THE CAUCHY PROBLEM FOR THE EULER EQUATIONS FOR COMPRESSIBLE FLUIDS

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Abstract. Some recent developments in the study of the Cauchy problem for the Euler equations for compressible fluids are reviewed. The local and global well-posedness for smooth solutions is presented, and the formation of singularity is exhibited; then the local and global well-posedness for discontinuous solutions, including the BV theory and the L^∞ theory, is extensively discussed. Some recent developments in the study of the Euler equations with source terms are also reviewed.

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1. Introduction

The Cauchy problem for the Euler equations for compressible fluids in d-space dimensions is the initial value problem for the system of $d+2$ conservation laws

$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \nabla \cdot \left(\dfrac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \\ \partial_t E + \nabla \cdot \left(\dfrac{\mathbf{m}}{\rho}(E+p) \right) = 0, \end{cases} \tag{1.1}$$

for $(\mathbf{x}, t) \in \mathbb{R}^{d+1}_+, \mathbb{R}^{d+1}_+ := \mathbb{R}^d \times (0, \infty)$, with initial data

$$(\rho, \mathbf{m}, E)|_{t=0} = (\rho_0, \mathbf{m}_0, E_0)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \tag{1.2}$$

where $(\rho_0, \mathbf{m}_0, E_0)(\mathbf{x})$ is a given vector function of $\mathbf{x} \in \mathbb{R}^d$.

System (1.1) is closed by the constitutive relations

$$p = p(\rho, e), \qquad E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\rho} + \rho e. \tag{1.3}$$

In (1.1) and (1.3), $\tau = 1/\rho$ is the deformation gradient (specific volume for fluids, strain for solids), $\mathbf{v} = (v_1, \cdots, v_d)^\top$ is the fluid velocity, with $\rho \mathbf{v} = \mathbf{m}$ the momentum vector, p is the scalar pressure, and E is the total energy, with e the internal energy which is a given function of (τ, p) or (ρ, p) defined through thermodynamical relations. The notation $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The other two thermodynamic variables are the temperature θ and the entropy S. If (ρ, S) are chosen as the independent variables, then the constitutive relations can be written into:

$$(e, p, \theta) = (e(\rho, S), p(\rho, S), \theta(\rho, S)), \tag{1.4}$$

governed by the First Law of Thermodynamics:

$$\theta dS = de + p d\tau = de - \frac{p}{\rho^2} d\rho. \tag{1.5}$$

For a polytropic gas,

$$p = R\rho\theta, \qquad e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v}, \tag{1.6}$$

and

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v}, \tag{1.7}$$

where $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas, $c_v > 0$ is the specific heat at constant volume, $\gamma > 1$ is the adiabatic exponent, and $\kappa > 0$ can be any constant under scaling.

As it will be shown in §4, no matter how smooth the Cauchy data (1.2) are, solutions of (1.1) generally develop singularities in a finite time. Hence, System (1.1) is complemented by the Clausius inequality

$$\partial_t(\rho a(S)) + \nabla \cdot (\mathbf{m} a(S)) \geq 0 \tag{1.8}$$

in the sense of distributions for any $a(S) \in C^1, a'(S) \geq 0$, in order to single out physically relevant discontinuous solutions, called entropy solutions.

The Euler equations for a compressible fluid that flows isentropically take the following simpler form:

$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \nabla \cdot \left(\dfrac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \end{cases} \tag{1.9}$$

where the pressure is regarded as a function of density, $p = p(\rho, S_0)$, with constant S_0. For a polytropic gas,

$$p(\rho) = \kappa_0 \rho^\gamma, \qquad \gamma > 1, \tag{1.10}$$

where $\kappa_0 > 0$ is any constant under scaling. This system can be derived as follows. It is well-known that, for smooth solutions of (1.1), the entropy $S(\rho, E)$ is conserved along fluid particle trajectories, i.e.,

$$\partial_t(\rho S) + \nabla \cdot (\mathbf{m}S) = 0. \tag{1.11}$$

If the entropy is initially a uniform constant and the solution remains smooth, then (1.11) implies that the energy equation can be eliminated, and the entropy S keeps the same constant in later time, in comparison with non-smooth solutions (entropy solutions) for which only $S(x,t) \geq \min S(x,0)$ is generally available (see [297]). Thus, under constant initial entropy, a smooth solution of (1.1) satisfies the equations in (1.9). Furthermore, it should be observed that solutions of System (1.9) are also a good approximation to solutions of System (1.1) even after shocks form, since the entropy increases across a shock to third-order in wave strength for solutions of (1.1) (cf. [120]), while in (1.9) the entropy is constant. Moreover, System (1.9) is an excellent model for isothermal fluid flow with $\gamma = 1$, and for shallow water flow with $\gamma = 2$.

In the one-dimensional case, System (1.1) in Eulerian coordinates is

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p\right) = 0, \\ \partial_t E + \partial_x \left(\frac{m}{\rho}(E + p)\right) = 0, \end{cases} \tag{1.12}$$

with $E = \frac{1}{2}\frac{m^2}{\rho} + \rho e$. The system above can be rewritten in Lagrangian coordinates in one-to-one correspondence so long as the fluid flow stays away from the vacuum $\rho = 0$:

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t (e + \frac{v^2}{2}) + \partial_x (pv) = 0, \end{cases} \tag{1.13}$$

with $v = m/\rho$, where the coordinates (x,t) are the Lagrangian coordinates, which are different from the Euler coordinates for (1.12); for simplicity of notations, we do not distinguish them. For the isentropic case, Systems (1.12) and (1.13) reduce to:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p\right) = 0, \end{cases} \tag{1.14}$$

and

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \end{cases} \tag{1.15}$$

respectively, where the pressure p is determined by (1.10) for the polytropic case, $p = p(\rho) = \tilde{p}(\tau), \tau = 1/\rho$.

The Cauchy problem for all the systems above fits into the following general conservation form:

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{u} \in \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.16}$$

with initial data:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(\mathbf{x}), \tag{1.17}$$

where $\mathbf{f} = (\mathbf{f}_1, \cdots, \mathbf{f}_d) : \mathbb{R}^n \to (\mathbb{R}^n)^d$ is a nonlinear mapping with $\mathbf{f}_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \cdots, d$. Besides (1.1)–(1.15), many partial differential equations arising in the physical or engineering sciences can be also formulated into the form (1.16) or its variants. The hyperbolicity of

System (1.16) requires that, for any $\omega \in S^{d-1}$, the matrix $(\nabla \mathbf{f}(\mathbf{u}) \cdot \omega)_{n\times n}$ have n real eigenvalues $\lambda_i(\mathbf{u}, \omega), i = 1, 2, \cdots, n$, and be diagonalizable.

One of the main difficulties in dealing with (1.16) and (1.17) is that solutions of the Cauchy problem (even those starting out from smooth initial data) generally develop singularities in a finite time, because of the physical phenomena of focusing and breaking of waves and the development of shock waves and vortices, among others. For this reason, attention focuses on solutions in the space of discontinuous functions. Therefore, one can not directly use the classical analytic techniques that predominate in the theory of partial differential equations of other types.

Another main difficulty is nonstrict hyperbolicity or resonance of (1.16), that is, there exist some $\omega_0 \in S^{d-1}$ and $\mathbf{u}_0 \in \mathbb{R}^d$ such that $\lambda_i(\mathbf{u}_0, \omega_0) = \lambda_j(\mathbf{u}_0, \omega_0)$ for some $i \neq j$. In particular, for the Euler equations, such a degeneracy occurs at the vacuum states or from the multiplicity of eigenvalues of the system.

The correspondence of (1.8) in the context of hyperbolic conservation laws is the Lax entropy inequality:

$$\partial_t \eta(\mathbf{u}) + \nabla \cdot \mathbf{q}(\mathbf{u}) \leq 0 \tag{1.18}$$

in the sense of distributions for any C^2 entropy-entropy flux pair $(\eta, \mathbf{q}) : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^d, \mathbf{q} = (q_1, \cdots, q_d)$, satisfying

$$\nabla^2 \eta(\mathbf{u}) \geq 0, \qquad \nabla q_i(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_i(\mathbf{u}), \quad i = 1, \cdots, d.$$

Most sections in this paper focus on the Cauchy problem for one-dimensional hyperbolic systems of n conservation laws

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{u} \in \mathbb{R}^n, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.19}$$

with Cauchy data:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x). \tag{1.20}$$

The Euler equations can describe more complicated physical fluid flows by coupling with other physical equations.

One of the most important examples is the Euler equations for nonequilibrium thermodynamic fluid flow. In local thermodynamic equilibrium as we discussed above, System (1.1) is closed by the constitutive relation (1.3). When the temperature varies over a wide range, the gas may not be in local thermodynamic equilibrium, and the pressure p may then be regarded as a function of only a part e of the specific internal energy, while another part q is governed by a rate equation:

$$\partial_t(\rho q) + \nabla_x \cdot (\mathbf{m} q) = \frac{Q(\rho, e) - q}{\epsilon\, s(\rho, e)}, \tag{1.21}$$

and

$$p = p(\rho, e), \qquad E = \frac{|\mathbf{m}|^2}{2\rho} + \rho(e + q), \tag{1.22}$$

where $\epsilon > 0$ is a parameter measuring the relaxation time, which is small in general, and $Q(\rho, e)$ and $s(\rho, e)$ are given functions of (ρ, e). The equations in (1.1) and (1.21) with (1.22) define the Euler equations for nonequilibrium fluids, which model the nonequilibrium thermodynamical process.

Another important example is the inviscid combustion equations that consist of the Euler equations in (1.1) adjoined with the continuum chemistry equation:

$$\partial_t(\rho Z) + \nabla \cdot (\mathbf{m} Z) = -\phi(\theta)\rho Z, \quad \phi(\theta) = K e^{-\theta_0/\theta}, \tag{1.23}$$

where θ_0 and K are some positive constants, Z denotes the mass fraction of unburnt gas so that $1 - Z$ is the mass fraction of burnt gas. Here we assume that there are only two species present, the unburnt gas and the burnt gas, and the unburnt gas is converted to the burnt gas through a one-step irreversible exothermic chemical reaction with an Arrhenius kinetic mechanism. As regards the equations in (1.1), a modification of the internal energy e is the only change in these equations. The internal energy of the mixture, $e(\rho, S, Z)$, is defined within a constant by

$$e(\rho, S, Z) = Ze_u(\rho, S) + (1 - Z)e_b(\rho, S),$$

with e_u and e_b the internal energies of the unburnt and burnt gas, respectively. For simplicity, we assume that both of the burnt and unburnt gas are ideal with the same γ-law so that

$$e_u(\rho, S) = c_v\theta + q_0, \qquad e_b = c_v\theta,$$

with $q_0 > 0$ the normalized energy of formation at some reference temperature for the unburnt gas for an exothermic reaction. Then

$$e(\rho, S, Z) = c_v\theta(\rho, S) + q_0 Z, \qquad \theta(\rho, S) = \frac{p(\rho, S)}{R\rho}. \tag{1.24}$$

Then the equations in (1.1) and (1.23) with (1.24) define the inviscid combustion equations, which model detonation waves in combustion.

This paper is organized as follows.

In §2, we present a local well-posedness theory for smooth solutions and then in §3 a global well-posedness theory for smooth solutions. In §4, we exhibit the formation of singularity in smooth solutions, the main feature of the Cauchy problem for the Euler equations. In §5, we present a local well-posedness theory for discontinuous entropy solutions.

From §6 to §10, we discuss global well-posedness theories for discontinuous entropy solutions.

In §6, we present a global theory for discontinuous entropy solutions of the Riemann problem, the simplest Cauchy problem with discontinuous initial data. First we recall two Lax's theorems for the local behavior of wave curves in the phase space and the existence of global solutions of the Riemann problem, respectively, for general one-dimensional conservation laws with small Riemann data. Then we discuss the construction of global Riemann solutions and their behavior for the isothermal, isentropic, and non-isentropic Euler equations in (1.12)–(1.15) with large Riemann data, respectively.

In §7, we focus on the global discontinuous solutions obtained from the Glimm scheme [130], called Glimm solutions. We first describe the Glimm scheme for hyperbolic conservation laws and a global well-posedness theory for the Glimm solutions, including the existence, decay, and L^1-stability of the Glimm solutions. The Glimm scheme is also applied to the construction of global entropy solutions of the isothermal Euler equations with large initial data. We also present an alternative method, the wave-front tracking method, to construct global discontinuous solutions, which can be identified with a trajectory of the standard Riemann semigroup, and to yield the L^1-stability of the solutions.

In §8, our focus is on general global discontinuous solutions in $L^\infty \cap BV_{loc}$ satisfying the Lax entropy inequality and without specific reference on the method for construction of the solutions. We first describe a theory of generalized characteristics and its direct applications to the decay problem of the discontinuous solutions under the assumption that the traces of the solutions along any space-like curves are functions of locally bounded variation. Then we study the uniqueness of Riemann solutions and the asymptotic stability of entropy solutions in BV for gas dynamics, without additional a priori information on the solutions besides the natural Lax entropy inequality.

In §9, our focus is on the one-dimensional system of the isentropic Euler equations and its global discontinuous solutions in L^∞ satisfying only the weak Lax entropy inequality. We first carefully analyze the system and its entropy-entropy flux pairs. Then we describe a general compactness framework, with a proof for the case $\gamma = 5/3$, for establishing the existence, compactness, and decay of entropy solutions in L^∞, and the convergence of finite-difference schemes including the Lax-Friedrichs scheme and the Godunov scheme. We discuss the stability of rarefaction waves and vacuum states even in a broader class of discontinuous entropy solutions in L^∞. We also record some related results for the system of elasticity and the non-isentropic Euler equations.

In §10, we discuss global discontinuous solutions for the multidimensional case. We describe a shock capturing difference scheme and its applications to the multidimensional Euler equations for compressible fluids with geometric structure. Then we present some classifications and phenomena of solution structures of the two-dimensional Riemann problem, especially wave interactions and elementary waves, for the Euler equations and some further results in this direction.

In §11, we consider the Euler equations for compressible fluids with source terms. Our focus is on two of the most important examples: relaxation effect and combustion effect. Some new phenomena are reviewed.

We remark that, in this paper, we focus only on some recent developments in the theoretical study of the Cauchy problem for the Euler equations for compressible fluids. We refer the reader to other papers in these volumes, as well as Glimm-Majda [134], Godlewski-Raviart [138], LeVeque [189], Lions [201], Perthame [255], Tadmor [296], Toro [306], and the references cited therein for related topics including various kinetic formulations and approximate methods for the Cauchy problem for the Euler equations.

2. Local Well-Posedness for Smooth Solutions

Consider the three-dimensional Euler equations in (1.1) and (1.7) for polytropic compressible fluids staying away from the vacuum, which are rewritten in terms of the density $\rho \in \mathbb{R}$, the velocity $\mathbf{v} \in \mathbb{R}^3$, and the entropy $S \in \mathbb{R}$ (taking $\kappa = c_v = 1$ without loss of generality) in the form:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \\ \partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0, \\ \partial_t S + \mathbf{v} \cdot \nabla S = 0, \end{cases} \tag{2.1}$$

with the equation of state: $p = p(\rho, S) = \rho^\gamma e^S$, $\gamma > 1$. System (2.1) is a 5×5 system of conservation laws. It can be written in terms of the variables $(p, \mathbf{v}, S)$ in the equivalent form in the region where the solution is smooth:

$$\begin{cases} \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \, \nabla \cdot \mathbf{v} = 0, \\ \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla p = 0, \\ \partial_t S + \mathbf{v} \cdot \nabla S = 0, \end{cases} \tag{2.2}$$

with $\rho = \rho(p, S) = p^{1/\gamma} e^{-S/\gamma}$.

The norm of the Sobolev space $H^s(\mathbb{R}^d)$ is denoted by

$$\|g\|_s^2 = \sum_{|\alpha| \le s} \int_{\mathbb{R}^d} |D^\alpha g|^2 dx.$$

For $g \in L^\infty([0, T]; H^s)$, define

$$|||g|||_{s,T} = \sup_{0 \le t \le T} \|g(\cdot, t)\|_s.$$

For the Cauchy problem of (2.2) with smooth initial data:

$$(p, \mathbf{v}, S)|_{t=0} = (p_0, \mathbf{v}_0, S_0)(\mathbf{x}), \tag{2.3}$$

the following local existence theorem of smooth solutions holds.

Theorem 2.1. *Assume* $(p_0, \mathbf{v}_0, S_0) \in H^s \cap L^\infty(\mathbb{R}^3)$ *with* $s > 5/2$ *and* $p_0(\mathbf{x}) > 0$. *Then there is a finite time* $T \in (0, \infty)$, *depending on the* H^s *and* L^∞ *norms of the initial data, such that the Cauchy problem* (2.2) *and* (2.3) *has a unique bounded smooth solution* $(p, \mathbf{v}, S) \in C^1(\mathbb{R}^3 \times [0, T])$, *with* $p(\mathbf{x}, t) > 0$ *for all* $(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, T]$, *and* $(p, \mathbf{v}, S) \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.

Consider the Cauchy problem (1.16) and (1.17) for a general hyperbolic system of conservation laws with the values of $\mathbf{u}$ lying in the state space G, an open set in $\mathbb{R}^n$. The state space G arises because physical quantities such as the density should be positive. Assume that (1.16) has the following structure of symmetric hyperbolic systems: For all $\mathbf{u} \in G$, there is a positive definite symmetric matrix $A_0(\mathbf{u})$ that is smooth in $\mathbf{u}$ and satisfies

$$c_0^{-1}\mathbf{I}_n \leq A_0(\mathbf{u}) \leq c_0\mathbf{I}_n \tag{2.4}$$

with a constant c_0 uniform for $\mathbf{u} \in G_1$, for any $G_1 \subset \overline{G_1} \Subset G$, such that $A_i(\mathbf{u}) = A_0(\mathbf{u})\nabla\mathbf{f}_i(\mathbf{u})$ is symmetric, where $\nabla\mathbf{f}_i(\mathbf{u})$, $i = 1, \cdots, d$, are the $n \times n$ Jacobian matrices and $\mathbf{I}_n$ is the $n \times n$ identity matrix. A consequence of this structure for (1.16) is that the linearized problem of (1.16) and (1.17) is well-posed (see Majda [223]). The matrix $A_0(\mathbf{u})$ is called the symmetrizing matrix of System (1.16). Multiplying (1.16) by the matrix $A_0(\mathbf{u})$ and denoting $A(\mathbf{u}) = (A_1(\mathbf{u}), \cdots, A_d(\mathbf{u}))$ yield the system:

$$A_0(\mathbf{u})\partial_t\mathbf{u} + A(\mathbf{u})\nabla\mathbf{u} = 0. \tag{2.5}$$

An important observation is that almost all equations of classical physics of the form (1.16) admit this structure. For example, the equations in (2.2) for polytropic gases are symmetrized by the 5×5 matrix

$$A_0(p, S) = \begin{pmatrix} (\gamma p)^{-1} & 0 & 0 \\ 0 & \rho(p, S)\mathbf{I_3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, Theorem 2.1 is a consequence of the following theorem on the local existence of smooth solutions, with the specific state space $G = \{(p, \mathbf{v}, S)^\top \,:\, p > 0\} \subset \mathbb{R}^5$, for the general symmetric hyperbolic system (1.16).

Theorem 2.2. *Assume that* $\mathbf{u}_0 : \mathbb{R}^d \to G$ *is in* $H^s \cap L^\infty$ *with* $s > \frac{d}{2} + 1$. *Then, for the Cauchy problem* (1.16) *and* (1.17), *there exists a finite time* $T = T(\|\mathbf{u}_0\|_s, \|\mathbf{u}_0\|_{L^\infty}) \in (0, \infty)$ *such that there is a unique bounded classical solution* $\mathbf{u} \in C^1(\mathbb{R}^d \times [0, T])$ *with* $\mathbf{u}(x, t) \in G$ *for* $(x, t) \in \mathbb{R}^d \times [0, T]$ *and* $\mathbf{u} \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$.

The proof of this theorem proceeds via a classical iteration scheme. An outline of the proof of Theorem 2.2 (thus Theorem 2.1) is given as follows.

To prove the existence of the smooth solution of (1.16) and (1.17), it is equivalent to construct the smooth solution of (2.5) and (1.17) by applying the symmetrizing matrix $A_0(\mathbf{u})$. Choose the standard mollifier $j(\mathbf{x}) \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp} j(\mathbf{x}) \subseteq \{\mathbf{x} :\ |\mathbf{x}| \leq 1\}$, $j(\mathbf{x}) \geq 0$, $\int_{\mathbb{R}^d} j(\mathbf{x})d\mathbf{x} = 1$, and set $j_\epsilon(\mathbf{x}) = \epsilon^{-d}j(\mathbf{x}/\epsilon)$. For $k = 0, 1, 2, \cdots$, take $\epsilon_k = 2^{-k}\epsilon_0$, where $\epsilon_0 > 0$ is a constant, and define $\mathbf{u}_0^k \in C^\infty(\mathbb{R}^d)$ by

$$\mathbf{u}_0^k(\mathbf{x}) = J_{\epsilon_k}\mathbf{u}_0(\mathbf{x}) = \int_{\mathbb{R}^d} j_{\epsilon_k}(\mathbf{x} - \mathbf{y})\mathbf{u}(\mathbf{y})d\mathbf{y}.$$

We construct the solution of (2.5) and (1.17) through the following iteration scheme: Set $\mathbf{u}^0(\mathbf{x}, t) = \mathbf{u}_0^0(\mathbf{x})$ and define $\mathbf{u}^{k+1}(\mathbf{x}, t)$, for $k = 0, 1, 2, \cdots$, inductively as the solution of the linear equations:

$$A_0(\mathbf{u}^k)\partial_t\mathbf{u}^{k+1} + A(\mathbf{u}^k)\nabla\mathbf{u}^{k+1} = 0, \quad \mathbf{u}^{k+1}|_{t=0} = \mathbf{u}_0^{k+1}(\mathbf{x}). \tag{2.6}$$

From the well-known properties of the mollification: $\|\mathbf{u}_0^k - \mathbf{u}_0\|_s \to 0$, as $k \to \infty$, and $\|\mathbf{u}_0^k - \mathbf{u}_0\|_0 \le C_0\epsilon_k\|\mathbf{u}_0\|_1$, for some constant C_0, it is evident that $\mathbf{u}^{k+1} \in C^\infty(\mathbb{R}^d \times [0, T_k])$ is well-defined on the time interval $[0, T_k]$. Here $T_k > 0$ denotes the largest time where the estimate $|||\mathbf{u}^k - \mathbf{u}_0^0|||_{s,T_k} \le C_1$ holds for some constant $C_1 > 0$. Then there is a constant $T_* > 0$ such that $T_k \ge T_*$ ($T_0 = \infty$) for $k = 0, 1, 2, \cdots$, which follows from the following estimates:

$$|||\mathbf{u}^{k+1} - \mathbf{u}_0^0|||_{s,T_*} \le C_1, \quad |||\mathbf{u}_t^{k+1}|||_{s-1,T_*} \le C_2, \tag{2.7}$$

for all $k = 0, 1, 2, \cdots$, with some constant $C_2 > 0$.

From (2.6), we obtain

$$A_0(\mathbf{u}^k)\partial_t(\mathbf{u}^{k+1} - \mathbf{u}^k) + A(\mathbf{u}^k)\nabla(\mathbf{u}^{k+1} - \mathbf{u}^k) = E_k, \tag{2.8}$$

where

$$E_k = -(A_0(\mathbf{u}^k) - A_0(\mathbf{u}^{k-1}))\partial_t\mathbf{u}^k - (A(\mathbf{u}^k) - A(\mathbf{u}^{k-1}))\nabla\mathbf{u}^k.$$

Use the standard energy estimate method for the linearized problem (2.8) to obtain

$$|||\mathbf{u}^{k+1} - \mathbf{u}^k|||_{0,T} \le Ce^{CT}\left(\|\mathbf{u}_0^{k+1} - \mathbf{u}_0^k\|_0 + T|||E_k|||_{0,T}\right).$$

The property of mollification, (2.7), and Taylor's theorem yield

$$\|\mathbf{u}_0^{k+1} - \mathbf{u}_0^k\|_0 \le C2^{-k}, \quad |||E_k|||_{0,T} \le C|||\mathbf{u}^k - \mathbf{u}^{k-1}|||_{0,T}.$$

For small T such that $C^2T\exp(CT) < 1$, one obtains

$$\sum_{k=1}^{\infty} |||\mathbf{u}^{k+1} - \mathbf{u}^k|||_{0,T} < \infty,$$

which implies that there exists $\mathbf{u} \in C([0, T]; L^2(\mathbb{R}^d))$ such that

$$\lim_{k\to\infty} |||\mathbf{u}^k - \mathbf{u}|||_{0,T} = 0. \tag{2.9}$$

From (2.7), we have $|||\mathbf{u}^k|||_{s,T} + |||\mathbf{u}_t^k|||_{s-1,T} \le C$, and $\mathbf{u}^k(x, t)$ belongs to a bounded set of G for $(x, t) \in \mathbb{R}^d \times [0, T]$. Then the interpolation inequalities imply that, for any r with $0 \le r < s$,

$$|||\mathbf{u}^k - \mathbf{u}^l|||_{r,T} \le C_s|||\mathbf{u}^k - \mathbf{u}^l|||_{0,T}^{1-r/s}|||\mathbf{u}^k - \mathbf{u}^l|||_{s,T}^{r/s} \le C|||\mathbf{u}^k - \mathbf{u}^l|||_{0,T}^{1-r/s}. \tag{2.10}$$

From (2.9) and (2.10), $\lim_{k\to\infty} |||\mathbf{u}^k - \mathbf{u}|||_{r,T} = 0$ for any $0 \le r < s$. Thus, choosing $r > \frac{d}{2} + 1$, Sobolev's lemma implies

$$\mathbf{u}^k \to \mathbf{u} \qquad \text{in} \quad C([0, t]; C^1(\mathbb{R}^d)). \tag{2.11}$$

From (2.8) and (2.11), one can conclude that $\mathbf{u}^k \to \mathbf{u}$ in $C([0, T]; C(\mathbb{R}^d))$, $\mathbf{u} \in C^1(\mathbb{R}^d \times [0, T])$, and $\mathbf{u}(\mathbf{x}, t)$ is the smooth solution of (1.16) and (1.17).

To prove $\mathbf{u} \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$, it is sufficient to prove $\mathbf{u} \in C([0, T]; H^s)$, since it follows from the equations in (2.5) that $\mathbf{u} \in C^1([0, T]; H^{s-1})$. The proof can be further reduced to verifying that $\mathbf{u}(\mathbf{x}, t)$ is strongly right-continuous at $t = 0$, since the same argument works for the strong right-continuity at any other $t \in [0, T)$, and the strong right-continuity on $[0, T)$ implies the strong left-continuity on $(0, T]$ because the equations in (2.5) are reversible in time.

Remark 2.1. Theorem 2.2 was established by Majda [223] which relies solely on the elementary linear existence theory for symmetric hyperbolic systems with smooth coefficients (Courant-Hilbert [77]), as we illustrated above. Moreover, a sharp continuation principle was also proved there: For $\mathbf{u}_0 \in H^s$, with $s > \frac{d}{2} + 1$, the interval $[0, T)$ with $T < \infty$ is the maximal interval of the classical H^s existence for (1.16) if and only if either $\|(\mathbf{u}_t, D\mathbf{u})\|_{L^\infty} \to \infty$ as $t \to T$, or, as $t \to T$, $\mathbf{u}(\mathbf{x}, t)$ escapes every compact subset $K \Subset G$. The first catastrophe in this principle is associated with the formation of shock waves in the smooth solutions, and the second is associated with a blow-up phenomenon.

Kato also gave a proof of Theorem 2.2, in [164], which uses the abstract semigroup theory of evolution equations to treat appropriate linearized problems. In [165], Kato also formulated and applied this basic idea in an abstract framework which yields the local existence of smooth solutions for many interesting equations of mathematical physics. See Crandall-Souganidis [78] for related discussions.

In [226], Makino-Ukai-Kawashima established the local existence of classical solutions of the Cauchy problem with compactly supported initial data for the multidimensional Euler equations, with the aid of the theory of quasilinear symmetric hyperbolic systems; in particular, they introduced a symmetrization which works for initial data having compact support or vanishing at infinity. There are also discussions on the local existence of smooth solutions of the three-dimensional Euler equations (2.1) in Chemin [35].

Remark 2.2. For the one-dimensional Cauchy problem (1.19) and (1.20), it is known from Friedrichs [121], Lax [175], and Li-Yu [195] that, if $\mathbf{u}_0(x)$ is in C^1 for all $x \in \mathbb{R}$ with finite C^1 norm, then there is a unique C^1 solution $\mathbf{u}(x,t)$, for $(x,t) \in \mathbb{R} \times [0,T]$, with sufficiently small T. As a consequence, the one-dimensional Euler equations in (1.12)-(1.15) admit a unique local C^1 solution provided that the initial data are in C^1 with finite C^1 norm and stay away from the vacuum.

3. Global Well-Posedness for Smooth Solutions

Consider the Cauchy problem for the one-dimensional isentropic Euler equations of gas dynamics in (1.14), for $x \in \mathbb{R}$ and $t > 0$, with initial data:

$$(\rho, m)|_{t=0} = (\rho_0, m_0)(x), \tag{3.1}$$

and γ-law for pressure:

$$p(\rho) = \rho^\gamma/\gamma, \quad \gamma > 1. \tag{3.2}$$

For the case $1 < \gamma \leq 3$, which is of physical significance, System (1.14) is genuinely nonlinear in the sense of Lax [181] in the domain $\{(x,t) : \rho(x,t) \geq 0\}$. For $\rho > 0$, consider the velocity $v = m/\rho$ and $v_0(x) = m_0(x)/\rho_0(x)$. The eigenvalues of (1.14) are

$$\lambda_1 = v - c, \quad \lambda_2 = v + c,$$

where $c = \rho^\theta$, with $\theta = \frac{\gamma-1}{2} \in (0,1]$, is the sound speed. The Riemann invariants of (1.14) are

$$w_1 = w_1(\rho, v) := v + \frac{\rho^\theta}{\theta}, \quad w_2 = w_2(\rho, v) := v - \frac{\rho^\theta}{\theta}.$$

Set

$$w_{10}(x) := w_1(\rho_0(x), v_0(x)), \quad w_{20}(x) := w_2(\rho_0(x), v_0(x))$$

as the initial values of the Riemann invariants. With the aid of the method of characteristics (see Lax [178]), the following global existence theorem of smooth solutions of (1.14) and (3.1) can be proved.

Theorem 3.1. *Suppose that the initial data $(\rho_0, v_0)(x)$, with $\rho_0(x) > 0$, are in $C^1(\mathbb{R})$, with finite C^1 norm and*

$$w_{10}'(x) \geq 0, \quad w_{20}'(x) \geq 0, \tag{3.3}$$

for all $x \in \mathbb{R}$. Then the Cauchy problem (1.14) and (3.1) has a unique global C^1 solution $(\rho, v)(x,t)$, with $\rho(x,t) > 0$ for all $x \in \mathbb{R}$ and $t > 0$.

