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CATEGORY-THEORETIC SOLUTION
OF
RECURSIVE DOMAIN EQUATIONS

by

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1. Introduction. The solution of a recursive domain equation, of the form

$$D \cong F(D) \qquad (1)$$

may be viewed as the finding of a fixpoint (up to isomorphism) of the functor F. This has led to the idea of formulating a category-theoretic analogue of Tarski's fixpoint theorem for lattices, as a basis for a general method of solution for this kind of equation; see especially Reynolds [1], Wand [2], Plotkin [3].

In seeking to adapt Tarski's results to the category situation, one would expect the role of continuous functions to be taken by colimit-preserving functors. In all the versions developed up till now, however, the major role is taken by a quite different notion of continuity of functors ("continuity on morphism-sets"), having to do with orderings of the hom-sets. Apparently it was thought that the requirement of preservation of (ω-)colimits was too difficult to handle ([2]).

The purpose of the present note is to show that, on the contrary, a much better organization of the theory can be achieved by taking preservation of ω-colimits (or ω-continuity, as we shall call it) as the basic notion) The main theorem (Theorem 1) now takes the form of an exact generalization of the lattice-theoretic fixpoint theorem, rather than only an analogue of it. More substantially, since the theorem applies to arbitrary categories admitting ω-colimits (no ordering of hom-sets is needed) the range of application is wider: see Sec. 2 below. Furthermore, the application of the theorem to a given class C of domains takes on a simpler form: instead of the manipulations with three distinct categories (K,KP,KR in Wand's notation) characteristic of the "continuity on morphism sets" approach, we consider just the category of "embeddings" in C (Definition 2)

In Section 2 we discuss the existence of ω-colimits in certain relevant categories. The material here is fairly standard. One point, however, is worth noting. We have not found the abstract approach of Wand [2] to be worth the effort it involves. In all the relevant categories, the objects are sets (with structure); and it involves little more than a routine verification to show that the set-theoretic inverse limit is also the colimit. The abstract method , however, brings with it a double complication when we try to apply it to a concrete category. We must first show that the set-theoretic inverse limit is the (category-theoretic) limit; and then, with the aid of special conditions (cf. Wand's "Condition A"), that the limit is also the colimit.

Section 3 concerns the ω-continuity of various useful functors. In addition to the ones usually discussed, we have included a brief treatment of Plotkin's powerdomain construction.

2. The fixpoint theorem. The "lattice-theoretic" fixedpoint theorem for continuous functions may be stated in a strong form as follows. Let $(P, \sqsubseteq)$ be a poset in which every increasing ω-chain has a lub, let $a \in P$ and let $f : P \to P$ be an ω-continuous function (that is, f preserves lubs of ω-chains) such that $a \sqsubseteq fa$. Then the set of post-fixedpoints of f greater than a (that is, $\{x \mid f(x) \sqsubseteq x \ \& \ a \sqsubseteq x\}$) has a least element, namely $\bigsqcup_n f^n(a)$ - which is, moreover a fixedpoint of f. We seek the appropriate generalization of this result to categories.

Definition 1. An ω-chain is a functor from ω (the natural numbers with the standard linear ordering) into K. K is said to admit ω-colimits if every

ω-chain in K has a colimit. A functor $F:K \to K'$ is <u>ω-continuous</u> if F transforms any colimit diagram for an ω-chain Γ in K into a colimit diagram (for $F\Gamma$) in K'.

An ω-chain in K may be pictured like this:

$$\Gamma: \quad a_0 \xrightarrow{g_0} a_1 \xrightarrow{g_1} \cdots$$

We adopt the viewpoint that a colimit (diagram) for Γ is an initial object in the category of canes from Γ. Corresponding to the sequence $\langle f^n(a) \rangle$ in the above formulation of the fixedpoint theorem, we will have an ω-chain

$$\Delta = a \xrightarrow{\theta} Fa \xrightarrow{F\theta} F^2a \cdots$$

where $F:K \to K$ is now an ω-continuous functor. Our program is to construct the category $PF(K,F,\theta)$ of "post-fixed-objects along θ" and "post-fixed-arrows" of F, and to show that - roughly speaking - colim Δ is initial in this category.

