1}\bar{\mathcal{M}}_{(\Sigma,\Sigma^0)}$ can be canonically embedded into $M_1 \otimes \mathbb{Q}$). Similarly, $\alpha^{-1}\pi^{-1}_{(\Sigma,\Sigma^0)} f_c^{-1}\bar{\mathcal{M}}_\Delta \to \alpha^{-1} f^{-1}\bar{\mathcal{M}}_{(\Delta,\Delta^0)}$ can be identified with the natural inclusion $Q = \delta^\vee \cap M_2 \hookrightarrow F'$. The homomorphism $\alpha^{-1} f^{-1}\bar{\mathcal{M}}_{(\Delta,\Delta^0)} \to \alpha^{-1}\bar{\mathcal{M}}_{(\Sigma,\Sigma^0)}$ can be naturally embedded into $L^\vee \otimes \mathbb{Q} : M_2 \otimes \mathbb{Q} \to M_1 \otimes \mathbb{Q}$. However, the assumption $L(\Sigma^0) \nsubseteq \Delta^0$ implies that $L^\vee(F') \nsubseteq F$, which gives rise to a contradiction. Next, consider the general case, i.e. where σ and δ are not necessarily full-dimensional. Choose splittings $N_i \cong N'_i \oplus N''_i$ $(i = 1, 2)$, $\sigma \cong \sigma' \oplus \{0\}$ and $\delta \cong \delta' \oplus \{0\}$ such that σ' and δ' are full-dimensional in $N'_{1,\mathbb{R}}$ and $N'_{2,\mathbb{R}}$, respectively. Note that $\mathscr{X}_{(\sigma,\sigma^0)} \cong \mathscr{X}_{(\sigma',\sigma'^0)} \times \operatorname{Spec} k[M''_1]$ and $\mathscr{X}_{(\delta,\delta^0)} \cong \mathscr{X}_{(\delta',\delta'^0)} \times \operatorname{Spec} k[M''_2]$. Consider the sequence of

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torus-equivariant morphisms

$$\mathscr{X}_{(\sigma',\sigma'^0)} \xrightarrow{i} \mathscr{X}_{(\sigma,\sigma^0)} \xrightarrow{f} \mathscr{X}_{(\delta,\delta^0)} \cong \mathscr{X}_{(\delta',\delta'^0)} \times \operatorname{Spec} k[M_2''] \xrightarrow{\mathrm{pr}_1} \mathscr{X}_{(\delta',\delta'^0)},$$

where i is determined by the natural inclusion $N_1' \hookrightarrow N_1' \oplus N_1''$ and pr_1 is the first projection. Notice that i and pr_1 naturally induce isomorphisms $i^*\mathcal{M}_{(\sigma,\sigma^0)} \xrightarrow{\sim} \mathcal{M}_{(\sigma',\sigma'^0)}$ and $\mathrm{pr}_1^*\mathcal{M}_{(\delta',\delta'^0)} \xrightarrow{\sim} \mathcal{M}_{(\delta,\delta^0)}$, respectively. Thus, the general case follows from the full-dimensional case. □

Remark 3.4.

(1) By Proposition 3.2, there exists the natural functor

$$c : \mathfrak{Torst} \to \mathrm{Simtoric}, \quad \mathscr{X}_{(\Sigma,\Sigma^0)} \mapsto X_\Sigma$$

which sends a torus-equivariant morphism $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ to $f_c : X_\Sigma \to X_\Delta$, where f_c is the unique morphism such that $f_c \circ \pi_{(\Sigma,\Sigma^0)} = \pi_{(\Delta,\Delta^0)} \circ f$.

(2) We can define a 2-functor

$$\Phi : \mathfrak{Torst} \to (\text{category of stacky fans}), \quad \mathscr{X}_{(\Sigma,\Sigma^0)} \mapsto (\Sigma, \Sigma^0)$$

as follows. For each torus-equivariant morphism $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$, the restriction of f to the torus $\operatorname{Spec} k[M_1] \subset \mathscr{X}_{(\Sigma,\Sigma^0)}$ induces the homomorphism $\Phi(f) : N_1 \to N_2$ that defines a morphism of stacky fans $\Phi(f) : (\Sigma, \Sigma^0) \to (\Delta, \Delta^0)$ via Proposition 3.2. For each 2-isomorphism morphism $g : f_1 \to f_2$ ($f_1, f_2 : \mathscr{X}_{(\Sigma,\Sigma^0)} \rightrightarrows \mathscr{X}_{(\Delta,\Delta^0)}$ are torus-equivariant morphisms), define $\Phi(g)$ to be $\mathrm{Id}_{\Phi(f_1)}$. (Note that $\Phi(f_1) = \Phi(f_2)$.)

In order to prove Theorem 1.2, we need to establish the following key proposition.

PROPOSITION 3.5. *Let $\xi : (\Sigma \text{ in } N_{1,\mathbb{R}}, \Sigma^0) \to (\Delta \text{ in } N_{2,\mathbb{R}}, \Delta^0)$ be a morphism of stacky fans. Let $(f, h_f) : (X_\Sigma, \mathcal{M}_\Sigma) \to (X_\Delta, \mathcal{M}_\Delta)$ be the morphism of log toric varieties induced by $\xi : \Sigma \to \Delta$. Let S be a k-scheme and let $(\alpha, h) : (S, \mathcal{N}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ be a Σ^0-FR morphism. Then there exist a fine log structure $\mathscr{A}$ on S and morphisms of log structures $a : \alpha^* f^* \mathcal{M}_\Delta \to \mathscr{A}$ and $\theta : \mathscr{A} \to \mathcal{N}$ which make the diagram*

$$\begin{tikzcd} \alpha^* f^* \mathcal{M}_\Delta \arrow[r, "a"] \arrow[d, "\alpha^* h_f"'] & \mathscr{A} \arrow[d, "\theta"] \\ \alpha^* \mathcal{M}_\Sigma \arrow[r, "h"] & \mathcal{N} \end{tikzcd}$$

commutative and make $(f \circ \alpha, a) : (S, \mathscr{A}) \to (X_\Delta, \mathcal{M}_\Delta)$ a Δ^0-FR morphism. The triple $(\mathscr{A}, a, \theta)$ is unique in the following sense: if there is another such triple $(\mathscr{A}', a', \theta')$, then there exists a unique isomorphism $\eta : \mathscr{A} \to \mathscr{A}'$ which makes the diagram (♣)

$$\begin{tikzcd} \alpha^* f^* \mathcal{M}_\Delta \arrow[rr, "a"] \arrow[dr, "a'"] \arrow[dd, "\alpha^* h_f"] & & \mathscr{A} \arrow[dl, "\eta"'] \arrow[dd, "\theta"] \\ & \mathscr{A}' \arrow[dr, "\theta'"] & \\ \alpha^* \mathcal{M}_\Sigma \arrow[rr, "h"] & & \mathcal{N} \end{tikzcd}$$

commutative.

We prove this first for the case of $S = \operatorname{Spec} R$ where R is a strictly Henselian local k-ring. Note that if $\mathcal{M}$ is a fine saturated log structure on $S = \operatorname{Spec} R$, then by [Ols03, Proposition 2.1] there exists a chart $\bar{\mathcal{M}}(S) \to \mathcal{M}$ on S. The chart induces an isomorphism $\bar{\mathcal{M}}(S) \oplus R^* \xrightarrow{\sim} \mathcal{M}(S)$. If a chart of $\mathcal{M}$ is fixed, we usually abuse notation and write $\bar{\mathcal{M}}(S) \oplus R^*$ for the log

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structure $\mathcal{M}$. Similarly, we write simply $\bar{\mathcal{M}}$ for $\bar{\mathcal{M}}(S)$. Before proving the proposition, we show the following lemma.

Lemma 3.6. *Set* $P := \alpha^{-1}\bar{\mathcal{M}}_\Sigma(S)$, $Q := \alpha^{-1}f^{-1}\bar{\mathcal{M}}_\Delta(S)$ *and* $\iota := \alpha^{-1}\bar{h}_f : Q \to P$. *Let* $\gamma : Q \xrightarrow{r} \mathbb{N}^r \xrightarrow{t} \mathbb{N}^r$ *be the composite map where* r *is the minimal free resolution and* t *is a map defined as follows: for the irreducible element* $e_\rho \in \mathbb{N}^r$ *that corresponds to a ray* ρ *in* Δ, t *sends* e_ρ *to* $n_\rho \cdot e_\rho$, *where* n_ρ *is the level of* Δ^0 *on* ρ. *Then, there exists a unique homomorphism of monoids* $l : \mathbb{N}^r \to \bar{\mathcal{N}}$ *such that the diagram*

$$\begin{tikzcd} Q \arrow[r, "\gamma"] \arrow[d, "\iota"'] & \mathbb{N}^r \arrow[d, "l"] \\ P \arrow[r, "\bar{h}"] & \bar{\mathcal{N}} \end{tikzcd}$$

commutes.

Proof. The uniqueness of l follows from the fact that $\bar{\mathcal{N}}$ is free and that $\gamma(Q)$ is close to $\mathbb{N}^r$. To show the existence of l, we may assume that Σ and Δ are cones. Set $\sigma = \Sigma$, $\sigma^0 = \Sigma^0$, $\delta = \Delta$ and $\delta^0 = \Delta^0$. Choose splittings $N_i \cong N_i' \oplus N_i''$ $(i = 1, 2)$, $\sigma \cong \sigma' \oplus \{0\}$ and $\delta \cong \delta' \oplus \{0\}$ such that $\dim N_1' = \dim \sigma'$ and $\dim N_2' = \dim \delta'$. Then the projections $N_i' \oplus N_i'' \to N_i'$ $(i = 1, 2)$ yield the commutative diagram of log schemes

$$\begin{tikzcd} (X_\sigma, \mathcal{M}_\sigma) \arrow[r, "(f, h_f)"] \arrow[d] & (X_\delta, \mathcal{M}_\delta) \arrow[d] \\ (X_{\sigma'}, \mathcal{M}_{\sigma'}) \arrow[r, "(g, h_g)"] & (X_{\delta'}, \mathcal{M}_{\delta'}) \end{tikzcd}$$

where the vertical arrows are strict morphisms induced by projections and (g, h_g) is the morphism induced by $\xi|_{N_1'} : N_1' \to N_2'$. Thus we may assume that σ and δ are full-dimensional in $N_{1,\mathbb{R}}$ and $N_{2,\mathbb{R}}$, respectively. Then $P' := \sigma^\vee \cap M_1$ and $Q' := \delta^\vee \cap M_2$ are sharp (i.e. unit-free). Let $R_1 : P' \to \mathbb{N}^m$ (respectively, $R_2 : Q' \to \mathbb{N}^n$) be the (σ, σ^0)-free (respectively, (δ, δ^0)-free) resolution. Let $f^\# : Q' \to P'$ be the homomorphism arising from f. Then, by the assumption that $\xi(\Sigma^0) \subset \Delta^0$, there exists a homomorphism $w : \mathbb{N}^n \to \mathbb{N}^m$ such that $w \circ R_2 = R_1 \circ f^\#$. Taking Lemma 2.10 into account, we see that our claim follows. □