Proof. First we show that, if $\rho_0(x) > 0$, no vacuum will be developed at any time $t > 0$ for the smooth solution. From the first equation of (1.14),

$$\frac{d}{dt}\rho = -\rho \partial_x v, \tag{3.4}$$

where

$$\frac{d}{dt} = \partial_t + v(x,t)\partial_x$$

denotes the directional derivative along the direction

$$\frac{dx}{dt} = v(x,t). \tag{3.5}$$

For any point $(\bar{x},\bar{t}) \in \mathbb{R}^2_+ := \{(x,t) : x \in \mathbb{R},\ t \in \mathbb{R}_+\}, \mathbb{R}_+ = (0,\infty)$, the integral curve of (3.5) through $(\bar{x},\bar{t})$ is denoted by $x = x(t;\bar{x},\bar{t})$. At $t = 0$, it passes through the point $(x_0(\bar{x},\bar{t}),0) := (x(0;\bar{x},\bar{t}),0)$. Along the curve $x = x(t;\bar{x},\bar{t})$, the solution of the ordinary differential equation (3.4) with initial data:

$$\rho|_{t=0} = \rho_0(x_0(\bar{x},\bar{t}))$$

is

$$\rho(\bar{x},\bar{t}) = \rho_0(x_0(\bar{x},\bar{t}))\exp\left(-\int_0^{\bar{t}} \partial_x v(x(t;\bar{x},\bar{t}),t)dt\right) > 0.$$

To prove the global existence of the C^1 solution $(\rho,v)(x,t)$, given the local existence from Remark 2.2, it is sufficient to prove the following uniform a priori estimate: For any fixed $T > 0$, if the Cauchy problem (1.14) and (3.1) has a unique C^1 solution $(\rho,v)(x,t)$ for $x \in \mathbb{R}$ and $t \in [0,T)$, then the C^1 norm of $(\rho,v)(x,t)$ is bounded on $\mathbb{R} \times [0,T]$.

For a smooth solution (ρ,v) of System (1.14), one can verify by straightforward calculations that the derivatives of the Riemann invariants w_1 and w_2 along the characteristics are zero:

$$w_1' = 0, \quad w_2^{\backprime} = 0, \tag{3.6}$$

where $' = \partial_t + \lambda_2\partial_x$ and $^{\backprime} = \partial_t + \lambda_1\partial_x$ are the differentiation operators along the characteristics. Differentiate the equation $w_1' = 0$ in (3.6) with respect to the spatial variable x to obtain

$$\partial^2_{tx}w_1 + \lambda_2\,\partial^2_{xx}w_1 + \partial_{w_1}\lambda_2\,(\partial_x w_1)^2 + \partial_{w_2}\lambda_2\,\partial_x w_1\,\partial_x w_2 = 0.$$

Since $0 = w_2^{\backprime} = w_2' - 2c\,\partial_x w_2$, by setting $r = \partial_x w_1$ and noticing

$$\lambda_2 = \lambda_2(w_1,w_2) = \frac{1+\theta}{2}w_1 + \frac{1-\theta}{2}w_2, \qquad \partial_x w_2 = \frac{w_2'}{2c},$$

one has

$$r' + \frac{1+\theta}{2}r^2 + \frac{1-\theta}{4c}w_2' r = 0.$$

Set

$$s = \frac{\theta-1}{2}\ln\rho = \frac{\theta-1}{2\theta}\ln(w_1-w_2).$$

Then

$$\partial_{w_2}s = \frac{1-\theta}{4c} \quad \text{and} \quad s' = w_2'\,\partial_{w_2}s = \frac{1-\theta}{4c}w_2'.$$

Thus

$$r' + \frac{1+\theta}{2}r^2 + s'r = 0.$$

Set

$$g = e^s r = \rho^{(\theta-1)/2}\partial_x w_1.$$

Then

$$g' = -\frac{1+\theta}{2}\left(\frac{\theta}{2}|w_1-w_2|\right)^{\frac{1-\theta}{2\theta}} g^2. \tag{3.7}$$

Similarly, for $h = \rho^{(\theta-1)/2}\partial_x w_2$, one has

$$h^{\backprime} = -\frac{1+\theta}{2}\left(\frac{\theta}{2}|w_1-w_2|\right)^{\frac{1-\theta}{2\theta}} h^2. \tag{3.8}$$

Let $x = x(\beta, t)$ be the forward characteristic passing through any fixed point $(\beta, 0)$ at $t = 0$, defined by

$$\frac{dx(\beta,t)}{dt} = \lambda_2(w_1(x(\beta,t),t), w_2(x(\beta,t),t)), \quad x(\beta,0) = \beta.$$

According to (3.6), w_1 is constant along characteristics, and thus $w_1(x(\beta,t),t) = w_1(\beta,0) = w_{10}(\beta)$ and $\sup|w_1(x,t)| = \sup|w_{10}(x)|$. Similarly, w_2 is constant along the backward characteristics corresponding to the eigenvalue λ_1, and $\sup|w_2(x,t)| = \sup|w_{20}(x)|$. For any given point $(x(\beta,t),t)$ on the forward characteristic $x = x(\beta,t)$, there exists a unique $\alpha = \alpha(\beta,t) \geq \beta$ such that $w_2(x(\beta,t),t) = w_{20}(\alpha)$. Therefore, along the characteristic $x = x(\beta,t)$, one has from (3.7) that

$$\begin{cases} \dfrac{dg(x(\beta,t),t)}{dt} = -\dfrac{1+\theta}{2}\left(\dfrac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} g(x(\beta,t),t)^2, \\ g|_{t=0} = \rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta). \end{cases} \tag{3.9}$$

Then

$$g(x(\beta,t),t) = \frac{\rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta)}{1 + \int_0^t K(\beta,\tau)d\tau}, \tag{3.10}$$

where

$$K(\beta,t) = \frac{1+\theta}{2}\left(\frac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} \rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta). \tag{3.11}$$

From (3.3), $K(\beta,t) \geq 0$. Thus, $g(x(\beta,t),t)$ is bounded, and

$$\partial_x w_1(x(\beta,t),t) = \left(\frac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} g(x(\beta,t),t)$$

is also bounded. Similarly, $\partial_x w_2$ is also bounded from (3.8). As a consequence, the C^1 norms of $\rho = (\theta(w_1 - w_2)/2)^{1/\theta}$ and $v = (w_1 + w_2)/2$ are bounded on $\mathbb{R} \times [0,T]$. The proof is complete. □

Remark 3.1. In the proof of Theorem 3.1, the second-order derivatives of the Riemann invariants are formally used. However, the final equality (3.10) does not involve these second-order derivatives. Some appropriate arguments of approximation or weak formulation can be used to show that the conclusion is still valid for C^1 solutions.

Remark 3.2. For the global existence of smooth solutions of general one-dimensional hyperbolic systems of conservation laws, we refer the reader to Li [194] which contains some results and discussions on this subject. Also see Lin [197, 198] and the references cited therein for the global existence of Lipschitz continuous solutions for the case that discontinuous initial data may not stay away from the vacuum. For the three-dimensional Euler equations for polytropic gases in (2.1), Serre and Grassin in [141, 142, 273] studied the existence of global smooth solutions under appropriate assumptions on the initial data for both isentropic and non-isentropic cases. It was proved in [141] that the three-dimensional Euler equations for a polytropic gas in (2.1) have global smooth solutions, provided that the initial entropy S_0 and the initial density ρ_0 are small enough and the initial velocity v_0 forces particles to spread out, which are of similar nature to the condition (3.3).

4. Formation of Singularities in Smooth Solutions

The formation of shock waves is a fundamental physical phenomenon manifested in solutions of the Euler equations for compressible fluids, which are a prototypical example of hyperbolic systems of conservation laws. This phenomenon can be explained by mathematical analysis by showing the finite-time formation of singularities in the solutions. For nonlinear scalar conservation laws, the development of shock waves can be explained through the intersection of characteristics; see the discussions in Lax [180, 181] and Majda [223]. For systems in one space dimension, this problem has been extensively studied by using the method of characteristics developed in Lax [178], John [161], Liu [206], Klainerman-Majda [170], Dafermos [83], etc. For systems with multidimensional space variables, the method of characteristics has not been proved tractable. An efficient method, involving the use of averaged quantities, was developed in Sideris [282] for hyperbolic systems of conservation laws and was further refined in Sideris [283] for the three-dimensional Euler equations. Also see Majda [223].

4.1. **One-Dimensional Euler Equations.**

Consider the Cauchy problem (1.14) and (3.1) for the one-dimensional Euler equations of isentropic gas dynamics. With the notations in §3, the following result on the formation of singularity in smooth solutions of (1.14) and (3.1) follows.

Theorem 4.1. *The lifespan of any smooth solution of* (1.14) *and* (3.1)*, staying away from the vacuum, is finite, for C^1 initial data $(\rho_0, v_0)(x)$, with $\rho_0(x) > 0$ and finite C^1 norm satisfying*

$$w_{10}'(\beta) < 0, \quad \text{or} \quad w_{20}'(\beta) < 0, \tag{4.1}$$

for some point $\beta \in \mathbb{R}$, Furthermore, if there exist two positive constants δ and ϵ such that

$$\min_x w_{10}(x) - \max_x w_{20}(x) := \delta > 0, \tag{4.2}$$

and, for some point $\beta \in \mathbb{R}$,

$$w_{10}'(\beta) \le -\epsilon, \quad \text{or} \quad w_{20}'(\beta) \le -\epsilon, \tag{4.3}$$

then the lifespan of any smooth solution of (1.14) *and* (3.1) *does not exceed*

$$T_* = \frac{2}{(1+\theta)\epsilon}\left(\frac{\theta}{2}\delta\right)^{\frac{\theta-1}{2\theta}} \|\rho_0\|_{C(\mathbb{R})}^{\frac{\theta-1}{2}}. \tag{4.4}$$

Proof. For a smooth solution $(\rho, v)(x,t)$ of System (1.14), one can verify, as in the proof of Theorem 3.1, that $\rho(x,t) > 0$, and

$$g' = \frac{1+\theta}{2}\left(\frac{\theta}{2}(w_1 - w_2)\right)^{\frac{1-\theta}{2\theta}} g^2,$$

with $g = -\rho^{(\theta-1)/2}\partial_x w_1$. By defining the characteristic $x = x(\beta, t)$ passing through the point $(\beta, 0), \beta \in \mathbb{R}$, as in the proof of Theorem 3.1, we have, as in (3.10) and (3.11),

$$g(x(\beta,t),t) = \frac{\rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta)}{1 + \int_0^t K(\beta,\tau)d\tau},$$

with

$$\begin{aligned} K(\beta,t) &= \frac{1+\theta}{2}\left(\frac{\theta}{2}|w_{10}(\beta) - w_{20}(\alpha(\beta,t))|\right)^{\frac{1-\theta}{2\theta}} \rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta) \\ &= \frac{1+\theta}{2}\rho(x(\beta,t),t)^{\frac{1-\theta}{2}} \rho_0(\beta)^{\frac{\theta-1}{2}} w_{10}'(\beta). \end{aligned}$$

If the smooth solution stays away from the vacuum, i.e., the density ρ has a positive lower bound, then one concludes that $g(x(\beta,t),t)$ will blow up at a certain finite time if $w'_{10}(\beta)<0$. Under the condition (4.2) and if $w'_{10}(\beta)\le -\epsilon$ in (4.3), $g(x(\beta,t),t)$ will blow up at some finite time which is less than or equal to T_* defined in (4.4). If $w'_{20}(\beta)\le 0$ or further $w'_{20}(\beta)\le -\epsilon$, similar consequence can be obtained from (3.8). This completes the proof of Theorem 4.1. □

Remark 4.1. The argument was developed in Lax [178] for 2×2 hyperbolic systems of conservation laws with genuine nonlinearity. The implication of the result is that the first derivatives of solutions blow up in a finite time, while the solutions stay themselves bounded and away from the vacuum. This is in agreement with the phenomenon of shock waves. See also Majda [223] and Lin [197, 198] for further discussions.

The formation of singularities for $n\times n$ genuinely nonlinear hyperbolic systems of one-dimensional conservation laws (1.19) was discussed in John [161]. It was shown in [161] that, if the initial data are sufficiently small (but not identically zero), then the first derivatives of the solution will become infinite in some finite time.

Theorem 4.2. *Consider the Cauchy problem* (1.19) *and* (1.20) *of $n\times n$ genuinely nonlinear hyperbolic systems. Assume the initial data $\mathbf{u}_0(x)$ are a C^2 function with compact support. Then there exists a positive constant δ such that, if $0<\sup_x|\mathbf{u}''_0(x)|\le\delta$, the solution $\mathbf{u}(x,t)$ cannot exist in the class C^2 for all positive t.*

This result was generalized in Liu [206] to include systems with linearly degenerate characteristic fields such as the Euler equations.

4.2. **Three-Dimensional Euler Equations.** Consider the Cauchy problem of the three-dimensional Euler equations for polytropic gases in (2.1) with smooth initial data:

$$(\rho,\mathbf{v},S)|_{t=0}=(\rho_0,\mathbf{v}_0,S_0)(\mathbf{x}),\quad \rho_0(\mathbf{x})>0,\quad \mathbf{x}\in\mathbb{R}^3, \tag{4.5}$$

satisfying

$$(\rho_0,\mathbf{v}_0,S_0)(\mathbf{x})=(\bar\rho,0,\bar S),\quad \text{for}\quad |\mathbf{x}|\ge R,$$

where $\bar\rho>0$, $\bar S$, and R are given constants. The equations in (2.1) possess a unique local C^1 solution $(\rho,\mathbf{v},S)(\mathbf{x},t)$ with $\rho(\mathbf{x},t)>0$ provided the initial data (4.5) are sufficiently regular (Theorem 2.1). The support of the smooth disturbance $(\rho_0(\mathbf{x})-\bar\rho,\ \mathbf{v}_0(\mathbf{x}),\ S_0(\mathbf{x})-\bar S)$ propagates with speed at most $\sigma=\sqrt{p_\rho(\bar\rho,\bar S)}$ (the sound speed), that is,

$$(\rho,\mathbf{v},S)(\mathbf{x},t)=(\bar\rho,0,\bar S),\quad \text{if}\quad |\mathbf{x}|\ge R+\sigma t. \tag{4.6}$$

The proof of this essential fact of finite speed of propagation for the three-dimensional case can be found in John [162], as well as in Sideris [282], established through local energy estimates. Take $\bar p=p(\bar\rho,\bar S)$. Define

$$P(t)=\int_{\mathbb{R}^3}\left(p(\mathbf{x},t)^{1/\gamma}-\bar p^{1/\gamma}\right)d\mathbf{x}=\int_{\mathbb{R}^3}\left(\rho(\mathbf{x},t)\exp(S(\mathbf{x},t)/\gamma)-\bar\rho\exp(\bar S/\gamma)\right)d\mathbf{x},$$

$$F(t)=\int_{\mathbb{R}^3}\mathbf{x}\cdot\rho\mathbf{v}(\mathbf{x},t)d\mathbf{x},$$

which, roughly speaking, measure the entropy and the radial component of momentum. The following theorem on the formation of singularities in solutions of (2.1) and (4.5) is due to Sideris [283].

Theorem 4.3. *Suppose that $(\rho,\mathbf{v},S)(\mathbf{x},t)$ is a C^1 solution of* (2.1) *and* (4.5) *for $0<t<T$, and*

$$P(0)\ge 0, \tag{4.7}$$

$$F(0)>\alpha\sigma R^4\max_{\mathbf{x}}\rho_0(\mathbf{x}), \tag{4.8}$$

where $\alpha = 16\pi/3$. *Then the lifespan* T *of the* C^1 *solution is finite.*

Proof. Set

$$M(t) = \int_{\mathbb{R}^3} (\rho(\mathbf{x}, t) - \bar{\rho}) d\mathbf{x}.$$

From the equations in (2.1), combined with (4.6), and integration by parts, one has

$$M'(t) = -\int_{\mathbb{R}^3} \nabla \cdot (\rho \mathbf{v}) d\mathbf{x} = 0, \quad P'(t) = -\int_{\mathbb{R}^3} \nabla \cdot (\rho \mathbf{v} \exp(S/\gamma)) d\mathbf{x} = 0,$$

which implies

$$M(t) = M(0), \quad P(t) = P(0); \tag{4.9}$$

and

$$F'(t) = \int_{\mathbb{R}^3} \mathbf{x} \cdot (\rho \mathbf{v})_t d\mathbf{x} = \int_{\mathbb{R}^3} \left(\rho |\mathbf{v}|^2 + 3(p - \bar{p}) \right) d\mathbf{x} = \int_{B(t)} \left(\rho |\mathbf{v}|^2 + 3(p - \bar{p}) \right) d\mathbf{x}, \tag{4.10}$$

where $B(t) = \{\mathbf{x} \in \mathbb{R}^3 \,:\, |\mathbf{x}| \le R + \sigma t\}$. From Hölder's inequality, (4.7), and (4.9), one has

$$\begin{aligned} \int_{B(t)} p \, d\mathbf{x} &\ge \frac{1}{|B(t)|^{\gamma-1}} \left(\int_{B(t)} p^{1/\gamma} d\mathbf{x} \right)^{\gamma} \\ &= \frac{1}{|B(t)|^{\gamma-1}} \left(P(0) + \int_{B(t)} \bar{p}^{1/\gamma} d\mathbf{x} \right)^{\gamma} \ge \int_{B(t)} \bar{p} \, d\mathbf{x}, \end{aligned}$$

where $|B(t)|$ denotes the volume of the set $B(t)$. Therefore, by (4.10),

$$F'(t) \ge \int_{\mathbb{R}^3} \rho |\mathbf{v}|^2 d\mathbf{x}. \tag{4.11}$$

By the Cauchy-Schwarz inequality and (4.9),

$$\begin{aligned} F(t)^2 &= \left(\int_{B(t)} \mathbf{x} \cdot \rho \mathbf{v} d\mathbf{x} \right)^2 \le \int_{B(t)} \rho |\mathbf{v}|^2 d\mathbf{x} \int_{B(t)} \rho |\mathbf{x}|^2 d\mathbf{x} \\ &\le (R + \sigma t)^2 \int_{B(t)} \rho |\mathbf{v}|^2 d\mathbf{x} \left(M(t) + \int_{B(t)} \bar{\rho} d\mathbf{x} \right) \\ &\le (R + \sigma t)^2 \int_{B(t)} \rho |\mathbf{v}|^2 d\mathbf{x} \left(\int_{B(t)} (\rho_0(\mathbf{x}) - \bar{\rho}) d\mathbf{x} + \int_{B(t)} \bar{\rho} d\mathbf{x} \right) \\ &\le \frac{4\pi}{3} (R + \sigma t)^5 \max_x \rho_0(\mathbf{x}) \int_{B(t)} \rho |\mathbf{v}|^2 d\mathbf{x}. \end{aligned}$$

Then (4.11) implies that

$$F'(t) \le \left(\frac{4\pi}{3} (R + \sigma t)^5 \max_{\mathbf{x}} \rho_0(\mathbf{x}) \right)^{-1} F(t)^2. \tag{4.12}$$

Since $F(0) > 0$ by (4.8), $F(t)$ remains positive for $0 < t < T$, as a consequence of (4.12). Dividing by $F(t)^2$ and integrating from 0 to T in (4.12) yields

$$F(0)^{-1} > F(0)^{-1} - F(T)^{-1} \ge (\alpha \sigma \max \rho_0)^{-1} \left(R^{-4} - (R + \sigma T)^{-4} \right).$$

Thus,

$$(R + \sigma T)^4 < R^4 F(0) / (F(0) - \alpha \sigma R^4 \max \rho_0).$$

This completes the proof of Theorem 4.3. $\square$

Remark 4.2. The method of the proof above, which is a refinement of Sideris [282], applies equally well in one and two space dimensions. In the isentropic case (S is a constant), the condition $P(0) \geq 0$ reduces to $M(0) \geq 0$.

Remark 4.3. To illustrate a way in which the conditions (4.7) and (4.8) may be satisfied, consider the initial data: $\rho_0 = \bar{\rho}$, $S_0 = \bar{S}$. Then $P(0) = 0$, and (4.8) holds if

$$\int_{|\mathbf{x}|<R} \mathbf{x} \cdot \mathbf{v}_0(x) d\mathbf{x} > \alpha\sigma R^4.$$

Comparing both sides, one finds that the initial velocity must be supersonic in some region relative to the sound speed at infinity. The formation of a singularity (presumably a shock wave) is detected as the disturbance overtakes the wave front forcing the front to propagate with supersonic speed.

Remark 4.4. Another result was established in Sideris [283] on the formation of singularities, without condition of largeness such as (4.8). The result says that, if $S_0(x) \geq \bar{S}$ and, for some $0 < R_0 < R$,

$$\begin{aligned} &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-1}(|\mathbf{x}| - r)^2(\rho_0(\mathbf{x}) - \bar{\rho}) d\mathbf{x} > 0, \\ &\int_{|\mathbf{x}|>r} |\mathbf{x}|^{-3}(|\mathbf{x}|^2 - r^2)\mathbf{x} \cdot \rho_0(\mathbf{x})\mathbf{v}_0(\mathbf{x}) d\mathbf{x} \geq 0, \end{aligned} \tag{4.13}$$

for $R_0 < r < R$, then the lifespan T of the C^1 solution of (2.1) and (4.5) is finite. The assumption (4.13) means that, in an average sense, the gas must be slightly compressed and outgoing directly behind the wave front. For the proof, some important technical points were adopted from Sideris [281] on the nonlinear wave equations in three dimensions.

Remark 4.5. The result in Theorem 4.3 indicates that the C^1 regularity of solutions breaks down in a finite time. It is believed that in fact only $\nabla\rho$ and $\nabla\mathbf{v}$ blow up in most cases; see a proof in Alinhac [2] for the case of axisymmetric initial data for the Euler equations for compressible fluids in two space dimensions.

4.3. **Other Results.** The method of characteristics has been used to establish the finite-time formation of singularities for one-dimensional hyperbolic systems of conservation laws and related equations; see Lax [178], John [161], Liu [206], Klainerman-Majda [170], Dafermos [83], Keller-Ting [169], Slemrod [287], Lin [197, 198], etc.

A technique was introduced in Dafermos [83] to monitor the time evolution of the spatial supremum norms of first derivatives and was further applied in Dafermos-Hsiao [90], Hrusa-Messaoudi [153], and Chen-Wang [320] for the problems with thermal diffusion.

Contrary to the formation of singularities, global smooth solutions may exist for conservation laws with certain dissipation mechanisms including friction damping, heat diffusion, and memory effects, provided the initial data are smooth and small. That is, the smoothing effect from the dissipation may prevent the development of shock waves in solutions with small smooth initial data. See the survey paper by Dafermos [82].

In the case of damping, this property has been justified for certain one-dimensional equations; see Nishida [242], Hsiao [154], and the references cited therein for the existence of global smooth solutions with small smooth initial data to the one-dimensional Euler equations with damping. For the multidimensional Euler equations, it has been proved by Sideris-Wang [286] that the damping can also prevent the formation of singularities in smooth solutions with small initial data. For related discussions, see Wang [319] for a spherically symmetric smooth Euler-Poisson flow and Guo [149] for a smooth irrotational Euler-Poisson flow in three space dimensions.

In the case of heat diffusion, the global existence of smooth solutions was established in Slemrod [288] for nonlinear thermoelasticity with smooth and small initial data.

Although the smoothing effect from damping or heat diffusion alone can prevent the breaking of smooth waves of small amplitude, the combined effect of damping and heat diffusion may still not be strong enough to prevent the formation of singularities in large smooth solutions, as shown in Chen-Wang [320]. A preliminary study of the so called critical threshold phenomena associated with the Euler-Poisson equations was made in Engelberg-Liu-Tadmor [110], where the answer to questions of global smoothness vs. finite-time breakdown depends on whether the initial configuration crosses an intrinsic critical threshold.

The damping induced by memory effects can also preserve the smoothness of small initial data; see Dafermos-Nohel [91] and MacCamy [220].

For multidimensional scalar conservation laws, the formation of shock waves was proved in Majda [223] by using characteristics for solutions with smooth initial data. Some general discussions on the formation of shock waves in plane wave solutions of multidimensional systems of conservation laws can also be found in Majda [223]. The method of Sideris [282, 283] has been effective for multidimensional systems of Euler equations. A similar technique was employed by Glassey [128] in the case of nonlinear Schrödinger equations (see also Strauss [294]). It has been adopted to prove the formation of singularities in solutions of many other multidimensional problems; see Makino-Ukai-Kawashima [226] and Rendall [263] for a compressible fluid body surrounded by the vacuum, Rammaha [261, 262] for two-dimensional Euler equations and magnetohydrodynamics, Perthame [254] for the Euler-Poisson equations for spherically symmetric flows, and Guo and Tahvildar-Zadeh [150] for the Euler-Maxwell equations for spherically symmetric plasma flows, etc.

For the multidimensional Euler equations for compressible fluids with smooth initial data that are a small perturbation of amplitude ϵ from a constant state, the lifespan of smooth solutions is at least $O(\epsilon^{-1})$ from the theory of symmetric hyperbolic systems (Friedrichs [122], Kato [163]). Results on the formation of singularities show that the lifespan of a smooth solution is no better than $O(\epsilon^{-2})$ in the two-dimensional case (Rammaha [261]) and $O(e^{\epsilon^{-2}})$ (Sideris [283]) in the three-dimensional case. See Alinhac [2] and Sideris [284, 285] for additional discussions in this direction.

There have been many studies on the blow-up of smooth solutions for nonlinear wave equations; see the results collected in Alinhac [3], John [162], and the references cited therein. Other related discussions about the formation of singularities for conservation laws can be found in Brauer [17], Chemin [35, 36], Kosinski [171], Wang [316], as well as the references cited therein.

5. Local Well-Posedness for Discontinuous Solutions

The formation of singularities, especially shock waves, discussed in §4 indicates that one should seek discontinuous entropy solutions of the Euler equations for general initial data. Usually, it is difficult to construct the discontinuous solutions especially in the multidimensional case. We focus on the local existence of discontinuous entropy solutions in this section.

We first consider the local existence of the simplest type of discontinuous solutions, i.e., the shock front solutions of the multidimensional Euler equations. Shock front solutions are the most important discontinuous nonlinear progressing wave solutions in compressible Euler flows and other systems of conservation laws. For a general multidimensional hyperbolic system of conservation laws (1.16), shock front solutions are discontinuous piecewise smooth entropy solutions with the following structure:

(a). There exists a C^2 space-time hypersurface $S(t)$ defined in $(\mathbf{x}, t)$ for $0 \leq t \leq T$ with space-time normal $(\nu_{\mathbf{x}}, \nu_t) = (\nu_1, \cdots, \nu_d, \nu_t)$ as well as two C^1 vector-valued functions: $\mathbf{u}^+(\mathbf{x},t)$ and $\mathbf{u}^-(\mathbf{x},t)$, defined on respective domains S^+ and S^- on either side of the hypersurface $S(t)$, and satisfying

$$\partial_t \mathbf{u}^\pm + \nabla\cdot \mathbf{f}(\mathbf{u}^\pm) = 0, \qquad \text{in } S^\pm; \tag{5.1}$$

(b). The jump across the hypersurface $S(t)$ satisfies the Rankine-Hugoniot condition:

$$\left\{\nu_t(\mathbf{u}^+ - \mathbf{u}^-) + \nu_{\mathbf{x}}\cdot(\mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-))\right\}\big|_S = 0. \tag{5.2}$$

For the quasilinear system (1.16), the surface S is not known in advance and must be determined as part of the solution of the problem; thus the equations in (5.1) and (5.2) describe a multidimensional, highly nonlinear, free-boundary value problem for the quasilinear system of conservation laws.

The initial data yielding shock front solutions are defined as follows. Let S_0 be a smooth hypersurface parametrized by α, and let $\nu(\alpha) = (\nu_1(\alpha), \cdots, \nu_n(\alpha))$ be a unit normal to S_0. Define the piecewise smooth initial values for respective domains S_0^+ and S_0^- on either side of the hypersurface S_0 as

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases} \mathbf{u}_0^+(\mathbf{x}), & \mathbf{x}\in S_0^+, \\ \mathbf{u}_0^-(\mathbf{x}), & \mathbf{x}\in S_0^-. \end{cases} \tag{5.3}$$

It is assumed that the initial jump in (5.3) satisfies the Rankine-Hugoniot condition, i.e., there is a smooth scalar function $\sigma(\alpha)$ so that

$$-\sigma(\alpha)(\mathbf{u}_0^+(\alpha) - \mathbf{u}_0^-(\alpha)) + \nu(\alpha)\cdot(\mathbf{f}(\mathbf{u}_0^+(\alpha)) - \mathbf{f}(\mathbf{u}_0^-(\alpha))) = 0, \tag{5.4}$$

and that $\sigma(\alpha)$ does not define a characteristic direction, i.e.,

$$\sigma(\alpha) \neq \lambda_i(\mathbf{u}_0^\pm), \quad \alpha\in\overline{S_0}, \quad 1\le i\le n, \tag{5.5}$$

where λ_i, $i = 1,\cdots,n$, are the eigenvalues of (1.16). It is natural to require that $S(0) = S_0$.

Consider the two-dimensional isentropic Euler equations in (1.9), away from the vacuum, which can be rewritten in the form:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \rho\ge 0, \quad \mathbf{v}\in\mathbb{R}^2, \quad \mathbf{x}\in\mathbb{R}^2, \quad t>0, \\ \partial_t(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) + \nabla p = 0, \quad p = p(\rho) = \rho^\gamma/\gamma, \quad \gamma>1, \end{cases} \tag{5.6}$$

with piecewise smooth initial data:

$$(\rho,\mathbf{v})|_{t=0} = \begin{cases} (\rho_0^+,\mathbf{v}_0^+)(\mathbf{x}), & \mathbf{x}\in S_0^+, \\ (\rho_0^-,\mathbf{v}_0^-)(\mathbf{x}), & \mathbf{x}\in S_0^-. \end{cases} \tag{5.7}$$

The following local existence of discontinuous entropy solutions is taken from Majda [222].

Theorem 5.1. *Assume that S_0 is a smooth closed curve and that $(\rho_0^+,\mathbf{v}_0^+)(\mathbf{x})$ belongs to the uniform local Sobolev space $H^s_{ul}(S_0^+)$, while $(\rho_0^-,\mathbf{v}_0^-)(\mathbf{x})$ belongs to the Sobolev space $H^s(S_0^-)$, for some fixed $s\ge 10$. Assume also that there is a function $\sigma(\alpha)\in H^s(S_0)$ so that* (5.4) *and* (5.5) *hold, and the compatibility conditions up to order $s-1$ are satisfied on S_0 by the initial data, together with the entropy condition*

$$\mathbf{v}_0^+\cdot\nu(\alpha) + (\rho_0^+)^\theta < \sigma(\alpha) < \mathbf{v}_0^-\cdot\nu(\alpha) + (\rho_0^-)^\theta, \quad \theta = (\gamma-1)/2, \tag{5.8}$$

and the stability condition

$$\frac{p(\rho_0^+) - p(\rho_0^-)}{\rho_0^+ - \rho_0^-} < (\rho_0^-)^{\gamma-1} + (\mathbf{v}_0^-\cdot\nu(\alpha) - \sigma(\alpha))^2. \tag{5.9}$$

Then there is a C^2 hypersurface $S(t)$ together with C^1 functions $(\rho^\pm, \mathbf{v}^\pm)(\mathbf{x}, t)$ defined for $t \in [0, T]$, with T sufficiently small, so that

$$(\rho, \mathbf{v})(\mathbf{x}, t) = \begin{cases} (\rho^+, \mathbf{v}^+)(\mathbf{x}, t), & (\mathbf{x}, t) \in S^+, \\ (\rho^-, \mathbf{v}^-)(\mathbf{x}, t), & (\mathbf{x}, t) \in S^- \end{cases} \tag{5.10}$$

is the discontinuous shock front solution of the Cauchy problem (5.6) *and* (5.7) *satisfying* (5.1) *and* (5.2).