Let F be an endofunctor of a category K. We say that an object x of K is a <u>post-fixedpoint</u> of F <u>via γ</u> if $\gamma:Fx \to x$; and an arrow $\pi:A \to B$ is a <u>post-fixed-arrow</u> of F <u>via γ,δ</u> if

$$\begin{array}{ccc} B & \xleftarrow{\delta} & FB \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle F\pi} \\ A & \xleftarrow{\gamma} & FA \end{array}$$

commutes. If now $\theta:a \to Fa$, define the category $PF(K,F,\theta)$ as follows: the objects are the triples $\langle \alpha,M,\gamma \rangle$, where M is a post-fixedpoint of F via γ, and $\alpha = \gamma\, F\alpha\, \theta$ (equivalently, the objects are the commuting squares

$$\begin{array}{ccc} a & \xrightarrow{\theta} & Fa \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle F\alpha} \\ M & \xleftarrow{\gamma} & FM \end{array}$$
)

while $Hom(\langle \alpha,M,\gamma \rangle,\langle \alpha',M',\gamma' \rangle)$ is the set of arrows (in K) $\pi:M \to M'$ such that (1) $\pi\alpha = \alpha'$, and (2) π is a post-fixed arrow of F via γ,γ'.

Suppose now that F is ω-continuous, and that λ is a colimit cone for

$\Delta = a \xrightarrow{\theta} Fa \xrightarrow{F\theta} \ldots$ (the vertex of λ being L). $F\lambda$ is a colimit cone for $F\Delta = Fa \xrightarrow{F\theta} F^2a \ldots$. $F\lambda$ extends trivially to a colimit cone λ^+ for Δ (put $\lambda_0^+ = T\lambda_0\ \theta$, $\lambda_{n+1} = T\lambda_n$). Let $\psi: L \rightarrow TL$ be the (unique) arrow from λ to λ^+; then ψ is an isomorphism (since λ, λ^+ are both initial), and ψ^{-1} is the unique arrow from λ^+ to λ. In particular, $<\lambda_0, L, \psi^{-1}>$ is an object of PF(K,F,θ). We have:

Theorem 1. Let K admit ω-colimits and F be ω-continuous. Then L is a fixedpoint of F via ψ^{-1}, and $<\lambda_0, L, \psi^{-1}>$ is initial in PF(K,F,θ).

Proof. That L is a fixedpoint of F via ψ^{-1} is contained in the preceding remarks. Turning to the initiality of $<\lambda_0, L, \psi^{-1}>$, we note first that any object $<\alpha, M, \gamma>$ of PF(K,F,θ) determines a cone μ from Δ to M, as follows:

$$\mu_0 = \alpha$$

$$\mu_{n+1} = \gamma\ F\mu_n \quad .$$

(In this sense, $<\lambda_0, L, \psi^{-1}>$ determines λ.) Moreover, any arrow $\pi: <\alpha, M, \gamma> \rightarrow <\alpha', M', \gamma'>$ is a morphism of the corresponding cones μ, μ' (That is, $\mu'_n = \pi\ \mu_n$, n=0,1,..). For $\mu'_0 = \alpha' = \pi\ \alpha = \pi\ \mu_0$; while (induction step) $\mu'_{n+1} = \gamma'\ F\ \mu'_n = \gamma'\ F(\pi\ \mu_n) = \pi\gamma\ F\ \mu_n = \pi\ \mu_{n+1}$. Hence, the only possible arrow from $<\lambda_0, L, \psi^{-1}>$ to $<\alpha, M, \gamma>$ is the unique arrow $\sigma: \lambda \rightarrow \mu$.