Proof of Proposition 3.5. By Lemma 3.6, there exists a unique homomorphism $l : \mathbb{N}^r \to \bar{\mathcal{N}}$. By [Ols03, Proposition 2.1], there exists a chart $c_{\bar{\mathcal{N}}} : \bar{\mathcal{N}} \to \mathcal{N}$. Then the maps $c_{\bar{\mathcal{N}}} \circ l$ and $c_{\bar{\mathcal{N}}} \circ l \circ \gamma$ induce the log structures $\mathbb{N}^r \oplus R^*$ and $Q \oplus R^*$, respectively. On the other hand, by [Ols03, Proposition 2.1], there exists a chart $c_Q' : Q \to \alpha^* f^* \mathcal{M}_\Delta$. Let $i : R^* \hookrightarrow \mathcal{N}$ be the canonical immersion. Let us denote by j the composite map

$$Q \xrightarrow{c_Q'} \alpha^* f^* \mathcal{M}_\Delta \xrightarrow{\alpha^* h_f} \mathcal{M}_\Sigma \xrightarrow{h} \mathcal{N} \xrightarrow{(c_{\bar{\mathcal{N}}} \oplus i)^{-1}} \bar{\mathcal{N}} \oplus R^* \xrightarrow{\mathrm{pr}_2} R^*$$

and define a chart $c_Q : Q \to \alpha^* f^* \mathcal{M}_\Delta$ by $Q \ni q \mapsto c_Q'(q) \cdot j(q)^{-1} \in \alpha^* f^* \mathcal{M}_\Delta$ (here $j(q)$ is viewed as an element in $\alpha^* f^* \mathcal{M}_\Delta$). We then have the commutative diagram

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$$\begin{tikzcd}
 & & & \bar{\mathcal{N}} \oplus R^* \arrow[rr, equal] \arrow[dd, "c_{\bar{\mathcal{N}}}\oplus i"'] & & \bar{\mathcal{N}} \oplus R^* \arrow[dd] \\
Q \oplus R^* \arrow[rr, equal] \arrow[dd, "c_Q\oplus i_\Delta"'] & & Q \oplus R^* \arrow[rr, "\gamma\oplus\mathrm{Id}_{R^*}"'] \arrow[dd] & & \mathbb{N}^r \oplus R^* \arrow[ur, "l\oplus\mathrm{Id}_{R^*}"] \arrow[dd] & \\
 & \alpha^*\mathcal{M}_\Sigma \arrow[rr, "h"] & & \mathcal{N} \arrow[rr] & & \bar{\mathcal{N}} \\
\alpha^* f^*\mathcal{M}_\Delta \arrow[ur, "\alpha^* h_f"] \arrow[rr] & & Q \arrow[rr, "\gamma"] & & \mathbb{N}^r \arrow[ur, "l"] &
\end{tikzcd}$$

where $i_\Delta : R^* \hookrightarrow \alpha^* f^* \mathcal{M}_\Delta$ is the canonical immersion. The map $c_Q \oplus i_\Delta$ is an isomorphism, and $(\gamma \oplus \mathrm{Id}_{R^*}) \circ (c_Q \oplus i_\Delta)^{-1} : \alpha^* f^* \mathcal{M}_\Delta \to \mathbb{N}^r \oplus R^*$ makes $f \circ \alpha : S \to X_\Delta$ a Δ^0-FR morphism. Thus we have the desired diagram. Next, we prove uniqueness. As above, let us fix the chart $c_{\bar{\mathcal{N}}} : \bar{\mathcal{N}} \to \mathcal{N}$. Suppose that for $\lambda = 1, 2$, there exist a Δ^0-FR morphism $a_\lambda : \alpha^* f^* \mathcal{M}_\Delta \to \mathscr{A}_\lambda$ and a morphism of log structures $\theta_\lambda : \mathscr{A}_\lambda \to \mathcal{N}$ such that $h \circ \alpha^* h_f = \theta_\lambda \circ a_\lambda$. By the same argument as for the proof of existence, we obtain a chart $c_Q : Q \to \alpha^* f^* \mathcal{M}_\Delta$ such that the image of the composite

$$Q \xrightarrow{c_Q} \alpha^* f^* \mathcal{M}_\Delta \xrightarrow{\alpha^* h_f} \alpha^* \mathcal{M}_\Sigma \xrightarrow{h} \mathcal{N} \xrightarrow{(c_{\bar{\mathcal{N}}} \oplus i)^{-1}} \bar{\mathcal{N}} \oplus R^* \xrightarrow{\mathrm{pr}_2} R^*$$

is trivial. By Lemma 2.11, c_Q can be extended to a chart $c_\lambda : \mathbb{N}^r = \bar{\mathscr{A}}_\lambda \to \mathscr{A}_\lambda$ such that $c_\lambda \circ \gamma = a_\lambda \circ c_Q$ for $\lambda = 1, 2$. Then the composite $\mathrm{pr}_2 \circ (c_{\bar{\mathcal{N}}} \oplus i)^{-1} \circ \theta_\lambda \circ c_\lambda : \mathbb{N}^r \to R^*$ induces a character $\mathrm{ch}_\lambda : (\mathbb{N}^r)^{\mathrm{gp}} / \gamma(Q)^{\mathrm{gp}} \to R^*$. Note that if i_λ denotes the canonical immersion $R^* \hookrightarrow \mathscr{A}_\lambda$ for $\lambda = 1, 2$, then $c_\lambda \oplus i_\lambda : \mathbb{N}^r \oplus R^* \to \mathscr{A}_\lambda$ is an isomorphism. Let us denote by $\eta : \mathscr{A}_1 \to \mathscr{A}_2$ an isomorphism of log structures, defined to be the composite $\mathscr{A}_1 \xrightarrow{(c_1 \oplus i_1)^{-1}} \mathbb{N}^r \oplus R^* \xrightarrow{\omega} \mathbb{N}^r \oplus R^* \xrightarrow{(c_2 \oplus i_2)} \mathscr{A}_2$, where $\omega : \mathbb{N}^r \oplus R^* \ni (n, u) \mapsto (n, \mathrm{ch}_1(n) \cdot \mathrm{ch}_2(n)^{-1} \cdot u) \in \mathbb{N}^r \oplus R^*$. Then it is easy to see that $\eta : \mathscr{A}_1 \to \mathscr{A}_2$ is a unique isomorphism which makes all diagrams commutative. $\square$

Next, consider the case of a general k-scheme S. First, we shall prove the uniqueness part. If there exists a diagram like (♣) but without η, then, by virtue of the spectrum of a strictly Henselian local k-ring, for every geometric point $\bar{s}$ on S there exists a unique homomorphism $\eta_{\bar{s}} : \mathscr{A}_{\bar{s}} \to \mathscr{A}'_{\bar{s}}$ which makes the diagram (♣) over $\operatorname{Spec} \mathscr{O}_{S,\bar{s}}$ commutative. Thus, to prove uniqueness, it suffices to show that $\eta_{\bar{s}}$ can be extended to an isomorphism on some étale neighborhood of $\bar{s}$ that makes the diagram (♣) commutative. To this end, put $Q = \alpha^{-1} f^{-1} \bar{\mathcal{M}}_{\Delta,\bar{s}}$ and choose a chart $c_Q : Q \to \alpha^* f^* \mathcal{M}_\Delta$ on some étale neighborhood U of $\bar{s}$ (the existence of such a chart follows from [Ols03, Proposition 2.1]). We view the monoid Q as a submonoid of $\mathbb{N}^r \cong \bar{\mathscr{A}}_{\bar{s}} \cong \bar{\mathscr{A}}'_{\bar{s}}$. Taking Lemma 2.11 and the existence of $\eta_{\bar{s}}$ into account, after shrinking U, if necessary, we can choose charts $\mathbb{N}^r \xrightarrow{c} \mathscr{A}$ and $\mathbb{N}^r \xrightarrow{c'} \mathscr{A}'$ on U such that the restriction of c (respectively, c') to Q is equal to the composite $a \circ c_Q$ (respectively, $a' \circ c_Q$) and $\theta \circ c = \theta' \circ c'$, with notation as in (♣). Then charts c and c' induce an isomorphism $\mathscr{A} \to \mathscr{A}'$ on U, which makes the diagram (♣) commutative.

We now prove the existence of a triple $(\mathscr{A}, a, \theta)$. For a geometric point $\bar{s}$ on S, consider the localization $S' = \operatorname{Spec} \mathscr{O}_{S,\bar{s}}$. Set $Q = \alpha^{-1} f^{-1} \bar{\mathcal{M}}_{\Delta,\bar{s}}$. Then, owing to the spectrum of a strictly

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Henselian local k-ring, there exist a log structure $\mathscr{A}$ on S', a Δ^0-FR morphism $a: \alpha^* f^* \mathcal{M}_{\Delta,\bar{s}} \to \mathscr{A}$, and a diagram

$$\begin{tikzcd} Q \arrow[r, "\bar{a}"] \arrow[d, "c"] & F \arrow[r, "\bar{\theta}"] \arrow[d, "c'"] & \bar{\mathcal{N}}_{\bar{s}} \arrow[d, "c''"] \\ \alpha^* f^* \mathcal{M}_{\Delta,\bar{s}} \arrow[r, "a"] & \mathscr{A} \arrow[r, "\theta"] & \mathcal{N}_{\bar{s}} \end{tikzcd}$$

of fine log structures on S' such that $\theta \circ a = h \circ h_f$. Here c, c' and c'' are charts, and $F = \bar{\mathscr{A}}$. To prove the existence on S, it suffices, by the uniqueness, to show only that we can extend the above diagram to some étale neighborhood of $\bar{s}$. In some étale neighborhood U of $\bar{s}$, there exist charts $\tilde{c}: Q \to \alpha^* f^* \mathcal{M}_\Delta$ and $\tilde{c''}: \bar{\mathcal{N}}_{\bar{s}} \to \mathcal{N}$ extending c and c'', respectively, such that the diagram

$$\begin{tikzcd} Q \arrow[r, "\bar{a}"] \arrow[d, "\tilde{c}"] & F \arrow[r, "\bar{\theta}"] & \bar{\mathcal{N}}_{\bar{s}} \arrow[d, "\tilde{c}''"] \\ \alpha^* f^* \mathcal{M}_\Delta \arrow[rr, "h \circ h_f"] & & \mathcal{N} \end{tikzcd}$$

commutes. Let $\tilde{\mathscr{A}}$ be the fine log structure associated to the prelog structure $F \xrightarrow{\bar{\theta}} \bar{\mathcal{N}}_{\bar{s}} \xrightarrow{\tilde{c}''} \mathcal{N} \to \mathscr{O}_U$. Then there exists a sequence of morphisms of log structures

$$\alpha^* f^* \mathcal{M}_\Delta \xrightarrow{\tilde{a}} \tilde{\mathscr{A}} \xrightarrow{\tilde{\theta}} \mathcal{N},$$

with $\tilde{a} \circ \tilde{\theta} = h \circ h_f$, which is an extension of $\alpha^* f^* \mathcal{M}_{\Delta,\bar{s}} \xrightarrow{a} \mathscr{A} \xrightarrow{\theta} \mathcal{N}_{\bar{s}}$. Since $\alpha^* f^* \mathcal{M}_\Delta \to \tilde{\mathscr{A}}$ has a chart by $Q \to F$, Lemma 2.10 allows us to conclude that $(f \circ \alpha, \tilde{a}: \alpha^* f^* \mathcal{M}_\Delta \to \tilde{\mathscr{A}}): (S, \tilde{\mathscr{A}}) \to (X_\Delta, \mathcal{M}_\Delta)$ is a Δ^0-FR morphism. This completes the proof of Proposition 3.5. □

Proof of Theorem 1.2. Let $\mathcal{H}om(\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathscr{X}_{\Delta,\Delta^0)})$ be the category of torus-equivariant 1-morphisms from $\mathscr{X}_{(\Sigma,\Sigma^0)}$ to $\mathscr{X}_{(\Delta,\Delta^0)}$ (whose morphisms are 2-isomorphisms). Let $\mathrm{Hom}((\Sigma, \Sigma^0), (\Delta, \Delta^0))$ be the discrete category arising from the set of morphisms from (Σ, Σ^0) to (Δ, Δ^0). We have to show that the natural map

$$\Phi: \mathcal{H}om(\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathscr{X}_{(\Delta,\Delta^0)}) \to \mathrm{Hom}((\Sigma, \Sigma^0), (\Delta, \Delta^0))$$

is an equivalence. This amounts to proving the following statement: if $F: (\Sigma, \Sigma^0) \to (\Delta, \Delta^0)$ is a map of stacky fans and $(f, h_f): (X_\Sigma, \mathcal{M}_\Sigma) \to (X_\Delta, \mathcal{M}_\Delta)$ denotes the torus-equivariant morphism (with the natural morphism of the log structures) of toric varieties induced by $F: \Sigma \to \Delta$, then there exists a torus-equivariant 1-morphism

$$(\tilde{f}, h_{\tilde{f}}): (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)}) \to (\mathscr{X}_{(\Delta,\Delta^0)}, \mathcal{M}_{(\Delta,\Delta^0)})$$