In Theorem 5.1, the uniform local Sobolev space $H^s_{ul}(S_0^+)$ is defined as follows: Let $w \in C_0^\infty(\mathbb{R}^d)$ be a function so that $w(\mathbf{x}) \geq 0$; and $w(\mathbf{x}) = 1$ when $|\mathbf{x}| \leq 1/2$, and $w(\mathbf{x}) = 0$ when $|\mathbf{x}| > 1$. Define

$$w_{r,\mathbf{y}}(\mathbf{x}) = w\left(\frac{\mathbf{x} - \mathbf{y}}{r}\right).$$

A vector function $\mathbf{u}$ is in H^s_{ul}, provided that there exists some $r > 0$ so that

$$\max_{\mathbf{y} \in \mathbb{R}^d} \|w_{r,\mathbf{y}} \mathbf{u}\|_{H^s} < \infty.$$

Remark 5.1. There are extensive studies in Majda [221, 222, 223] on the local existence and stability of shock front solutions. The compatibility conditions in Theorem 5.1 are defined in [222] and needed in order to avoid the formation of discontinuities in higher derivatives along other characteristic surfaces emanating from S_0. Once the main condition in (5.4) is satisfied, the compatibility conditions are automatically guaranteed for a wide class of initial data. Theorem 5.1 can be extended to the full Euler equations in three space dimensions ($d = 3$) in (1.1) (see Majda [222]). See Métivier [229] for the uniform existence time of shock front solutions in the shock strength. Also see Blokhin-Trokhinin [14] in this volume for further discussions.

The proof of Theorem 5.1 can be found in [222]. The idea of the proof is similar to that of the proof of Theorem 2.2, but the technical details are quite different due to the unusual features of the problem considered in Theorem 5.1. The shock front solutions are defined as the limit of a convergent classical iteration scheme based on a linearization by using the theory of linearized stability for shock fronts developed in [221]. The technical condition $s \geq 10$, instead of $s > 1 + d/2 = 2$ ($d = 2$), is required because pseudo-differential operators are needed in the proof of the main estimates. Some improved technical estimates regarding the dependence of operator norms of pseudo-differential operators on their coefficients would lower the value of s.

For the one-dimensional Euler equations in (1.12), away from the vacuum, $m = \rho v$ and

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, & x \in \mathbb{R}, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, \\ \partial_t E + \partial_x(v(E + p)) = 0, & E = \frac{1}{2}\rho v^2 + \rho e, \end{cases} \tag{5.11}$$

some stronger existence results of local discontinuous solutions can be found in [148, 195] for the Cauchy problem with piecewise smooth initial data

$$(\rho, v, e)|_{t=0} = \begin{cases} (\rho_0^+, v_0^+, e_0^+)(x), & x > 0, \\ (\rho_0^-, v_0^-, e_0^-)(x), & x < 0, \end{cases} \tag{5.12}$$

where $(\rho_0^\pm, v_0^\pm, e_0^\pm)(x)$ are bounded smooth functions for $x \geq 0$ and $x \leq 0$, respectively, and $(\rho_0^+, v_0^+, e_0^+)(0) \neq (\rho_0^-, v_0^-, e_0^-)(0)$. Then the following theorem holds.

Theorem 5.2. *Suppose that the amplitude $|(\rho_0^+ - \rho_0^-, v_0^+ - v_0^-, e_0^+ - e_0^-)(0)|$ is sufficiently small, then the Cauchy problem* (5.11) *and* (5.12) *has a unique piecewise smooth solution $(\rho, v, e)(x, t)$ for $x \in \mathbb{R}$ and $t \in [0, T]$, for sufficiently small T.*

Remark 5.2. For the one-dimensional Euler equations for (isentropic or non-isentropic) polytropic gases (2.1) or (1.14) ($d = 1$), the assumption of small amplitude is not needed. See [148, 195] for the proofs of Theorem 5.2 and related results.

Remark 5.3. The piecewise smooth solution $(\rho, v, e)(x, t)$ of the Cauchy problem (5.11) and (5.12) possesses a structure in a neighborhood of the origin similar to the solution of the corresponding Riemann problem of (5.11) with initial data

$$(\rho, v, e)|_{t=0} = \begin{cases} (\rho_0^+, v_0^+, e_0^+)(0), & x > 0, \\ (\rho_0^-, v_0^-, e_0^-)(0), & x < 0. \end{cases} \tag{5.13}$$

See §6, as well as Chang-Hsiao [33], Courant-Friedrichs [76], Dafermos [88], Serre [277], and Smoller [291], for the discussion of the solution structure of the Riemann problem.

Remark 5.4. There are some discussions in [76, 195] on the local existence of spherically symmetric discontinuous solutions with spherically symmetric initial data. See §10.1 for some recent results on the global existence of spherically symmetric discontinuous entropy solutions.

6. Global Discontinuous Solutions I: Riemann Solutions

In this section, we present a global theory of discontinuous entropy solutions of the Riemann problem, the simplest Cauchy problem with discontinuous initial data.

6.1. The Riemann Problem and Lax's Theorems.

We first introduce two Lax's Theorems for the local behavior of wave curves in the phase space and the existence of global entropy solutions of the Riemann problem, respectively, for one-dimensional strictly hyperbolic systems of conservation laws (1.19) with Riemann data:

$$\mathbf{u}|_{t=0} = \mathbf{u}_0(x) := \begin{cases} \mathbf{u}_L, & x < 0, \\ \mathbf{u}_R, & x > 0, \end{cases} \tag{6.1}$$

where $\mathbf{u}_L$ and $\mathbf{u}_R$ are two constant states. This theorem applies to the Euler equations with small Riemann data.

Since both System (1.19) and the Riemann initial data (6.1) are invariant under uniform stretching of coordinates: $(x, t) \to (\alpha x, \alpha t)$, the Cauchy problem (1.19) and (6.1) admits self-similar solutions, defined on the space-time plane, and constant along straight-line rays emanating from the origin:

$$\mathbf{u}(x, t) = \mathbf{R}(x/t), \quad x \in \mathbb{R},\ t \in \mathbb{R}_+, \tag{6.2}$$

where $\mathbf{R}(\xi)$ is a bounded measurable function in $\xi \in \mathbb{R}$, which satisfies the ordinary differential equation:

$$\frac{d(\mathbf{f}(\mathbf{R}(\xi)) - \xi\mathbf{R}(\xi))}{d\xi} + \mathbf{R}(\xi) = 0 \tag{6.3}$$

in the sense of distributions.

To solve the Riemann problem, it is more instructive to present first the rarefaction curves and the shock curves in the phase space.

Rarefaction Curves. Given a state $\mathbf{u}_-$, we consider possible states $\mathbf{u}$ that can be connected to the state $\mathbf{u}_-$, on the right, by a centered rarefaction wave of the ith characteristic field, which is genuinely nonlinear, that is, $\nabla\lambda_i \cdot \mathbf{r}_i = 1$, where $\nabla\mathbf{f} \cdot \mathbf{r}_i = \lambda_i\mathbf{r}_i$, $1 \le i \le n$.

Consider the self–similar Lipschitz solutions $\mathbf{V}(\xi), \xi = x/t$, of the Riemann problem (1.19) and (6.1) as above. Then we have

$$\begin{cases} \xi = \lambda_i(\mathbf{V})(\xi), \\ (\nabla \mathbf{f}(\mathbf{V}(\xi)) - \xi \mathbf{I})\mathbf{V}'(\xi) = 0, \end{cases} \tag{6.4}$$

with boundary condition:

$$\mathbf{V}|_{\xi=\lambda_i(\mathbf{u}_-)} = \mathbf{u}_-, \tag{6.5}$$

and, on the i-centered rarefaction waves,

$$\frac{\partial \mathbf{V}}{\partial x} = \frac{1}{t}\frac{d\mathbf{V}}{d\xi} = \frac{1}{t}\mathbf{r}_i(\mathbf{V}(x/t)). \tag{6.6}$$

Then we conclude:

Proposition 6.1. *Let the ith characteristic field of System* (1.19) *be genuinely nonlinear in $\mathcal{N} \subset \mathbb{R}^n$. Let $\mathbf{u}_-$ be any point in $\mathcal{N}$. Then there exists a one-parameter family of states $\mathbf{u} = \mathbf{u}(\epsilon), \epsilon \geq 0, \mathbf{u}(0) = \mathbf{u}_-$, which can be connected to $\mathbf{u}_-$ on the right by an i-centered rarefaction wave. The parametrization can be chosen so that $\dot{\mathbf{u}}(0) = \mathbf{r}_i(\mathbf{u}_-)$ and $\ddot{\mathbf{u}}(0) = \dot{\mathbf{r}}_i(\mathbf{u}_-)$.*

Shock Curves. Given a state $\mathbf{u}_-$, we consider possible states $\mathbf{u}$ that can be connected to the state $\mathbf{u}_-$, on the right, by a shock or contact discontinuity. The Rankine-Hugoniot condition for discontinuities with speed σ is

$$\sigma[\mathbf{u}] = [\mathbf{f}(\mathbf{u})]. \tag{6.7}$$

Here and in what follows we use the notation $[H] = H_+ - H_-$, where H_- and H_+ are the values of any function H on the left-hand side and the right-hand side of the discontinuity, respectively. A discontinuity satisfying (6.7) is called an i-shock if it satisfies the Lax entropy conditions:

$$\lambda_{i-1}(\mathbf{u}_-) < \sigma < \lambda_i(\mathbf{u}_-), \quad \lambda_i(\mathbf{u}) < \sigma < \lambda_{i+1}(\mathbf{u}). \tag{6.8}$$

First we consider the case which the i-field is genuinely nonlinear. Given $\mathbf{u}_- \in \mathcal{N}$, we can view (6.7) as n-equations for the $(n+1)$-unknowns $\mathbf{u}$ and σ.

Proposition 6.2. *Let the ith characteristic field of System* (1.19) *be genuinely nonlinear in $\mathcal{N}$. Let $\mathbf{u}_-$ be any point in $\mathcal{N}$. Then there exists a one-parameter family of states $\mathbf{u} = \mathbf{u}(\epsilon), \epsilon \leq 0, \mathbf{u}(0) = \mathbf{u}_-$, which can be connected to $\mathbf{u}_-$ on the right by an i-shock. The parametrization can be chosen so that $\dot{\mathbf{u}}(0) = \mathbf{r}_i(\mathbf{u}_-)$ and $\ddot{\mathbf{u}}(0) = \dot{\mathbf{r}}_i(\mathbf{u}_-)$, and $\sigma(0) = \lambda_i(\mathbf{u}_-), \dot{\sigma}(0) = \frac{1}{2}$.*

Contact Discontinuities. If the ith characteristic field is linearly degenerate, then λ_i is an i-Riemann invariant.

Proposition 6.3. *Let the ith characteristic field of System* (1.19) *be linearly degenerate. If $\mathbf{u}_-$ and $\mathbf{u}_+$ have the same i-Riemann invariants with respect to the linearly degenerate field, then they are connected to each other by a contact discontinuity of speed $\sigma = \lambda_i(\mathbf{u}_-) = \lambda_i(\mathbf{u}_+)$.*

Propositions 6.1, 6.2, and 6.3 can be combined into the following Lax's theorem [177] (also see [181]).

Theorem 6.1. *Given a state $\mathbf{u}_-$, it can be connected to a one-parameter family of states $\mathbf{u}_+ = \mathbf{u}(\epsilon), -\epsilon_0 < \epsilon < \epsilon_0$, on the right of $\mathbf{u}_-$ through a centered i-wave, i.e. an i-shock, or an i-rarefaction wave, or an i-contact discontinuity; $\mathbf{u}(\epsilon)$ is* **twice** *continuously differentiable with respect to ϵ.*

Then, using Theorem 6.1 and the implicit function theorem leads to the Lax's existence theorem [177] (also see [181]) for the Riemann problem (1.19) and (6.1).

Theorem 6.2. *Assume that System* (1.19) *is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. For sufficiently small* $|\mathbf{u}_L - \mathbf{u}_R|$*, there exists a unique self-similar solution* (6.2) *of the Riemann problem* (1.19) *and* (6.1)*, with small total variation. This solution comprises* $n+1$ *constant states* $\mathbf{u}_L = \mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_{n-1}, \mathbf{u}_n = \mathbf{u}_R$*. When the* i*th characteristic field is linearly degenerate,* $\mathbf{u}_i$ *is joined to* $\mathbf{u}_{i-1}$ *by an* i*-contact discontinuity; when the* i*th characteristic field is genuinely nonlinear,* $\mathbf{u}_i$ *is joined to* $\mathbf{u}_{i-1}$ *by either an* i*-centered rarefaction wave or an* i*-compressive shock.*

6.2. **Isothermal Euler Equations.** Consider the isothermal Euler equations in (1.15), that is, $\gamma = 1$ and

$$p = \tilde{p}(\tau) = 1/\tau, \tag{6.9}$$

with Riemann data:

$$(\tau, v)|_{t=0} = \begin{cases} (\tau_L, v_L), & x < 0, \\ (\tau_R, v_R), & x > 0. \end{cases} \tag{6.10}$$

System (1.15) and (6.9) has the eigenvalues $\pm 1/\tau$ and the Riemann invariants $v \pm \ln\tau$. The shock curves S_i, $i = 1, 2$, and rarefaction curves R_i, $i = 1, 2$, with left state (τ_-, v_-) have the following forms, respectively:

$$\begin{aligned} S_i(\tau_-, v_-): &\quad v - v_- = (-1)^i\, 2\sinh\frac{q - q_-}{2}, \quad q > q_-, \\ R_i(\tau_-, v_-): &\quad v - v_- = (-1)^i (q - q_-), \quad q < q_-, \end{aligned}$$

where $q = -\ln\tau$ and $q_- = -\ln\tau_-$. Define a function

$$W(s) = \begin{cases} s, & s \le 0, \\ 2\sinh\frac{s}{2}, & s \ge 0. \end{cases} \tag{6.11}$$

Then the equations for the i-wave curves, $i = 1, 2$, can be rewritten into the form:

$$i\text{-wave curve:} \quad v - v_- = (-1)^i W(q - q_-). \tag{6.12}$$

The function $W(s)$ in (6.11) satisfies $W'(s) > 0$, i.e., $W(s)$ is increasing. It is easy to verify that

$$\begin{cases} W(s_1 + s_2) \ge W(s_1) + W(s_2), & \text{for } s_1, s_2 \ge 0, \\ W(s_1 + s_2) = W(s_1) + W(s_2), & \text{for } s_1, s_2 \le 0. \end{cases} \tag{6.13}$$

For any s, let $s^\pm = (|s| \pm s)/2$. Then

$$W(s^+) + W(s^-) = W(|s|) \ge W(s). \tag{6.14}$$

If (v_m, q_m) is the intermediate state in the Riemann problem of (1.15) and (6.9) connecting the two states $(v_L, q_L) = (v_1, q_1)$ and $(v_R, q_R) = (v_2, q_2)$, then

$$W(q_m - q_1) + W(q_m - q_2) = v_1 - v_2. \tag{6.15}$$

Without ambiguity, we denote

$$D(q_1, q_2) := |q_1 - q_m| + |q_2 - q_m|, \tag{6.16}$$

although D also depends on v_1 and v_2. Then we have

Proposition 6.4. *For any* q_i, $i = 1, 2, 3$,

$$D(q_1, q_3) \le D(q_1, q_2) + D(q_2, q_3). \tag{6.17}$$

Proof. Let q_{ij} be the intermediate states between q_i and q_j, $i, j = 1, 2, 3$, $i \neq j$. Then, from (6.15), one has

$$W(q_{13} - q_1) + W(q_{13} - q_3) = W(q_{12} - q_1) + W(q_{12} - q_2) + W(q_{23} - q_2) + W(q_{23} - q_3).$$

Set $x = q_{13} - q_1$, $y = q_{13} - q_3$, $a = q_{12} - q_1$, $b = q_{12} - q_2$, $c = q_{23} - q_2$, and $d = q_{23} - q_3$. Then $x - y = a - b + c - d$, and

$$W(x) + W(y) = W(a) + W(b) + W(c) + W(d). \tag{6.18}$$

If $xy \leq 0$, then

$$\begin{aligned} D(q_1, q_3) &= |x| + |y| = |x - y| = |a - b + c - d| \\ &\leq |a| + |b| + |c| + |d| \leq D(q_1, q_2) + D(q_2, q_3). \end{aligned}$$

If $x > 0$ and $y > 0$, by (6.14) and (6.18),

$$\begin{aligned} W(x) + W(y) \leq \big(&W(a^+) + W(b^-) + W(c^+) + W(d^-)\big) \\ &+ \big(W(a^-) + W(b^+) + W(c^-) + W(d^+)\big), \end{aligned}$$

and then either

$$W(x) \leq W(a^+) + W(b^-) + W(c^+) + W(d^-), \tag{6.19}$$

or

$$W(y) \leq W(a^-) + W(b^+) + W(c^-) + W(d^+). \tag{6.20}$$

If (6.19) is true, then, by (6.13), $W(x) \leq W(a^+ + b^- + c^+ + d^-)$. The monotonicity property of W yields $x \leq a^+ + b^- + c^+ + d^-$, and thus

$$\begin{aligned} D(q_1, q_3) &= x + y = 2x - (x - y) \leq 2(a^+ + b^- + c^+ + d^-) - (a - b + c - d) \\ &\leq |a| + |b| + |c| + |d| = D(q_1, q_2) + D(q_2, q_3). \end{aligned}$$

Similarly, if (6.20) is true, then (6.17) also holds. □

The Riemann solutions of (1.15), (6.9), and (6.10) have three constant states (τ_j, v_j), $j = 1, 2, 3$, connected by two of the elementary waves: 1-wave (S_1 wave or R_1 wave) and 2-wave (S_2 wave or R_2 wave).

Remark 6.1. The proof of (6.17) given above is due to Poupaud-Rascle-Vila [258].

6.3. **Isentropic Euler Equations.** Consider the Riemann problem for the isentropic Euler equations in (1.14) with Riemann data:

$$(\rho, m)|_{t=0} = \begin{cases} (\rho_L, m_L), & x < 0, \\ (\rho_R, m_R), & x > 0, \end{cases} \tag{6.21}$$

which may contain the vacuum states, where $\rho_J \geq 0$ and m_J are the constants, and $\left|\frac{m_J}{\rho_J}\right| \leq C_0 < \infty$, $J = L, R$. As usual, assume that the pressure function $p(\rho)$ satisfies that, when $\rho > 0$,

$$p(\rho) > 0,\ \ p'(\rho) > 0 \,(\text{hyperbolicity}),\ \ \rho p''(\rho) + 2p'(\rho) > 0 \,(\text{genuine nonlinearity}), \tag{6.22}$$

and, when ρ tends to zero,

$$p(\rho),\ p'(\rho) \to 0, \tag{6.23}$$

which is different from the isothermal case.

The eigenvalues of System (1.14) are

$$\lambda_i = m/\rho + (-1)^i \sqrt{p'(\rho)}, \quad i = 1, 2, \tag{6.24}$$

and the corresponding right-eigenvectors are

$$\mathbf{r}_i = \alpha_i(\rho)(1, \lambda_i)^\top, \quad \alpha_i(\rho) = (-1)^i \frac{2\rho\sqrt{p'(\rho)}}{\rho p''(\rho) + 2p'(\rho)}, \tag{6.25}$$

so that $\nabla\lambda_i \cdot \mathbf{r}_i = 1, i = 1, 2$. The Riemann invariants are

$$w_i = \frac{m}{\rho} + (-1)^{i-1} \int_0^\rho \frac{\sqrt{p'(s)}}{s} ds, \quad i = 1, 2. \tag{6.26}$$

From (6.23) and (6.24),

$$\lambda_2 - \lambda_1 = 2\sqrt{p'(\rho)} \to 0, \qquad \rho \to 0.$$

Therefore, System (1.14) is strictly hyperbolic in the nonvacuum states $\{(\rho, v) : \ \rho > 0, |v| \leq C_0\}$. However, strict hyperbolicity fails near the vacuum states $\{(\rho, m/\rho) : \ \rho = 0, |m/\rho| \leq C_0\}$.

Shock Wave Curves. From the Rankine–Hugoniot condition (6.7) and the Lax entropy condition (6.8), we obtain that the i–shock wave curves $S_i(\rho_-, m_-), i = 1, 2$, are

$$S_i(\rho_-, m_-) : \qquad m - m_- = \frac{m_-}{\rho_-}(\rho - \rho_-) + (-1)^i \sqrt{\frac{\rho}{\rho_-}\frac{p(\rho) - p(\rho_-)}{\rho - \rho_-}}(\rho - \rho_-),$$
$$(-1)^i(\rho - \rho_-) < 0, \ \rho_- > 0.$$

It is easy to check that the curves $S_i(\rho_-, m_-), i = 1, 2$, are concave and convex, respectively, with respect to (ρ_-, m_-) in the $\rho - m$ plane.

Rarefaction Wave Curves. Given a state (ρ_-, m_-), the i-centered rarefaction wave curves $R_i(\rho_-, m_-), i = 1, 2$, are

$$R_i(\rho_-, m_-) : \qquad m - m_- = \frac{m_-}{\rho_-}(\rho - \rho_-) + (-1)^i \rho \int_{\rho_-}^\rho \frac{\sqrt{p'(s)}}{s} ds,$$
$$(-1)^i(\rho - \rho_-) > 0.$$

Then the curves $R_i, i = 1, 2$, are concave and convex, respectively, in the $\rho - m$ plane.

For the Riemann problem (1.14) and (6.21) satisfying (6.22) and (6.23), there exists a unique, globally defined, piecewise smooth entropy solution $\mathbf{R}(x/t)$, which may contain the vacuum states on the upper half-plane $t > 0$, satisfying

$$w_1(\mathbf{R}(x/t)) \leq w_1(\mathbf{u}_R), \ \ w_2(\mathbf{R}(x/t)) \geq w_2(\mathbf{u}_L), \ \ w_1(\mathbf{R}(x/t)) - w_2(\mathbf{R}(x/t)) \geq 0.$$

These Riemann solutions can be constructed for the case: $w_i(\mathbf{u}_R) \geq w_i(\mathbf{u}_L), i = 1, 2$, as follows.

If $\rho_L > 0$ and $\rho_R = 0$, then there exists a unique v_c such that

$$\mathbf{R}(x/t) = \begin{cases} \mathbf{u}_L, & x/t < \lambda_1(\mathbf{u}_L), \\ \mathbf{V}_1(x/t), & \lambda_1(\mathbf{u}_L) \leq x/t \leq v_c, \\ \text{vacuum}, & x/t > v_c, \end{cases}$$

where $\mathbf{V}_1(\xi)$ is the solution of the boundary value problem

$$\mathbf{V}_1'(\xi) = \mathbf{r}_1(\mathbf{V}_1(\xi)), \quad \xi > \lambda_1(\mathbf{u}_L); \qquad \mathbf{V}_1|_{\xi=\lambda_1(\mathbf{u}_L)} = \mathbf{u}_L. \tag{6.27}$$

If $\rho_L = 0$ and $\rho_R > 0$, then there exists a unique $\tilde{v}_c$ such that

$$\mathbf{R}(x/t) = \begin{cases} \text{vacuum}, & x/t < \tilde{v}_c, \\ \mathbf{V}_2(x/t), & \tilde{v}_c \leq x/t \leq \lambda_2(\mathbf{u}_R), \\ \mathbf{u}_R, & x/t > \lambda_2(\mathbf{u}_R), \end{cases}$$

where $\mathbf{V}_2(\xi)$ is the solution of the boundary value problem

$$\mathbf{V}_2'(\xi) = \mathbf{r}_2(\mathbf{V}_2(\xi)), \quad \xi < \lambda_2(\mathbf{u}_R); \qquad \mathbf{V}_2|_{\xi=\lambda_2(\mathbf{u}_R)} = \mathbf{u}_R. \tag{6.28}$$

If $\rho_L, \rho_R > 0$, there are two subcases:

(a). There exist unique $v_{c_1}, v_{c_2}, v_{c_1} < v_{c_2}$, such that the Riemann solution has the form:

$$\mathbf{R}(x/t) := \begin{cases} \mathbf{u}_L, & x/t < \lambda_1(\mathbf{u}_L), \\ \mathbf{V}_1(x/t), & \lambda_1(\mathbf{u}_L) \le x/t \le v_{c_1}, \\ \text{vacuum}, & v_{c_1} < x/t < v_{c_2}, \\ \mathbf{V}_2(x/t), & v_{c_2} \le x/t \le \lambda_2(\mathbf{u}_R), \\ \mathbf{u}_R, & x/t > \lambda_2(\mathbf{u}_R), \end{cases} \tag{6.29}$$

where $\mathbf{V}_1(\xi)$ and $\mathbf{V}_2(\xi)$ are the solutions of the boundary value problems (6.27) and (6.28), respectively.

(b). There exists a unique $\mathbf{u}_c = (\rho_c, m_c), \rho_c > 0$, such that the Riemann solution has the form:

$$\mathbf{R}(x/t) := \begin{cases} \mathbf{u}_L, & x/t < \lambda_1(\mathbf{u}_L), \\ \mathbf{V}_1(x/t), & \lambda_1(\mathbf{u}_L) \le x/t \le \lambda_1(\mathbf{u}_c), \\ \mathbf{u}_c, & \lambda_1(\mathbf{u}_c) < x/t < \lambda_2(\mathbf{u}_c), \\ \mathbf{V}_2(x/t), & \lambda_2(\mathbf{u}_c) \le x/t \le \lambda_2(\mathbf{u}_R), \\ \mathbf{u}_R, & x/t > \lambda_2(\mathbf{u}_R), \end{cases}$$

where $\mathbf{V}_1(\xi)$ and $\mathbf{V}_2(\xi)$ are the solutions of the boundary value problems (6.27) and (6.28), respectively.

For the subcase (a), although the Riemann data are nonvacuum states at $t = 0$, the vacuum states occur in the Riemann solutions instantaneously as t becomes positive. Therefore, the vacuum states are generic in inviscid compressible fluid flow (except the isothermal case).

Riemann solutions for the other cases can be constructed similarly. See Chang-Hsiao [33], Dafermos [88], Serre [277], and Smoller [291] for the details.

Proposition 6.5. *The regions*

$$\sum(w_1^0, w_2^0) = \{(\rho, m) \ : \ w_1 \le w_1^0, w_2 \ge w_2^0, w_1 - w_2 \ge 0\}$$

are invariant regions of the Riemann problem (1.14) *and* (6.21). *That is, if the Riemann data lie in* $\sum(w_1^0, w_2^0)$, *the solution of the Riemann problem also lies in* $\sum(w_1^0, w_2^0)$.

This can be checked directly from the explicit formulas known for the Riemann solutions.

6.4. **Non-Isentropic Euler Equations.** For convenience, in this section we focus on the non-isentropic Euler equations in (1.13), (1.6), and (1.7) in Lagrangian coordinates. We first analyze the global behavior of shock curves in the phase space and the singularity of centered rarefaction waves in the physical plane, and then construct global solutions of the Riemann problem (6.1) for (1.13), (1.6), and (1.7). These are essential for determining the uniqueness of Riemann solutions with arbitrarily large oscillation in §8.2.

Shock Curves. The Rankine-Hugoniot condition (6.7) for a discontinuity with speed σ for (1.13) is

$$\sigma[v] = [p], \qquad \sigma[\tau] = -[v], \qquad \sigma[e + \frac{1}{2}v^2] = [pv]. \tag{6.30}$$

If $\sigma = 0$, then the discontinuity is a contact discontinuity which corresponds to the second characteristic field.

If $\sigma \neq 0$, then the discontinuity is a shock, which corresponds to either the first or third characteristic field.