It remains to show that σ is indeed an arrow of PF(K,F,θ), that is, that σ is a post-fixed-arrow of F via ψ^{-1}, γ. By the preceding remarks, the (unique) arrow from λ^+ to μ is $\sigma\psi^{-1}$. But we can also show that it is $\gamma\ F\sigma$:

$$\mu_0 = \alpha = \gamma\ F\alpha\ \theta = \gamma\ F(\sigma\ \lambda_0)\ \theta = \gamma\ F\sigma\ \lambda_0^+$$

$$\mu_{n+1} = \gamma\ F\ \mu_n = \gamma\ F(\sigma\ \lambda_n) = \gamma\ F\sigma\ \lambda_{n+1}^+ \quad .$$

Hence $\sigma\ \psi^{-1} = \gamma\ F\sigma$; the proof is complete.

This theorem is formally similar to Wand's Theorem 3. The content (and proof) is different, due to the changed notion of continuity of functors.

Also, we have brought in the terminology of post-fixedpoints (and arrows), in the hope of making the result more intuitive.

In most applications of the theorem, a is initial in K. (The important exception is the construction of models of the λ-calculus via the domain equation $D = [D \to D]$) In this case, the conclusion of the theorem can be simplified:

Corollary 1. Under the hypotheses of Theorem 1, suppose that a is initial in K. For any post-fixedpoint M of F via γ, there is a unique $\sigma: L \to M$ such that σ is a post-fixed-arrow of F via ψ^{-1}, γ.

The following lemma will be useful (via Corollary 2) in establishing ω-continuity of functors:

Lemma 1. Let K' admit ω-colimits, and $F: K \to K'$. Then F is ω-continuous iff the following condition holds for every ω-chain Δ in K with colimit cone λ: if μ is a colimit cone for $F\Delta$ in K', the arrow ψ from μ to $F\lambda$ is an isomorphism.

Proof. The necessity of the condition is obvious. For sufficiency, assume the condition satisfied. Then there is an arrow from $F\lambda$ to any cone ν from $F\Delta$, viz. $\sigma\psi^{-1}$, where $\sigma: \mu \to \nu$; and there is at most one such arrow, for if $\gamma_1, \gamma_2: F\lambda \to \nu$ are distinct, then so are $\gamma_1\psi, \gamma_2\psi: \mu \to \nu$. (Alternative proof: verify that ψ is an isomorphism in the category of cones from $F\Delta$.)

3. Existence of ω-colimits

Definition 2. An ω-cpo is a poset in which every (ascending) ω-chain has a lub. A map $f: D \to D'$, where D, D' are ω-cpo's, is ω-continuous if $f(\bigsqcup_i a_i) = \bigsqcup_i f(a_i)$ for every ω-chain $\langle a_i \rangle_i$ in D. An ω-continuous map $f: D \to D'$ is an embedding if f possesses an ω-continuous right adjoint $f': D' \to D$. $\omega\text{-CPO}^E$ is the category of ω-cpo's with embeddings as arrows.

Some of these concepts were already introduced informally at the beginning of Sec. 2. A more familiar characterization of embeddings is as follows: $f:D \to D'$ is an embedding iff there is an (ω-continuous) $f':D' \to D$ such that $f' f = I_D$ and $f f' \sqsubseteq I_{D'}$, (so that f, f' form a projection pair). The prefix ω- will usually be omitted (from the terms introduced in Definition 2). In fact, nothing hinges on our use of ω-cpo's rather than the more restricted class of cpo's (in which <u>all</u> directed sets have lubs). The point simply that the ω-notions generalize more readily to categories: see remarks at the end of this section.

Usually, the categories in which we need to solve recursive domain equations are full subcategories of CPO^E. There follows a concise treatment of the construction of ω-colimits in CPO^E. This is little more than a summary of (parts of) the existing proofs for complete lattices (Reynolds, Wand).