(with the natural morphism of log structures) such that $(f, h_f) \circ (\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)}) = (\pi_{(\Delta,\Delta^0)}, h_{(\Delta,\Delta^0)}) \circ (\tilde{f}, h_{\tilde{f}})$, and it is unique up to a unique isomorphism. By Proposition 3.5, for each object $(\alpha, h): (S, \mathcal{N}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ in $\mathscr{X}_\Sigma(\Sigma^0)$ we can choose a pair $((f \circ \alpha, g), \xi_{(\alpha,h)})$, where $(f \circ \alpha, g): (S, \mathcal{M}) \to (X_\Delta, \mathcal{M}_\Delta)$ is an object in $\mathscr{X}_\Delta(\Delta^0)$, i.e. a Δ^0-FR morphism, and $\xi_{(\alpha,h)}: \mathcal{M} \to \mathcal{N}$ is a homomorphism of log structures such that the diagram

$$\begin{tikzcd} \alpha^* f^* \mathcal{M}_\Delta \arrow[r, "g"] \arrow[d, "\alpha^* h_f"'] & \mathcal{M} \arrow[d, "\xi_{(\alpha,h)}"] \\ \alpha^* \mathcal{M}_\Sigma \arrow[r, "h"] & \mathcal{N} \end{tikzcd}$$

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commutes. For each object $(\alpha, h) : (S, \mathcal{N}) \to (X_\Sigma, \mathcal{M}_\Sigma) \in \mathrm{Ob}(\mathscr{X}_\Sigma(\Sigma^0))$, we choose such a pair $(\tilde f((\alpha, h)) : (S, \mathcal{M}) \to (X_\Delta, \mathcal{M}_\Delta) \in \mathrm{Ob}(\mathscr{X}_\Delta(\Delta^0)),\ \xi_{(\alpha,h)} : \mathcal{M} \to \mathcal{N})$. By the axiom of choice, there exists a function $\mathrm{Ob}(\mathscr{X}_\Sigma(\Sigma^0)) \to \mathrm{Ob}(\mathscr{X}_\Delta(\Delta^0)),\ (\alpha, h) \mapsto \tilde f((\alpha, h))$. Let $(\alpha_i, h_i) : (S_i, \mathcal{N}_i) \to (X_\Sigma, \mathcal{M}_\Sigma)$ be a Σ^0-FR morphism for $i = 1, 2$. For each morphism $(q, e) : (S_1, \mathcal{N}_1) \to (S_2, \mathcal{N}_2)$ in $\mathscr{X}_\Sigma(\Sigma^0)$, define $\tilde f((q, e)) : \tilde f((S_1, \mathcal{N}_1)) := (S_1, \mathcal{M}_1)_{/(X_\Delta, \mathcal{M}_\Delta)} \to \tilde f((S_2, \mathcal{N}_2)) := (S_2, \mathcal{M}_2)_{/(X_\Delta, \mathcal{M}_\Delta)}$ to be $(q, \tilde f(e)) : (S_1, \mathcal{M}_1) \to (S_2, \mathcal{M}_2)$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{M}_1 \arrow[rr, "\xi_{(\alpha_1,h_1)}"] & & \mathcal{N}_1 \\
q^*\mathcal{M}_2 \arrow[ur, "\tilde f(e)"] \arrow[rr, "q^*\xi_{(\alpha_2,h_2)}"] & & q^*\mathcal{N}_2 \arrow[ur, "e"] & \\
 & \alpha_1^* f^* \mathcal{M}_\Delta \arrow[ul, "q^*g_2"] \arrow[uu, "g_1"'] \arrow[rr] & & \alpha_1^* \mathcal{M}_\Sigma \arrow[ul, "q^*h_2"'] \arrow[uu, "h_1"']
\end{tikzcd}$$

commutes. Here, the $\xi_{(\alpha_i,h_i)}$ are the homomorphisms chosen as above, and the uniqueness of such a homomorphism $\tilde f(e)$ follows from Proposition 3.5. This yields a functor $\tilde f : \mathscr{X}_\Sigma(\Sigma^0) \to \mathscr{X}_\Delta(\Delta^0)$ with a homomorphism of log structures $\xi : \tilde f^* \mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ (determined by the collection $\{\xi_{(\alpha,h)}\}$); it gives rise to a lifted morphism $(\tilde f, \xi) : (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)}) \to (\mathscr{X}_{(\Delta,\Delta^0)}, \mathcal{M}_{(\Delta,\Delta^0)})$. Because $\mathcal{M}_{(\Sigma,\Sigma^0)} \subset \mathscr{O}_{\mathscr{X}_{(\Sigma,\Sigma^0)}}$ on $\mathscr{X}_{(\Sigma,\Sigma^0),\mathrm{et}}$, we have that $\xi : \tilde f^* \mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ is the homomorphism $h_{\tilde f}$ induced by $\tilde f$ (see § 2.4). In addition, if $\tilde f' : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ is another lifting of f and $\zeta : \tilde f \to \tilde f'$ is a 2-isomorphism, then ζ induces an isomorphism $\sigma : \tilde f^* \mathcal{M}_{(\Delta,\Delta^0)} \to \tilde f'^* \mathcal{M}_{(\Delta,\Delta^0)}$ such that $h_{\tilde f} = \xi = h_{\tilde f'} \circ \sigma$, where $h_{\tilde f'} : \tilde f'^* \mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ is the homomorphism induced by $\tilde f'$. Therefore, for another lifting $\tilde f' : \mathscr{X}_{(\Sigma,\Sigma^0)} = \mathscr{X}_\Sigma(\Sigma^0) \to \mathscr{X}_\Delta(\Delta^0) = \mathscr{X}_{(\Delta,\Delta^0)}$ of f, the existence and uniqueness of the 2-isomorphism $\zeta : \tilde f \to \tilde f'$ follows from Proposition 3.5. Indeed, let $(\alpha, h) : (S, \mathcal{N}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ be a Σ^0-FR morphism and set $\tilde f((\alpha, h)) = \{(f \circ \alpha, g) : (S, \mathcal{M}) \to (X_\Delta, \mathcal{M}_\Delta)\}$ and $\tilde f'((\alpha, h)) = \{(f \circ \alpha, g') : (S, \mathcal{M}') \to (X_\Delta, \mathcal{M}_\Delta)\}$ (these are Δ^0-FR morphisms). Then we have the commutative diagram

$$\begin{tikzcd}
\alpha^* f^* \mathcal{M}_\Delta \arrow[rr, "g"] \arrow[dr, "g'"] \arrow[dd, "\alpha^* h_f"] & & \mathcal{M} \arrow[dd, "h_{\tilde f}"] \\
 & \mathcal{M}' \arrow[dr, "h_{\tilde f'}"] & \\
\alpha^* \mathcal{M}_\Sigma \arrow[rr, "h"] & & \mathcal{N}
\end{tikzcd}$$

where $h_{\tilde f} : \mathcal{M} \to \mathcal{N}$ (respectively, $h_{\tilde f'} : \mathcal{M}' \to \mathcal{N}$) denotes the homomorphism induced by $h_{\tilde f} : \tilde f^* \mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$ (respectively, $h_{\tilde f'} : \tilde f'^* \mathcal{M}_{(\Delta,\Delta^0)} \to \mathcal{M}_{(\Sigma,\Sigma^0)}$) (there is some abuse of notation here). By Proposition 3.5, there exists a unique isomorphism of log structures $\sigma_{(\alpha,h)} : \mathcal{M} \to \mathcal{M}'$ which fits into the above diagram. Then we can easily see that the collection $\{\sigma_{(\alpha,h)}\}_{(\alpha,h) \in \mathscr{X}_\Sigma(\Sigma^0)}$ defines a 2-isomorphism $\tilde f \to \tilde f'$. Conversely, by the above observation and Proposition 3.5, a 2-isomorphism $\tilde f \to \tilde f'$ must be $\{\sigma_{(\alpha,h)}\}_{(\alpha,h) \in \mathscr{X}_\Sigma(\Sigma^0)}$ and thus the uniqueness follows. Finally, we show that $\tilde f$ is torus-equivariant (see Definition 2.6). This follows from the uniqueness (up to a unique isomorphism) of a lifting $\mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M_1] \to \mathscr{X}_{(\Delta,\Delta^0)}$ of the torus-equivariant morphism $X_\Sigma \times \operatorname{Spec} k[M_1] \xrightarrow{a} X_\Sigma \xrightarrow{f} X_\Delta$. Here a is the torus action, and by a lifting of $f \circ a$ we mean a functor which commutes with $f \circ a$ via coarse moduli maps. Thus, the proof of Theorem 1.2 is complete. $\square$

Theorem 1.2 and its proof imply the following results.

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Corollary 3.7. *Let $f : X_\Sigma \to X_\Delta$ be a torus-equivariant morphism of simplicial toric varieties. Then a (not necessarily torus-equivariant) functor $\tilde{f} : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ such that $\pi_{(\Delta,\Delta^0)} \circ \tilde{f} = f \circ \pi_{(\Sigma,\Sigma^0)}$ is unique up to a unique isomorphism (if it exists).*

Proof. This follows immediately from the proof of Theorem 1.2. □

Corollary 3.8. *Let $f : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ be a (not necessarily torus-equivariant) functor. Then f is torus-equivariant if and only if the induced morphism $f_c : X_\Sigma \to X_\Delta$ of toric varieties is torus-equivariant.*

Proof. The 'only if' part follows from Proposition 3.2, and the proof of Theorem 1.2 gives the 'if' part. □

Corollary 3.9. *Let Σ and Δ be simplicial fans and let (Δ, Δ^0) be a stacky fan that is an extension of Δ. Let $F : \Sigma \to \Delta$ be a homomorphism of fans and let $f : X_\Sigma \to X_\Delta$ be the associated morphism of toric varieties. Then there exist a stacky fan (Σ, Σ^0) that is an extension of Σ and a torus-equivariant morphism $\tilde{f} : \mathscr{X}_{(\Sigma,\Sigma^0)} \to \mathscr{X}_{(\Delta,\Delta^0)}$ such that $\pi_{(\Delta,\Delta^0)} \circ \tilde{f} = f \circ \pi_{(\Sigma,\Sigma^0)}$. Moreover, if we fix such a stacky fan (Σ, Σ^0), then $\tilde{f}$ is unique up to a unique isomorphism.*

Proof. Theorem 1.2 immediately implies our assertion because we can choose a free-net Σ^0 such that $F(\Sigma^0) \subset \Delta^0$. □

Corollary 3.10. *Let (Σ, Σ^0) be a stacky fan in $N_\mathbb{R}$ and let $\mathscr{X}_{(\Sigma,\Sigma^0)}$ be the associated toric algebraic stack. Then there exists a smooth surjective torus-equivariant morphism*

$$p : X_\Delta \longrightarrow \mathscr{X}_{(\Sigma,\Sigma^0)},$$

where X_Δ is a quasi-affine smooth toric variety. Furthermore, X_Δ can be explicitly constructed.