The Lax entropy inequality (6.8) and the Rankine-Hugoniot condition (6.30) imply that, on a 1-shock,

$$[p] > 0, \qquad [\tau] < 0, \qquad [v] < 0,$$

and, on a 3-shock,

$$[p] < 0, \qquad [\tau] > 0, \qquad [v] < 0.$$

From (6.30), we have

$$e - e_- + \frac{1}{2}(p + p_-)(\tau - \tau_-) = 0. \tag{6.31}$$

Set $s = \frac{p}{p_-}$. Then (6.31) becomes

$$p\tau = p_-\tau_- \left(1 - \frac{\gamma - 1}{2}(s+1)(\frac{\tau}{\tau_-} - 1)\right),$$

which implies

$$\frac{\tau}{\tau_-} = \frac{s + \beta}{\beta s + 1}, \quad \text{with} \quad \beta = \frac{\gamma + 1}{\gamma - 1}. \tag{6.32}$$

Note that

$$[v] = -\sigma[\tau] = -\sqrt{-[p][\tau]}.$$

Then, denoting the sound speed by c, i.e., $c = \sqrt{\gamma p \tau}$, one has

$$v - v_- = (-1)^{\frac{i-1}{2}} c_- \sqrt{\frac{2}{\gamma(\gamma - 1)}} \frac{1 - s}{\sqrt{\beta s + 1}}. \tag{6.33}$$

Let $s = e^{-x}$. From (6.32) and (6.33), we obtain that the i-shock is determined by

$$\frac{p}{p_-} = e^{-x}, \qquad (-1)^{\frac{i-1}{2}} x \le 0, \tag{6.34}$$

$$\frac{\tau}{\tau_-} = \frac{1 + \beta e^x}{\beta + e^x}, \tag{6.35}$$

$$\frac{v - v_-}{c_-} = (-1)^{\frac{i-1}{2}} \sqrt{\frac{2}{\gamma(\gamma - 1)}} \frac{1 - e^{-x}}{\sqrt{1 + \beta e^{-x}}}, \tag{6.36}$$

with speed

$$\sigma = (-1)^{\frac{i+1}{2}} \frac{c_-}{\tau_-} \sqrt{\frac{1 + \beta e^{-x}}{\beta + 1}}. \tag{6.37}$$

Now we choose the speed σ as a parameter for the shock curve, that is, x is a function of σ: $x = x(\sigma)$, and compute the derivatives of $x(\sigma)$ in $\sigma < 0$ (1-shock) and in $\sigma > 0$ (3-shock). We use the notations $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{d\sigma}$. Since

$$\sigma^2 = \frac{c_-^2}{\tau_-^2} \frac{1 + \beta e^{-x(\sigma)}}{\beta + 1}, \tag{6.38}$$

we take the derivative on both sides of (6.38) in σ and use (6.38) to deduce

$$\dot{x}(\sigma) = (-1)^{\frac{i-1}{2}} 2 \frac{\beta + 1}{\beta} \frac{\tau_-}{c_-} e^{x(\sigma)} \sqrt{\frac{1 + \beta e^{-x(\sigma)}}{\beta + 1}}.$$

We take the second-order derivative on both sides of (6.38) in σ to have

$$\dot{x}(\sigma)^2 - \ddot{x}(\sigma) = \frac{\beta}{2(\beta + e^{x(\sigma)})} \dot{x}(\sigma)^2 > 0.$$

Then

$$\ddot{x}(\sigma) = \frac{e^{x(\sigma)} + \beta/2}{e^{x(\sigma)} + \beta} \dot{x}(\sigma)^2 > 0.$$

We take the third-order derivative on both sides of (6.38) in σ to have

$$\dddot{x}(\sigma) - 3\dot{x}(\sigma)\ddot{x}(\sigma) + \dot{x}(\sigma)^3 = 0.$$

On the other hand, we have from (6.34) that

$$p' = -p, \;\; p'' = p, \;\; p''' = -p,$$

and then

$$\dot{p} = -p\dot{x}, \quad \ddot{p} = p((\dot{x})^2 - \ddot{x}) > 0, \quad \dddot{p} = p(-(\dot{x})^3 + 3\dot{x}\ddot{x} - \dddot{x}) = 0.$$

From (6.35), we similarly have

$$\dot{\tau} = \tau'\dot{x}, \quad \ddot{\tau} = 3\tau'((\dot{x})^2 - \ddot{x}), \quad \dddot{\tau} = \frac{6\beta\tau'\dot{x}}{\beta + e^x}((\dot{x})^2 - \ddot{x}).$$

Furthermore, we note that $\frac{S}{c_v} = \ln(\frac{1}{\kappa}p\tau^\gamma)$. Then

$$\begin{aligned}
\frac{\dot{S}}{c_v} &= \frac{\dot{p}}{p} + \gamma\frac{\dot{\tau}}{\tau} = -\frac{\beta(e^x - 1)^2\dot{x}}{(\beta + e^x)(1 + \beta e^x)}, \\
\frac{\ddot{S}}{c_v} &= \frac{\ddot{p}}{p} - \frac{\dot{p}^2}{p^2} + \gamma\frac{\ddot{\tau}}{\tau} - \gamma\frac{\dot{\tau}^2}{\tau^2} = \frac{(\dot{x})^2}{(\beta + e^x)^2(1 + \beta e^x)^2} P(e^x),
\end{aligned}$$

where

$$P(y) = \beta(y - 1)\left(-\beta y^3 - (\frac{3}{2}\beta^2 + \beta + 2)y^2 - \frac{1}{2}(\beta^2 + 5\beta)y + \frac{\beta}{2}\right), \quad y > 0.$$

The following proposition is taken from Chen-Frid-Li [52].

Proposition 6.6. *Along any shock curve, $S = S(\sigma)$ satisfies*

$$2\dot{S}(\sigma) + \sigma\ddot{S}(\sigma) \le 0.$$

Proof. This can be seen from a direct calculation, which yields

$$2\dot{S}(\sigma) + \sigma\ddot{S}(\sigma) = \frac{c_v\dot{x}(\sigma)(1 - e^{x(\sigma)})}{(\beta + e^{x(\sigma)})(1 + \beta e^{x(\sigma)})^2} Q(e^{x(\sigma)}),$$

while

$$Q(y) = -2\beta y^3 - (\beta^2 + 2\beta + 4)y^2 - 3\beta(\beta + 1)y - \beta < 0.$$

Since $\dot{x}(\sigma)(1 - e^{x(\sigma)})$ is always nonnegative, the result follows. □

Rarefaction Waves. Consider the self–similar solutions $\mathbf{V}(\xi) = (\tau, v, e + \frac{v}{2})(\xi), \xi = x/t$, of (1.13) with left state $\mathbf{u}_- = (\tau_-, v_-, e_- + \frac{v_-^2}{2})$. Then we have

$$\begin{cases}
\xi = \lambda_i(\mathbf{V})(\xi), \quad i = 1, 3, \\
\frac{dv}{d\xi} + \xi\frac{d\tau}{d\xi} = 0, \\
\frac{de}{d\xi} + p\frac{d\tau}{d\xi} = 0,
\end{cases}$$

with boundary condition $\mathbf{V}|_{\xi=\lambda_i(\mathbf{u}_-)} = \mathbf{u}_-$ and, on the i-centered rarefaction waves,

$$\frac{\partial\mathbf{V}}{\partial x} = \frac{1}{t}\frac{d\mathbf{V}}{d\xi} = \frac{1}{t}\mathbf{r}_i(\mathbf{V}(\frac{x}{t})), \quad i = 1, 3. \tag{6.46}$$

In particular, we have

$$\frac{\partial\mathbf{W}}{\partial x} = \frac{1}{t}\tilde{\mathbf{r}}_i(\mathbf{W}(\frac{x}{t})), \quad i = 1, 3, \tag{6.47}$$

where $\mathbf{W}=(\tau,v,S)$ and $\tilde{\mathbf{r}}_i=\frac{2\sqrt{-p_v(v,S)}}{p_{vv}(v,S)}(\sqrt{-p_v(v,S)},(-1)^{\frac{i-1}{2}},0)^\top$.

Similar to the argument for shocks, we can also obtain centered rarefaction wave curves in the phase space for the first and third characteristic fields.

For a rarefaction wave $\mathbf{V}(x/t)$ with right state $\mathbf{u}_+$, denoting

$$w_i=v+(-1)^{\frac{i-1}{2}}\int_\tau^\infty\sqrt{-p_\tau(s,S_\pm)}ds, \qquad i=1,3,$$

with $w_1(\mathbf{u}_-)-w_3(\mathbf{u}_+)>0$, one has

$$\begin{cases} w_1(\mathbf{u}_-)\le w_1(\mathbf{V}(x/t))\le w_1(\mathbf{u}_+), \;\; w_3(\mathbf{u}_-)\le w_3(\mathbf{V}(x/t))\le w_3(\mathbf{u}_+),\\ w_1(\mathbf{V}(x/t))-w_3(\mathbf{V}(x/t))>0, \qquad S(x/t)=S_+=S_-. \end{cases}$$

These rarefaction waves are identical to those of the isentropic case with the 2-field in the isentropic case corresponding to the 3-field in the non-isentropic case.

Solvability. For the Riemann problem (1.13) and (6.1), we have

Proposition 6.7. *Given states $\mathbf{W}_L=(v_L,\tau_L,S_L)$ and $\mathbf{W}_R=(v_R,\tau_R,S_R)$, there exists a unique global Riemann solution $\mathbf{R}(x/t)$ in the class of self-similar piecewise smooth solutions consisting of shocks, rarefaction waves, and contact discontinuities, provided that the Riemann data satisfy*

$$v_R-v_L<\frac{2}{\gamma-1}(c(\tau_L,S_L)+c(\tau_R,S_R)), \tag{6.49}$$

where $c(\tau,S)=\tau\sqrt{-p_\tau(\tau,S)}$.

The proof of Proposition 6.7 can be found in [290, 291, 33]. The condition (6.49) is necessary and sufficient for Riemann solutions staying away from the vacuum; without this condition, Riemann solutions may contain δ-masses at the vacuum states and become measure solutions (see Wagner [314] and Chen-Frid [51]).

7. Global Discontinuous Solutions II: Glimm Solutions

We now discuss the Glimm solutions that are the entropy solutions, obtained via the Glimm random choice method, of the Cauchy problem for hyperbolic systems of conservation laws, which apply to the Euler equations for compressible fluids. A related method, the wave-front tracking algorithm, is also discussed.

7.1. The Glimm Scheme and Existence.

We first discuss the Glimm scheme in [130] which uses the solutions of the Riemann problem to construct a global entropy solution in BV of the Cauchy problem (1.19) and (1.20) for hyperbolic systems of n conservation laws, provided that $\mathbf{u}_0(x)$ has small total variation on $\mathbb{R}$. For the isothermal Euler equations, the Glimm scheme yields a global entropy solution with initial data of arbitrarily large total variation.

Glimm Scheme. Assume that System (1.19) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate in a neighborhood of a constant state $\bar{\mathbf{u}}$. Denote by $\lambda_1(\mathbf{u})<\cdots<\lambda_n(\mathbf{u})$ the eigenvalues of the Jacobian matrix $\nabla\mathbf{f}(\mathbf{u})$. The solution $\mathbf{u}(x,t)$ of the Cauchy problem is obtained as the limit of the approximate solutions $\mathbf{u}^h(x,t)$, when $h\to 0$, constructed by the Glimm scheme, as described below.

Fix $h>0$, a space-step size, and determine the corresponding time-step size $\Delta t=h/\Lambda$ satisfying the Courant-Friedrichs-Lewy condition, where Λ is an upper bound of the characteristic speeds $|\lambda_i|,\ i=1,2,\cdots,n$. Then we partition the upper half-plane $\mathbb{R}^2_+:=\{(x,t)\,:\,x\in\mathbb{R},\,t\ge 0\}$ into the strips $S^k=\{(x,t)\,:\,x\in\mathbb{R},\,k\Delta t\le t<(k+1)\Delta t\}$, $k\in\mathcal{Z}_+$, and identify the mesh points $(jh,k\Delta t)$ with $k\in\mathcal{Z}_+, j\in\mathcal{Z}$, and $j+k$ even.

Choose any random sequence of numbers $a = \{a_0, a_1, a_2, \cdots\} \subset (-1,1)$ which is equidistributed in $(-1,1)$ in the following sense: for any subinterval $I \subset (-1,1)$ of length $|I|$,

$$\lim_{l\to\infty} \frac{2}{l} N_l = |I|$$

uniformly with respect to I, where N_l is the number of indices $k \le l$ with $a_k \in I$. Set the sampling points as $P_j^k = ((j + a_k)h, k\Delta t)$ with $j + k$ odd.

Denote the approximate solution by $\mathbf{u}^h(x,t)$. It is defined by induction on $k = 0, 1, 2, \cdots$ in each strip S^k. Define $\mathbf{u}_j^0 = \mathbf{u}_0((j + a_0)h)$ and

$$\mathbf{u}_j^k = \mathbf{u}^h((j + a_k)h - 0, k\Delta t - 0)$$

for $j + k$ odd and $k \ge 1$. Set $\mathbf{u}^h(x, k\Delta t) = \mathbf{u}_j^k$ for $x \in ((j-1)h, (j+1)h)$ with $j + k$ odd. Define the solution $\mathbf{u}^h(x,t)$ for $x \in [(j-1)h, (j+1)h]$, $t \in [k\Delta t, (k+1)\Delta t)$, $j + k$ even, as the solution of the Riemann problem of the system with initial data

$$\mathbf{u}|_{t=k\Delta t} = \begin{cases} \mathbf{u}_{j-1}^k, & x < jh, \\ \mathbf{u}_{j+1}^k, & x > jh. \end{cases}$$

Then $\mathbf{u}^h(x,t)$ is well defined: it is the exact entropy solution in each strip S^k, it is continuous at the interfaces $x = jh$, $k\Delta t \le t < (k+1)\Delta t$ with $j + k$ odd, and it experiences jump discontinuities across the lines $t = k\Delta t$, $k = 0, 1, 2, \cdots$. The waves emanating from the neighboring discontinuing mesh points $(jh, k\Delta t)$ and $((j+2)h, k\Delta t)$, $j + k$ even, do not intersect.

If it is proved that $\mathbf{u}^h(x,t)$ is uniformly bounded in h in $\mathbb{R}_+^2$, Λ can be chosen, and the Glimm approximate solutions are constructed for all $t \ge 0$. Then the limit of the approximate solutions is the entropy solution of the Cauchy problem (1.19) and (1.20) as in the following theorem.

Theorem 7.1. *Assume that System* (1.19) *is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate in a neighborhood of a constant state* $\bar{\mathbf{u}}$. *Then there exist two positive constants* δ_1 *and* δ_2 *such that, for initial data* $\mathbf{u}_0$ *satisfying*

$$\|\mathbf{u}_0 - \bar{\mathbf{u}}\|_{L^\infty(\mathbb{R})} \le \delta_1, \quad \mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0) \le \delta_2, \tag{7.1}$$

the Cauchy problem (1.19) *and* (1.20) *has a global entropy solution* $\mathbf{u}(x,t)$ *for* $(x,t) \in \mathbb{R}_+^2$, *satisfying the entropy inequality* (1.18) *(*$d = 1$*) in the sense of distributions for any convex entropy-entropy flux pair and*

$$\|\mathbf{u}(\cdot,t) - \bar{\mathbf{u}}\|_{L^\infty(\mathbb{R})} \le C_0 \|\mathbf{u}_0 - \bar{\mathbf{u}}\|_{L^\infty(\mathbb{R})}, \quad \textit{for any} \quad t \in [0,\infty), \tag{7.2}$$

$$\mathrm{TV}_{\mathbb{R}}(\mathbf{u}(\cdot,t)) \le C_0 \mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0), \quad \textit{for any} \quad t \in [0,\infty), \tag{7.3}$$

$$\|\mathbf{u}(\cdot,t_1) - \mathbf{u}(\cdot,t_2)\|_{L^1(\mathbb{R})} \le C_0 |t_1 - t_2|\, \mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0), \quad \textit{for any} \quad t_1, t_2 \in [0,\infty), \tag{7.4}$$

for some constant $C_0 > 0$.

In order to show that the approximate solutions $\mathbf{u}^h(x,t)$ converge to a solution of the Cauchy problem (1.19) and (1.20), it is required to establish

(i) The compactness of the approximate solutions in order to ensure that a convergent subsequence (still denoted by) $\mathbf{u}^h(x,t)$ may be selected such that $\mathbf{u}^h(x,t) \to \mathbf{u}(x,t)$, a.e. for $(x,t) \in \mathbb{R}_+^2$;

(ii) The consistency of the scheme in order to guarantee that the limit $\mathbf{u}(x,t)$ is indeed a solution of the Cauchy problem (1.19) and (1.20).

For the compactness of the Glimm approximate solutions under the assumption (7.1), the following estimates can be established:

$$\|\mathbf{u}^h(\cdot,t)-\bar{\mathbf{u}}\|_{L^\infty(\mathbb{R})}\le C_0\|\mathbf{u}_0-\bar{\mathbf{u}}\|_{L^\infty(\mathbb{R})},\quad \text{for any}\quad t\in[0,\infty), \tag{7.5}$$

$$\mathrm{TV}_{\mathbb{R}}(\mathbf{u}^h(\cdot,t))\le C_0\,\mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0),\quad \text{for any}\quad t\in[0,\infty), \tag{7.6}$$

$$\|\mathbf{u}^h(\cdot,t_1)-\mathbf{u}^h(\cdot,t_2)\|_{L^1(\mathbb{R})}\le C_0(|t_1-t_2|+h)\,\mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0),\quad \text{for any}\quad t_1,t_2\in[0,\infty), \tag{7.7}$$

for some constant $C_0>0$. Estimate (7.5) guarantees that the approximate solutions $\mathbf{u}^h(x,t)$ can be constructed globally for all t over $[0,\infty)$ if δ_1 in (7.1) is sufficiently small. These compactness estimates imply that the family of approximate solutions $\mathbf{u}^h(x,t)$ has uniformly bounded variation and thus converges almost everywhere, by the Helly theorem, to a function $\mathbf{u}(x,t)$ in BV. It can be shown that, for any equidistributed random sequence of numbers $a=\{a_0,a_1,a_2,\cdots\}\subset(-1,1)$, the limit function $\mathbf{u}(x,t)$ is an entropy solution of the Cauchy problem (1.19) and (1.20), which also satisfies the entropy condition.

For the compactness estimates, (7.7) is an immediate consequence of (7.6) since the waves emanating from each mesh point propagate with speed not exceeding Λ. To establish (7.6), one first notes that, for any $t\in(k\Delta t,(k+1)\Delta t)$, $\mathrm{TV}_{\mathbb{R}}(\mathbf{u}(\cdot,t))$ is constant and can be measured by the sum of the strengths of waves that emanate from the mesh points $(jh,k\Delta t)$ with $j+k$ even. To estimate how the sum of wave strengths changes from the strip S^k to the strip S^{k+1}, consider the family of diamond shaped regions $\Diamond_{jk}$, $j+k$ odd, with vertices P_j^k, P_{j+1}^{k+1}, P_j^{k+2}, and P_{j-1}^{k+1}. A wave fan of n waves $(\varepsilon_1,\cdots,\varepsilon_n)$ emanates from the mesh point P_j^{k+1} inside $\Diamond_{jk}$. Through the side of $\Diamond_{jk}$ connecting the two vertices P_j^k and P_{j-1}^{k+1}, there crosses a fan of waves $(\alpha_1,\cdots,\alpha_n)$ which is part (possibly none or all, as some of the components α_i could be zero) of the wave fan emanating from the mesh point P_{j-1}^k, and through the side of $\Diamond_{jk}$ connecting the two vertices P_j^k and P_{j+1}^{k+1} there crosses a fan of waves $(\beta_1,\cdots,\beta_n)$ which is part (possibly none or all, as some of the components β_i could be zero) of the wave fan emanating from the mesh point P_{j+1}^k. Indeed, the wave fan $(\varepsilon_1,\cdots,\varepsilon_n)$ approximates the wave pattern that would have resulted if the wave fans $(\alpha_1,\cdots,\alpha_n)$ and $(\beta_1,\cdots,\beta_n)$ had been allowed to propagate beyond $t=(k+1)\Delta t$ and thus interact. It can be shown that the strengths of incoming and outgoing waves are related by

$$\sum_{i=1}^n|\varepsilon_i|=\sum_{i=1}^n(|\alpha_i|+|\beta_i|)+O(Q_{jk}) \tag{7.8}$$

with $Q_{jk}=\sum_{i,j}\{|\alpha_i||\beta_j|\ :\ \alpha_i\,\text{and}\,\beta_j\,\text{interacting}\}$. If the quadratic term Q_{jk} were not present, the total variation of $\mathbf{u}^h(\cdot,t)$, as measured by the strengths of waves, would not increase from S^k to S^{k+1}.

The effect of the quadratic term can be controlled as follows. Consider the polygonal curve J_k whose arcs connect nodes P_j^k, P_{j-1}^{k+1}, and P_{j+1}^{k+1}, $j+k$ odd. Define the Glimm functional associated with the curve J_k as

$$\mathcal{F}(J_k)=\mathcal{L}(J_k)+M\mathcal{Q}(J_k), \tag{7.9}$$

where

$$\mathcal{L}(J_k):=\sum\{|\alpha|\ :\ \text{any wave}\ \alpha\ \text{crossing}\,J_k\}$$

is the linear part measuring the total variation,

$$\mathcal{Q}(J_k):=\sum\{|\alpha||\beta|\ :\ \alpha,\beta\,\text{interacting waves crossing}\,J_k\}$$

is the quadratic part measuring the potential wave interaction, and M is a large positive constant. The functional $\mathcal{F}(J_k)$ is well defined and essentially equivalent to $\mathrm{TV}_{\mathbb{R}}(\mathbf{u}^h(\cdot,t))$ for $k\Delta t\le t<(k+1)\Delta t$. It can be shown from (7.8) that $\mathcal{F}(J_k)$ is nonincreasing in k as long as the total variation remains small, which implies the estimate (7.6).

For the details of the proof of Theorem 7.1, see Glimm [130] and Liu [211, 214]; also see Dafermos [88], Serre [277], and Smoller [291]. For extensions of the Glimm scheme to nonhomogeneous balance laws, see Dafermos-Hsiao [89], Liu [207], and Chen-Wagner [64].

The proof of Theorem 7.1 is based on the estimate showing that the effect of interactions is of second-order for the general system of n conservation laws, that is, the change in magnitude of waves due to interaction is of second-order in the magnitude of waves before interaction. For a system of two conservation laws, there exists a coordinate system of Riemann invariants, and the effect of interaction is of third-order, that is, the system is uncoupled modulo the third-order of the total variation of the solution. Therefore, Theorem 7.1 holds for the initial data of small oscillation but of larger total variation in the case of two conservation laws. This, in particular, applies to the isentropic Euler equations in (1.14) and (1.15). For the isothermal Euler equations, $\gamma = 1$, the condition of small oscillation can also be removed.

Isothermal Gas Dynamics. For the Euler equations for isothermal gas dynamics, global entropy solutions can be constructed by the Glimm scheme with any large initial data of bounded variation due to the special structure of the wave curves (6.12).

For (1.15) in Lagrangian coordinates, the one-dimensional isothermal motion of gases has the equation of state (6.9). Consider the Cauchy problem of (1.15) and (6.9) for $x \in \mathbb{R}$ and $t \geq 0$ with initial data:

$$(\tau, v)|_{t=0} = (\tau_0, v_0)(x), \quad x \in \mathbb{R}. \tag{7.10}$$

Then we have the following theorem due to Nishida [242].

Theorem 7.2. *Suppose that $\tau_0(x)$ and $v_0(x)$ are bounded functions with bounded variation over $\mathbb{R}$ and $\inf_{x\in\mathbb{R}} \tau_0(x) > 0$. Then the Cauchy problem* (1.15)*,* (6.9)*, and* (7.10) *has a global entropy solution $(\tau, v)(x,t)$ with bounded total variation in $x \in \mathbb{R}$ for any $t \geq 0$.*

Proof. The solution $(\tau, v)(x, t)$ in Theorem 7.2 is obtained as the limit of the approximate solutions $(\tau^h, v^h)(x, t)$ constructed by the Glimm scheme as in Theorem 7.1. In order to prove Theorem 7.2, it suffices to show that there exists a constant $C_0 > 0$ such that

$$\mathrm{TV}_{\mathbb{R}}(\tau^h, v^h)(\cdot, t) \leq C_0 \mathrm{TV}_{\mathbb{R}}(\tau_0, v_0). \tag{7.11}$$

To establish the compactness estimate (7.11), one needs to use the special structure of the wave curves of (1.15) and (6.9), established in §6.2, and show that the linear part $\mathcal{L}$ of the Glimm functional is decreasing. To see this, we first notice that the Riemann solution of (1.15), (6.9), and (6.10) has three constant states (τ_i, v_i), $i = 1, 2, 3$, connected by two of the elementary waves: 1-wave (S_1 wave or R_1 wave) and 2-wave (S_2 wave or R_2 wave). Denote these two waves by a vector $\alpha = (\alpha_1, \alpha_2)$, and denote the strength of i-wave, $i = 1, 2$, by $|\alpha_1| = |q_2 - q_1|$, $|\alpha_2| = |q_3 - q_2|$, and $|\alpha| = |\alpha_1| + |\alpha_2|$.

The approximate solutions (τ^h, v^h) will be estimated along the piecewise linear curves J defined as follows. Let the curve J_0 be composed of the all segments joining P_j^0 to P_{j+1}^1 and P_{j+1}^1 to P_{j+2}^0 for all odd j. An immediate successor curve J_2 of curve J_1 is composed of the same line segments except two segments joining P_j^k to P_{j+1}^{k-1} and P_{j+1}^{k-1} to P_{j+2}^k, which are replaced by those joining P_j^k to P_{j+1}^{k+1} and P_{j+1}^{k+1} to P_{j+2}^k. Then all curves J are obtained by taking successively immediate successors, starting out from the curve J_0. Define the functional $\mathcal{L}(J)$ on the approximate solutions restricted to each curve J by

$$\mathcal{L}(J) = \sum |\alpha|,$$

where the summation is taken over all vectors of two elementary waves $\alpha = (\alpha_1, \alpha_2)$ in the approximate solutions crossing the curve J. If J_2 is an immediate successor of the curve J_1, Proposition 6.4 (i.e. (6.17)) implies $\mathcal{L}(J_2) \le \mathcal{L}(J_1)$. By induction, $\mathcal{L}(J) \le \mathcal{L}(J_0)$ for any curve J, which implies

$$\mathrm{TV}_{\mathbb{R}}(q^h(\cdot,t)) \le \mathrm{TV}_{\mathbb{R}}(q_0)$$

for any $t \ge 0$, with $q_0(x) = \ln \tau_0(x)$, and thus $|q^h(x,t)| \le K$ for some positive constant K, since $\tau_0(x) \in L^\infty(\mathbb{R})$. Then

$$\mathrm{TV}_{\mathbb{R}}(\tau^h(\cdot,t)) \le C_1 \mathrm{TV}_{\mathbb{R}}(q^h(\cdot,t)) \le C_1 \mathrm{TV}_{\mathbb{R}}(q_0),$$

and one has

$$\mathrm{TV}_{\mathbb{R}}(v^h(\cdot,t)) \le C_2 \mathrm{TV}_{\mathbb{R}}(q^h(\cdot,t)) \le C_2 \mathrm{TV}_{\mathbb{R}}(q_0),$$

from the equations of elementary wave curves (6.12) and $|W'(s)| \le C_2$ for $|s| \le K$. Therefore,

$$\mathrm{TV}_{\mathbb{R}}(\tau^h, v^h)(\cdot,t) \le C_3 \mathrm{TV}_{\mathbb{R}}(q_0) \le C_0 \mathrm{TV}_{\mathbb{R}}(\tau_0, v_0).$$

This completes the proof of Theorem 7.2. □

Theorem 7.2 was originally established by Nishida [242]. For extensions to other isothermal flows, see [258] for the Euler-Poisson flow and [224] for the spherically symmetric Euler flow.

For non-isentropic gas dynamics (1.13), consider the following Cauchy problem:

$$(\tau, v, S)|_{t=0} = (\tau_0, v_0, S_0)(x). \tag{7.12}$$

The following existence theorem is due to Liu [212] (also see Temple [302]).

Theorem 7.3. *Let $K \subset \{(\tau, v, S) : \tau > 0\}$ be a compact set in $\mathbb{R}_+ \times \mathbb{R}^2$, and let $N \ge 1$ be any positive constant. Then there exists a constant $C_0 = C_0(K, N)$, independent of $\gamma \in (1, 5/3]$, such that, for every initial data $(\tau_0, v_0, S_0)(x) \in K$ with $\mathrm{TV}_{\mathbb{R}}(\tau_0, v_0, S_0) \le N$, when*

$$(\gamma - 1)\mathrm{TV}_{\mathbb{R}}(\tau_0, v_0, S_0) \le C_0, \tag{7.13}$$

for any $\gamma \in (1, 5/3]$, the Cauchy problem (1.13) and (7.12) has a global entropy solution $(\tau, v, S)(x,t)$ which is bounded and satisfies

$$\mathrm{TV}_{\mathbb{R}}(\tau, v, S)(\cdot, t) \le C\ \mathrm{TV}_{\mathbb{R}}(\tau_0, v_0, S_0),$$

for some constant $C > 0$ independent of γ.

For the isentropic case: $S =$ constant, the existence result of Theorem 7.3 was proved in Nishida-Smoller [245] (also see DiPerna [99]). For extensions to the initial-boundary value problems, see [246, 213]. A similar theorem to Theorem 7.3, for general pressure law, was established in Temple [302]. In the direction of relaxing the requirement of small total variation, see Peng [253], Temple-Young [303, 304], and Schochet [269]. For additional further discussions and references to the Glimm scheme, see Dafermos [88] and Serre [277].

7.2. **Decay of Solutions.** In this section we discuss the decay properties of Glimm solutions in BV of hyperbolic systems of conservation laws (1.19).

Any system of two conservation laws (1.19) ($n = 2$) is endowed with a coordinate system (w_1, w_2) of Riemann invariants corresponding to the two eigenvalues λ_1 and λ_2. The system is genuinely nonlinear if $\partial_{w_i}\lambda_i \ne 0$, $i = 1, 2$. The isentropic Euler equations staying away from the vacuum are an important example of a 2×2 genuinely nonlinear and strictly hyperbolic system. We focus our attention on the decay properties of Glimm solutions with large total variation for genuinely nonlinear and strictly hyperbolic systems of two conservation laws, which are valid, in particular, for the isentropic Euler equations away from the vacuum.

First, for the Glimm solution $\mathbf{u}(x,t)$ of (1.19), one has the following decay law:

$$\mathrm{TV}_{\mathbb{R}}(\mathbf{u}(\cdot,t)) \le C t^{-1/2}, \tag{7.14}$$

for some constant $C > 0$, which holds for any genuinely nonlinear system of two conservation laws with initial data of small oscillation (see Glimm-Lax [133]) and of n conservation laws with initial data of small total variation (see Liu [204]). DiPerna also proved in [103] that the total variation decays to zero, with no rate of convergence, for a more general system of n conservation laws which admits linearly degenerate characteristic fields such as the non-isentropic Euler equations staying away from the vacuum ($n = 3$).

For any genuinely nonlinear and strictly hyperbolic system of two conservation laws, the Glimm solution with periodic initial data decays to the mean-value of the initial data over the period, and with initial data of compact support decays to an N-wave.

Periodic Solutions. First we consider Glimm solutions of the system of two conservation laws (1.19) with periodic initial data. The following fundamental decay behavior is due to Glimm and Lax [133].

Theorem 7.4. *For the genuinely nonlinear and strictly hyperbolic system of two conservation laws* (1.19) *with* $n = 2$*, if the initial data* $\mathbf{u}_0 \in L^\infty(\mathbb{R})$ *have small oscillation and are periodic with period* L*, then there exists a solution* $\mathbf{u}(x,t)$ *which is periodic with respect to* x *with period* L *for all* $t > 0$ *and satisfies*

$$\mathrm{TV}_{[x,x+L]}(\mathbf{u}(\cdot,t)) \le \frac{CL}{t}, \quad \text{for any} \quad x \in \mathbb{R}, \tag{7.15}$$

$$|\mathbf{u}(x,t) - \bar{\mathbf{u}}| \le \frac{CL}{t}, \tag{7.16}$$

where $\bar{\mathbf{u}}$ *is the mean-value of* $\mathbf{u}_0(x)$ *over the space period and* $C > 0$ *is some constant.*

To illustrate the ideas involved in the proof of this theorem, we first consider the scalar conservation law ([133, 180]):

$$\partial_t u + \partial_x f(u) = 0, \tag{7.17}$$

where $f(u)$ is strictly convex, $f''(u) \ge c_0 > 0$, and thus $f'(u)$ is strictly monotone increasing. Any differentiable solution $u(x,t)$ is constant along the characteristic $x = x(t)$ defined by

$$\frac{dx}{dt} = f'(u(x,t)). \tag{7.18}$$

The characteristics are straight lines and generally intersect. At a point of intersection, the solution becomes discontinuous. Along the curve of discontinuity with propagation speed σ, the Lax entropy condition

$$f'(u_-) > \sigma > f'(u_+) \tag{7.19}$$

is satisfied, which implies that

$$u_- > u_+, \tag{7.20}$$

since $f'(u)$ is increasing, where u_- and u_+ are the values of the solution $u(x,t)$ on the left-side and right-side of curve of discontinuity, respectively.