<u>Theorem 2</u>. Let $\Delta = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \dots$ be an ω-chain in CPO^E. Let D be the inverse limit $\{\langle x_n\rangle_{n\in\omega} \mid x_n \in D_n \ \&\ f'_n(x_{n+1}) = x_n\}$ with the induced (componentwise) partial ordering. Then D, together with the maps $i_n : D_n \to D$ given by $i_n(x_n) = \langle f_{nm}(x_n)\rangle_{m\in\omega}$, is a colimit for Δ. (Here we have used the standard notation

$$f_{nm} = \begin{cases} f'_m \cdots f'_{n-1}, & m < n \\ I_{D_n}, & m = n \\ f_{m-1} \cdots f_n, & m > n \end{cases} \quad)$$

<u>Proof</u>. This falls into a series of lemmas and sublemmas. For the easiest cases we just quote the main relevant fact(s), rather than giving a detailed proof. Certain elementary facts are used implicitly, for example: for $n \geq m$, f_{mn} is an embedding, with $f'_{mn} = f_{nm}$.

(1) D is a cpo. (Continuity of the f'_n.)

(2) Each i_n is an embedding, with adjoint i'_n given by: $i'_n(\langle x_m \rangle_{m \in \omega}) = x_n$.

(i) $\langle f_{nm}(x_n) \rangle_{m \in \omega} \in D$ (since $f'_m f_{n,m+1} = f_{nm}$)

(ii) $i'_n \circ i_n = I_{D_n}$

(iii) $i_n \circ i'_n(\langle x_m \rangle) \sqsubseteq \langle x_m \rangle$ $(x_n = f_{pn}(x_p)$ $(n \leq p)$ $f_{np}(x_n) \sqsubseteq x_p$).

(3) Given a cone λ from Δ to L, we have the mediating map $\psi : D \to L$:
$\langle x_m \rangle_m \to \bigsqcup_m \lambda_m(x_m)$, with adjoint $\psi' : L \to D : y \to \langle \lambda'_n(y) \rangle_n$.

(i) ψ is well-defined, i.e. $\langle \lambda_m(x_m) \rangle_m$ is increasing. (Since λ is a cone, $\lambda_m(x_m) = \lambda_{m+1} \circ f_m(x_m) = \lambda_{m+1} \circ f_m f'_m(x_{m+1}) \sqsubseteq \lambda_{m+1}(x_{m+1})$).

(ii) ψ' is well-defined, i.e. $\psi'(y) \in D$. This is equivalent to $f'_n \lambda'_{n+1}(y) = \lambda'_n(y)$. But this follows, by taking adjoints, from $\lambda_{n+1} f_n = \lambda_n$.

(iii) $\psi' \circ \psi = I_D$. For $(\psi' \circ \psi(\langle x_m \rangle_m))_n = \lambda'_n(\bigsqcup_m \lambda_m(x_m)) =$

$= \bigsqcup_m \bigsqcup_n \lambda'_n \circ \lambda_m(x_m) = \bigsqcup_m \bigsqcup_n f_{mn} \lambda'_m \circ \lambda_m(x_n) = \bigsqcup_m \bigsqcup_n f_{mn}(x_n) = x_n$.

(iv) $\psi \psi' \sqsubseteq I_L$. For $\psi \circ \psi' = \bigsqcup \lambda_n \circ \lambda'_n$, and each $\lambda_n \lambda'_n \sqsubseteq I_L$.

(v) $\lambda_n = \psi i_n$. For $\psi \circ i_n(x_n) = \bigsqcup_m \bigsqcup_n \lambda_m \circ f_{nm}(x_n) = \lambda_n(x_n)$.

(4) There is at most one mediating map $\varphi : D \to L$. For if $\lambda_n = \varphi i_n$, then $i'_n \varphi'_n(y) = \lambda'_n(y)$, so that $\varphi'(y)$ must be $\langle \lambda'_n(y) \rangle_n$. Hence φ', and so φ, is uniquely determined.

<u>Corollary 2</u>. λ is a colimiting cone for Δ iff $\bigsqcup \lambda_n \circ \lambda'_n = I_L$.

<u>Proof</u>. By Lemma 1, λ is colimiting iff ψ is an isomorphism. This holds iff $\psi \psi' = I$. Now $\psi \circ \psi' = \bigsqcup \lambda_n \circ \lambda'_n$.