Proof. Without loss of generality, we may suppose that rays of Σ span the vector space $N_\mathbb{R}$. Set $\tilde{N} = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} \cdot e_\rho$. Define a homomorphism of abelian groups $\eta : \tilde{N} \longrightarrow N$ by $e_\rho \mapsto P_\rho$, where P_ρ is the generator of Σ^0 on ρ (see Definition 2.1). Let Δ be a fan in $\tilde{N}_\mathbb{R}$ that consists of cones γ such that γ is a face of the cone $\bigoplus_{\rho\in\Sigma(1)} \mathbb{R}_{\geq 0} \cdot e_\rho$ and $\eta_\mathbb{R}(\gamma)$ lies in Δ. If Δ^0_{can} denotes the canonical free-net (see Definition 2.1), then η induces the morphism of stacky fans $\eta : (\Delta, \Delta^0_{\mathrm{can}}) \to (\Sigma, \Sigma^0)$. Note that $\mathscr{X}_{(\Delta,\Delta^0_{\mathrm{can}})}$ is the quasi-affine smooth toric variety X_Δ. Let $p : X_\Delta \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ be the torus-equivariant morphism induced by η (see Theorem 1.2). Since the composite $q := \pi_{(\Sigma,\Sigma^0)} \circ p : X_\Delta \to \mathscr{X}_{(\Sigma,\Sigma^0)} \to X_\Sigma$ is surjective and $\pi_{(\Sigma,\Sigma^0)}$ is the coarse moduli map, it follows from [LM00, Proposition 5.4(ii)] that p is surjective. It remains to show that p is smooth; this is an application of Kato's notion of *log smoothness*. From the construction of η and Lemmas 2.10 and 3.3, we can easily see that the induced morphism $(q, h_q) : (X_\Delta, \mathcal{M}_\Delta) \to (X_\Sigma, \mathcal{M}_\Sigma)$ is a Σ^0-FR morphism. Moreover, by [Kat88, Theorem 3.5], (q, h_q) is log smooth ($\mathrm{ch}(k) = 0$). Using the modular interpretation of $\mathscr{X}_\Sigma(\Sigma^0)$ (see § 2), there exists a 1-morphism $p' : X_\Delta \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ which corresponds to (q, h_q). Theorem 1.2 and Corollary 3.7 then imply that p' coincides with p, and thus the morphism $(p, h_p) : (X_\Delta, \mathcal{M}_\Delta) \to (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)})$ is a strict morphism. Here h_p is the homomorphism induced by p. The following lemma implies that p is smooth. □

Lemma 3.11. *Let $(X, \mathcal{M})$ be a log scheme and $(f, h) : (X, \mathcal{M}) \to (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)})$ a strict 1-morphism. If the composite*

$$(\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)}) \circ (f, h) : (X, \mathcal{M}) \to (X_\Sigma, \mathcal{M}_\Sigma)$$

is formally log smooth, then $f : X \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ is formally smooth.

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Proof. It suffices to show the lifting property as given in [Ols03, Definition 4.5]. Let $i : T_0 \to T$ be a closed immersion of schemes defined by a square zero ideal. Let $a_0 : T_0 \to X$ and $b : T \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ be a pair of 1-morphisms such that $f \circ a_0 \cong b \circ i$. We have to show that there exists a 1-morphism $a : T \to X$ such that $a \circ i = a_0$ and $f \circ a \cong b$. There exists the commutative diagram

$$\begin{tikzcd}
(T_0, a_0^*\mathcal{M}) \arrow[r, "a_0"] \arrow[d, "i"] & (X, \mathcal{M}) \arrow[d, "{(f,h)}"] \\
(T, b^*\mathcal{M}_{(\Sigma,\Sigma^0)}) \arrow[r, "b"] \arrow[dr] & (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)}) \arrow[d, "{(\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)})}"] \\
 & (X_\Sigma, \mathcal{M}_\Sigma)
\end{tikzcd}$$

where i, b and a_0 denote induced strict morphisms (we abuse notation here). Since $(\pi_{(\Sigma,\Sigma^0)}, h_{(\Sigma,\Sigma^0)}) \circ (f, h)$ is formally log smooth, there exists a morphism

$$(a, v) : (T, b^*\mathcal{M}_{(\Sigma,\Sigma^0)}) \to (X, \mathcal{M})$$

such that $a_0 = a \circ i$. Thus it suffices to prove that $b \cong f \circ a$, which is equivalent to showing that (a, v) is a strict morphism, since $\mathscr{X}_\Sigma(\Sigma^0) = \mathscr{X}_{(\Sigma,\Sigma^0)}$ is the moduli stack of Δ^0-FR morphisms into $(X_\Sigma, \mathcal{M}_\Sigma)$. To see this, we need to show that for any geometric point $\bar{t} \to T$, $\bar{v}_{\bar{t}} : (a^{-1}\bar{\mathcal{M}})_{\bar{t}} \to (b^*\mathcal{M}_{(\Sigma,\Sigma^0)}/\mathscr{O}_T^*)_{\bar{t}}$ is an isomorphism: let $\iota : P \to \mathbb{N}^r$ be an injective homomorphism of monoids such that $\iota(P)$ is close to $\mathbb{N}^r$, and let $e : \mathbb{N}^r \to \mathbb{N}^r$ be an endomorphism such that $\iota = e \circ \iota$; then e is an isomorphism and the lemma is proved. $\square$

4. A geometric characterization theorem

The aim of this section is to give proofs of Theorems 1.3 and 1.1. In this section, except in Lemma 4.1, we work over an algebraically closed base field k of characteristic zero.

Lemma 4.1. *Let $\mathcal{S}$ be a normal Deligne–Mumford stack that is locally of finite type and separated over a locally noetherian scheme. Let $p : \mathcal{S} \to S$ be a coarse moduli map. Then S is normal.*

Proof. Our assertion is étale local on S, hence we may assume that S is the spectrum of a strictly Henselian local ring. Set $S = \operatorname{Spec} O$. In this situation, by [AV02, Lemma 2.2.3] there exist a normal strictly Henselian local ring R, a finite group G and an action $m : \operatorname{Spec} R \times G \to \operatorname{Spec} R$ such that the quotient stack $[\operatorname{Spec} R/G]$ is isomorphic to $\mathcal{S}$ and $R^G = O$ (here R^G is the invariant ring). Let $n : \operatorname{Spec} A \to \operatorname{Spec} O$ be the normalization of $\operatorname{Spec} O$ in the function field $\mathbb{Q}(O)$. Let us denote by $q : \operatorname{Spec} R \to \operatorname{Spec} O$ (respectively, $\mathrm{pr}_1 : \operatorname{Spec} R \times G \to \operatorname{Spec} R$) the composite $\operatorname{Spec} R \to \mathcal{S} \to S = \operatorname{Spec} O$ (respectively, the natural projection). Then, by universality of the normalization, there exists a unique morphism $\tilde{q} : \operatorname{Spec} R \to \operatorname{Spec} A$ such that $n \circ \tilde{q} = q$. Note that $\tilde{q} \circ \mathrm{pr}_1$ (respectively, $\tilde{q} \circ m$) is the unique lifting of $q \circ \mathrm{pr}_1$ (respectively, $q \circ m$). Since $q \circ \mathrm{pr}_1 = q \circ m$, we have $\tilde{q} \circ \mathrm{pr}_1 = \tilde{q} \circ m$. This implies that $A \subset O = R^G$, and thus we conclude that S is normal. $\square$

Proposition 4.2. *Let $(\mathcal{X}, \iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}, a : \mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X})$ be a toric triple over k. Then the complement $\mathscr{D} := \mathcal{X} - \mathbb{G}_m^d$ with reduced closed substack structure is a divisor with normal crossings, and the coarse moduli space X is a simplicial toric variety over k.*

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Proof. First, we shall prove that X is a toric variety over k. Observe that the coarse moduli scheme X is a normal variety over k, i.e. it is normal, of finite type and separated over k. Indeed, according to the Keel–Mori theorem, X is locally of finite type and separated over k. Since $\mathcal{X}$ is of finite type over k and the underlying continuous morphism $|\mathcal{X}| \to |X|$ (cf. [LM00, Définition (5.2)]) of the coarse moduli map is a homeomorphism, X is of finite type over k by [LM00, Corollaire (5.6.3)]. Since $\mathcal{X}$ is smooth over k, by Lemma 4.1 X is normal. Since $\iota : \mathbb{G}_m^d \hookrightarrow \mathcal{X}$, the coarse moduli space X contains $\mathbb{G}_m^d$ as a dense open subset. The torus action $\mathcal{X} \times \mathbb{G}_m^d \to \mathcal{X}$ gives rise to a morphism of coarse moduli spaces $a_0 : X \times \mathbb{G}_m^d \to X$, because $X \times \mathbb{G}_m^d$ is a coarse moduli space for $\mathcal{X} \times \mathbb{G}_m^d$. Moreover, by the universality of coarse moduli spaces, it is an action of $\mathbb{G}_m^d$ on X. Therefore X is a toric variety over k.

Next, we prove that the complement $\mathscr{D}$ is a divisor with normal crossings. Set $\mathbb{G}_m^d = \operatorname{Spec} k[M]$ (with $M = \mathbb{Z}^d$) and $X_\Sigma = X$, where Σ is a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$ ($N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$). Let $\bar{x} \to X_\Sigma$ be a geometric point on X_Σ and put $\mathscr{O} := \mathscr{O}_{X_\Sigma, \bar{x}}$ (the étale stalk). Consider the pull-back $\mathcal{X}_{\mathscr{O}} := \mathcal{X} \times_{X_\Sigma} \mathscr{O} \to \operatorname{Spec} \mathscr{O}$ by $\operatorname{Spec} \mathscr{O}_{X_\Sigma, \bar{x}} \to X_\Sigma$. Clearly, our assertion is an étale local issue on X_Σ and thus it suffices to show that $\mathscr{D}$ defines a divisor with normal crossings on $\mathcal{X}_{\mathscr{O}}$. By [AV02, Lemma 2.2.3], there exist a strictly Henselian local k-ring R and a finite group Γ acting on $\operatorname{Spec} R$ such that $\mathcal{X}_{\mathscr{O}} \cong [\operatorname{Spec} R/\Gamma]$. We have a sequence of morphisms

$$\operatorname{Spec} R \xrightarrow{p} [\operatorname{Spec} R/\Gamma] \xrightarrow{\pi} \operatorname{Spec} \mathscr{O}.$$

The composite $q := \pi \circ p$ is a finite surjective morphism. If U denotes the open subscheme of $\operatorname{Spec} \mathscr{O}$ which is induced by the torus embedding $\mathbb{G}_m^d \subset X_\Sigma$, then the restriction $q^{-1}(U) \to U$ is a finite étale surjective morphism. Let us denote by $\mathcal{M}_{\mathscr{O}}$ the pull-back of the canonical log structure $\mathcal{M}_\Sigma$ on X_Σ to $\operatorname{Spec} \mathscr{O}$. Then, by virtue of the log Nagata–Zariski purity theorem [Moc99, Theorem 3.3] (see also [Hos06, Remark 1.10]), the complement $\operatorname{Spec} R - q^{-1}(U)$ (or, equivalently, $\mathscr{D}$) defines a log structure on $\operatorname{Spec} R$ (we denote this log structure by $\mathcal{M}_R$), and the finite étale surjective morphism $q^{-1}(U) \to U$ extends to a Kummer log étale surjective morphism

$$(q, h) : (\operatorname{Spec} R, \mathcal{M}_R) \to (\operatorname{Spec} \mathscr{O}, \mathcal{M}_{\mathscr{O}}).$$