Let $x_1(t)$ and $x_2(t)$ be a pair of characteristics for $0 \le t \le T$. Then there is a whole one-parameter family of characteristics connecting the points of the interval $[x_1(0), x_2(0)]$ at $t = 0$ with points of the interval $[x_1(T), x_2(T)]$ at $t = T$. Since $u(x,t)$ is constant along these characteristics, $u(x,0)$ on the interval $[x_1(0), x_2(0)]$ and $u(x,T)$ on the interval $[x_1(T), x_2(T)]$ are equivariant, i.e., they take on the same values in the same order, and thus the total increasing and decreasing variations of $u(x,t)$ on these two intervals are the same. Denote by $D(t) = x_2(t) - x_1(t) > 0$ the width of the strip bounded by x_1 and x_2. Then, from (7.18), $D'(t) = f'(u_2) - f'(u_1)$, where u_1 and u_2 are constant along the characteristics $x_1(t)$ and $x_2(t)$, respectively, and

$$D(T) = D(0) + (f'(u_2) - f'(u_1))T. \tag{7.21}$$

Suppose that there is a shock y present in $u(x,t)$ between the characteristics x_1 and x_2. Since the characteristics on either side of a shock run into the shock according to (7.19), for any given time T, there exist two characteristics y_1 and y_2 intersecting the shock y at exactly time T. Assume that there are no other shocks present. Then the increasing variations of $u(x,t)$ on the intervals $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are independent of t. From (7.20), $u(x,t)$ decreases across shocks, and then the increasing variation of $u(x,t)$ over $[x_1(T), x_2(T)]$ equals the sum of the increasing variations of $u(x,t)$ over $[x_1(0), y_1(0)]$ and over $[y_2(0), x_2(0)]$. This sum is in general less than the increasing variation of $u(x,t)$ over $[x_1(0), x_2(0)]$. Thus, if shocks are present, the total increasing variation of $u(x,t)$ between two characteristics decreases with time.

To give a quantitative estimate of this decrease, we assume for simplicity that $u_0(x)$ is piecewise monotone. Let I_0 be any interval of the x-axis. Subdivide it into subintervals $[y_{j-1}, y_j]$, $j = 1, \cdots, N$, in such a way that $u(x,0)$ is alternatively increasing and decreasing on the subintervals. Denote by $y_j(t)$ the characteristic issuing from the jth point y_j with the understanding that, if $y_j(t)$ runs into a shock, $y_j(t)$ is continued as that shock. Then, for all $t > 0$, $u(x,t)$ is alternately increasing and decreasing on the intervals $(y_{j-1}(t), y_j(t))$, i.e., increasing for j odd and decreasing for j even. Since $f'(u)$ is an increasing function and $u(x,t)$ decreases across shocks, the total increasing variation $\mathrm{TV}^+(T)$ of $f'(u(x,t))$ across the interval $I(T) = [y_0(T), y_N(T)]$ is

$$\mathrm{TV}^+(T) = \sum_{j\,\mathrm{odd}} \left(f'(u_j(T)) - f'(u_{j-1}(T))\right), \tag{7.22}$$

where $u_{j-1}(T) = u(y_{j-1}(T)+, T)$ and $u_j(T) = u(y_j(T)-, T)$. Denote, as before, by $x_{j-1}(t)$, $x_j(t)$ the characteristics starting out inside y_{j-1}, y_j, which intersect $y_{j-1}(t)$, $y_j(t)$, respectively, at $t = T$. Then $u_j(t)$ is the constant value of $u(x,t)$ on $x_j(t)$. Set $D_j(t) = x_j(t) - x_{j-1}(t)$. Then, by (7.21),

$$D_j(T) = D_j(0) + \left(f'(u_j(T)) - f'(u_{j-1}(T))\right) T.$$

Take the sum over odd j to get, from (7.22),

$$\sum_{j\,\mathrm{odd}} D_j(T) = \sum_{j\,\mathrm{odd}} D_j(0) + \mathrm{TV}^+(T)T. \tag{7.23}$$

Since the intervals $[x_{j-1}(T), x_j(T)]$ are disjoint and lie in $I(T)$, their total length cannot exceed the length $|I(T)|$ of $I(T)$, and then

$$\mathrm{TV}^+(T) \le \frac{|I(T)|}{T}. \tag{7.24}$$

Suppose that the solution $u(x,t)$ is periodic in x with period L. Take I_0 to be an interval of length L, then $I(t)$ has length L for all $t > 0$. From the strict convexity $f''(u) > c_0 > 0$, (7.24) implies that the increasing variation per period of $u(x,t)$ itself does not exceed $L(c_0T)^{-1}$. Since $u(x,t)$ is periodic, its decreasing and increasing variations are equal and serve as a bound for the oscillation of $u(x,t)$, especially for the deviation of $u(x,t)$ from its mean-value over period $\bar{u} = \frac{1}{L}\int_0^L u(x,t)dx$. Therefore, the total variation of $u(x,t)$ per period at time t does not exceed $2L(c_0T)^{-1}$ and $|u(x,t) - \bar{u}| \le (c_0T)^{-1}$.

To generalize this idea to a system of two conservation laws, we first note that there exist Riemann invariants w_1 and w_2 which are functions of $\mathbf{u}(x,t)$ and satisfy the following equations:

$$\partial_t w_i + \lambda_j \partial_x w_i = 0, \quad i \ne j,$$

where $\lambda_i, i = 1, 2$, are the eigenvalues of System (1.19) with $n = 2$ which can be considered as functions of the Riemann invariants $\mathbf{w} = (w_1, w_2)$. Along 1-characteristics: $dx/dt = \lambda_1$ and along 2-characteristics: $dx/dt = \lambda_2$, w_2 and w_1 are constant, respectively. If $x_j(t), j = 1, 2$, are two 1-characteristics, $dx_j/dt = \lambda_{1,j}$, then $\mathbf{u}(x,0)$ along the interval $[x_1(0), x_2(0)]$ and $\mathbf{u}(x,T)$ along the interval $[x_1(T), x_2(T)]$ are equivariant. The 1-characteristics are no longer straight lines, but in general they intersect if the system is genuinely nonlinear. The 1-shocks satisfy the following Lax entropy condition:

$$\lambda_1(\mathbf{u}_-) > \sigma > \lambda_1(\mathbf{u}_+),$$

which is the analogue of the condition (7.19) and implies that 1-characteristics drawn in the direction of increasing time t run into 1-shocks. Thus, as before, the presence of the 1-shocks decreases the total variation of w_2. Similarly, the presence of the 2-shocks decreases the total variation of w_1. To estimate the decrease of the total variation of w_1, the effect of 1-shocks on the total variation of w_1 has to be considered. It is known that, across weak 1-shocks, Δw_1 is proportional to $(\Delta w_2)^3$, where $\Delta w_j, j = 1, 2$, denote the change in $w_j, j = 1, 2$, respectively. Then the change in total variation of w_1 due to 1-shocks does not exceed $O(\epsilon)\mathrm{TV}(u_0)^2$, where ϵ is the oscillation of the solution. The width $D(t) = x_2(t) - x_1(t)$ of a strip bounded by 1-characteristics $x_j(t), j = 1, 2$, satisfies

$$D'(t) = \lambda_{1,2} - \lambda_{1,1} = \overline{\partial_{w_1}\lambda_1}(w_{1,2} - w_{1,1}) + \overline{\partial_{w_2}\lambda_1}(w_{2,2} - w_{2,1}),$$

according to the mean-value theorem, where $\lambda_{i,j} := \lambda_i(\mathbf{w}(x_j(t), t))$ and $w_{i,j} := w_i(x_j(t), t)$. If the oscillation ϵ of the solution is small, then $\overline{\partial_{w_j}\lambda_1} = O(\epsilon), j = 1, 2$. The quantities $w_{2,j}, j = 1, 2$, are independent of t, but $w_{1,j}, j = 1, 2$, are not. This difficulty can be overcome by measuring the width of the strip, bounded by the 1-characteristics not between points with the same t coordinates but between points which lie approximately on the same 2-characteristics. Since w_1 is constant along 2-characteristics, $w_{1,2} - w_{1,1}$ is small in the above equation on $D(t)$. After constructing approximate characteristics, one can derive the approximate conservation laws of the increasing and decreasing variations of $w_j, j = 1, 2$, which are formulated as a balance between the amount of shock wave and rarefaction wave of either family entering and leaving a region, the amount of rarefaction and shock wave of the same family cancelling each other in the region, and a correction term accounting for the interaction between waves belonging to different families. Finally, the inequalities for the variations of $w_j, j = 1, 2$, can be obtained by passage to the limit. See Glimm-Lax [133] for the details of proof.

N-waves. Now we consider the Glimm solutions of System (1.19) with initial data supported on a compact set, i.e.,

$$\mathbf{u}_0(x) = 0, \quad \text{if} \quad |x| > R, \tag{7.25}$$

for some constant $R > 0$. The solution may decay to an N-wave. In the case of scalar conservation laws, the solution with initial data of compact support approaches an N-wave in L^1 as $t \to \infty$; see Lax [177], DiPerna [101], and Dafermos [88], as well as a different proof of this result by Keyfitz [259] for piecewise smooth solutions. An N-wave consists of a rarefaction wave bracketed by two shock waves. It propagates at a constant speed while its support expands at the rate $t^{1/2}$. The L^∞-norm of an N-wave decays at the rate $t^{-1/2}$, but its L^1-norm remains constant with time. For systems of n conservation laws, it has been conjectured by Lax [177] that, if the initial data have compact support, then the asymptotic form of the solution consists of n distinct N-waves, each propagating at one of the n distinct characteristic speeds $\lambda_i(0)$ of zero state. This conjecture has been proved for the case of two conservation laws ($n = 2$) with initial data of large total variation (DiPerna [101]) and for the the case of n conservation laws with initial data of small total variation (Liu [204]). The primary mechanisms of decay of solutions are the spreading of rarefaction waves and the cancellation of shock and rarefaction waves of the same kind.

For a genuinely nonlinear and strictly hyperbolic system of two conservation laws with the eigenvalues λ_i and the Riemann invariants w_i, satisfying $\partial_{w_i}\lambda_i \neq 0$, $i = 1, 2$, define the

N-waves:

$$N_i(x,t;p_i,q_i) = \begin{cases} \frac{1}{k_i}\left(\frac{x}{t} - \lambda_i(0,0)\right), & -(p_i k_i t)^{1/2} < x - \lambda_i(0,0)t < (q_i k_i t)^{1/2}, \\ 0, & \text{otherwise}, \end{cases}$$

for $k_i = \partial_{w_i}\lambda_i(0,0)$ and some constants $p_i, q_i > 0$, $i = 1, 2$. One has the following decay behavior in L^1 due to DiPerna [101].

Theorem 7.5. *For the genuinely nonlinear and strictly hyperbolic system of two conservation laws* (1.19) *with* $n = 2$*, if the initial data* $\mathbf{u}_0(x) \in L^\infty(\mathbb{R})$ *have small oscillations and compact support, then there exist positive constants* p_i *and* q_i *such that*

$$\|w_i(\cdot,t) - N_i(\cdot,t;p_i,q_i)\|_{L^1(\mathbb{R})} \le Ct^{-1/6}, \quad i = 1, 2,$$

for some constant $C > 0$.

For the BV solutions constructed by the Glimm scheme to any genuinely nonlinear and strictly hyperbolic system of n conservation laws with initial data of small total variation, Liu proved in [204] that the solution also decays to the N-waves at the rate $t^{-1/6}$ if the initial data have compact support; the generalization of this result to systems with linearly degenerate characteristic fields was given in Liu [205].

Decay properties for general BV solutions to systems of two conservation laws were obtained by Dafermos [88] by using the theory of generalized characteristics under the assumption that the traces of the solutions along any space-like curve are functions of locally bounded variation (see §8.1). See also Greenberg [143] on the decay of special solutions for a class of two conservation laws generated by a second-order wave equation, and Greenberg-Rascle [144] for an interesting example of periodic solutions in both space and time when the flux-function is C^1 but not C^2. For periodic entropy solutions only in L^∞, an analytical framework has been established in Chen-Frid [46] (also see §9.6).

7.3. **L^1-Stability of Glimm Solutions.** We now discuss the stability of solutions to the Cauchy problem (1.19) and (1.20). The existence proof in Theorem 7.1 based on compactness arguments does not provide information on this issue. By monitoring the time evolution of a certain functional, it can be shown that the Glimm solutions depend continuously on their initial data.

Let $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ be two approximate solutions of (1.19) constructed by the Glimm scheme, with small total variation. As in Theorem 7.1, it is assumed that System (1.19) is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. We discuss how the distance $\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^1(\mathbb{R})}$ changes in time. Denote by

$$s \mapsto R_i(s)(\mathbf{u}_-), \quad s \mapsto S_i(s)(\mathbf{u}_-), \quad i = 1, \cdots, n, \tag{7.26}$$

the i-rarefaction and i-shock curve of (1.19) through the state $\mathbf{u}_-$, parametrized by arclength, and set

$$\Upsilon_i(s)(\mathbf{u}_-) = \begin{cases} R_i(s)(\mathbf{u}_-), & s \ge 0, \\ S_i(s)(\mathbf{u}_-), & s < 0. \end{cases} \tag{7.27}$$

For any fixed point (x,t), consider the scalar function $q_i(x,t)$, which can be regarded intuitively as the strength of the i-shock wave in the jump $(\mathbf{u}(x,t), \mathbf{v}(x,t))$, defined implicitly by

$$\mathbf{v}(x,t) = S_n(q_n(x,t)) \circ \cdots \circ S_1(q_1(x,t))(\mathbf{u}(x,t)). \tag{7.28}$$

It is clear that

$$C_1^{-1}|\mathbf{u}(x,t) - \mathbf{v}(x,t)| \le \sum_{i=1}^n |q_i(x,t)| \le C_1|\mathbf{u}(x,t) - \mathbf{v}(x,t)| \tag{7.29}$$

for some constant $C_1 > 0$. For each $i = 1, \cdots, n$, define

$$\mathcal{W}_i(x,t) = \Big(\sum\nolimits_- + \sum\nolimits_+ + \sum\nolimits_0\Big)\left(|\alpha(\mathbf{u}(x,t))| + |\alpha(\mathbf{v}(x,t))|\right). \tag{7.30}$$

In (7.30), $\sum_-$ sums the strengths $|\alpha(\mathbf{u}(x,t))|$ (and $|\alpha(\mathbf{v}(x,t))|$) of all k_α-waves $x_\alpha(t) < x$ of $\mathbf{u}(x,t)$ (and $\mathbf{v}(x,t)$) with $i < k_\alpha \le n$, respectively; $\sum_+$ sums the strengths $|\alpha(\mathbf{u}(x,t))|$ (and $|\alpha(\mathbf{v}(x,t))|$) of all k_α-waves $x_\alpha > x$ of $\mathbf{u}(x,t)$ (and $\mathbf{v}(x,t)$) with $1 \le k_\alpha < i$, respectively; and $\sum_0$ sums the strengths $|\alpha(\mathbf{u}(x,t))|$ (and $|\alpha(\mathbf{v}(x,t))|$) of all i-waves, here $k_\alpha = i$, with $x_\alpha < x$ (and $x_\alpha > x$) of $\mathbf{u}(x,t)$ (and $\mathbf{v}(x,t)$ if $q_i(x,t) < 0$, or with $x_\alpha > x$ (and $x_\alpha < x$) of $\mathbf{u}(x,t)$ (and $\mathbf{v}(x,t)$) if $q_i(x,t) > 0$, respectively. Define a functional, equivalent to the L^1 distance of $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$, as

$$\Phi(\mathbf{u},\mathbf{v})(t) = \sum_{i=1}^{n} \int_{\mathbb{R}} |q_i(x,t)|(1 + K_1(\mathcal{F}_{\mathbf{u}}(nN\Delta t) + \mathcal{F}_{\mathbf{v}}(nN\Delta t)) + K_2\mathcal{W}_i(x,t))dx,$$

for each $t \in (nN\Delta t, (n+1)N\Delta t)$, where K_1 and K_2 are sufficiently large positive constants, N is a constant in the wave tracing method, $\mathcal{F}_{\mathbf{u}}$ and $\mathcal{F}_{\mathbf{v}}$ are the Glimm functionals defined in (7.9) for $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$, respectively, valued at the end time $t = nN\Delta t$. The definition of this functional is given by Liu-Yang [215], and is similar to those used in Bressan-Liu-Yang [25] and Hu-LeFloch [155] for the solutions constructed by the wave-front tracking algorithm. The key estimate is that the functional $\Phi(\mathbf{u},\mathbf{v})(t)$ can be controlled by its initial value $\Phi(\mathbf{u},\mathbf{v})(0)$, up to a certain error term which approaches zero as the mesh size tends to zero. From Theorem 7.1, there exist subsequences of the approximate solutions which converge to the exact Glimm solutions, locally in the L^1 norm. Therefore, one has the following theorem on the L^1-stability of Glimm solutions to the Cauchy problem of the genuinely nonlinear and strictly hyperbolic system (1.19) with initial data (1.20):

Theorem 7.6. *If the initial data $\mathbf{u}_0(x)$ and $\mathbf{v}_0(x)$ have sufficiently small total variation and $\mathbf{u}_0 - \mathbf{v}_0 \in L^1(\mathbb{R})$, then, for the corresponding exact Glimm solutions $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ of the Cauchy problem* (1.19) *and* (1.20)*, there exists a constant $C > 0$ such that*

$$\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^1(\mathbb{R})} \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\mathbb{R})}, \tag{7.31}$$

for all $t > 0$.

An immediate consequence of this theorem is that the whole sequence of the approximate solutions constructed by the Glimm scheme converges to a unique entropy solution of (1.19) and (1.20) as the mesh size tends to zero. See also Bressan [19] for the uniqueness of limits of Glimm's random choice method.

The details of the proof of Theorem 7.6 can be found in Liu-Yang [215, 216, 217].

7.4. **Wave-Front Tracking Algorithm and L^1-Stability.** Assume that System (1.19) is strictly hyperbolic, with eigenvalues $\lambda_1(\mathbf{u}) < \cdots < \lambda_n(\mathbf{u})$, and each characteristic field is either genuinely nonlinear or linearly degenerate. The Glimm scheme has been the basic tool for the construction and analysis of entropy solutions to systems of conservation laws. An alternative method for constructing approximate solutions is the wave-front tracking algorithm, which generates entropy solutions of the Cauchy problem (1.19) and (1.20) with initial data of small total variation and provides an alternative proof of Theorem 7.1.

The entropy solutions of (1.19) and (1.20) obtained by the wave-front tracking algorithm, the same as those obtained by the Glimm scheme, are L^1-stable, i.e., the solutions depend Lipschitz continuously on the initial data in the L^1 norm, based on a priori estimates on the distance between two approximate solutions.

Wave-Front Tracking Algorithm. The wave-front tracking algorithm generates piecewise constant approximate solutions of the Cauchy problem (1.19) and (1.20). A wave-front tracking ϵ-approximate solution is, roughly speaking, a piecewise constant function

$\mathbf{u} = \mathbf{u}(x,t)$ whose jumps occur along finitely many segments $x = x_\alpha(t)$ in the x-t plane and can be classified as shocks, rarefactions, and non-physical waves. At each time $t > 0$, these jumps should approximately satisfy the Rankine-Hugoniot condition:

$$\sum_\alpha |x'_\alpha(t)(\mathbf{u}(x_\alpha+,t) - \mathbf{u}(x_\alpha-,t)) - (\mathbf{f}(\mathbf{u}(x_\alpha+,t)) - \mathbf{f}(\mathbf{u}(x_\alpha-,t)))| = O(\epsilon),$$

as well as the following condition:

$$\sum_\alpha \{(q(\mathbf{u}(x_\alpha+,t)) - q(\mathbf{u}(x_\alpha-,t))) - x'_\alpha(t)(\eta(\mathbf{u}(x_\alpha+,t)) - \eta(\mathbf{u}(x_\alpha-,t)))\} \le O(\epsilon),$$

for any entropy-entropy flux pair (η, q) with convex η. The small parameter ϵ controls three types of errors: errors in the speeds of shock and rarefaction fronts, the maximum strength of rarefaction fronts, and the total strength of all non-physical waves. The notations in (7.26) and (7.27) will be adopted in this section.

Definition 7.1. *Given $\epsilon > 0$, a function $\mathbf{u} : [0,\infty) \to L^1(\mathbb{R};\mathbb{R}^n)$ is called an ϵ-approximate solution of the Cauchy problem* (1.19) *and* (1.20) *if the following conditions are satisfied:*

(1). *The function $\mathbf{u}(x,t)$ is piecewise constant with discontinuities along finitely many lines in the x-t plane. The step function $\mathbf{u}(x,0)$ of the approximate solution $\mathbf{u}(x,t)$ at $t = 0$ approximates the initial data $\mathbf{u}_0(x)$ in L^1 within distance ϵ:*

$$\|\mathbf{u}(\cdot,0) - \mathbf{u}_0(\cdot)\|_{L^1} < \epsilon. \tag{7.32}$$

Only finitely many wave-front interactions occur, each involving exactly two incoming fronts. Jumps can be of three types: shocks (or contact discontinuities), rarefaction waves, and non-physical waves.

(2). *Along each shock (or contact discontinuity) $x = x_\alpha(t)$, the values $\mathbf{u}_\pm = \mathbf{u}(x_\alpha\pm,t)$ are related by $\mathbf{u}_+ = S_{k_\alpha}(s_\alpha)(\mathbf{u}_-)$ for some $k_\alpha \in \{1,\cdots,n\}$ and some wave size s_α satisfying $s_\alpha < 0$ if the k_α-th characteristic field is genuinely nonlinear. Moreover, the speed of the shock front $\sigma(\mathbf{u}_+,\mathbf{u}_-)$ with left and right states $\mathbf{u}_\pm$ satisfies*

$$|x'_\alpha(t) - \sigma(\mathbf{u}_+,\mathbf{u}_-)| \le \epsilon.$$

(3). *Along each rarefaction front $x = x_\alpha(t)$, one has $\mathbf{u}_+ = R_{k_\alpha}(s_\alpha)(\mathbf{u}_-)$ with $s_\alpha \in (0,\epsilon]$ and $|x'_\alpha(t) - \lambda_{k_\alpha}(\mathbf{u}_+)| \le \epsilon$, for some genuinely nonlinear field k_α.*

(4). *All non-physical fronts $x = x_\alpha(t)$ have the same speed $x'_\alpha(t) = \bar\lambda$, where $\bar\lambda$ is a fixed constant strictly greater than all characteristic speeds. The total strength of all non-physical fronts in $\mathbf{u}(x,t)$ remains uniformly small in the sense:*

$$\sum |\mathbf{u}(x_\alpha+,t) - \mathbf{u}(x_\alpha-,t)| \le \epsilon$$

for all $t \ge 0$, where the sum is taken over all non-physical fronts.

The algorithm for constructing these wave-front tracking approximations is described below. The basic ideas were introduced in Dafermos [80] for scalar conservation laws and DiPerna [102] for 2×2 systems, then extended in Bressan [18] to general $n \times n$ systems (see also Risebro [266]).

The construction starts at time $t = 0$ by taking a piecewise constant function $\mathbf{u}(x,0)$ approximating $\mathbf{u}_0(x)$ satisfying (7.32) and $\mathrm{TV}(\mathbf{u}(\cdot,0)) \le \mathrm{TV}(\mathbf{u}_0)$. Let $x_1 < \cdots < x_N$ be the points where $\mathbf{u}(\cdot,0)$ is discontinuous. For each $\alpha = 1,\cdots,N$, the Riemann problem generated by the jump $\mathbf{u}(x_\alpha\pm,0)$ is approximately solved on a forward neighborhood of $(x_\alpha,0)$ in the x-t plane by a function of the form $\mathbf{u}(x,t) = \phi((x-x_\alpha)/t)$ with $\phi : \mathbb{R} \to \mathbb{R}^n$ piecewise constant. More precisely, if the exact solution of the Riemann problem contains only shocks and contact discontinuities, then we let $\mathbf{u}(x,t)$ be the exact solution which is piecewise constant. If centered rarefaction waves are present, they are approximated by a centered rarefaction fan containing several small jumps traveling with a speed close to the characteristic speed. Suppose that the first set of interactions between two or more wave-fronts occurs at a time t_1. Since $\mathbf{u}(\cdot, t_1)$ is still a piecewise constant function, the corresponding Riemann problems can again be approximately solved within the class of piecewise constant functions. The solution $\mathbf{u}(x,t)$ is then continued up to a time t_2, where the second set of wave interactions takes place, etc.

However, it is observed that, at a generic interaction point, there will be two incoming fronts, while the number of outgoing fronts is n if all waves generated by the Riemann problem are shocks or contact discontinuities, or even larger if rarefaction waves are present. In turn, these outgoing wave-fronts may quickly interact with several other fronts, generating more and more lines of discontinuity. Therefore, for general $n \times n$ systems, the number of wave-fronts may approach infinity in a finite time, which causes the breakdown of the construction.

To avoid this breakdown, the algorithm must be modified, which can be achieved by using two different procedures for solving a Riemann problem within the class of piecewise constant functions: (1) an accurate Riemann solver which introduces several new wave-fronts; and (2) a simplified Riemann solver which involves a minimum number of outgoing fronts. Although the number of wave-fronts could approach infinity within a finite time if all Riemann problems were solved accurately, the new fronts generated by further interactions are very small since the total variation remains small. When their size becomes smaller than a threshold parameter $\nu > 0$, a simplified Riemann solver is used, which generates one single new non-physical front with very small amplitude and traveling with a fixed speed $\bar{\lambda}$ strictly larger than all characteristic speeds, that is, all new waves are lumped together in a single non-physical front. The total number of fronts thus remains bounded for all times. We now describe these two procedures which will be used to solve the Riemann problem of (1.19) and (1.20) at a given point $(\bar{x}, \bar{t})$ with

$$\mathbf{u}(x,\bar{t}) = \begin{cases} \mathbf{u}_-, & x < \bar{x}, \\ \mathbf{u}_+, & x > \bar{x}. \end{cases} \tag{7.33}$$

The accurate Riemann solver is as follows. Given $\mathbf{u}_-$ and $\mathbf{u}_+$ in (7.33), one first determines the states $\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_n$ and parameter values $s_1, \cdots, s_n$ such that, using the notations in (7.26) and (7.27),

$$\mathbf{u}_0 = \mathbf{u}_-, \quad \mathbf{u}_n = \mathbf{u}_+, \qquad \mathbf{u}_i = \Upsilon_i(s_i)(\mathbf{u}_{i-1}), \quad i = 1, \cdots, n.$$

These states $\mathbf{u}_0, \mathbf{u}_1, \cdots, \mathbf{u}_n$ are the constant states present in the exact solution of the Riemann problem. If all jumps $(\mathbf{u}_{i-1}, \mathbf{u}_i)$ were shocks or contact discontinuities, then the Riemann problem would have a piecewise constant solution with at most n lines of discontinuity. In the general case, the exact solution of (7.33) is not piecewise constant because of the presence of rarefaction waves. These will be approximated by piecewise constant rarefaction fans, inserting additional states $\mathbf{u}_{i,j}$ as follows. Let $\delta > 0$ be a fixed small constant. If the i-th characteristic field is genuinely nonlinear and $s_i > 0$, consider the integer

$$N_i = 1 + [s_i/\delta], \tag{7.34}$$

where $[s_i/\delta]$ denotes the largest integer less than or equal to s_i/δ. For $j = 1, \cdots, N_i$, define

$$\mathbf{u}_{i,j} = \Upsilon_i(js_i/N_i)(\mathbf{u}_{i-1}), \quad x_{i,j}(t) = \bar{x} + (t - \bar{t})\lambda_i(\mathbf{u}_{i,j}).$$

If the i-th characteristic field is genuinely nonlinear and $s_i \le 0$, or if the i-th characteristic field is linearly degenerate (with s_i arbitrary), define $N_i = 1$ and

$$\mathbf{u}_{i,1} = \mathbf{u}_i, \quad x_{i,1}(t) = \bar{x} + (t - \bar{t})\sigma_i(\mathbf{u}_{i-1}, \mathbf{u}_i),$$

with $\sigma_i(\mathbf{u}_{i-1}, \mathbf{u}_i)$ the Rankine-Hugoniot speed of a jump connecting $\mathbf{u}_{i-1}$ with $\mathbf{u}_i$ so that $\sigma_i(\mathbf{u}_{i-1}, \mathbf{u}_i)(\mathbf{u}_i - \mathbf{u}_{i-1}) = \mathbf{f}(\mathbf{u}_i) - \mathbf{f}(\mathbf{u}_{i-1})$. Then, define an approximate solution to the

Riemann problem (7.33) as

$$\mathbf{u}(x,t)=\begin{cases}\mathbf{u}_-, & x<x_{1,1}(t),\\ \mathbf{u}_{i,j}, & x_{i,j}(t)<x<x_{i,j+1}(t), \quad j=1,\cdots,N_i-1,\\ \mathbf{u}_i, & x_{i,N_i}(t)<x<x_{i+1,1}(t),\\ \mathbf{u}_+, & x>x_{n,N_n}(t).\end{cases} \tag{7.35}$$

Thus each centered i-rarefaction wave is here divided into N_i-1 equal parts and replaced by a rarefaction fan containing N_i wave-fronts. The strength of each one of these fronts is less than δ because of (7.34).

The simplified Riemann solver is as follows. The first case is that i_1 and i_2 are the families of two incoming wave-fronts with $i_1\ge i_2$, $i_1, i_2\in\{1,\cdots,n\}$. In this case, let $\mathbf{u}_L, \mathbf{u}_M$, and $\mathbf{u}_R$ be the left, middle, and right states before the interaction, related by

$$\mathbf{u}_M=\Upsilon_{i_1}(s_1)(\mathbf{u}_L), \quad \mathbf{u}_R=\Upsilon_{i_2}(s_2)(\mathbf{u}_M).$$

Define the auxiliary right state:

$$\tilde{\mathbf{u}}_R=\begin{cases}\Upsilon_{i_1}(s_1)\circ\Upsilon_{i_2}(s_2)(\mathbf{u}_L), & i_1>i_2,\\ \Upsilon_{i_1}(s_1+s_2)(\mathbf{u}_R), & i_1=i_2.\end{cases} \tag{7.36}$$

Let $\tilde{\mathbf{u}}(x,t)$ be the piecewise constant solution of the Riemann problem with data $\mathbf{u}_L$, $\mathbf{u}_R$, constructed as in (7.35). Because of (7.36), the piecewise constant function $\tilde{\mathbf{u}}(x,t)$ contains exactly two wave-fronts of size s_1, s_2, if $i_1>i_2$, or a single wave-front of size s_1+s_2 if $i_1=i_2$. In general, $\tilde{\mathbf{u}}_R\neq\mathbf{u}_R$. Let the jump $(\tilde{\mathbf{u}}_R,\mathbf{u}_R)$ travel with a fixed speed $\bar{\lambda}$ strictly bigger than all characteristic speeds. In a forward neighborhood of the point $(\bar{x},\bar{t})$, we thus define an approximate solution $\mathbf{u}(x,t)$ as

$$\mathbf{u}(x,t)=\begin{cases}\tilde{\mathbf{u}}(x,t), & x-\bar{x}<\bar{\lambda}(t-\bar{t}),\\ \mathbf{u}_R, & x-\bar{x}>\bar{\lambda}(t-\bar{t}).\end{cases} \tag{7.37}$$

This simplified Riemann solver introduces a new non-physical wave-front, traveling with constant speed $\bar{\lambda}$. In turn, this front may interact with other (physical) fronts. One more case of interaction thus needs to be considered, that is, a non-physical front hits a wave-front of the i-characteristic field for some $i\in\{1,\cdots,n\}$ from the left. Let $\mathbf{u}_L, \mathbf{u}_M$, and $\mathbf{u}_R$ be the left, middle, and right states before the interaction. If $\mathbf{u}_R=\Upsilon_i(s)(\mathbf{u}_M)$, define

$$\tilde{\mathbf{u}}_R=\Upsilon_i(s)(\mathbf{u}_L). \tag{7.38}$$

Let $\tilde{\mathbf{u}}(x,t)$ be the solution to the Riemann problem with data $\mathbf{u}_L$ and $\tilde{\mathbf{u}}_R$, constructed as in (7.35). Because of (7.38), $\tilde{\mathbf{u}}(x,t)$ will contain a single wave-front belonging to the i-th field with size s. Since in general $\tilde{\mathbf{u}}_R\neq\mathbf{u}_R$, we let the jump $(\tilde{\mathbf{u}}_R,\mathbf{u}_R)$ travel with the fixed speed $\bar{\lambda}$. In a forward neighborhood of the point $(\bar{x},\bar{t})$, the approximate solution $\mathbf{u}(x,t)$ is thus defined again according to (7.37). By construction, all non-physical fronts travel with the same speed $\bar{\lambda}$. The above cases therefore cover all possible interactions between two wave-fronts.