If K is any full subcategory of CPO^E, it follows from Theorem 2 that K admits colimits provided that the inverse limit of any ω-chain in K is an object of K. In particular, this holds for the <u>algebraic</u> cpo's, which may be introduced - using sequences instead of directed sets, as in Definition 2 - as follows:

Definition 3 A cpo D is countably algebraic if there is a countable subset B of D such that (1) every $x \in D$ is the lub of an increasing sequence of elements of B, and (2) for any increasing sequence $\langle e_i \rangle_i$ in B and any $a \in B$, if $a \sqsubseteq \bigsqcup_i e_i$ then $a \sqsubseteq e_i$ for some i.

The qualifier "countably" will usually be omitted. One readily shows that the elements of B are exactly the finite (=isolated) elements of D, and that the definition agrees with that given by Plotkin (for "algebraic ipo") except for one (minor) point: we do not require algebraic cpo's to have least elements.

For the proof that $ACPO^E$ (the category of algebraic cpo's and embeddings) is closed under the inverse limit construction, see [3]. One other elementary fact which we shall need is the following: if $(Q, \leq)$ is a countable preordered set, then the collection $\bar{Q}$ of all directed subsets of Q, ordered by inclusion, is an algebraic cpo; the finite elements of $\bar{Q}$ are the sets of the form $[a] = \{b \mid b \leq a\}$, for $a \in Q$.

The basic notion of this section is that of an ω-complete poset. As has already been hinted, however, the definitions and results can be formulated more generally, in terms of (ω-complete) categories. A generalization of this kind has been worked out by Lehmann [4], with a view to applications to the semantics of non-deterministic programs; in this approach the semantic domains are themselves categories. In a slightly different direction, [5] introduces a notion of "algebraic category", got by generalizing Definition 3; we return to this point in Sec.4. It should be emphasized that Theorem 1 applies without modification in the more general situation; this is an important advantage which the formulation in terms of ω-continuity has over that in terms of local continuity (continuity on morphism sets).

4. Continuity of special functors. The functor F in the domain equation (1) will typically be composed out of the following basic functors: +,x, → and (in case we follow Plotkin's "powerdomain" approach [3] to non-deterministic semantics) . The definition, and proof of ω-continuity, of + and are entirely straightforward. In the present section we consider → and .

The functor →: $(CPO^E)^2 \to CPO^E$ is defined on objects by

$$\to(\langle D, D'\rangle) = [D \to D']$$

(the cpo of continuous functions from D to D'), and on arrows by

$$\to(\langle p,q\rangle) = \lambda f. q \circ f \circ p'$$

$\to(\langle p,q\rangle)$ is an embedding, with adjoint $\lambda g. q' \circ g \circ p$.

Theorem 3. → is ω-continuous.

Proof. Suppose that Δ is a cone in $(CPO^E)^2$; that is, Δ is in effect a pair of cones $D_0 \to D_1 \to \dots$, $E_0 \to E_1 \to \dots$ in CPO^E. Let these cones have colimits D,E via the embeddings $d_n: D_n \to D$, $e_n: E_n \to E$. $\to(\Delta)$ is a cone with vertex $[D \to E]$ and embeddings $\varphi_n: [D_n \to E_n] \to [D \to E]: f \to e_n f d_n'$; we have to show that this cone is colimiting. But this follows by Corollary 2, since

$$\begin{aligned} \bigsqcup \varphi_n \circ \varphi' &= \lambda g. \bigsqcup e_n \circ e_n' \circ g \circ d_n \circ d_n' \\ &= \lambda g. (\bigsqcup e_n \circ e_n') \circ g \circ (\bigsqcup d_n \circ d_n') && \text{(by continuity of } \circ \text{)} \\ &= \lambda g\, g && \text{(by Corollary 2)} \\ &= I . \end{aligned}$$

Turning to the powerdomain, let D be an algebraic cpo, and let M(D) be the set of non-empty finite sets of finite elements of D. M(D) is given the preorder $\sqsubseteq_M$ (the "Milner ordering") defined as follows:

$$A \sqsubseteq_M B \quad =_{df} \quad \forall a \in A\, \exists b \in B.\ a \sqsubseteq b \ \ \&\ \ \forall b \in B\, \exists a \in A.\ a \sqsubseteq b$$