Let $\hat{\mathscr{O}}$ be the completion of $\mathscr{O}$ along its maximal ideal. Let us denote by

$$(\hat{q}, \hat{h}) : (T, \mathcal{M}_R|_T) \to (\operatorname{Spec} \hat{\mathscr{O}}, \mathcal{M}_{\hat{\mathscr{O}}} := \mathcal{M}_{\mathscr{O}}|_{\hat{\mathscr{O}}})$$

the pull-back of (q, h) by $\operatorname{Spec} \hat{\mathscr{O}} \to \operatorname{Spec} \mathscr{O}$. Then, by [Kat94, Theorem 3.2], the log scheme $(\operatorname{Spec} \hat{\mathscr{O}}, \mathcal{M}_{\mathscr{O}}|_{\hat{\mathscr{O}}})$ is isomorphic to

$$\operatorname{Spec}(P \to k(\bar{x})[[P]][[\mathbb{N}^l]]),$$

where $k(\bar{x})$ is the residue field of $\bar{x} \to X_\Sigma$, $P := \bar{\mathcal{M}}_{\hat{\mathscr{O}}, \bar{x}} \to k(\bar{x})[[P]][[\mathbb{N}^l]]$ ($p \mapsto p$), and l is a non-negative integer. (Strictly speaking, [Kat94] treats only the case of Zariski log structures, but the same proof applies to étale log structures.) By taking a connected component of T if necessary, we may assume that T is connected. Note that, since the connected scheme T is finite over $\operatorname{Spec} \hat{\mathscr{O}}$, T is the spectrum of a strictly Henselian local k-ring. Let Q be the stalk of $\bar{\mathcal{M}}_R|_T$ at a geometric point $\bar{t} \to T$ lying over the closed point t of T. Then, by [Hos06, Proposition A.4], the Kummer log étale cover $(\hat{q}, \hat{h})$ has the form

$$\operatorname{Spec}(Q \to \mathbb{Z}[Q] \otimes_{\mathbb{Z}[P]} k(\bar{x})[[P]][[\mathbb{N}^l]]) \longrightarrow \operatorname{Spec}(P \to k(\bar{x})[[P]][[\mathbb{N}^l]]),$$

defined by $P = \bar{\mathcal{M}}_{\hat{\mathscr{O}}, \bar{x}} \to (\bar{\mathcal{M}}_R|_T)_{\bar{t}} = Q$, $\mathrm{Id}_{\mathbb{N}^l} : \mathbb{N}^l \to \mathbb{N}^l$ and the natural map $Q \to \mathbb{Z}[Q] \otimes_{\mathbb{Z}[P]} k(\bar{x})[[P]][[\mathbb{N}^l]]$. (Note that $\mathbb{Z}[Q] \otimes_{\mathbb{Z}[P]} k(\bar{x})[[P]] \cong k(\bar{x})[[Q]]$ because $Q \to P$ is Kummer.) Since T

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is regular, Q is free, i.e. $Q \cong \mathbb{N}^r$ for some non-negative integer r. This implies that $\mathscr{D}$ is a divisor with normal crossings.

Finally, we shall show that the toric variety X_Σ is simplicial. To this end, let us assume that Σ is not simplicial and show that this assumption gives rise to a contradiction. By the assumption, there exists a geometric point $\alpha : \bar{x} \to X_\Sigma$ such that the number of irreducible components of the complement $D := X_\Sigma - \mathbb{G}_m^d$ on which the point $\bar{x}$ lies is greater than the rank $\mathrm{rk}\,\bar{\mathcal{M}}^{\mathrm{gp}}_{\Sigma,\bar{x}}$ of $\bar{\mathcal{M}}^{\mathrm{gp}}_{\Sigma,\bar{x}}$. Let r be the number of irreducible components of D on which the point $\bar{x}$ lies. Put $P := \bar{\mathcal{M}}_{\Sigma,\bar{x}}$. By the same argument as above, there exist a strictly Henselian local k-ring R and a sequence of Kummer log étale covers

$$(\operatorname{Spec} R, p^*\mathcal{M}_{\mathscr{D}}) \xrightarrow{p} (\mathcal{X} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}, \mathcal{M}_{\mathscr{D}}) \to (\operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}, \mathcal{M}_\Sigma|_{\operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}}),$$

where $\mathcal{M}_{\mathscr{D}}$ is the log structure induced by $\mathscr{D}$ and the left morphism is a strict morphism. Moreover, the pull-back of the composite $\operatorname{Spec} R \to \operatorname{Spec} \mathscr{O}_{\Sigma,\bar{x}}$ (which is a finite morphism) by the completion $\operatorname{Spec} \hat{\mathscr{O}}_{\Sigma,\bar{x}} \to \operatorname{Spec} \mathscr{O}_{\Sigma,\bar{x}}$ along the maximal ideal is of the form

$$\operatorname{Spec} k(\bar{x})[[\mathbb{N}^r]][[\mathbb{N}^l]] \to \operatorname{Spec} k(\bar{x})[[P]][[\mathbb{N}^l]],$$

because $\pi^{-1}(D)_{\mathrm{red}} = \mathscr{D}$ and $\mathscr{D}$ is a normal crossing divisor on the smooth stack. Here $\pi : \mathcal{X} \to X_\Sigma$ is the coarse moduli map, and for each irreducible component C of D, $\pi^{-1}(C)_{\mathrm{red}}$ is an irreducible component of $\mathscr{D}$ because the underlying continuous map $|\pi| : |\mathcal{X}| \to |X|$ (cf. [LM00, Définition (5.2)]) is a homeomorphism. However,

$$\dim \operatorname{Spec} k(\bar{x})[[\mathbb{N}^r]][[\mathbb{N}^l]] > \dim \operatorname{Spec} k(\bar{x})[[P]][[\mathbb{N}^l]] = \mathrm{rk}\, P^{\mathrm{gp}} + l,$$

which is a contradiction. □

Proof of Theorem 1.3. We shall construct a morphism from $\mathcal{X}$ to some toric algebraic stack and show that it is an isomorphism with the desired properties.

Step 1. We first construct a morphism from $\mathcal{X}$ to some toric algebraic stack. Set $\mathbb{G}_m^d = \operatorname{Spec} k[M]$ (with $M = \mathbb{Z}^d$) and $N = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. By Proposition 4.2, we can put $X = X_\Sigma$, where Σ is a simplicial fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$; let us denote by $\pi : \mathcal{X} \to X_\Sigma$ the coarse moduli map. By [Iwa07a, Theorem 3.3(2)] and [Iwa07a, Proposition 4.4], there exists a morphism $\phi : \mathcal{X} \to \mathscr{X}_{(\Sigma,\Sigma^0_{\mathrm{can}})}$ such that $\pi \cong \pi_{(\Sigma,\Sigma^0_{\mathrm{can}})} \circ \phi$. For a ray $\rho \in \Sigma(1)$, we denote by $V(\rho)$ the corresponding irreducible component of $D = X_\Sigma - \operatorname{Spec} k[M]$, where $\operatorname{Spec} k[M] \subset X_\Sigma$ is the torus embedding, i.e. the torus-invariant divisor corresponding to ρ. Then $\mathscr{V}(\rho) := \pi^{-1}_{(\Sigma,\Sigma^0_{\mathrm{can}})}(V(\rho))_{\mathrm{red}}$ (respectively, $\mathscr{W}(\rho) := \pi^{-1}(V(\rho))_{\mathrm{red}}$) is an irreducible component of the normal crossing divisor $\mathscr{X}_{(\Sigma,\Sigma^0_{\mathrm{can}})} - \operatorname{Spec} k[M]$ (respectively, $\mathscr{D} = \mathcal{X} - \mathbb{G}_m^d$). Since $\pi_{(\Sigma,\Sigma^0_{\mathrm{can}})}$ and π are coarse moduli maps, we have $\phi^{-1}(\mathscr{V}(\rho))_{\mathrm{red}} = \mathscr{W}(\rho)$. For each ray $\rho \in \Sigma(1)$, let $n_\rho \in \mathbb{N}$ be the natural number such that

$$\phi^{-1}(\mathscr{V}(\rho)) = n_\rho \cdot \mathscr{W}(\rho).$$

Let (Σ, Σ^0) be the stacky fan whose level on each ray ρ is n_ρ. If $\mathcal{M}_{\mathscr{D}}$ denotes the log structure associated to $\mathscr{D}$, the morphism of log stacks $(\pi, h_\pi) : (\mathcal{X}, \mathcal{M}_{\mathscr{D}}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ (see § 2.4) is a Σ^0-FR morphism since $\mathscr{D}$ is a normal crossing divisor and $(\pi_{(\Sigma,\Sigma^0_{\mathrm{can}})}, h_{(\Sigma,\Sigma^0_{\mathrm{can}})}) : (\mathscr{X}_{(\Sigma,\Sigma^0_{\mathrm{can}})}, \mathcal{M}_{(\Sigma,\Sigma^0_{\mathrm{can}})}) \to (X_\Sigma, \mathcal{M}_\Sigma)$ is a Σ^0_{can}-FR morphism. Then there exists a strict morphism of log stacks

$$\Phi : (\mathcal{X}, \mathcal{M}_{\mathscr{D}}) \longrightarrow (\mathscr{X}_{(\Sigma,\Sigma^0)}, \mathcal{M}_{(\Sigma,\Sigma^0)})$$

over $(X_\Sigma, \mathcal{M}_\Sigma)$, which is associated to the Σ^0-FR morphism (π, h_π). By the construction of Φ, the restriction of Φ to $\mathbb{G}_m^d \subset \mathcal{X}$ induces an isomorphism $\mathbb{G}_m^d \xrightarrow{\sim} \operatorname{Spec} k[M] \subset \mathscr{X}_{(\Sigma,\Sigma^0)}$ of group k-schemes.

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We will prove that Φ is an isomorphism in steps 2 and 3.

Step 2. Observe that, in order to show that Φ is an isomorphism, it suffices to prove that for each closed point $\bar{x} := \operatorname{Spec} k \to X_\Sigma$, the pull-back $\Phi_{\bar{x}} : \mathcal{X} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}} \to \mathscr{X}_{(\Sigma,\Sigma^0)} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}$ by $\operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}} \to X_\Sigma$ is an isomorphism (here $\mathscr{O}_{X_\Sigma,\bar{x}}$ is the étale stalk). Indeed, assume that $\Phi_{\bar{x}}$ is an isomorphism for every closed point $\bar{x} = \operatorname{Spec} k \to X_\Sigma$. Then, by [Con07, Theorem 2.2.5], Φ is representable. Moreover, Φ is finite; this can be seen as follows. Note that $\mathcal{X}$ is separated over k, thus Φ is separated. In addition, Φ is clearly of finite type. Since π and $\pi_{(\Sigma,\Sigma^0)}$ are coarse moduli maps (and, in particular, proper), it follows from [Ols06, Proposition 2.7] that Φ is a proper and quasi-finite surjective morphism, i.e. a finite surjective morphism (see [LM00, Corollary A.2.1]). Whether or not Φ is an isomorphism is an étale local issue on X_Σ; thus, by [DG61, ch. IV 8.8.2.4], we can conclude that Φ is an isomorphism because Φ is a finite representable morphism. Therefore, we need to prove that $\Phi_{\bar{x}} : \mathcal{X} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}} \to \mathscr{X}_{(\Sigma,\Sigma^0)} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}$ is an isomorphism for each closed point $\bar{x} = \operatorname{Spec} k \to X_\Sigma$. For simplicity, put $\mathscr{O} := \mathscr{O}_{X_\Sigma,\bar{x}}$, $\mathcal{X}' := \mathcal{X} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}$ and $\mathscr{X}'_{(\Sigma,\Sigma^0)} := \mathscr{X}_{(\Sigma,\Sigma^0)} \times_{X_\Sigma} \operatorname{Spec} \mathscr{O}_{X_\Sigma,\bar{x}}$. Set $\alpha : \operatorname{Spec} \mathscr{O} \to X_\Sigma$. Write $\mathcal{M}$, $\mathcal{M}'_{\mathscr{D}}$ and $\mathcal{M}'_{(\Sigma,\Sigma^0)}$ for the log structures $\alpha^*\mathcal{M}_\Sigma$, $(\alpha \times_{X_\Sigma} \mathcal{X})^*\mathcal{M}_{\mathscr{D}}$ and $(\alpha \times_{X_\Sigma} \mathscr{X}_{(\Sigma,\Sigma^0)})^*\mathcal{M}_{(\Sigma,\Sigma^0)}$ on $\operatorname{Spec} \mathscr{O}$, $\mathcal{X}'$ and $\mathscr{X}'_{(\Sigma,\Sigma^0)}$, respectively. Clearly, we may assume that Σ is a simplicial cone σ and that $X_\Sigma = \operatorname{Spec} k[P] \times_k \mathbb{G}_m^l$, where $P = (\sigma^\vee \cap M)/(\text{invertible elements})$ and l is a non-negative integer. In addition, by replacing σ with a face if necessary, we can suppose that the closed point $\bar{x}$ lies on the torus orbit of the point $(o, 1) \in \operatorname{Spec} k[P] \times_k \mathbb{G}_m^l$. Here $o \in \operatorname{Spec} k[P]$ is the origin and $1 \in \mathbb{G}_m^l$ is the unit point. Thus we may assume that $\bar{x} = (o, 1)$.