A threshold parameter $\nu>0$ is used to determine which Riemann solver is used at any given interaction. The accurate method is used at time $t=0$ and at every interaction where the product of the strengths of the incoming waves is $|s_1s_2|\ge\nu$; while the simplified method is used at every interaction involving a non-physical wave-front and also at interactions with $|s_1s_2|<\nu$. In the above, it is assumed that only two wave-fronts interact at any given point, which can always be achieved by an arbitrarily small change in the speed of one of the interacting fronts. It should also be adopted that, in the accurate Riemann solver, rarefaction fronts of the same field of one of the incoming fronts are never partitioned (even if their strength is bigger than δ). This guarantees that every wave-front can be uniquely continued forward in time, unless it gets completely cancelled by interacting with another front of the same field and opposite sign.

The above construction of an approximate solution involves three parameters: a fixed speed $\bar{\lambda}$ strictly larger than all characteristic speeds, a small constant $\delta > 0$ controlling the maximum strength of rarefaction fronts, and a threshold parameter $\nu > 0$ determining whether the accurate or the simplified Riemann solver is used. This wave-front tracking algorithm generates alternatively an entropy solution of the Cauchy problem (1.19) and (1.20) in Theorem 7.1.

Theorem 7.7. *Let $\mathbf{u}_0(x)$ have small total variation over $\mathbb{R}$. For any fixed small $\epsilon > 0$, approximate $\mathbf{u}_0(x)$ by some step function $\mathbf{u}_0^\epsilon(x)$ such that*

$$\|\mathbf{u}_0^\epsilon - \mathbf{u}_0\|_{L^1(\mathbb{R})} \le \epsilon, \quad \mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0^\epsilon) \le \mathrm{TV}_{\mathbb{R}}(\mathbf{u}_0).$$

Then, for the fixed speed $\bar{\lambda}$ independent of ϵ and strictly larger than all characteristic speeds, small $\delta_\epsilon = \epsilon > 0$ controlling the maximum strength of rarefaction fronts, and a threshold parameter $\nu_\epsilon > 0$ determining whether the accurate or the simplified Riemann solver is used and depending on ϵ and on the number of jumps of $\mathbf{u}_0^\epsilon(x)$, the wave-front tracking algorithm with initial data $\mathbf{u}_0^\epsilon(x)$ generates the global ϵ-approximate solutions $\mathbf{u}^\epsilon(x,t)$ which have a subsequence converging, a.e. on $\mathbb{R}^2_+$, to an entropy BV solution $\mathbf{u}(x,t)$ of the Cauchy problem (1.19) *and* (1.20) *with the estimates* (7.3) *and* (7.4).

To prove Theorem 7.7, the argument used in the proof of Theorem 7.1 can be applied with some modification. The proof consists of two steps.

The first step is to show that the ϵ-approximate solution $\mathbf{u}^\epsilon(x,t)$ is defined for all $t \ge 0$, which can be achieved by showing two facts: the total variation of $\mathbf{u}^\epsilon(\cdot,t)$ remains uniformly bounded and the number of wave-fronts in $\mathbf{u}^\epsilon(\cdot,t)$ remains finite. To derive the bound of the total variation of $\mathbf{u}^\epsilon(\cdot,t)$, as in (7.9), introduce the total strength of waves $\mathrm{L}(t)$ in $\mathbf{u}^\epsilon(x,t)$ cross the t-time line:

$$\mathrm{L}_{\mathbf{u}^\epsilon}(t) = \sum_\alpha |s_\alpha|, \tag{7.39}$$

where the summation is taken over all wave fronts of $\mathbf{u}^\epsilon(x,t)$; and the wave interaction potential $\mathrm{Q}(t)$ cross the t-time line:

$$\mathrm{Q}_{\mathbf{u}^\epsilon}(t) = \sum_{\alpha,\beta\in\mathrm{A}} |s_\alpha s_\beta|, \tag{7.40}$$

where the summation runs over all pairs of approaching waves. For a non-physical front $x = x_\alpha(t)$, we simply call $s_\alpha = |\mathbf{u}(x_\alpha(t)+,t) - \mathbf{u}(x_\alpha(t)-,t)|$ the strength of the non-physical front at $x_\alpha(t)$. For convenience, non-physical fronts are regarded as belonging to a fictitious linearly degenerate $(n+1)$-th characteristic field. Two fronts of the families k_α, $k_\beta \in \{1,\cdots,n{+}1\}$, located respectively at x_α, x_β with $x_\alpha < x_\beta$ ($k_\alpha = n{+}1$ if x_α is non-physical), are approaching if either $k_\alpha > k_\beta$, or $k_\alpha = k_\beta$, and at least one of them is a genuinely nonlinear shock. The total strength L of waves stays constant along time intervals between consecutive collisions of fronts and only changes across points of wave interaction. The wave interaction potential Q also stays constant along time intervals between consecutive collisions. It can be proved that, for a suitably large constant M, the quantity

$$\mathrm{F}_{\mathbf{u}^\epsilon}(t) = \mathrm{L}_{\mathbf{u}^\epsilon}(t) + M\mathrm{Q}_{\mathbf{u}^\epsilon}(t) \tag{7.41}$$

analogous to the Glimm functional, bounding the total variation of $\mathbf{u}^\epsilon(x,t)$, is non-increasing in time. The key observation is that $\mathrm{Q}(t)$ is positive and decreasing after each interaction. The number of physical fronts can grow only at times t where the accurate Riemann solver is used. The set of times where the accurate solver is used can be proved finite. Thus the number of physical fronts is finite. In turn, a new non-physical front can be generated only when two physical fronts interact. Since any two physical fronts interact at most once, it follows that the number of non-physical fronts also remains finite.

The second step is to show that the limit of the approximate solutions is an entropy solution. By Helly's compactness theorem, the estimate on the total variation of $\mathbf{u}^\epsilon(x,t)$ implies that there exists a subsequence (still denoted) $\mathbf{u}^\epsilon(x,t)$ converging to some function $\mathbf{u}(x,t)$ in L^1_{loc}, as $\epsilon \to 0$. To prove that $\mathbf{u}(x,t)$ is an entropy solution of (1.19) and (1.20), one needs to verify that both the maximum size of rarefaction fronts and the total strength of non-physical fronts in $\mathbf{u}^\epsilon(x,t)$ tend to zero as $\epsilon \to 0$, which follows from the construction and the interaction estimates. The approximate solutions $\mathbf{u}^\epsilon(x,t)$ satisfy the entropy inequality with an error tending to zero as $\epsilon \to 0$, which shows that the solution satisfies also the entropy condition. See Bressan [18] for the details of the proof.

L^1-Stability. As in Theorem 7.1, the existence proof of Theorem 7.7 provides no clue on the stability of solutions of the Cauchy problem (1.19) and (1.20). By monitoring the time evolution of a certain functional, it can be shown that the ϵ-approximate solutions constructed by the wave-front tracking algorithm depend continuously on their initial data up to a certain error of order ϵ. This shows, by passing to the limit $\epsilon \to 0$, that the front-tracking approximations converge to a unique limit, and the solution depends Lipschitz continuously on the initial data.

Suppose that System (1.19) is strictly hyperbolic and genuinely nonlinear. Let $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ be two ϵ-approximate solutions of (1.19) and (1.20) with small total variation. For any fixed point (x,t), define the scalar function $q_i(x,t)$ as in (7.28). Define the functional

$$\Psi(\mathbf{u},\mathbf{v})(t) = \sum_{i=1}^{n} \int_{\mathbb{R}} |q_i(x,t)| \left(1 + K_1(\mathrm{Q}_{\mathbf{u}}(t) + \mathrm{Q}_{\mathbf{v}}(t)) + K_2 W_i(x,t)\right) dx,$$

where K_1 and K_2 are sufficiently large positive constants, $\mathrm{Q}_{\mathbf{u}}(t)$ and $\mathrm{Q}_{\mathbf{v}}(t)$ are the wave interaction potentials for $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ respectively defined in (7.40), and $W_i(x,t)$ is defined as in (7.30). Notice that the strengths of non-physical fronts do enter in the definition of Q, but play no role in the definition of W_i. If the total variations of $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ are sufficiently small such that $0 \le K_1\left(\mathrm{Q}_{\mathbf{u}}(t) + \mathrm{Q}_{\mathbf{v}}(t)\right) + K_2 W_i(x,t) \le 1$ for all i, then

$$C_1^{-1}\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^1(\mathbb{R})} \le \Psi(\mathbf{u},\mathbf{v})(t) \le 2C_1\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^1(\mathbb{R})}. \tag{7.42}$$

Bressan-Liu-Yang [25] indicates that, if $\mathrm{F}_{\mathbf{u}}$ and $\mathrm{F}_{\mathbf{v}}$ defined in (7.41) are sufficiently small, then the functional $\Psi(\mathbf{u},\mathbf{v})(t)$ is almost decreasing in t, that is,

$$\Psi(\mathbf{u},\mathbf{v})(t_1) - \Psi(\mathbf{u},\mathbf{v})(t_2) \le C_2\epsilon(t_1 - t_2), \quad 0 \le t_2 < t_1. \tag{7.43}$$

For small constant $\delta > 0$, with the notation in (7.41), define the domain

$$\mathfrak{D} = \mathrm{CL}\{\mathbf{u} \in L^1(\mathbb{R};\mathbb{R}^n) \ : \ \mathbf{u}(x,t)\,\text{is piecewise constant}, \ \mathrm{F}_{\mathbf{u}}(t) < \delta\},$$

where CL denotes the closure in $L^1(\mathbb{R})$. Estimate (7.43) implies the L^1-stability of entropy solutions obtained by the wave-front tracking method.

Theorem 7.8. *For any initial data $\mathbf{u}_0 \in \mathfrak{D}$ with δ sufficiently small, as $\epsilon \to 0$, any subsequence of the approximate solutions $\mathbf{u}^\epsilon(x,t)$ constructed by the wave-front tracking algorithm for the Cauchy problem* (1.19) *and* (1.20) *converges to a unique limit $\mathbf{u}(x,t)$. The map $(\mathbf{u}_0,t) \mapsto S_t(\mathbf{u}_0) = \mathbf{u}(\cdot,t)$ defines a uniformly Lipschitz continuous semigroup whose trajectories are entropy solutions of* (1.19) *and* (1.20). *If $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ are two such entropy solutions of* (1.19) *and* (1.20) *with initial data $\mathbf{u}_0(x)$ and $\mathbf{v}_0(x)$, respectively, then*

$$\|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot,t)\|_{L^1(\mathbb{R})} \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1(\mathbb{R})}, \tag{7.44}$$

for some constant $C > 0$.

With the assumption (7.43), Theorem 7.8 can be proved as follows. For a given $\mathbf{u}_0 \in \mathfrak{D}$, consider any sequences $\{\mathbf{u}^l(x,t)\}$, $l = 1, 2, \cdots$, and $\{\mathbf{u}^k(x,t)\}$, $k = 1, 2, \cdots$, of the ϵ_l-approximate solutions and ϵ_k-approximate solutions of (1.19) and (1.20), respectively, with

$$\|\mathbf{u}^l(\cdot,0) - \mathbf{u}_0(\cdot)\|_{L^1} \le \epsilon_l, \quad \lim_{l\to\infty} \epsilon_l = 0, \quad \mathrm{F}_{\mathbf{u}^l}(t) < \delta,$$
$$\|\mathbf{u}^k(\cdot,0) - \mathbf{u}_0(\cdot)\|_{L^1} \le \epsilon_k, \quad \lim_{k\to\infty} \epsilon_k = 0, \quad \mathrm{F}_{\mathbf{u}^k}(t) < \delta,$$

for any $t > 0$. From (7.42) and (7.43), for any $l, k \ge 1$, and $t > 0$,

$$\begin{aligned}\|\mathbf{u}^l(\cdot,t) - \mathbf{u}^k(\cdot,t)\|_{L^1} &\le C_1\Psi(\mathbf{u}^l,\mathbf{u}^k)(t) \le C_1\Psi(\mathbf{u}^l,\mathbf{u}^k)(0) + C_1C_2t\max\{\epsilon_l,\epsilon_k\}\\ &\le 2C_1^2\|\mathbf{u}^l(\cdot,0) - \mathbf{u}^k(\cdot,0)\|_{L^1} + C_1C_2t\max\{\epsilon_l,\epsilon_k\}.\end{aligned}$$

As $l, k \to \infty$, the right-hand side tends to zero, the two sequences have the same limit, and thus any sequence of ϵ-approximate solutions converges to a unique limit. The semigroup property $S_{t_2}(S_{t_1}\mathbf{u}_0) = S_{t_1+t_2}\mathbf{u}_0$ follows immediately from the uniqueness. Let $\mathbf{u}_0(x)$ and $\mathbf{v}_0(x)$ be the initial data of the entropy solutions $\mathbf{u}(x,t)$ and $\mathbf{v}(x,t)$ which are the limits of the corresponding ϵ_j-approximate solutions $\mathbf{u}^j(x,t)$ and $\mathbf{v}^j(x,t)$ of (1.19) and (1.20), respectively, with $\|\mathbf{u}^j(\cdot,0) - \mathbf{u}_0(\cdot)\|_{L^1} < \epsilon_j$, $\|\mathbf{v}^j(\cdot,0) - \mathbf{v}_0(\cdot)\|_{L^1} < \epsilon_j$, and $\lim_{j\to\infty}\epsilon_j = 0$. From (7.42) and (7.43), one has

$$\begin{aligned}\|\mathbf{u}^j(\cdot,t) - \mathbf{v}^j(\cdot,t)\|_{L^1} &\le C_1\Psi(\mathbf{u}^j,\mathbf{v}^j)(t) \le C_1\Psi(\mathbf{u}^j,\mathbf{v}^j)(0) + C_1C_2t\epsilon_j\\ &\le 2C_1^2\|\mathbf{u}^j(\cdot,0) - \mathbf{v}^j(\cdot,0)\|_{L^1} + C_1C_2t\epsilon_j.\end{aligned}$$

Taking $j \to \infty$ yields (7.44).

Theorem 7.8 was established in Bressan-Liu-Yang [25], where the proof of (7.43) can be found, and in Hu-LeFloch [155], where Haar's method was extended to nonlinear systems of conservation laws, independently. A sharper version of the L^1-continuous dependence estimate, containing dissipation terms in the left-hand side of (7.44), was later established by Dafermos [88] (for scalar equations) and Goatin-LeFloch [137] (for systems). For other related results and discussions, see [20, 21, 88, 216] and the references therein.

The approach for Theorem 7.8 in [25] provides a much simpler proof of the existence of a Lipschitz semigroup, called the standard Riemann semigroup [21] generated by the $n \times n$ systems of conservation laws (1.19) and (1.20), which is defined as a continuous map $S : \mathfrak{D} \times [0,\infty) \to \mathfrak{D}$ such that, for some Lipschitz constant L, denoting $S_t(\cdot) = S(\cdot, t)$,

(1). $S_0\mathbf{u}_0 = \mathbf{u}_0$, $S_{t_2}S_{t_1}\mathbf{u}_0 = S_{t_1+t_2}\mathbf{u}_0$;
(2). For all $\mathbf{u}_0, \mathbf{v}_0 \in \mathfrak{D}$, $t_1, t_2 \ge 0$, $\|S_{t_1}\mathbf{u}_0 - S_{t_2}\mathbf{u}_0\|_{L^1} \le L(\|\mathbf{u}_0 - \mathbf{v}_0\|_{L^1} + |t_1 - t_2|)$;
(3). If $\mathbf{u}_0 \in \mathfrak{D}$ is piecewise constant, then, for $t > 0$ sufficiently small, the function $\mathbf{u}(\cdot,t) = S_t\mathbf{u}_0$ coincides with the solution of (1.19) and (1.20) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

For any initial data $\mathbf{u}_0 \in \mathfrak{D}$ with δ sufficiently small, the solution $\mathbf{u}(x,t)$ as the limit of the ϵ-approximate solutions constructed by the wave-front tracking algorithm can be identified with a trajectory of the standard Riemann semigroup [21], which also indicates that the limit of the ϵ-approximate solutions by the wave-front tracking algorithm is unique. As discussed earlier, the results in Bressan [19] and Liu-Yang [217] also imply the uniqueness of limits of Glimm's random choice method.

For initial data $\mathbf{u}_0(x)$ which are small BV perturbation of a large Riemann data, some progress has been made in Lewicka [190] and Lewicka and Trivisa [191].

There are some recent developments on uniform BV estimates for artificial viscosity approximations for hyperbolic systems of conservation laws with initial data of small total variation, as well as the L^1-stability of BV solutions constructed by the vanishing viscosity method; see Bianchini and Bressan [12, 13].

This uniqueness property can be extended to any solutions satisfying certain extra regularity condition as stated in the following theorem.

Theorem 7.9. *Any solution* $\mathbf{u}(x,t)$ *of the Cauchy problem* (1.19) *and* (1.20)*, with* $\mathbf{u}(\cdot,t) \in \mathfrak{D}$ *for all* $t \geq 0$*, which satisfies the following tame oscillation condition:*

$$|\mathbf{u}(x\pm, t+h) - \mathbf{u}(x\pm, t)| \leq \beta\, \mathrm{TV}_{[x-\lambda h, x+\lambda h]}(\mathbf{u}(\cdot,t)) \tag{7.45}$$

for all $x \in \mathbb{R}$*,* $t \geq 0$*, and any* $h > 0$*, with* λ *and* β *some positive constants, coincides with the trajectory of the standard Riemann semigroup* S_t *emanating from the initial data:* $\mathbf{u}(\cdot,t) = S_t\mathbf{u}_0(\cdot)$*. In particular,* $\mathbf{u}(x,t)$ *is uniquely determined by its initial data.*

The solutions constructed by either Glimm's random choice method or the wave-front tracking algorithm satisfy the tame oscillation condition (7.45). Such a uniqueness result of entropy solutions to systems was established first by Bressan-LeFloch [23] under a stronger assumption, called the tame variation condition. By improving upon these arguments, Theorem 7.9 was established by Bressan-Goatin [22]. The tame oscillation condition (7.45) can be also replaced by the assumption that the trace of solutions along space-like curves has local bounded variation (see Bressan-Lewicka [24]). Also see Hu-LeFloch [155] for a different approach based on Harr's method, and Baiti-LeFloch-Piccoli [5] for some further generalization. For other discussions about the wave-front tracking algorithm, standard Riemann semigroup, uniqueness, and related topics, we refer to Bressan [20], Dafermos [88], and LeFloch [187] which provide extensive discussions and references.

8. Global Discontinuous Solutions III: Entropy Solutions in BV

In this section we focus on general global discontinuous solutions in $L^\infty \cap BV_{loc}$ satisfying the Lax entropy inequality and without specific reference on the method for construction of the solutions.

8.1. Generalized Characteristics and Decay.

Consider the BV entropy solutions of (1.19) having bounded variation in the sense of Tonelli-Cesari, i.e., functions whose first-order distributional derivatives are locally Borel measures (Volpert [312]). The notion of characteristics for classical solutions can be extended to generalized characteristics for BV entropy solutions. The generalized characteristics provide a powerful tool for studying the structure and behavior of BV entropy solutions.

Suppose that System (1.19) is strictly hyperbolic with n real distinct eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\mathbf{u}(x,t)$ is a BV entropy solution of (1.19) for $(x,t) \in \mathbb{R}^2_+$. The domain $\mathbb{R}^2_+$ can be written as $\mathcal{C} \cup \mathcal{J} \cup \mathcal{I}$ with $\mathcal{C}$, $\mathcal{J}$, and $\mathcal{I}$ pairwise disjoint, where $\mathcal{C}$ is the set of points of approximate continuity of $\mathbf{u}(x,t)$, $\mathcal{J}$ is the set of points of approximate jump discontinuity (shock set) of $\mathbf{u}(x,t)$, and $\mathcal{I}$ denotes the set of irregular points of $\mathbf{u}(x,t)$. The one-dimensional Hausdorff measure of $\mathcal{I}$ is zero. The shock set $\mathcal{J}$ is essentially the (at most) countable union of C^1 arcs. With any point $(x,t) \in \mathcal{J}$ are associated distinct one-sided approximate limits $\mathbf{u}_\pm$ and a shock speed σ related by the Rankine-Hugoniot condition (6.7) and satisfying the Lax entropy condition (6.8). To handle shock waves in solutions, we employ the concept of generalized characteristics introduced by Dafermos (cf. [104]).

The generalized characteristics are defined in Filippov's sense of differential inclusion [118] as follows.

Definition 8.1. *A generalized* i*-characteristic for* (1.19) *on an interval* $[t_1, t_2]$*,* $0 \leq t_1 < t_2 < \infty$*, associated with the solution* $\mathbf{u}(x,t)$*, is a Lipschitz function* $\xi : [t_1, t_2] \to \mathbb{R}$ *such that*

$$\xi'(t) \in [\lambda_i(\mathbf{u}(\xi(t)+, t)), \lambda_i(\mathbf{u}(\xi(t)-, t))],$$

for almost all $t \in [t_1, t_2]$.

Generalized characteristics propagate with either classical characteristic speed or shock speed, as indicated in the following proposition.

Proposition 8.1. *Let $\xi(t)$ be a generalized i-characteristic on $[t_1, t_2]$. Then, for almost all $t \in [t_1, t_2]$, $\xi(t)$ propagates with classical i-characteristic speed if $(\xi(t), t) \in \mathcal{C}$ and with i-shock speed if $(\xi(t), t) \in \mathcal{J}$.*

Proposition 8.2. *Given any point $(\bar{x}, \bar{t})$ of the upper half-plane, there exists at least one generalized i-characteristic, defined on $[0, \infty)$, passing through $(\bar{x}, \bar{t})$. The set of i-characteristics passing through $(\bar{x}, \bar{t})$ spans a funnel-shaped region bordered by a minimal i-characteristic and a maximal i-characteristic (possibly coinciding). Furthermore, if $\xi(t)$ denotes the minimal or the maximal backward i-characteristic issuing from $(\bar{x}, \bar{t})$, then*

$$\mathbf{u}(\xi(t)+, t) = \mathbf{u}(\xi(t)-, t), \quad \xi'(t) = \lambda_i(\mathbf{u}(\xi(t)\pm, t)),$$

for almost all $t \in [0, \bar{t}]$.

Definition 8.2. *A minimal (or maximal) i-divide, associated with the solution $\mathbf{u}(x, t)$, is a Lipschitz function $\phi : [0, \infty) \to \mathbb{R}$ with the property that $\phi(t) = \lim_{k\to\infty} \xi_k(t)$, uniformly on compact subsets of $[0, \infty)$, where $\xi_k(t)$ is the minimal (or maximal) backward i-characteristic emanating from some point (x_k, t_k) with $t_k \to \infty$, as $k \to \infty$. Two minimal (or maximal) i-divides $\phi_1(t)$ and $\phi_2(t)$, with $\phi_1(t) \le \phi_2(t)$, $0 \le t < \infty$, are disjoint if the set $\{(x, t) : 0 \le t < \infty, \phi_1(t) < x < \phi_2(t)\}$ does not intersect the graph of any minimal (or maximal) i-divide.*

The graphs of any two minimal (or maximal) i-characteristics may run into each other but they cannot cross. Then the graph of a minimal (or maximal) backward i-characteristic cannot cross the graph of any minimal (or maximal) i-divide and the graphs of any two minimal (or maximal) i-divides cannot cross. Any minimal (or maximal) i-divide divides the upper half-plane into two parts in such a way that no forward i-characteristic may cross from the left to the right (or from the right to the left). The concept of i-divide plays a central role in the investigation of the large-time behavior of solutions with periodic initial data through the approach of generalized characteristics. The set of minimal or maximal i-divides associated with a particular solution may be empty, but it is nonempty if the solution is periodic.

Proposition 8.3. *If $\phi(t)$ is any minimal or maximal i-divide, then*

$$\mathbf{u}(\phi(t)+, t) = \mathbf{u}(\phi(t)-, t), \quad \phi'(t) = \lambda_i(\mathbf{u}(\phi(t)\pm, t)),$$

for almost all $t \in [0, \infty)$. In particular, $\phi(t)$ is a generalized i-characteristic on $[0, \infty)$. Furthermore, if $\{\phi_k(t)\}$ is a sequence of minimal (or maximal) i-divides converging to some function $\phi(t)$ uniformly on compact subsets of $[0, \infty)$, then $\phi(t)$ is a minimal (or maximal) i-divide.

Proposition 8.4. *The set of minimal (or maximal) i-divides associated with any solution $\mathbf{u}(x, t)$, periodic in x, with period P, is not empty. The union of the graphs of these i-divides is invariant under the translation by P in the x-direction.*

The above theory of generalized characteristics follows Dafermos [84, 86, 88]. The proofs of these propositions and further discussions can be found in these references. A closely related alternative definition of generalized characteristics was given in Glimm-Lax [133] which are Lipschitz curves propagating with either classical characteristic speed or shock speed, constructed as limits of families of approximate characteristics. The following result is due to DiPerna [104].

Proposition 8.5. *Let* (1.19) *be an $n \times n$ strictly hyperbolic system endowed with a strictly convex entropy. Suppose $\mathbf{u}(x, t)$ is an $L^\infty \cap BV_{loc}$ entropy solution of* (1.19) *and* (6.1) *for $(x, t) \in \mathbb{R}^2_+$. Let $x^n_{max}(t)$ denote the maximal forward n-characteristic through $(0, 0)$. Let*

$x^1_{min}(t)$ *denote the minimal forward 1-characteristic passing through* $(0,0)$. *Then* $\mathbf{u}(x,t) = \mathbf{u}_L$*, for a.e.* (x,t) *with* $x < x^1_{min}(t)$*, and* $\mathbf{u}(x,t) = \mathbf{u}_R$*, for a.e.* (x,t) *with* $x > x^n_{max}(t)$.

Using the theory of generalized characteristics, Dafermos in [84, 86, 88] proved a series of decay properties for general BV solutions to hyperbolic systems of two conservation laws. For this purpose, the following structural condition on the BV solution $\mathbf{u}(x,t)$ is imposed: *The traces of the Riemann invariants* w_1 *and* w_2 *along any space-like curve are functions of locally bounded variation.*
Here, a space-like curve relative to the BV solution $\mathbf{u}(x,t)$ is a Lipschitz curve, with graph embedded in the upper half-plane, such that, for each point $(\bar{x},\bar{t})$ on the graph of the curve, the set $\{(x,t) : 0 \le t < \bar{t}, \zeta(t) < x < \xi(t)\}$ of points confined between the maximal backward 2-characteristic ζ and the minimal backward 1-characteristic ξ, emanating from the point $(\bar{x},\bar{t})$, has empty intersection with the graph of the curve. Under this condition, one has the following results on the regularity and decay of the BV entropy solutions to hyperbolic systems of two conservation laws, which are due to Dafermos [81, 84, 86, 88].

Theorem 8.1. *Suppose that* $\mathbf{u}(x,t)$ *is a BV entropy solution of the genuinely nonlinear and strictly hyperbolic system* (1.19) *with* $n = 2$. *Then any point of approximate continuity is a point of continuity of* $\mathbf{u}(x,t)$*, any point of approximate jump discontinuity is a point of classical jump discontinuity of* $\mathbf{u}(x,t)$*, the set of irregular points is (at most) countable, and any irregular point is the focus of a centered compression wave of either, or both, characteristic fields, and/or a point of interaction of shocks of the same or opposite characteristic fields.*

If the initial data $(w_1,w_2)(x,0)$ belong to $L^1(\mathbb{R})$ with small oscillation, then the solution $(w_1,w_2)(x,t)$ to the genuinely nonlinear and strictly hyperbolic system (1.19) with $n = 2$ decays, as $t \to \infty$, at the rate $O(t^{-1/2})$ (Dafermos [84]), which is an analogue for the scalar conservation laws (Lax [181]).

For solutions with periodic initial data, one has the following decay property (Dafermos [86]).

Theorem 8.2. *Suppose that* $\mathbf{u}(x,t)$ *is a BV entropy solution of the genuinely nonlinear and strictly hyperbolic system* (1.19) *with* $n = 2$*, and the initial data* $\mathbf{u}_0(x)$ *are periodic with period* P *and mean zero. Then the upper half-plane is partitioned by minimal (or maximal) divides of the first (or second) characteristic field, along which the Riemann invariant* w_1 *(or* w_2*) of the first (or second) field decays to zero,* $O(t^{-2})$*, as* $t \to \infty$. *If* $\phi(t)$ *and* $\psi(t)$ *are any two adjacent 1- (or 2-) divides, then* $\psi(t) - \phi(t)$ *approaches a constant at the rate* $O(t^{-1})$*, as* $t \to \infty$*, and there is a 1- (or 2-) characteristic* $\chi(t)$ *between* $\phi(t)$ *and* $\psi(t)$ *such that, as* $t \to \infty$*,* $\chi(t) = (\psi(t) + \phi(t))/2 + o(1)$*, and*

$$\partial_{w_i}\lambda_i(0,0)\, w_i(x,t) = \begin{cases} \frac{x-\phi(t)}{t} + o(\frac{1}{t}), & \phi(t) < x < \chi(t), \\ \frac{x-\psi(t)}{t} + o(\frac{1}{t}), & \chi(t) < x < \psi(t), \end{cases} \qquad i = 1,2. \tag{8.1}$$

The proof of Theorem 8.2 is based on the analysis of the large-time behavior of divides. Assume $\phi(t)$ is a minimal 1-divide, say the limit of a sequence $\{\xi_k(t)\}$ of minimal backward 1-characteristics emanating from some points $\{(x_k,t_k)\}$ with $t_k \to \infty$ as $k \to \infty$. Consider the traces of w_1 and w_2 along $\xi_k(t)$: $w_{1k}(t) := w_1(\xi_k(t)-,t)$ and $w_{2k}(t) := w_2(\xi_k(t)+,t)$. The total variation of w_{2k} and the supremum of $|w_{2k}|$ over any interval $[t,t+1] \subset [0,t_k]$ are $O(t^{-1})$, uniformly in k. Then w_{1k} is a nonincreasing function whose oscillation over $[t,t+1]$ is $O(t^{-3})$ uniformly in k since $w_{1k}(t-) - w_{1k}(t+) \le C|w_{2k}(t) - w_{2k}(t+)|^3$ (see [84]). Thus, for any $t \in [0,t_k]$, $w_{1k} = O(t^{-2}) + O(t_k^{-1})$ uniformly in k, and it can be concluded that, for almost all $t \in [0,\infty)$, $w_1(\phi(t)\pm,t)$ is a nonincreasing function which decays to zero, $O(t^{-2})$, as $t \to \infty$. Further analysis of divides leads to (8.1). See Dafermos [86, 88] for the details.