It was shown in [5] that the powerdomain of D, as defined by Plotkin, is isomorphic to $\overline{M(D)}$; we will define (D) as $\overline{M(D)}$. To describe the action of $\mathcal{P}$ on arrows of $ACPO^E$, we proceed as follows. Given any continuous $f:D \to D'$ (D,D' algebraic), we define the "extension" $\hat{f}$ of f to (D) by:

$$\hat{f}(X) \quad = \quad \bigcup_{A \in X} [f(A)]$$

(of course, the operation [ ] is here taken w.r.t. the preorder $\sqsubseteq_M$ of $M(D')$)
The following properties are immediate: f is a continuous function; $I_D = I_{\mathcal{P}(D)}$; $\widehat{f\,g} = \hat{f}\,\hat{g}$; $\hat{}$ is monotone (that is, $f \sqsubseteq g \twoheadrightarrow \hat{f} \sqsubseteq \hat{g}$). It follows that if f is an embedding, then $\hat{f}$ is an embedding (with adjoint $\hat{f'}$). Thus, if we define $\mathcal{P}$ on arrows by: $\mathcal{P}(f) = \hat{f}$, then $\mathcal{P}:ACPO^E \to ACPO^E$ is indeed a functor.

It will be useful to have the property of local continuity for this functor:

Lemma 2. If $f_n:D \to D'$ is an increasing sequence of continuous functions (D,D' algebraic), then $(\bigsqcup_n f_n)^\wedge = \bigsqcup_n \hat{f}_n$.

Proof. The conclusion of the lemma is equivalent to the following statement: for any $A \in M(D)$, $[(\bigsqcup \hat{f}_n)(a)] = \bigcup_n [\hat{f}_n(A)]$. The right-to-left inclusion here is trivial. For the left-to-right inclusion, suppose that $B \sqsubseteq_M (\bigsqcup f_n)(A)$. Since the elements of B are finite, we can choose (1) k such that $\forall a \in f_k(A) \exists b \in B.\ b \sqsubseteq a$, and (2) l such that $\forall b \in B\ \exists a \in f_l(A).\ b \sqsubseteq a$. Setting $m = \max(k,l)$, we have $B \sqsubseteq_M f_m(A)$, and so $B \in \bigcup_n [f_n(A)]$.

Theorem 4. is ω-continuous.

Proof. Again using Corollary 2, it suffices to show that if $\bigsqcup_n f_n \circ f'_n = I_D$ (where each $f_n:D_n \to D$ is an embedding), then $\bigsqcup_n \hat{f}_n \circ \hat{f}'_n = I_{\mathcal{P}(D)}$.
Now

$$\bigsqcup_n \hat{f}_n \cdot \hat{f}'_n = \bigsqcup_n (f_n \circ f'_n)^\wedge = (\bigsqcup_n f_n\ f'_n)^\wedge \qquad \text{(by Lemma 2)}$$
$$= \hat{I}_D = I_{(D)} .$$

The functors $\rightarrow$ and $\mathcal{P}$ have been defined for different categories. If we wish to solve a recursive domain equation which involves both functors, we need to have a single category which is closed under both of them. For this purpose Plotkin introduces the <u>SFP objects</u> - these are the cpo's which are colimits of ω-chains of <u>finite</u> cpo's. Every SFP object is algebraic, by the remarks following Definition 3 (closure under inverse limit construction of $ACPO^E$). Thus SFP^E (category of SFP objects and embeddings) is a full subcategory of $ACPO^E$. If D,E are finite, then $\mathcal{P}(D)$ and $D \rightarrow E$ are (trivially) finite. It follows by Theorems 3,4 that SFP^E is closed under both constructions.

It remains only to show that SFP^E admits ω-colimits. A proof of this result is given in [3]. An alternative proof - which proceeds by way of showing that SFP^E is an "algebraic category" - may be found in [5].

R E F E R E N C E S

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