Step 3. Here we prove that $\Phi_{\bar{x}}$ is an isomorphism. To this end, we first give an explicit representation of $(\mathcal{X}', \mathcal{M}'_{\mathscr{D}})$ as a form of quotient stack. By [AV02, Lemma 2.2.3] and [Ols06, Theorem 2.12], there exist a d-dimensional strictly Henselian regular local k-ring R (here $d := \dim X_\Sigma$), a finite group Γ acting on R which is isomorphic to the stabilizer group of any geometric point on $\mathcal{X}$ lying over $\bar{x}$, and an isomorphism

$$\mathcal{X}' \cong [\operatorname{Spec} R/\Gamma]$$

over $\operatorname{Spec} \mathscr{O}$. Furthermore, the action of Γ on the closed point of $\operatorname{Spec} R$ is trivial, and the invariant ring R^Γ is the image of $\mathscr{O} \hookrightarrow R$. Note that if $\operatorname{Aut}_{\operatorname{Spec} \mathscr{O}}(\operatorname{Spec} R)$ denotes the group of automorphisms of $\operatorname{Spec} R$ over $\operatorname{Spec} \mathscr{O}$, the natural homomorphism of groups $\Gamma \to \operatorname{Aut}_{\operatorname{Spec} \mathscr{O}}(\operatorname{Spec} R)$ is injective because $\mathcal{X}'$ is generically representable. Let us denote by $p : \operatorname{Spec} R \to [\operatorname{Spec} R/\Gamma]$ the natural projection and put $\mathcal{M}'_{\mathscr{D},R} := p^*\mathcal{M}'_{\mathscr{D}}$. Consider the composite $\operatorname{Spec} R \to [\operatorname{Spec} R/\Gamma] \to \operatorname{Spec} \mathscr{O}$. This composite induces the morphism of log schemes

$$(f, h) : (\operatorname{Spec} R, \mathcal{M}'_{\mathscr{D},R}) \to (\operatorname{Spec} \mathscr{O}, \mathcal{M}),$$

whose underlying morphism $\operatorname{Spec} R \to \operatorname{Spec} \mathscr{O}$ is finite and surjective. Let W be the open subscheme $\alpha^{-1}(\mathbb{G}_m^d) \subset \operatorname{Spec} \mathscr{O}$ ($\mathbb{G}_m^d \subset X_\Sigma$). Then the restriction $f^{-1}(W) \to W$ is a finite étale surjective morphism. By virtue of the log Nagata–Zariski purity theorems ([Moc99, Theorem 3.3] and [Hos06, Remark 1.10]), (f, h) is a Kummer log étale cover ($\operatorname{ch}(k) = 0$). Now put $\mathscr{O} = k\{P, t_1, \ldots, t_l\} \subset k[[P]][[t_1, \ldots, t_l]]$, where $k\{P, t_1, \ldots, t_l\}$ is the (strict) Henselization of the Zariski stalk of the origin of $\operatorname{Spec} k[P, t_1, \ldots, t_l]$. Consider the homomorphism $P = \bar{\mathcal{M}}_{\bar{s}} \to F := \bar{\mathcal{M}}'_{\mathscr{D},R,\bar{t}}$, where $\bar{s} \to \operatorname{Spec} \mathscr{O}$ and $\bar{t} \to \operatorname{Spec} R$ are geometric points lying over the closed points of $\operatorname{Spec} \mathscr{O}$ and $\operatorname{Spec} R$, respectively. Then, by [Hos06, Proposition A.4], (f, h) is of the form

$$\operatorname{Spec}(F \to k[F] \otimes_{k[P]} k\{P, t_1, \ldots, t_l\}) \longrightarrow \operatorname{Spec}(P \to k\{P, t_1, \ldots, t_l\}),$$

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where the underlying morphism and homomorphism of log structures are naturally induced by $P \to F$ and $t_i \mapsto t_i$. Here $F \to k[F] \otimes_{k[P]} k\{P, t_1, \ldots, t_l\}$ and $P \to k\{P, t_1, \ldots, t_l\}$ are the natural homomorphisms. As observed in [Sti02, Proposition 3.1.9], the group $\mathrm{Aut}_{(\mathrm{Spec}\,\mathscr{O},\mathcal{M})}((\mathrm{Spec}\,R, \mathcal{M}'_{\mathscr{D},R}))$ of automorphisms of $(\mathrm{Spec}\,R, \mathcal{M}'_{\mathscr{D},R})$ over $(\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$ is naturally isomorphic to $G := \mathrm{Hom}_{\mathrm{group}}(F^{\mathrm{gp}}/P^{\mathrm{gp}}, k^*)$. Here an element $g \in G$ acts on $k[F] \otimes_{k[P]} k\{P, t_1, \ldots, t_l\}$ by $f \mapsto g(f) \cdot f$ for any $f \in F$. The natural forgetting homomorphism $\mathrm{Aut}_{(\mathrm{Spec}\,\mathscr{O},\mathcal{M})}((\mathrm{Spec}\,R, \mathcal{M}'_{\mathscr{D},R})) \to \mathrm{Aut}_{\mathrm{Spec}\,\mathscr{O}}(\mathrm{Spec}\,R)$ is an isomorphism. The injectivity is clear from the action of G, and the surjectivity follows from the fact that $\mathcal{M}'_{\mathscr{D},R} = \{s \in \mathscr{O}_{\mathrm{Spec}\,R} |\ \ s \text{ is invertible on } f^{-1}(W)\}$ and that $\mathcal{M} = \{s \in \mathscr{O}_{\mathrm{Spec}\,\mathscr{O}} |\ \ s \text{ is invertible on } W\}$. Furthermore, since the category of Kummer log étale coverings is a Galois category (see, e.g., [Hos06, Theorem A.1]), the injective morphism $\Gamma \to G = \mathrm{Aut}_{\mathrm{Spec}\,\mathscr{O}}(\mathrm{Spec}\,R) = \mathrm{Aut}_{(\mathrm{Spec}\,\mathscr{O},\mathcal{M})}((\mathrm{Spec}\,R, \mathcal{M}'_{\mathscr{D},R}))$ is surjective and hence bijective. Indeed, if it were not surjective, then the Kummer log étale cover (or its underlying morphism) $(\mathrm{Spec}\,R, \mathcal{M}'_{\mathscr{D},R})/\Gamma \to (\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$ would not be an isomorphism and thus we would obtain a contradiction to $R^\Gamma = \mathscr{O}$. Since any local ring that is finite over a Henselian local ring is also Henselian, we have $k[F] \otimes_{k[P]} k\{P, t_1, \ldots, t_l\} \cong k\{F, t_1 \ldots, t_l\}$, where $k\{F, t_1, \ldots, t_l\}$ is the (strict) Henselization of the Zariski stalk of the origin of $\mathrm{Spec}\,k[F, t_1, \ldots, t_l]$. Hence there exists an isomorphism of log stacks

$$(\mathcal{X}', \mathcal{M}'_{\mathscr{D}}) \cong ([\mathrm{Spec}\,k\{F, t_1, \ldots, t_l\}/G], \mathcal{M}_F)$$

over $(\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$, where $\mathcal{M}_F$ is the log structure on $[\mathrm{Spec}\,k\{F, t_1, \ldots, t_l\}/G]$ induced by the natural chart $F \to k\{F, t_1, \ldots, t_l\}$. In particular, the morphism

$$([\mathrm{Spec}\,k\{F, t_1, \ldots, t_l\}/G], \mathcal{M}_F) \to (\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$$

is isomorphic to $(\mathcal{X}', \mathcal{M}'_{\mathscr{D}})$ as a Σ^0-FR morphism over $(\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$. By using this form, we will prove next that $\Phi_{\bar{x}}$ is an isomorphism. Note that the morphism $\Phi_{\bar{x}} : \mathcal{X}' \to \mathscr{X}'_{(\Sigma,\Sigma^0)}$ over $\mathrm{Spec}\,\mathscr{O}$ is the morphism associated to the Σ^0-FR morphism $(\mathcal{X}', \mathcal{M}'_{\mathscr{D}}) \to (\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$. Thus, what we have to show is that $[\mathrm{Spec}\,k\{F, t_1, \ldots, t_l\}/G]_{/\mathrm{Spec}\,\mathscr{O}}$ is the stack whose objects over $S \to \mathrm{Spec}\,\mathscr{O}$ are Σ^0-FR morphisms $(S, \mathcal{N}) \to (\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$ and whose morphisms are strict $(\mathrm{Spec}\,\mathscr{O}, \mathcal{M})$-morphisms between them (see § 2.3). By Proposition 2.12, the stack $([\mathrm{Spec}\,k[F]/G] \times_k \mathbb{G}_m^l)_{/\mathrm{Spec}\,k[P]\times_k\mathbb{G}_m^l}$ represents the stack whose objects over $S \to \mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l$ are Σ^0-FR morphisms $(S, \mathcal{N}) \to (\mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l, \mathcal{N}_P)$ and whose morphisms are strict $(\mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l, \mathcal{N}_P)$-morphisms between them. Here we abuse notation and write $\mathcal{N}_P$ for the log structure associated to the natural map $P \to k[P] \otimes_k \Gamma(\mathbb{G}_m^l, \mathscr{O}_{\mathbb{G}_m^l})$ (i.e. the canonical log structure on $X_\Sigma = \mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l$); G acts on $\mathrm{Spec}\,k[F]$ in the same way as above. Consider the cartesian diagram

$$\begin{tikzcd}
{[\mathrm{Spec}\,k\{F, t_1, \ldots, t_l\}/G]} \arrow[r] \arrow[d] & {[\mathrm{Spec}\,k[F]/G] \times_k \mathbb{G}_m^l} \arrow[d] \\
\mathrm{Spec}\,k\{P, t_1, \ldots, t_l\} \arrow[r] & \mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l
\end{tikzcd}$$

where the lower horizontal arrow is $\alpha : \mathrm{Spec}\,\mathscr{O} \to \mathrm{Spec}\,k[P] \times_k \mathbb{G}_m^l$. This diagram implies our assertion, so we conclude that $\Phi_{\bar{x}}$ is an isomorphism.