Now we consider System (1.19) with initial data of compact support (7.25). The BV entropy solution decays to an N-wave as follows (Dafermos [84, 88]).

Theorem 8.3. *Suppose that* $\mathbf{u}(x,t)$ *is an entropy BV solution of the genuinely nonlinear and strictly hyperbolic system* (1.19) *with* $n = 2$*, and the initial data* $\mathbf{u}_0(x)$ *have compact support* (7.25) *and small oscillation. Then the minimal i-characteristics* $\phi_i^-(t)$ *issuing from the point* $(-R, 0)$ *and the maximal i-characteristics* $\phi_i^+(t)$ *issuing from the point* $(R, 0)$*,* $i = 1, 2$*, satisfy, for t large,*

$$\phi_1^-(t) = \lambda_1(0,0)t - (p_-t)^{1/2} + O(1), \quad \phi_1^+(t) = \lambda_1(0,0)t + (p_+t)^{1/2} + O(t^{1/4}),$$
$$\phi_2^-(t) = \lambda_2(0,0)t - (q_-t)^{1/2} + O(t^{1/4}), \quad \phi_2^+(t) = \lambda_2(0,0)t + (q_+t)^{1/2} + O(1),$$

for some nonnegative constants $p_\pm$ *and* $q_\pm$*, and*

$$TV_{[\phi_1^-(t),\phi_2^+(t)]}(w_1, w_2)(\cdot, t) = O(t^{-1/2}); \tag{8.2}$$

and, if $p_+ > 0$ *and* $q_- > 0$*, then*

$$\|w_i(\cdot,t) - N_i(\cdot,t)\|_{L^1(\mathbb{R})} = O(t^{-1/4}), \quad i = 1, 2, \tag{8.3}$$

as $t \to \infty$*, with the N-waves* $N_1(x,t)$ *and* $N_2(x,t)$ *defined by*

$$N_1(x,t) = \begin{cases} \frac{1}{\partial_{w_1}\lambda_1(0,0)}\left(\frac{x}{t} - \lambda_1(0,0)\right), & -(p_-t)^{1/2} \le x - \lambda_1(0,0)t \le (p_+t)^{1/2}, \\ 0, & \textit{otherwise}, \end{cases}$$
$$N_2(x,t) = \begin{cases} \frac{1}{\partial_{w_2}\lambda_2(0,0)}\left(\frac{x}{t} - \lambda_2(0,0)\right), & -(q_-t)^{1/2} \le x - \lambda_2(0,0)t \le (q_+t)^{1/2}, \\ 0, & \textit{otherwise}. \end{cases}$$

The main ingredients of the proof of Theorem 8.3 include the following estimates: $\mathbf{u}(x,t) = 0$ for any $t > 0$ and $x \notin (\phi_1^-(t), \phi_2^+(t))$; for large t, $\lambda_1(w_1(x,t), 0) = x/t + O(t^{-1})$ for $x \in (\phi_1^-(t), \phi_1^+(t))$ and $\lambda_2(0, w_2(x,t)) = x/t + O(t^{-1})$ for $x \in (\phi_2^-(t), \phi_2^+(t))$; and for large t, $0 \le -w_1(x,t) \le C(x - \lambda_1(0,0)t)^{-3/2}$ for $x > \phi_1^+(t)$ and $p_+ > 0$, and $0 \le -w_2(x,t) \le C(\lambda_2(0,0)t - x)^{-3/2}$ for $x < \phi_2^-(t)$ and $q_- > 0$. These estimates indicate that, as $t \to \infty$, the two characteristic fields decouple and each one develops an N-wave profile, of width $O(t^{1/2})$ and strength $O(t^{-1/2})$, which propagates into the rest state at the characteristic speed. See Dafermos [84, 88] for the details of the proof.

From Theorem 8.3, we see that the total variation of the solution decays to zero as $O(t^{-1/2})$ (Glimm-Lax [133]). The solution decays to N-waves at the rate $O(t^{-1/4})$ slower than the rate $O(t^{-1/2})$ for scalar conservation laws, due to the interaction of the characteristic fields in the systems, a phenomenon which is not present in a single conservation law. However, an improvement in this uniform rate may be possible, while, in the scalar case, simple examples show that the decay rate $O(t^{-1/2})$ cannot be improved. See also Greenberg [143] on the decay of special solutions for a class of two conservation laws generated by a second-order wave equation.

8.2. **Uniqueness of Riemann Solutions.**

In this section we prove the uniqueness of Riemann solutions of the Riemann problem (1.13) and (6.1) in the class of entropy solutions in BV without extra regularity condition on the solutions. Without loss of generality, we assume that the classical Riemann solution has the following generic form:

$$\mathbf{R}(x/t) = \begin{cases} \mathbf{u}_L, & x/t < \sigma_1, \\ \mathbf{u}_M, & \sigma_1 < x/t < 0, \\ \mathbf{u}_N, & 0 < x/t \le \lambda_3(\mathbf{u}_N), \\ \mathbf{V}_3(x/t), & \lambda_3(\mathbf{u}_N) < x/t < \lambda_3(\mathbf{u}_R), \\ \mathbf{u}_R, & x/t \ge \lambda_3(\mathbf{u}_R), \end{cases} \tag{8.4}$$

where $\sigma_1 = \sigma_1(\mathbf{u}_L, \mathbf{u}_M)$ is the shock speed, determined by (6.30), and $\mathbf{V}_3(\xi)$ is the solution of the boundary value problem:

$$\begin{cases} \frac{d\mathbf{V}_3(\xi)}{d\xi} = \mathbf{r}_3(\mathbf{V}_3(\xi)), \qquad \xi < \lambda_3(\mathbf{u}_R), \\ \mathbf{V}_3|_{\xi=\lambda_3(\mathbf{u}_R)} = \mathbf{u}_R. \end{cases} \tag{8.5}$$

The 1-shock connecting $\mathbf{u}_L$ and $\mathbf{u}_M$ satisfies the Lax entropy condition: $\lambda_1(\mathbf{u}_M) < \sigma_1 < \lambda_1(\mathbf{u}_L) < 0$. The states $\mathbf{u}_M$ and $\mathbf{u}_N$ are also completely determined by the shock curve formula (6.34)–(6.36) and (8.5). The best way to see this fact is first to recall that S is increasing across 1-shock waves and is constant over rarefaction curves, since S is a Riemann invariant of the first and third fields (see [290, 291]). Similarly, v and p are both constant over the wave curves of the second (linearly degenerate) field. Hence, in the space (v, p, S), we can project the curves S_1 and R_3 on the plane (v, p), find the intersection point (v_M, p_M) of these projected curves, and immediately obtain the two intersection points $(v_M, p_M, S_M), (v_M, p_M, S_N)$, of the line $\{(v, p, S) \;:\; v = v_M, p = p_M\}$ with the 1-shock curve S_1 and the 3-rarefaction curve R_3 in the phase space.

We now state and prove the uniqueness theorem in Chen-Frid-Li [52].

Theorem 8.4. *Let* $\mathbf{u}(x,t) = (\tau, v, e + \frac{v^2}{2})(x,t)$ *be an entropy solution of* (1.13) *and* (6.1) *in* $\Pi_T := \{(x,t) \;:\; 0 \le t \le T\}$ *for some* $T \in (0,\infty)$*, which belongs* $BV_{loc}(\Pi_T; \mathcal{D})$ *with* $\mathcal{D} \subset \{(\tau, v, e + \frac{v^2}{2}) \;:\; \tau > 0\} \subset \mathbb{R}^3$ *bounded. Then* $\mathbf{u}(x,t) = \mathbf{R}(x/t)$*, for a.e.* $(x,t) \in \Pi_T$*.*

Proof. Step 1. Consider the auxiliary function in Π_T:

$$\tilde{\mathbf{u}}(x,t) = \begin{cases} \mathbf{u}_L, & x < x(t), \\ \mathbf{u}_M, & x(t) < x < \max\{x(t), \sigma_1 t\}, \\ \mathbf{R}(x/t), & x > \max\{x(t), \sigma_1 t\}, \end{cases}$$

where $x(t)$ is the minimal 1-characteristic of $\mathbf{u}(x,t)$, and $x = \sigma t$ is the line of 1-shock in $\mathbf{R}(x/t)$. One of the main ingredients in the proof is to use the state variables $\mathbf{W} = (\tau, v, S)$ as the basic variable, rather than the conserved variables $\mathbf{u}(x,t)$, and we let $\tilde{\mathbf{W}}(x,t)$ denote $R(x/t)$ in these state variables. Motivated by a procedure introduced by Dafermos (*cf.* [87, 104]), we use the quadratic entropy-entropy flux pairs obtained from (η_*, q_*):

$$\alpha(\mathbf{W}, \tilde{\mathbf{W}}) = \eta_*(\mathbf{W}) - \eta_*(\tilde{\mathbf{W}}) - \nabla\eta_*(\tilde{\mathbf{W}}) \cdot (\mathbf{W} - \tilde{\mathbf{W}}), \tag{8.6}$$

$$\beta(\mathbf{W}, \tilde{\mathbf{W}}) = q_*(\mathbf{W}) - q_*(\tilde{\mathbf{W}}) - \nabla\eta_*(\tilde{\mathbf{W}}) \cdot (\mathbf{f}(\mathbf{W}) - \mathbf{f}(\tilde{\mathbf{W}})). \tag{8.7}$$

We then consider the measures

$$\mu = \partial_t \alpha(\mathbf{W}(x,t), \tilde{\mathbf{W}}(x,t)) + \partial_x \beta(\mathbf{W}(x,t), \tilde{\mathbf{W}}(x,t)), \quad (x,t) \in \Pi_T,$$

$$\nu = \partial_t \eta_*(\mathbf{W}(x,t)) + \partial_x q_*(\mathbf{W}(x,t)) - \partial_S \eta_*(\tilde{\mathbf{W}}(x,t)) \partial_t S(x,t), \;\; (x,t) \in \Pi_T - \{\ell_T \cup L_T\},$$

where $\ell_t = \{(0,s) \;:\; 0 \le s \le t\}$ and $L_t = \{(x(s), s) \;:\; 0 \le s \le t\}$.

Then the uniqueness problem essentially reduces to analyzing the measure μ over the region, where the Riemann solution is a rarefaction wave, and over the curve $(x(t), t)$, which for simplicity may be taken as the jump set of $\tilde{\mathbf{W}}(x,t)$.

Step 2. The first important fact is that $\mu\{\ell_T\} = 0$, since $\mu\{\ell_T\} = \int_{\ell_T} [\beta(\mathbf{W}, \tilde{\mathbf{W}})] \, d\mathcal{H}^1$ and $[\beta(\mathbf{W}, \tilde{\mathbf{W}})] = 0$, $\mathcal{H}^1$-a.e. over ℓ_T. The latter follows from $\beta(\mathbf{W}, \tilde{\mathbf{W}}) = (v - \bar{v})(p - \bar{p})$ and the fact that v, p, $\bar{v}$, $\bar{p}$ cannot change across the jump discontinuities of $\mathbf{W}$ and $\tilde{\mathbf{W}}$ over ℓ_T, because of the Rankine-Hugoniot relation (6.7).

Let

$$\Omega_3 := \{\, (x,t) \;:\; \lambda_3(\mathbf{u}_N) < x/t < \lambda_3(\mathbf{u}_R), \; 0 < t \le T \,\}$$

denote the rarefaction wave region of the classical Riemann solution. Over the region Ω_3, $\tilde{\mathbf{W}} = \bar{\mathbf{W}}$, and μ satisfies

$$\mu = \partial_t \alpha(\mathbf{W}, \bar{\mathbf{W}}) + \partial_x \beta(\mathbf{W}, \bar{\mathbf{W}}) = \nu - \nabla^2 \eta_*(\bar{\mathbf{W}})(\partial_x \bar{\mathbf{W}}, Q\mathbf{f}(\mathbf{W}, \bar{\mathbf{W}})), \tag{8.8}$$

where we used the fact that $\nabla^2 \eta_* \nabla \mathbf{f}$ is symmetric, and $Q\mathbf{f}(\mathbf{W}, \bar{\mathbf{W}}) = \mathbf{f}(\mathbf{W}) - \mathbf{f}(\bar{\mathbf{W}}) - \nabla \mathbf{f}(\bar{\mathbf{W}}) \cdot (\mathbf{W} - \bar{\mathbf{W}})$ is the quadratic part of $\mathbf{f}$ at $\bar{\mathbf{W}}$. Since $\tilde{\mathbf{l}}_3(\bar{\mathbf{W}}) = \tilde{\mathbf{r}}_3(\bar{\mathbf{W}}) \nabla^2 \eta_*(\bar{\mathbf{W}})$ is a left-eigenvector of $\nabla \mathbf{f}(\bar{\mathbf{W}})$ corresponding to the eigenvalue $\lambda_3(\bar{\mathbf{W}})$, and

$$\frac{\partial \bar{\mathbf{W}}(x,t)}{\partial x} = \frac{1}{t} \tilde{\mathbf{r}}_3(\bar{\mathbf{W}}(x,t)), \qquad \text{for } (x,t) \in \Omega_3.$$

Then, for any Borel set $E \subset \Omega_3$, we have

$$\mu(E) = \nu(E) - \int_E \frac{1}{t} \tilde{\mathbf{l}}_3(\bar{\mathbf{W}}) Q\mathbf{f}(\mathbf{W}, \bar{\mathbf{W}}) \, dx dt.$$

The fact $\tilde{\mathbf{l}}_3(\bar{\mathbf{W}}) Q\mathbf{f}(\mathbf{W}, \bar{\mathbf{W}}) \geq 0$ yields

$$\mu(\Omega_3) \leq 0.$$

Step 3. Using the Gauss-Green formula for BV functions and the finiteness of propagation speed of the solutions yields

$$\mu\{\Pi_t\} = \int_{-\infty}^{\infty} \alpha(\mathbf{W}(x,t), \tilde{\mathbf{W}}(x,t)) \, dx. \tag{8.10}$$

On the other hand, since μ reduces to the measure ν on the open sets where $\tilde{\mathbf{W}}$ is a constant, and $\tilde{\mathbf{W}} = \bar{\mathbf{W}}$ over Ω_3,

$$\mu\{\Pi_t\} = \mu\{L_t\} + \mu\{\Omega_3(t)\} + \nu\big\{\Pi_t - \big(L_t \cup \ell_t \cup \Omega_3(t)\big)\big\}, \tag{8.11}$$

where we have used the fact that $\mu\{\ell_t\} = 0$. Hence, it suffices to show

$$\mu\{L_t\} \leq 0. \tag{8.12}$$

Thus, we consider the functional

$$D(\sigma, \mathbf{W}_-, \mathbf{W}_+, \tilde{\mathbf{W}}_-, \tilde{\mathbf{W}}_+) = \sigma[\alpha(\mathbf{W}, \tilde{\mathbf{W}})] - [\beta(\mathbf{W}, \tilde{\mathbf{W}})].$$

Step 4. We now prove

$$D(\sigma, \mathbf{W}_-, \mathbf{W}_+, \tilde{\mathbf{W}}_-, \tilde{\mathbf{W}}_+) \leq 0, \tag{8.13}$$

if $\mathbf{W}_-$ and $\mathbf{W}_+$ are connected by a 1-shock of speed $\sigma = x'(t)$, $\tilde{\mathbf{W}}_-$ and $\tilde{\mathbf{W}}_+$ are connected by a 1-shock of speed $\bar{\sigma}$, and also $\mathbf{W}_- = \tilde{\mathbf{W}}_-$. Using Proposition 8.5, it is then clear that (8.13) immediately implies (8.12). Thus, when $\mathbf{W}_- = \tilde{\mathbf{W}}_-$, a careful calculation shows that

$$\begin{aligned} D(\sigma, \mathbf{W}_-, \mathbf{W}_+, \tilde{\mathbf{W}}_-, \tilde{\mathbf{W}}_+) =& d(\sigma, \mathbf{W}_-, \mathbf{W}_+) - d(\bar{\sigma}, \mathbf{W}_-, \tilde{\mathbf{W}}_+) - (\sigma - \bar{\sigma}) \alpha(\mathbf{W}_-, \tilde{\mathbf{W}}_+) \\ & - \partial_S \eta(\tilde{\mathbf{W}}_+) \Big(\sigma(S_- - S_+) - \bar{\sigma}(S_- - \tilde{S}_+) \Big), \end{aligned} \tag{8.14}$$

where $d(\sigma, \mathbf{W}_-, \mathbf{W}_+) = \sigma[\eta(\mathbf{W})] - [q(\mathbf{W})]$, and $(\eta, q) = (\eta_*, q_*)$ is the energy-energy flux pair in (1.13). From the Rankine-Hugoniot relation (6.30), we may view the state $\mathbf{W}_+ = (\tau_+, v_+, S_+)$ connected on the right by a 1-shock to a state $\mathbf{W}_- = (\tau_-, v_-, S_-)$ as parametrized by the shock speed σ, with $\sigma \leq \lambda_1(\mathbf{W}_-) < 0$.

According to the parametrization, we set $\mathbf{W}_+ = \mathbf{W}_+(\sigma)$ and $\tilde{\mathbf{W}}_+ = \mathbf{W}_+(\bar{\sigma})$ in (8.14), and define

$$h(\sigma) := d(\sigma, \mathbf{W}_-, \mathbf{W}_+(\sigma)) = \sigma[\eta(\mathbf{W})] - [q(\mathbf{W})].$$

Then, using Proposition 6.6 and making a careful calculation yield

$$D(\sigma, \mathbf{W}_-, \mathbf{W}_+, \tilde{\mathbf{W}}_-, \tilde{\mathbf{W}}_+) \leq h(\sigma) - h(\bar{\sigma}) - \dot{h}(\bar{\sigma})(\sigma - \bar{\sigma}).$$

On the other hand, (6.30) implies $h(\sigma) = 0$ for all σ; thereby, (8.13) holds.

Step 5. Now, by (8.10), we conclude that $\mathbf{W}(x,t) = \tilde{\mathbf{W}}(x,t)$, a.e. in Π_T. In particular, $\tilde{\mathbf{W}}(x,t)$ is an entropy solution of (1.13) and (6.1), and then the Rankine-Hugoniot relation (6.30) implies that $\tilde{\mathbf{W}}(x,t)$ must coincide with the classical Riemann solution $\bar{\mathbf{W}}(x,t)$. This concludes the proof. □

8.3. **Large-Time Stability of Entropy Solutions.** In this section we follow the framework established in Chen-Frid [46] to show that the uniqueness of the classical Riemann solution $\mathbf{R}(\xi)$, corresponding to the Riemann data (6.1), implies the large-time stability of entropy solutions $\mathbf{u}(x,t) \in L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of the Cauchy problem (1.13) and

$$\mathbf{u}|_{t=0} = \mathbf{R}_0(x) + \mathbf{P}_0(x), \qquad \mathbf{P}(x) \in L^1 \cap L^\infty(\mathbb{R}), \tag{8.15}$$

whose local total variation satisfies a certain natural growth condition.

Theorem 8.5. *Let $\mathcal{S}(\mathbb{R}^2_+)$ denote a class of functions defined on $\mathbb{R}^2_+$. Assume that the Cauchy problem* (1.19), (8.15), *and* (6.1) *satisfies the following.*

(i) System (1.19) *has a strictly convex entropy;*

(ii) The Riemann solution is unique in the class $\mathcal{S}(\mathbb{R}^2_+)$;

(iii) Given any entropy solution $\mathbf{u} \in \mathcal{S}(\mathbb{R}^2_+)$ of (1.19) *and* (8.15), *the sequence $\mathbf{u}^T(x,t) = \mathbf{u}(Tx,Tt)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$, and any limit function of its subsequences is still in $\mathcal{S}(\mathbb{R}^2_+)$.*

Then the Riemann solution $\mathbf{R}(x/t)$ is asymptotically stable in $\mathcal{S}(\mathbb{R}^2_+)$ with respect to the corresponding initial perturbation $\mathbf{P}_0(x)$:

$$ess \lim_{t\to\infty} \int_{-L}^{L} |\mathbf{u}(\xi t,t) - \mathbf{R}(\xi)|d\xi = 0, \qquad \text{for any } L > 0. \tag{8.16}$$

System (1.13) has a strictly convex entropy $S(\tau, v, e + \frac{v^2}{2})$ in $\mathcal{D}$, and hence the condition (i) follows.

We choose $\mathcal{S}(\mathbb{R}^2_+)$ as the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ satisfying a natural growth condition of local total variation:

$$\begin{cases} \text{There exists } c_0 > 0 \text{ such that, for all } c \ge c_0, \text{ there is } C > 0 \text{ depending} \\ \text{only upon } c \text{ such that } \mathrm{TV}(\mathbf{u}\,|\,\mathcal{K}_{c,T}) \le CT, \quad \text{for any } T > 0, \end{cases} \tag{8.17}$$

where $\mathcal{K}_{c,T} = \{(x,t) \in \mathbb{R}^2_+ \,:\, |x| \le ct,\ t \in (0,T)\}$. Such a condition is natural, since any solution obtained by the Glimm method or related methods satisfies (8.17).

For such solutions and for any $T > 0$, $\mathbf{u}^T(x,t)$ also satisfies (8.17) with the same constant C depending only upon c. Furthermore, the sequence $\mathbf{u}^T(x,t)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$. Then the condition (iii) follows.

Therefore, the uniqueness result established in §8.2 yields the large-time stability of entropy solutions satisfying (8.17).

Theorem 8.6. *Any Riemann solution of System* (1.13), *staying away from the vacuum, with large Riemann initial data* (6.1) *is large-time asymptotically stable in the sense of* (8.16) *in the class of entropy solutions in $L^\infty \cap BV_{loc}(\mathbb{R}^2_+)$ of* (1.13) *with large initial perturbation* (8.15) *satisfying* (8.17).

Remark 8.1. A uniqueness theorem of Riemann solutions was first established by DiPerna [104] for 2×2 strictly hyperbolic and genuinely nonlinear systems in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with small oscillation. In [48], Chen and Frid established the uniqueness and stability of Riemann solutions, with shocks of small strength, for the 3×3 system of Euler equations with general equation of state in the class of entropy solutions in $L^\infty \cap BV_{loc}$ with small oscillation. However, the uniqueness result presented here neither imposes smallness on the oscillation nor the extra regularity of the solutions, as well as does not require specific reference to any particular method for constructing the entropy solutions. In this connection, we recall that, for System (1.13) for polytropic gases, there are many existence results of solutions in $L^\infty \cap BV_{loc}$ via the Glimm scheme [130], especially when the adiabatic exponent $\gamma > 1$ is close to one (see, *e.g.*, [212, 302, 253]). We also refer the reader to Dafermos [87] for the stability of Lipschitz solutions for hyperbolic systems of conservation laws.

9. Global Discontinuous Solutions IV: Entropy Solutions in L^∞

In this section we extensively discuss the Cauchy problem for the one-dimensional isentropic Euler equations in (1.14) and show the existence, compactness, decay, and stability of global entropy solutions in L^∞. In the study of entropy solutions to the Euler equations, several numerical approximate schemes or methods have played an important role. As an example, we show here the convergence of the Lax-Friedrichs scheme and the Godunov scheme for the Cauchy problem.

9.1. **Isentropic Euler Equations.** Consider the Cauchy problem for the isentropic Euler equations in (1.14) with initial data:

$$(\rho, m)|_{t=0} = (\rho_0, m_0)(x), \tag{9.1}$$

where ρ and m are in the physical region $\{(\rho, m) : \rho \geq 0, |m| \leq C_0\rho\}$ for some $C_0 > 0$. For $\rho > 0$, $v = m/\rho$ is the velocity. The pressure function $p(\rho)$ is a smooth function in $\rho > 0$ (nonvacuum states) satisfying (6.22) when $\rho > 0$, and

$$p(0) = p'(0) = 0, \quad \lim_{\rho \to 0} \frac{\rho p^{(j+1)}(\rho)}{p^{(j)}(\rho)} = c_j > 0, \; j = 0, 1. \tag{9.2}$$

More precisely, we consider a general situation of pressure law that there exist a sequence of exponents

$$1 < \gamma := \gamma_1 < \gamma_2 < \cdots < \gamma_J \leq \frac{3\gamma - 1}{2} < \gamma_{J+1}$$

and a function $P(\rho)$ such that

$$p(\rho) = \sum_{j=1}^{J} \kappa_j \rho^{\gamma_j} + \rho^{\gamma_{J+1}} P(\rho); \qquad P(\rho), \; \rho^3 P'''(\rho) \text{ are bounded as } \rho \to 0, \tag{9.3}$$

for some $\kappa_j, j = 1, \cdots, J$, with $\kappa_1 = \frac{(\gamma-1)^2}{4\gamma}$ after renormalization.

For a polytropic gas obeying the γ-law (1.10), or a mixed ideal polytropic fluid,

$$p(\rho) = \kappa_1 \rho^{\gamma_1} + \kappa_2 \rho^{\gamma_2}, \qquad \kappa_2 > 0,$$

the pressure function clearly satisfies (6.22) and (9.3).

System (1.14) is strictly hyperbolic at the nonvacuum states $\{(\rho, v) : \rho > 0, |v| \leq C_0\}$, and strict hyperbolicity fails at the vacuum states $\{(\rho, m/\rho) : \rho = 0, |m/\rho| \leq C_0\}$.

9.2. **Entropy-Entropy Flux Pairs.** A pair of mappings $(\eta, q) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is called an entropy-entropy flux pair (or entropy pair for short) of System (1.14) if it satisfies the hyperbolic system:

$$\nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla \mathbf{f}(\rho, m). \tag{9.4}$$

Furthermore, $\eta(\rho, m)$ is called a weak entropy if

$$\eta\Big|_{\substack{\rho=0, \\ v=m/\rho \text{ fixed}}} = 0. \tag{9.5}$$

For example, the mechanical energy (a sum of the kinetic and internal energy) and the mechanical energy flux

$$\eta_*(\rho,m)=\frac{m^2}{2\rho}+\rho\int_0^\rho\frac{p(s)}{s^2}ds,\quad q_*(\rho,m)=\frac{m^3}{2\rho^2}+m\int_0^\rho\frac{p'(s)}{s}ds \tag{9.6}$$

form a special entropy pair; $\eta_*(\rho,m)$ is convex for any $\gamma>1$ and strictly convex (even at the vacuum states) if $\gamma\le 2$, in any bounded region in $\rho\ge 0$.

Definition 9.1. *A bounded measurable function $\mathbf{u}(x,t)=(\rho,m)(x,t)$ is an entropy solution of* (1.14)*,* (6.22)*,* (9.1)*, and* (9.2) *in $\mathbb{R}^2_+$ if $\mathbf{u}(x,t)$ satisfies the following:*

(i) *There exists $C>0$ such that*

$$0\le\rho(x,t)\le C,\qquad |m(x,t)/\rho(x,t)|\le C;$$

(ii) *The entropy inequality holds in the sense of distributions in $\mathbb{R}^2_+$, i.e., for any weak entropy pair $(\eta,q)(\mathbf{u})$ with convex $\eta(\mathbf{u})$ and any nonnegative function $\phi\in C^1_0(\mathbb{R}\times[0,\infty))$,*

$$\int_0^\infty\int_{-\infty}^\infty(\eta(\mathbf{u})\partial_t\phi+q(\mathbf{u})\partial_x\phi)\,dx\,dt+\int_{-\infty}^\infty\eta(\mathbf{u}_0)(x)\phi(x,0)dx\ge 0. \tag{9.7}$$

Notice that $\eta(\mathbf{u})=\pm\mathbf{u}$ are both trivial convex entropy functions so that (9.7) implies that $\mathbf{u}(x,t)$ is a weak solution in the sense of distributions.

In the coordinates (ρ,v), any weak entropy function $\eta(\rho,v)$ is governed by the second-order linear wave equation

$$\begin{cases}\eta_{\rho\rho}-k'(\rho)^2\eta_{vv}=0, & \rho>0,\\ \eta|_{\rho=0}=0,\end{cases} \tag{9.8}$$

with $k(\rho)=\int_0^\rho\frac{p'(s)}{s}ds$.

In the Riemann invariant coordinates $\mathbf{w}=(w_1,w_2)$ defined in (6.26), any entropy function $\eta(\mathbf{w})$ is governed by

$$\eta_{w_1w_2}+\frac{\Lambda(w_1-w_2)}{w_1-w_2}(\eta_{w_1}-\eta_{w_2})=0, \tag{9.9}$$

where

$$\Lambda(w_1-w_2)=-k(\rho)k'(\rho)^{-2}k''(\rho),\quad\text{with }\rho=k^{-1}(\frac{w_1-w_2}{2}). \tag{9.10}$$

The corresponding entropy flux function $q(\mathbf{w})$ is

$$q_{w_j}(\mathbf{w})=\lambda_i(\mathbf{w})\eta_{w_j}(\mathbf{w}),\qquad i\ne j. \tag{9.11}$$

In general, any weak entropy pair (η,q) can be represented by

$$\eta(\rho,v)=\int_{\mathbb{R}}\chi(\rho,v;s)a(s)ds,\quad q(\rho,v)=\int_{\mathbb{R}}\sigma(\rho,v;s)b(s)ds, \tag{9.12}$$

for any continuous function $a(s)$ and related function $b(s)$, where the weak entropy kernel and entropy flux kernel are determined by

$$\begin{cases}\chi_{\rho\rho}-k'(\rho)^2\chi_{vv}=0,\\ \chi(0,v;s)=0,\quad \chi_\rho(0,v;s)=\delta_{v=s},\end{cases} \tag{9.13}$$

and

$$\begin{cases}\sigma_{\rho\rho}-k'(\rho)^2\sigma_{vv}=\frac{p''(\rho)}{\rho}\chi_v,\\ \sigma(0,v;s)=0,\quad \sigma_\rho(0,v;s)=v\delta_{v=s},\end{cases} \tag{9.14}$$

with $\delta_{v=s}$ the Delta function concentrated at the point $v=s$.