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Step 4. In this final step, we show that the diagram

$$\begin{tikzcd} \mathcal{X} \times \mathbb{G}_m^d \arrow[r, "m"] \arrow[d, "\Phi\times\Phi_0"] & \mathcal{X} \arrow[d, "\Phi"] \\ \mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M] \arrow[r, "a_{(\Sigma,\Sigma^0)}"] & \mathscr{X}_{(\Sigma,\Sigma^0)} \end{tikzcd}$$

commutes. Let $\Psi : \mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M] \to \mathcal{X} \times \mathbb{G}_m^d$ be a functor such that $(\Phi \times \Phi_0) \circ \Psi \cong \mathrm{Id}$ and $\Psi \circ (\Phi \times \Phi_0) \cong \mathrm{Id}$. Notice that both $\Phi \circ m \circ \Psi$ and $a_{(\Sigma,\Sigma^0)}$ are liftings of the torus action $X_\Sigma \times \operatorname{Spec} k[M] \to X_\Sigma$. We then have $\Phi \circ m \circ \Psi \cong a_{(\Sigma,\Sigma^0)}$, because a lifting (as a functor) $\mathscr{X}_{(\Sigma,\Sigma^0)} \times \operatorname{Spec} k[M] \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ of the torus action $X_\Sigma \times \operatorname{Spec} k[M] \to X_\Sigma$ is unique up to a unique isomorphism (see Corollary 3.7). Thus $\Phi \circ m \cong a_{(\Sigma,\Sigma^0)} \circ (\Phi \times \Phi_0)$, and this completes the proof of Theorem 1.3. $\square$

Proof of Theorem 1.1. This follows from Theorems 1.2 and 1.3. $\square$

Let $\mathscr{X}$ be an algebraic stack. For a point $a : \operatorname{Spec} K \to \mathscr{X}$ with an algebraically closed field K, the stabilizer group scheme is defined to be $\mathrm{pr}_1 : \operatorname{Spec} K \times_{(a,a),\mathscr{X}\times\mathscr{X},\Delta} \mathscr{X} \to \operatorname{Spec} K$, where Δ is diagonal. If $\mathscr{X}$ is Deligne–Mumford, then the stabilizer group scheme is a finite group. The proof of Theorem 1.3 immediately implies the following.

COROLLARY 4.3. *Let $\mathscr{X}$ be a smooth Deligne–Mumford stack that is separated and of finite type over k. Suppose that there exists a coarse moduli map $\pi : \mathscr{X} \to X_\Sigma$ to a toric variety such that π is an isomorphism over T_Σ. Let $V(\rho)$ denote the torus-invariant divisor corresponding to a ray ρ, and suppose that the order of stabilizer group of the generic point on $\pi^{-1}(V(\rho))$ is n_ρ. Then there exists an isomorphism $\mathscr{X} \to \mathscr{X}_{(\Sigma,\Sigma^0)}$ over X_Σ, where the level of Σ^0 on ρ is n_ρ for each ρ.*

Proof. By the proof of Theorem 1.3, there exist some stacky fan (Σ, Σ^0) and an isomorphism $\mathscr{X} \cong \mathscr{X}_{(\Sigma,\Sigma^0)}$ over X_Σ. Moreover, if the level of Σ^0 on ρ is n, then by [Iwa07a, Proposition 4.13] the stabilizer group of the generic point on the torus-invariant divisor on $\mathscr{X}_{(\Sigma,\Sigma^0)}$ corresponding to ρ is of the form $\mu_n = \operatorname{Spec} K[X]/(X^n - 1)$. Our claim follows. $\square$

Remark 4.4. By virtue of Theorem 1.3, we can handle toric triples, regardless of their constructions, by using the machinery of toric algebraic stacks [Iwa07a] and various approaches (see § 5).

One reasonable generalization of a toric triple to the case of positive characteristics might be a smooth tame Artin stack with finite diagonal that is of finite type over an algebraically closed field and which satisfies properties (i), (ii) and (iii) given in the introduction. (For the definition of tameness, see [AOV08]; since the stabilizer group of each point on a toric algebraic stack is diagonalizable, every toric algebraic stack is a tame Artin stack.) Indeed, toric algebraic stacks as defined in [Iwa07a] are toric triples in this sense, in arbitrary characteristic. We conjecture that the geometric characterization theorem holds also for positive characteristics.

Remark 4.5. Let us denote by $\mathfrak{Torst}$ the 2-category of toric algebraic stacks or, equivalently (by Theorem 1.1), the 2-category of toric triples (see § 1). Let us denote by Smtoric (respectively, Simtoric) the category of smooth (non-singular) toric varieties (respectively, simplicial toric varieties) whose morphisms are torus-equivariant. From the results obtained so far, we have

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the following commutative diagram (picture).

$$\begin{tikzcd}
& \mathfrak{Torst} \arrow[rr, "\sim"] \arrow[dd] & & \text{(category of stacky fans)} \arrow[dd] \\
\text{Smtoric} \arrow[ur, "\iota"] \arrow[rr, "\sim"] \arrow[dr, "i"'] & & \text{(category of non-singular fans)} \arrow[ur, "a"] \arrow[dr, "b"'] & \\
& \text{Simtoric} \arrow[rr, "\sim"] & & \text{(category of simplicial fans)}
\end{tikzcd}$$

Here $a(\Sigma) = (\Sigma, \Sigma^0_{\text{can}})$ and $b(\Sigma) = \Sigma$ for a non-singular fan Σ, the functors ι and i are natural inclusion functors, and the functors ι, i, a and b are *fully faithful*. All horizontal arrows are *equivalences*.

5. Related work

In this section we discuss the relationship of our results with [BCS05], [FMN07] and [Per08]. We work over the complex number field $\mathbb{C}$. If no confusion seems likely to arise, we refer to toric triples as toric stacks.

First, let us recall the stacky fans introduced in [BCS05]. Let N be a finitely generated abelian group, Σ a simplicial fan in $N \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\bar{N}$ the lattice, i.e. the image of $N \to N \otimes_{\mathbb{Z}} \mathbb{Q}$. For any $b \in N$, we denote by $\bar{b}$ the image of b in $\bar{N}$. Let $\{\rho_1, \ldots, \rho_r\}$ be the set of rays of Σ, and let $\{b_1, \ldots, b_r\}$ be the set of elements of N such that each $\bar{b}_i$ spans ρ_i. The set $\{b_1, \ldots, b_r\}$ gives rise to the homomorphism $\beta : \mathbb{Z}^r \to N$. The triple $\mathbf{\Sigma} = (N, \Sigma, \beta : \mathbb{Z}^r \to N)$ is called a stacky fan. If N is free, then we say that $\mathbf{\Sigma}$ is reduced. Every stacky fan $\mathbf{\Sigma}$ has the natural underlying reduced stacky fan $\mathbf{\Sigma}_{\text{red}} = (\bar{N}, \Sigma, \bar{\beta} : \mathbb{Z}^r \to \bar{N})$, where $\bar{N} = N/(\text{torsion})$ is the lattice and $\bar{\beta}$ is defined to be the composite $\mathbb{Z}^r \to N \to \bar{N}$.

Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a reduced stacky fan. Let $\beta_{\geq 0} : \mathbb{Z}^r_{\geq 0} \to N$ be the map induced by the restriction of β. The intersection $\beta_{\geq 0}(\mathbb{Z}^r_{\geq 0}) \cap |\Sigma|$ forms a free-net of Σ. The pair $(\Sigma, \beta_{\geq 0}(\mathbb{Z}^r_{\geq 0}) \cap |\Sigma|)$ is a stacky fan in the sense of Definition 2.1. Conversely, every stacky fan (Σ, Σ^0) according to Definition 2.1 is obtained from a unique reduced stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$. This gives rise to a one-to-one bijective correspondence between reduced stacky fans and stacky fans in the sense of Definition 2.1. To avoid confusion, in this section a stacky fan in the sense of Definition 2.1 will be referred to as a 'framed stacky fan'.

Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. Assume that the rays span the vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$. In [BCS05], by modelling the construction of D. Cox [Cox95a], the toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is constructed as a quotient stack $[Z/G]$. There exists a coarse moduli map $\pi(\mathbf{\Sigma}) : \mathcal{X}(\mathbf{\Sigma}) \to X_{\Sigma}$.

Suppose that $\mathbf{\Sigma}$ is reduced and let (Σ, Σ^0) be the corresponding framed stacky fan. Then we have the following result.

Proposition 5.1. *There exists an isomorphism*

$$\mathscr{X}_{(\Sigma,\Sigma^0)} \xrightarrow{\sim} \mathcal{X}(\mathbf{\Sigma})$$

of algebraic stacks over X_{Σ}.

Proof. We will prove this proposition by applying the geometric characterization of Theorem 1.3 and Corollary 4.3. Let d be the rank of N. The stack $\mathcal{X}(\mathbf{\Sigma})$ is a smooth d-dimensional

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Deligne–Mumford stack that is separated and of finite type over $\mathbb{C}$, and its coarse moduli space is the toric variety X_Σ (see [BCS05, Lemma 3.1 and Propositions 3.2 and 3.7]). From the quotient construction, $\mathcal{X}(\mathbf{\Sigma})$ has a torus embedding $\mathbb{G}_m^d \to \mathcal{X}(\mathbf{\Sigma})$. Owing to Corollary 4.3, to prove our proposition it suffices to check that the order of the stabilizer group at the generic point of $\pi(\mathbf{\Sigma})^{-1}(V(\rho_i))$ is equal to the level of Σ^0 on ρ_i; here $V(\rho_i)$ is the torus-invariant divisor corresponding to ρ_i. To this end, we may assume that Σ is a complete fan; then [BCS05, Proposition 4.7] implies that the order of the stabilizer group at the generic point of $\pi(\mathbf{\Sigma})^{-1}(V(\rho_i))$ is the level n_{ρ_i} of ρ_i. □

Remark 5.2. The explicit construction of $\mathcal{X}(\mathbf{\Sigma})$ plays no essential role in the proof of Proposition 5.1, and the proof uses only some intrinsic properties. This shows the flexibility of our results. If a new approach to this subject (i.e. a new construction) is proposed in the future, the category of toric triples should provide a useful bridge. We believe that it is good to have various approaches at one's disposal and to be free to choose whichever approach is most suited to a given situation.

Let $\mathbf{\Sigma} = (N = N' \oplus \mathbb{Z}/w_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/w_t\mathbb{Z}, \Sigma, \beta)$ be a stacky fan such that N' is a free abelian group, and let $\mathbf{\Sigma}_{\mathrm{red}}$ be the associated reduced stacky fan. There is a morphism $\mathcal{X}(\mathbf{\Sigma}) \to \mathcal{X}(\mathbf{\Sigma}_{\mathrm{red}})$ which is a finite abelian gerbe. This structure is obtained by a simple technique called 'taking nth roots of an invertible sheaf', which is explained as follows. Recall the notion of the stack of roots of an invertible sheaf (see, for example, [Cad07]). Let X be an algebraic stack and $\mathcal{L}$ an invertible sheaf on X. Let $P : X \to B\mathbb{G}_m$ be the morphism to the classifying stack of $\mathbb{G}_m$ that corresponds to $\mathcal{L}$. Let l be a positive integer and $f_l : B\mathbb{G}_m \to B\mathbb{G}_m$ the morphism associated to $l : \mathbb{G}_m \to \mathbb{G}_m :\ g \mapsto g^l$. Then the stack of lth roots of $\mathcal{L}$ is defined to be $X \times_{P,B\mathbb{G}_m,f_l} B\mathbb{G}_m$. The reason for this stack being called the 'stack of lth roots' is that it has the following modular interpretation: objects of $X \times_{P,B\mathbb{G}_m,f_l} B\mathbb{G}_m$ over a scheme S are triples $(S \to X, \mathcal{M}, \phi : \mathcal{M}^{\otimes l} \to \mathcal{L})$, where $\mathcal{M}$ is an invertible sheaf on S and ϕ is an isomorphism. A morphism of triples is defined in a natural manner. The first projection $X \times_{P,B\mathbb{G}_m,f_l} B\mathbb{G}_m \to X$ forgets the data $\mathcal{M}$ and ϕ. We write $X(\mathcal{L}^{1/l})$ for the collection of stacks of lth roots of $\mathcal{L}$. In [JT07, Proposition 2.9 and Remark 2.10] and [Per08, Proposition h3.1] it was observed, and shown, that $\mathcal{X}(\mathbf{\Sigma})$ is a finite abelian gerbe over $\mathcal{X}(\mathbf{\Sigma}_{\mathrm{red}})$ which is obtained by using the stacks of roots of invertible sheaves. In light of Proposition 5.1, this can be stated as follows.

COROLLARY 5.3. *Let (Σ, Σ^0) be the framed stacky fan that corresponds to $\mathbf{\Sigma}_{\mathrm{red}}$. Let $b_{i,j} \in \mathbb{Z}/w_j\mathbb{Z}$ be the image of $b_i \in N$ in $\mathbb{Z}/w_j\mathbb{Z}$. (We may regard $b_{i,j}$ as an element of $\{0, \ldots, w_j - 1\}$.) Let $\mathcal{N}_i$ be an invertible sheaf on $\mathscr{X}_{(\Sigma,\Sigma^0)}$ which is associated to the torus-invariant divisor corresponding to ρ_i. Let $\mathcal{L}_j = \otimes_i \mathcal{N}_i^{\otimes b_{i,j}}$. Then $\mathcal{X}(\mathbf{\Sigma})$ is isomorphic to*

$$\mathscr{X}_{(\Sigma,\Sigma^0)}(\mathcal{L}_1^{1/w_1}) \times_{\mathscr{X}_{(\Sigma,\Sigma^0)}} \cdots \times_{\mathscr{X}_{(\Sigma,\Sigma^0)}} \mathscr{X}_{(\Sigma,\Sigma^0)}(\mathcal{L}_r^{1/w_r}).$$

It is known that every separated normal Deligne–Mumford stack is a gerbe over a Deligne–Mumford stack that is generically a scheme. We will consider an intrinsic characterization of toric Deligne–Mumford stacks in the sense of [BCS05] from the viewpoint of gerbes. Since the construction in [BCS05] employed the idea of Cox, we need to impose the assumption that the rays $\{\rho_1, \ldots, \rho_r\}$ span the vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$. In order to fit in with [BCS05], we consider the following condition on toric stacks (toric triples): a toric stack (triple) $\mathcal{X}$ is said to be *full* if $\mathcal{X}$ has no splitting $\mathcal{X}' \times \mathbb{G}_m^p$ such that $\mathcal{X}'$ is a toric stack and p is a positive integer.

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Let $\mathscr{X}$ be an algebraic stack. We say that an algebraic stack $\mathscr{Y} \to \mathscr{X}$ is a *polyroots gerbe* over $\mathscr{X}$ if it has the form of the composite

$$\mathscr{X}(\mathcal{L}_1^{1/n_1})(\mathcal{L}_2^{1/n_2})\cdots(\mathcal{L}_k^{1/n_k}) \to \mathscr{X}(\mathcal{L}_1^{1/n_1})(\mathcal{L}_2^{1/n_2})\cdots(\mathcal{L}_{k-1}^{1/n_{k-1}}) \to \cdots \to \mathscr{X},$$

where $\mathcal{L}_i$ is an invertible sheaf on $\mathscr{X}(\mathcal{L}_1^{1/n_1})(\mathcal{L}_2^{1/n_2})\cdots(\mathcal{L}_{i-1}^{1/n_{i-1}})$ for $i \geq 2$ and $\mathcal{L}_1$ is an invertible sheaf on $\mathscr{X}$. For example, if $\mathcal{L}_a$ and $\mathcal{L}_b$ are invertible sheaves on $\mathscr{X}$ and $\mathscr{Y} = \mathscr{X}(\mathcal{L}_a^{1/n}) \times_{\mathscr{X}} \mathscr{X}(\mathcal{L}_b^{1/n'})$, then $\mathscr{Y}$ is the composite $\mathscr{X}(\mathcal{L}_a^{1/n})(\mathcal{L}_b|_{\mathscr{X}(\mathcal{L}_a)}^{1/n'}) \to \mathscr{X}(\mathcal{L}_a) \to \mathscr{X}$ and therefore is a polyroots gerbe over $\mathscr{X}$.

Proposition 5.4. *A toric Deligne–Mumford stack in the sense of [BCS05] is precisely characterized as a polyroots gerbe over a full toric stack (i.e. a full toric triple in our sense).*

Proof. Note first that by Corollary 5.3 every toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is a polyroots gerbe over some toric stack $\mathscr{X}_{(\Sigma,\Sigma^0)}$, so $\mathcal{X}(\mathbf{\Sigma})$ is a polyroots gerbe over $\mathscr{X}_{(\Sigma,\Sigma^0)}$. Since the condition on Σ that rays span the vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$ is equivalent to the condition that $\mathscr{X}_{(\Sigma,\Sigma^0)}$ is full, every toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is a polyroots gerbe over a full toric stack (triple). Thus we will prove the converse; it suffices to show that for any toric Deligne–Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ and any invertible sheaf $\mathcal{L}$ on it, the stack $\mathcal{X}(\mathbf{\Sigma})(\mathcal{L}^{1/n})$ is also a toric Deligne–Mumford stack in the sense of [BCS05]. Here n is a positive integer. Note that $\mathcal{X}(\mathbf{\Sigma})$ is the quotient stack $[Z/G]$, where G is a diagonalizable group and Z is an open subset of an affine space $\mathbb{A}^q$ such that the codimension of the complement $\mathbb{A}^q - Z$ is greater than one. Therefore, the Picard group $\mathbb{A}^q$ is naturally isomorphic to that of Z, and so every invertible sheaf on Z is trivial, i.e. every principal $\mathbb{G}_m$-bundle on Z is trivial. Hence every principal $\mathbb{G}_m$-bundle on $[Z/G]$ has the form $[Z \times \mathbb{G}_m/G] \to [Z/G]$, where the action of G on $Z \times \mathbb{G}_m$ arises from the action of G on Z and some character $\lambda : G \to \mathbb{G}_m$. Let $\alpha : [Z/G] \to B\mathbb{G}_m$ be the morphism induced by $Z \to \operatorname{Spec}\mathbb{C}$ and $\lambda : G \to \mathbb{G}_m$. Notice that $\alpha : [Z/G] \to B\mathbb{G}_m$ is the composite $[Z/G] \to BG \to B\mathbb{G}_m$, where the first morphism is induced by the G-equivariant morphism $Z \to \operatorname{Spec}\mathbb{C}$ and the second morphism is induced by $\lambda : G \to \mathbb{G}_m$. Let $l : \mathbb{G}_m \to \mathbb{G}_m$, $g \mapsto g^l$, and let $\tilde{G} := G \times_{\lambda,\mathbb{G}_m,l} \mathbb{G}_m$. Then we obtain the diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & B\tilde{G} \arrow[r] \arrow[d] & B\mathbb{G}_m \arrow[d, "f_l"] \\
{[Z/G]} \arrow[r] & BG \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

where the square on the left is a cartesian diagram and the vertical morphism on the right is induced by $l : \mathbb{G}_m \to \mathbb{G}_m$. Then, by [Jia08, Corollary 1.2], $\mathcal{G}$ is a toric Deligne–Mumford stack. Since $[Z \times \mathbb{G}_m/G] \cong [Z/G] \times_{\alpha,B\mathbb{G}_m} \operatorname{Spec}\mathbb{C}$, the morphism α corresponds to the principal $\mathbb{G}_m$-bundle $[Z \times \mathbb{G}_m/G] \to [Z/G]$. Thus, if $\mathcal{G} \cong [Z/G] \times_{\alpha,B\mathbb{G}_m,f_l} B\mathbb{G}_m$, our claim follows. It therefore suffices to check that the square on the right is also a cartesian diagram. Indeed, there exists a natural isomorphism $B\mu_l \cong \operatorname{Spec}\mathbb{C} \times_{B\mathbb{G}_m,l} B\mathbb{G}_m$, and so we have $B\mu_l \cong \operatorname{Spec}\mathbb{C} \times_{BG} (BG \times_{B\mathbb{G}_m,f_l} B\mathbb{G}_m)$. Moreover, there exists a natural isomorphism $B\mu_l \cong \operatorname{Spec}\mathbb{C} \times_{BG} B\tilde{G}$, because the kernel of $\tilde{G} \to G$ is μ_l. Consider the natural morphism $B\tilde{G} \to BG \times_{B\mathbb{G}_m,f_l} B\mathbb{G}_m$ over BG. Its pull-back by the flat surjective morphism $\operatorname{Spec}\mathbb{C} \to BG$ is an isomorphism $B\mu_l \to B\mu_l$. Hence $B\tilde{G} \cong BG \times_{B\mathbb{G}_m,f_l} B\mathbb{G}_m$ and our proof is complete. □

Now we discuss how our work relates to [FMN07]. In [FMN07], Fantechi, Mann and Nironi generalized the notion of toric triples defined in this paper. In order to fit in with the framework of [BCS05], they introduced the 'DM torus', a torus with a trivial gerbe structure, and considered

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actions of DM tori on algebraic stacks. Following the point of view that 'toric objects' should be characterized by torus embeddings and actions, they discuss a geometric characterization of toric Deligne–Mumford stacks in the sense of [BCS05] by means of smooth Deligne–Mumford stacks with DM torus embeddings and actions (see [FMN07, Theorem II]). Specifically, the embeddings and actions of DM tori provide gerbe structures on toric triples, as discussed above.

An older version of this paper was posted on the arXiv server in December 2006, containing proofs of the main results presented here; [FMN07] appeared on the arXiv in August 2007.

Finally, we discuss the relation of our work to [Per08]. In [Per08], Perroni studied 2-isomorphism classes of all 1-morphisms between toric Deligne–Mumford stacks in the sense of [BCS05]. The method and description are parallel to those in [Cox95b, § 3]. Let $\mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{X}(\mathbf{\Sigma}')$ be toric Deligne–Mumford stacks, and suppose that $\mathcal{X}(\mathbf{\Sigma})$ is proper over $\mathbb{C}$. Perroni gave a description of 2-isomorphism classes of 1-morphisms from $\mathcal{X}(\mathbf{\Sigma})$ to $\mathcal{X}(\mathbf{\Sigma}')$ in terms of homogeneous polynomials of $\mathcal{X}(\mathbf{\Sigma})$ (for details, see [Per08, § 5]). Assume that $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$ are reduced. If the morphism $\tilde{f}:\mathcal{X}(\mathbf{\Sigma})\to\mathcal{X}(\mathbf{\Sigma}')$ associated to a system of homogeneous polynomials (cf. [Cox95b, Theorem 5.1]) induces a torus-equivariant morphism $f:X_\Sigma\to X_{\Sigma'}$, then by Corollary 3.7 and 3.8 $\tilde{f}$ is a torus-equivariant morphism. In other words, if (Σ,Σ^0) and $(\Sigma',{\Sigma'}^0)$ denote framed stacky fans corresponding to $\mathbf{\Sigma}$ and $\mathbf{\Sigma}'$, respectively, then the morphism $\Sigma\to\Sigma'$ of fans corresponding to $f:X_\Sigma\to X_{\Sigma'}$ induces $(\Sigma,\Sigma^0)\to(\Sigma',{\Sigma'}^0)$, and through isomorphisms $\mathscr{X}_{(\Sigma,\Sigma^0)}\cong\mathcal{X}(\mathbf{\Sigma})$ and $\mathscr{X}_{(\Sigma',{\Sigma'}^0)}\cong\mathcal{X}(\mathbf{\Sigma}')$ the morphism $\mathscr{X}_{(\Sigma,\Sigma^0)}\to\mathscr{X}_{(\Sigma',{\Sigma'}^0)}$ associated to $(\Sigma,\Sigma^0)\to(\Sigma',{\Sigma'}^0)$ can be identified with $\tilde{f}$.

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