The equations in (9.8)–(9.9) and (9.13)–(9.14) belong to the class of Euler-Poisson-Darboux type equations. The main difficulty comes from the singular behavior of $\Lambda(w_1-w_2)$

near the vacuum. In view of (9.10), the derivative of $\Lambda(w_1-w_2)$ in the coefficients of (9.9) may blow up like $(w_1-w_2)^{-(\gamma-1)/2}$ when $w_1-w_2\to 0$ in general, and its higher derivatives may be more singular, for which the classical theory of Euler-Poisson-Darboux equations does not apply (cf. [11, 324, 325]). However, for a gas obeying the γ-law,

$$\Lambda(w_1-w_2)=\lambda:=\frac{3-\gamma}{2(\gamma-1)},$$

the simplest case, which excludes such a difficulty. In particular, for this case, the weak entropy kernel is

$$\chi(\rho,v;s)=[(w_1(\rho,v)-s)(s-w_2(\rho,v))]_+^{\lambda}.$$

A mathematical theory for dealing with such a difficulty for the singularities can be found in Chen-LeFloch [57, 58, 59]. Now we list several important entropy pairs and their properties. First, we have

Proposition 9.1. *For the general pressure law* (6.22), (9.2), *and* (9.3), *any weak entropy* $\eta(\rho,m)$ *satisfies that, when* $(\rho,m)\in\mathcal{D}_M:=\{0\le\rho\le M, |m|\le M\rho\}$,

$$|\nabla\eta(\rho,m)|\le C_M,\qquad |\nabla^2\eta(\rho,m)|\le C_M\nabla^2\eta_*(\rho,m).$$

The equations in (1.14) have several important entropy pairs from (9.9)–(9.12). As an example, we give their formulae for the case $\gamma=5/3$.

(i) Goursat entropy wave $G_0=(\eta_0,q_0)$:

$$\eta_0(\mathbf{w})=w_1w_2X(\mathbf{w}),\qquad q_0(\mathbf{w})=\lambda_2\eta_0+\tau_0,\;\;\tau_0:=\frac{1}{3}w_1^2w_2X(\mathbf{w}),\tag{9.15}$$

where $X(\mathbf{w})$ is the characteristic function with $X(\mathbf{w})=1$, when $w_1>0>w_2$; and $X(\mathbf{w})=0$, otherwise.

(ii) Goursat entropy wave $G_1=(\eta_1,q_1)$:

$$\eta_1(\mathbf{w})=(w_1+w_2)X(\mathbf{w}),\qquad q_1(\mathbf{w})=\lambda_2\eta_1+\tau_1,\;\;\tau_1:=\frac{1}{3}w_1(w_1+2w_2)X(\mathbf{w}).\tag{9.16}$$

(iii) Lax entropy waves $G_{\pm k}=(\eta_{\pm k},q_{\pm k})$ for $k\gg 1$:

$$\eta_k(\mathbf{w})=e^{kw_1}\rho^{1/3}\left(1+O(\frac{1}{k})\right),\qquad q_k=\eta_k\left(\lambda_2+O(\frac{1}{k})\right),\tag{9.17}$$

and

$$\eta_{-k}(\mathbf{w})=e^{-kw_2}\rho^{1/3}\left(1+O(\frac{1}{k})\right),\qquad q_{-k}=\eta_{-k}\left(\lambda_1+O(\frac{1}{k})\right).\tag{9.18}$$

(iv) Entropy wave sequence $G_\ell=(\eta_\ell,q_\ell)$:

$$\begin{cases}\eta_\ell(\mathbf{w}):=\eta(\mathbf{w};\psi_\ell)=(w_1-w_2)(\psi_\ell(w_1)+\psi_\ell(w_2))-2\int_{w_2}^{w_1}\psi_\ell(x)dx;\\ q_\ell(\mathbf{w}):=q(\mathbf{w};\psi_\ell)=\lambda_2\eta_\ell+\tau_\ell,\\ \qquad\tau_\ell(\mathbf{w};\psi_\ell):=\frac{2}{3}\int_{w_2}^{w_1}(x-w_2)(\psi_\ell(x)+\psi_\ell(w_2))dx-\frac{4}{3}\int_{w_2}^{w_1}(w_1-x)\psi_\ell(x)dx,\end{cases}\tag{9.19}$$

where $\psi_\ell(s)=\ell\psi(\ell s),\psi(s)\in C_0^\infty(-1,1),\int_{-1}^1\psi(s)ds=0$, and $supp\,\psi\subset[-1+\epsilon/2,1-\epsilon/2],\epsilon<1/4$.

Then we have

Proposition 9.2. *For* $(\rho,m)\in\mathcal{D}_M$, *there exists* $C=C_M>0$ *such that*

$$|\eta_0q_\ell-\eta_\ell q_0|\le\frac{C}{\ell},\qquad |\eta_1q_\ell-\eta_\ell q_1|\le C.$$

Proposition 9.3. *For $\psi(s) \in C_0^\infty(-1,1)$ with $\int_{-1}^1 \psi(s)ds = 0$, choose $\hat\psi(t) = t\psi(t) + \int_{-1}^t \psi(s)ds$, which implies $\int_{-1}^1 \hat\psi(s)ds = 0$. Define*

$$B_\ell(\mathbf{w};\psi) = \eta_\ell \hat q_\ell - \hat\eta_\ell q_\ell,$$

where

$$(\eta_\ell, q_\ell) = (\eta(\mathbf{w};\psi_\ell), q(\mathbf{w};\psi_\ell)), \qquad (\hat\eta_\ell, \hat q_\ell) = (\eta(\mathbf{w};\hat\psi_\ell), q(\mathbf{w};\hat\psi_\ell)). \tag{9.20}$$

Then, for $(\rho, m) \in \mathcal{D}_M$,

$$B_\ell(\mathbf{w};\psi) = \begin{cases} \ell(w_1-w_2)^2 A(\ell w_j) + (w_1-w_2)B_\ell^j(\mathbf{w}) + \frac{C_\ell^j(\mathbf{w})}{\ell}, & \text{in } S_{w_j}^{\epsilon,\ell},\ j=1,2, \\ O(\frac{1}{\ell}), \quad \text{otherwise}, \end{cases} \tag{9.21}$$

where

$$\begin{aligned} &A(x) = \frac{2}{3}\left(\int_{-1}^x \psi(s)ds\right)^2, \quad |B_\ell^j(\mathbf{w})| \le C\left(|\psi(\ell w_j)| + |\int_{-1}^{\ell w_j}\psi(s)ds|\right), \\ &|C_\ell^j(\mathbf{w})| \le C < \infty, \qquad j = 1,2, \\ &S_{w_j}^{\epsilon,\ell} = \{\mathbf{w} : |w_j| \le \frac{1-\epsilon}{\ell}\}. \end{aligned} \tag{9.22}$$

9.3. **Compactness Framework.** We now establish the following compactness framework.

Theorem 9.1. *Consider the Euler equations* (1.14) *for compressible fluids under the assumptions* (6.22) *and* (9.3)*. Let $(\rho^\epsilon, m^\epsilon)(x,t)$ be a sequence of functions satisfying*

$$0 \le \rho^\epsilon(x,t) \le C, \qquad |m^\epsilon(x,t)| \le C\,\rho^\epsilon(x,t), \qquad \text{for a.e. } (x,t), \tag{9.23}$$

such that, for any weak entropy pair (η, q),

$$\partial_t \eta(\rho^\epsilon, m^\epsilon) + \partial_x q(\rho^\epsilon, m^\epsilon) \quad \text{is compact in } H_{loc}^{-1}(\mathbb{R}_+^2). \tag{9.24}$$

Then the sequence $(\rho^\epsilon, m^\epsilon)(x,t)$ is compact in $L_{loc}^1(\mathbb{R}_+^2)$.

Proof. We now give a sketch of the proof for the case $\gamma = 5/3$.

Step 1. First, with the aid of the div-curl lemma and the Young measure representation theorem (see Murat [238, 239] and Tartar [299]; also see Chen [39]), the conditions (9.23)–(9.24) imply that there exists a family of probability measures $\{\nu_{x,t} \in \mathrm{P}rob.(\mathbb{R}_+ \times \mathbb{R})\}$, uniquely determined by $(\rho^\epsilon, m^\epsilon)(x,t)$, such that

$$\operatorname{supp}\nu_{x,t} \subset \mathcal{D}_M, \tag{9.25}$$

and, for any continuous or bounded measurable weak entropy pairs $(\eta_j, q_j), j = 1,2$,

$$\left\langle \nu, \begin{vmatrix} \eta_1 & q_1 \\ \eta_2 & q_2 \end{vmatrix} \right\rangle = \begin{vmatrix} <\nu, \eta_1> & <\nu, q_1> \\ <\nu, \eta_2> & <\nu, q_2> \end{vmatrix}, \qquad a.e. \tag{9.26}$$

For simplicity, we often drop the index (x,t) of $\nu_{x,t}$. Then the compactness problem reduces to the question whether the Young measures are Delta masses concentrated at $\mathbf{u}(x,t) = (\rho, m)(x,t) = w^* - \lim_{\epsilon\to 0}(\rho^\epsilon, m^\epsilon)(x,t)$, that is,

$$\nu_{x,t} = \delta_{\mathbf{u}(x,t)}. \tag{9.27}$$

To achieve (9.27), it suffices to show

$$\operatorname{supp}\nu \subset V \cup P, \tag{9.28}$$

where $V = \{\mathbf{w} : \rho = 0\}$, the vacuum set, and $P = (w_1^0, w_2^0) = \mathbf{w}(\rho^0, v^0), \rho^0 > 0$, is the vertex of the smallest triangle K containing $\operatorname{supp}\nu - V$ in the $\mathbf{w}$-coordinates.

This can be seen as follows. If (9.28) holds, then there are only three possibilities:

(i) $\operatorname{supp}\nu = \{P\}$;

(ii) $\operatorname{supp}\nu \subset V$;
(iii) $\nu = \nu|_V + \alpha\delta_P, \qquad \alpha \neq 0, 1.$

It is clear that (i) and (ii) imply (9.27). For (iii), we choose $(\eta_1, q_1) = (\rho, m)$ and $(\eta_2, q_2) = (m, \frac{m^2}{\rho} + p(\rho))$ in (9.26) to have

$$\alpha\rho^0 p(\rho^0) = \alpha^2\rho^0 p(\rho^0),$$

which implies that $\alpha = \alpha^2$ since $\rho^0 > 0$. That is, either $\alpha = 0$ or $\alpha = 1$, which is a contradiction.

Step 2. To achieve (9.28), it suffices to prove

$$\lim_{\ell\to\infty}\sum_{i=1}^{2} < \nu|_{S^{\epsilon,\ell}_{w_i}}, \ell(w_1 - w_2)^2 >= 0. \tag{9.29}$$

This can be seen as follows. Set $\tilde\nu = (w_1 - w_2)^2\nu$, a weighted measure. Define

$$P_{w_i}\tilde\nu(a, b) = \tilde\nu\{\mathbf{w} \mid a < w_i < b\},$$

an orthogonal projection of $\tilde\nu$ onto the segment parallel to the w_i-axis. If we can prove that the following Lebesgue derivatives of Radon measures are zero for $i = 1, 2$, that is,

$$DP_{w_i}\tilde\nu(w_i) = 0, \qquad w_2^0 < w_i < w_1^0, \quad i = 1, 2, \tag{9.30}$$

then we conclude

$$\tilde\nu(K - P) = 0,$$

that is,

$$\nu(K - (V \cup P)) = 0,$$

which implies (9.28).

By Galilean invariance, it suffices for (9.30) to show that

$$DP_{w_i}\tilde\nu(0) = 0, \quad i = 1, 2, \tag{9.31}$$

with $w_2^0 < 0 < w_1^0$, which is equivalent to (9.29).

Step 3. Claim: If $\operatorname{supp}\nu \cap (K - V) \neq \emptyset$, then $P \in \operatorname{supp}\nu$.
If $P \notin \operatorname{supp}\nu$, then there exists $\delta > 0$ such that $B_{2\delta}(P) \cap \operatorname{supp}\nu = \emptyset$. In (9.26), we choose $(\eta_1, q_1) = (\eta_k, q_k)$ and $(\eta_2, q_2) = (\eta_{-k}, q_{-k})$ to have

$$\frac{< \nu, \eta_k q_{-k} - \eta_{-k} q_k >}{< \nu, \eta_k >< \nu, \eta_{-k} >} = \frac{< \nu, q_k >}{< \nu, \eta_k >} - \frac{< \nu, q_{-k} >}{< \nu, \eta_{-k} >}. \tag{9.32}$$

Observe that, as k is sufficiently large,

$$| < \nu, \eta_k q_{-k} - \eta_{-k} q_k > | \leq C e^{k(w_1^0 - w_2^0 - \sqrt{2}\delta)},$$

and

$$| < \nu, \eta_k > | \geq c_0 e^{k(w_1^0 - \frac{\delta}{2})}, \qquad | < \nu, \eta_{-k} > | \geq c_0 e^{-k(w_2^0 + \frac{\delta}{2})}.$$

We conclude from (9.32) that

$$\lim_{k\to\infty}\left(\frac{< \nu, q_k >}{< \nu, \eta_k >} - \frac{< \nu, q_{-k} >}{< \nu, \eta_{-k} >}\right) = 0. \tag{9.33}$$

Define the probability measures $\mu_k^{\pm} \in Prob.(\mathbb{R}^2)$:

$$< \mu_k^{\pm}, h >= \frac{< \nu, h\eta_{\pm k} >}{< \nu, \eta_{\pm k} >}, \qquad h \in C_0(\mathbb{R}^2),$$

as k is sufficiently large. Then $\|\mu_k^{\pm}\|_{\mathcal{M}} = 1$, and there exists a subsequence $\{\mu_{k_j}\}_{j=1}^{\infty}$ such that

$$w^* - \lim_{j\to\infty}\mu_{k_j}^{\pm} = \mu^{\pm},$$

and

$$\operatorname{supp}\mu^+ \subset \{w_1 = w_1^0\} \cap K, \qquad \operatorname{supp}\mu^- \subset \{w_2 = w_2^0\} \cap K.$$

Notice that $\lambda_{1w_1} = \lambda_{2w_2} = \frac{1}{3} > 0$. We have

$$\lim_{j\to\infty} \frac{<\nu, q_{k_j}>}{<\nu, \eta_{k_j}>} = <\mu^+, \lambda_2> \geq \lambda_2(P) > \lambda_1(P) \geq <\mu^-, \lambda_1> = \lim_{j\to\infty} \frac{<\nu, q_{-k_j}>}{<\nu, \eta_{-k_j}>},$$

which is a contradiction to (9.33).

Step 4. We now show that there exists $C > 0$, independent of ℓ, such that

$$|<\nu, \eta_\ell>| + |<\nu, q_\ell>| \leq C.$$

If not, there exists a subsequence $\{\ell_j\}_{j=1}^\infty$ such that

$$\lim_{j\to\infty} |<\nu, \eta_{\ell_j}>| = \infty, \quad \text{and/or} \quad \lim_{j\to\infty} |<\nu, q_{\ell_j}>| = \infty.$$

For concreteness, we assume

$$\lim_{j\to\infty} \frac{<\nu, q_{\ell_j}>}{<\nu, \eta_{\ell_j}>} = \alpha \in (-\infty, \infty).$$

Consider the commutativity relations

$$<\nu, q_0> - \frac{<\nu, q_{\ell_j}>}{<\nu, \eta_{\ell_j}>} <\nu, \eta_0> = \frac{<\nu, \eta_{\ell_j} q_0 - \eta_0 q_{\ell_j}>}{<\nu, \eta_{\ell_j}>},$$

and

$$<\nu, q_1> - \frac{<\nu, q_{\ell_j}>}{<\nu, \eta_{\ell_j}>} <\nu, \eta_1> = \frac{<\nu, \eta_{\ell_j} q_1 - \eta_1 q_{\ell_j}>}{<\nu, \eta_{\ell_j}>}.$$

Let $j \to \infty$ and use Proposition 9.2. Then

$$\begin{cases} <\nu, q_0> - \alpha <\nu, \eta_0> = 0, \\ <\nu, q_1> - \alpha <\nu, \eta_1> = 0, \end{cases}$$

which implies

$$0 = \begin{vmatrix} <\nu, \eta_0> & <\nu, q_0> \\ <\nu, \eta_1> & <\nu, q_1> \end{vmatrix} = <\nu, \eta_0 q_1 - \eta_1 q_0> = \frac{1}{3} <\nu, (w_1 w_2)^2 X(\mathbf{w})> .$$

This implies that $P \notin \operatorname{supp}\nu$, which is a contradiction to Step 3.

Step 5. Claim: For (η_ℓ, q_ℓ) and $(\hat{\eta}_\ell, \hat{q}_\ell)$ defined in (9.20),

$$\lim_{\ell\to\infty} <\nu, \eta_\ell \hat{q}_\ell - \hat{\eta}_\ell q_\ell> = 0.$$

If not, there exists a subsequence such that

$$\lim_{j\to\infty} <\nu, \eta_{\ell_j} \hat{q}_{\ell_j} - \hat{\eta}_{\ell_j} q_{\ell_j}> \neq 0.$$

Step 4 indicates that there further exists a subsequence (still denoted) $\{\ell_j\}$ such that

$$\lim_{j\to\infty} (<\nu, \eta_{\ell_j}>, <\nu, q_{\ell_j}>, <\nu, \hat{\eta}_{\ell_j}>, <\nu, \hat{q}_{\ell_j}>) \quad \text{exists.}$$

Proposition 9.2 and the identity (9.25) imply

$$\begin{cases} -<\nu, \eta_0> \lim_{j\to\infty} <\nu, q_{\ell_j}> + <\nu, q_0> \lim_{j\to\infty} <\nu, \eta_{\ell_j}> = 0, \\ -<\nu, \eta_0> \lim_{j\to\infty} <\nu, \hat{q}_{\ell_j}> + <\nu, q_0> \lim_{j\to\infty} <\nu, \hat{\eta}_{\ell_j}> = 0. \end{cases}$$

Since $<\nu, \eta_0> > 0$ from Step 3, we have

$$0 = \begin{vmatrix} \lim_{j\to\infty} <\nu, \eta_{\ell_j}> & \lim_{j\to\infty} <\nu, q_{\ell_j}> \\ \lim_{j\to\infty} <\nu, \hat{\eta}_{\ell_j}> & \lim_{j\to\infty} <\nu, \hat{q}_{\ell_j}> \end{vmatrix} = \lim_{j\to\infty} <\nu, \eta_{\ell_j} \hat{q}_{\ell_j} - \hat{\eta}_{\ell_j} q_{\ell_j}>,$$

which is a contradiction.

Step 6. Proposition 9.3 and Step 5 imply that

$$\sum_{i=1}^{2} < \nu|_{S^{\epsilon,\ell}_{w_i}}, \ell(w_1-w_2)^2 A(\ell w_i) + (w_1-w_2)B^i_\ell(\mathbf{w}) > \to 0, \qquad \ell\to\infty. \tag{9.34}$$

Choose $\psi(s)=a_{\frac{2-\epsilon}{4}}(s+\frac{2-\epsilon}{4})-a_{\frac{2-\epsilon}{4}}(s-\frac{2-\epsilon}{4})$, where

$$a_\delta(s)=\delta a(\frac{s}{\delta}), \quad a(s)=\begin{cases} e^{\frac{1}{|s|^2-1}}, & |s|\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\int_{-1}^{1}\psi(s)ds=0$, and

$$\Big(\int_{-1}^{x}\psi(s)ds\Big)^2 \ge c_\epsilon>0, \qquad x\in[-1+\epsilon,1-\epsilon]. \tag{9.35}$$

Combining (9.34) with (9.21) and (9.35) yields (9.29). This completes the proof. □

Remark 9.1. The proof of Theorem 9.1 is taken from Chen [37] and Ding-Chen-Luo [96]. For a gas obeying the γ-law, the case $\gamma=\frac{N+2}{N}, N\ge 5$ odd, was first treated by DiPerna [106], while the case $1<\gamma\le 5/3$ was first solved by Chen [37] and Ding-Chen-Luo [96]. Finally, motivated by a kinetic formulation, the cases $\gamma\ge 3$ and $5/3<\gamma<3$ were treated by Lions-Perthame-Tadmor [203] and Lions-Perthame-Souganidis [202], respectively, where their analysis applies to the whole interval $1<\gamma<3$. For the general pressure law (6.22) and (9.3), Theorem 9.1 is due to Chen-LeFloch [57, 59].

9.4. **Convergence of the Lax-Friedrichs Scheme and the Godunov Scheme.** We now apply the compactness framework established in Theorem 9.1 to show the convergence of the Lax-Friedrichs scheme [176] for the Cauchy problem (1.14), (6.22), (9.1), and (9.3) under the assumptions:

$$0\le\rho_0(x)\le C_0, \qquad |m_0(x)|\le C_0\,\rho_0(x), \qquad \text{for a.e. } x \text{ and some } C_0>0. \tag{9.36}$$

The convergence proof for the Godunov scheme [139] is similar (see [97, 40]).

As every difference scheme, the Lax-Friedrichs scheme satisfies the property of propagation with finite speed, which is an advantage over the vanishing viscosity method: the convergence result applies without assumption on the decay of initial data at infinity. We now construct the family of Lax-Friedrichs approximate solutions $(\rho^h,m^h)(x,t)$, similar to these in §7.1 for the Glimm scheme. We also set $v^h=m^h/\rho^h$ when $\rho^h>0$ and $v^h=0$ otherwise. The Lax-Friedrichs scheme is based on a regular partition of the half-plane $t\ge 0$ defined by $t_k=k\,\Delta t$, $x_j=j\,h$ for $k\in\mathcal{Z}_+$, $j\in\mathcal{Z}$, where Δt and h are the sizes of time-step and space-step, respectively. It is assumed that the ratio $\Delta t/h$ is constant and satisfies the Courant-Friedrichs-Lewy stability condition:

$$\frac{\Delta t}{h}\|\lambda_j(\rho^h,v^h)\|_{L^\infty}<1.$$

In the first strip $\{(x,t)\ :\ x_{j-1}<x<x_{j+1},\ 0\le t<\tau,\ j \text{ odd}\}$, we define $(\rho^h,m^h)(x,t)$ by solving a sequence of Riemann problems for (1.14) corresponding to the Riemann data:

$$(\rho^h,m^h)(x,0)=\begin{cases}(\rho^0_{j-1},m^0_{j-1}), & x<x_j,\\ (\rho^0_{j+1},m^0_{j+1}), & x>x_j,\end{cases}$$

with

$$(\rho^0_{j+1},m^0_{j+1})=\frac{1}{2\,h}\int_{x_j}^{x_{j+2}}(\rho_0,m_0)(x)\,dx.$$

Recall that the Riemann problem is uniquely solvable (see §6.3).

If $(\rho^h, m^h)(x,t)$ is known for $t < t_k$, we set

$$(\rho_j^k, m_j^k) = \frac{1}{2\,h}\int_{x_{j-1}}^{x_{j+1}} (\rho^h, m^h)(x, t_k - 0)\, dx.$$

In the strip $\{(x,t) \,:\, x_j < x < x_{j+2}, t_k < t < t_{k+1},\ j+k = \text{even}\}$, we define $(\rho^h, m^h)(x,t)$ by solving the Riemann problems with the data:

$$(\rho^h, m^h)(x, t_k) = \begin{cases} (\rho_j^k, m_j^k), & x < x_{j+1}, \\ (\rho_{j+2}^k, m_{j+2}^k), & x > x_{j+1}. \end{cases}$$

This completes the construction of the Lax-Friedrichs approximate solutions $(\rho^h, m^h)(x,t)$.

Theorem 9.2. *Let $(\rho_0, m_0)(x)$ be the Cauchy data satisfying* (9.36). *Extracting a subsequence, if necessary, the Lax-Friedrichs (or Godunov) approximate solutions $(\rho^h, m^h)(x,t)$ converge strongly almost everywhere to a limit $(\rho, m) \in L^\infty(\mathbb{R}^2_+)$ which is an entropy solution of the Cauchy problem* (1.14) *and* (9.1).

The following two propositions will be used in the proof of Theorem 9.2.

Proposition 9.4. *For any $w_1^0 > w_2^0$, the region*

$$\sum(w_1^0, w_2^0) = \big\{(\rho, m) \,:\, w_1 \le w_1^0, w_2 \ge w_2^0, w_1 - w_2 \ge 0\big\}$$

is also invariant for the Lax-Friedrichs approximate solutions, where $w_i, i = 1, 2,$ are the Riemann invariants.

Proof. Proposition 6.5 indicates that $\sum(w_1^0, w_2^0)$ is an invariant region for the Riemann solutions. Since the set $\sum(w_1^0, w_2^0)$ is convex in the (ρ, m)-plane, it follows from Jensen's inequality that, for any function satisfying $\{(\rho, m)(x) \,:\, a \le x \le b\} \subset \sum(w_1^0, w_2^0)$ for some (w_1^0, w_2^0),

$$(\bar\rho, \bar m) := \frac{1}{b-a}\int_a^b (\rho, m)(x) dx \in \sum(w_1^0, w_2^0).$$

Therefore, $\sum(w_1^0, w_2^0)$ is also an invariant region for the Lax-Friedrichs scheme. □

In particular, Proposition 9.4 shows that the approximate density function $\rho^h(x,t)$ remains nonnegative, and both $\rho^h(x,t)$ and $m^h(x,t)/\rho^h(x,t)$ are uniformly bounded so it is indeed possible to construct the approximate solutions globally, as described earlier.

Consider the entropy pair (η_*, q_*) defined from the kinetic and internal energy by (9.6).

Proposition 9.5. *For any weak entropy pair (η, q) and any invariant region $R(w_0, z_0)$, there exists a constant $C > 0$ such that, for any solution $(\rho, m)(x,t)$ of the Riemann problem with initial data in $R(w_0, z_0)$,*

$$\big|x'(t)\,[\eta(\rho, m)](t) - [q(\rho, m)](t)\big| \;\le\; C\ \left(x'(t)\,[\eta_*(\rho, m)](t) - [q_*(\rho, m)](t)\right),$$

where $x'(t)$ is the speed of any shock located at $x(t)$ in the Riemann solution $(\rho, m)(x,t)$.

The proof given in [37, 96] for the γ-law case extends immediately to the general pressure law.

Proof of Theorem 9.2. Since the scheme satisfies the property of propagation with finite speed, we can assume without loss of generality that the initial data have compact support. To establish the strong convergence of the scheme, it suffices to check that the sequence $\mathbf{u}^h(x,t) = (\rho^h, m^h)(x,t)$ satisfies the compactness framework in Theorem 9.1. The L^∞ bound is a direct corollary of Proposition 9.4. We will prove (9.24).

Consider the weak entropy dissipation measures $\partial_t\eta(\mathbf{u}^h)+\partial_x q(\mathbf{u}^h)$ associated with a weak entropy pair (η, q). Using the Gauss-Green formula, for any test-function $\varphi(x,t)$ compactly supported in $\mathbb{R}\times[0,T]$ with $T\equiv K\Delta t$ for some integer K, one has

$$\int_{\mathbb{R}}\int_0^T (\eta(\mathbf{u}^h)\partial_t\varphi + q(\mathbf{u}^h)\partial_x\varphi)\,dxdt = M^h(\varphi)+S^h(\varphi)+L_1^h(\varphi)+L_2^h(\varphi), \tag{9.37}$$

where

$$\begin{aligned}
M^h(\varphi) &:= \int_{\mathbb{R}} \eta(\mathbf{u}^h(x,T))\,\varphi(x,T)\,dx - \int_{\mathbb{R}} \eta(\mathbf{u}^h(x,0))\,\varphi(x,0)\,dx,\\
S^h(\varphi) &:= \int_0^T \sum_{\text{shocks } x(t)} \big(x'(t)\,[\eta](t)-[q](t)\big)\,\varphi(x(t),t)\,dt,\\
L_1^h(\varphi) &:= \sum_{j,k}\varphi_j^k \int_{x_{j-1}}^{x_{j+1}} \big(\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k)\big)\,dx,\\
L_2^h(\varphi) &:= \sum_{j,k}\int_{x_{j-1}}^{x_{j+1}} \big(\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k)\big)\,(\varphi(x,t_k)-\varphi_j^k)\,dx.
\end{aligned} \tag{9.38}$$

Here the same notations as in Proposition 9.5 are used, and $\mathbf{u}_-^k(x) := \mathbf{u}^h(x, t_k-)$ and $\varphi_j^k := \varphi(x_j, t_k)$.

Since each $\mathbf{u}^h(x,t)$ has compact support, we may substitute $(\eta,q)=(\eta_*,q_*)$ and $\varphi\equiv 1$ in the formulas (9.37)–(9.38) to obtain

$$\begin{aligned}
&\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}} (\eta_*(\mathbf{u}_-^k)-\eta_*(\mathbf{u}_j^k))\,dx + \int_0^T \sum_{\text{shocks } x(t)} \big(x'(t)\ [\eta_*](t)-[q_*](t)\big)\,dt\\
&\le \int_{\mathbb{R}} \eta_*(\mathbf{u}_0(x))\,dx,
\end{aligned} \tag{9.39}$$

while

$$\begin{aligned}
&\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}} (\eta_*(\mathbf{u}_-^k)-\eta_*(\mathbf{u}_j^k))\,dx\\
&=\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\int_0^1 (\mathbf{u}_-^k-\mathbf{u}_j^k)^\top\nabla^2\eta_*\Big(\mathbf{u}_j^k+\tau(\mathbf{u}_-^k-\mathbf{u}_j^k)\Big)(\mathbf{u}_-^k-\mathbf{u}_j^k)\,(1-\tau)\,d\tau dx,
\end{aligned} \tag{9.40}$$

where the summations are over all $k\le K$. In view of Proposition 9.5, the entropy inequality, $x'(t)\,[\eta_*](t)-[q_*](t)\ge 0$, is satisfied for the shocks. On the other hand, η_* is convex in the conservative variables (ρ, m). Estimates (9.39)–(9.40) yield

$$\int_0^T \sum_{\text{shocks } x(t)} \Big(x'(t)[\eta_*](t)-[q_*](t)\Big)\,dt \,\le\, C, \tag{9.41}$$

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\int_0^1 (\mathbf{u}_-^k-\mathbf{u}_j^k)^\top\nabla^2\eta_*\Big(\mathbf{u}_j^k+\tau(\mathbf{u}_-^k-\mathbf{u}_j^k)\Big)(\mathbf{u}_-^k-\mathbf{u}_j^k)\,(1-\tau)\,d\tau dx \,\le\, C. \tag{9.42}$$

Then we observe the following:

(i) For $1<\gamma\le 2$, the entropy η_* is uniformly convex so that the Hessian matrix $\nabla^2\eta_*$ is bounded below by a positive constant, which implies

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}} |\mathbf{u}_-^k-\mathbf{u}_j^k|^2\,dx \,\le\, C. \tag{9.43}$$

(ii) For $\gamma > 2$, the estimate (9.42) implies

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\left(\frac{\rho_-^k}{2}\big(\frac{m_-^k}{\rho_-^k}-\frac{m_j^k}{\rho_j^k}\big)^2+\int_0^1\frac{p'(\rho_j^k+\tau(\rho_-^k-\rho_j^k))}{\rho_j^k+\tau(\rho_-^k-\rho_j^k)}\,(1-\tau)\,d\tau(\rho_-^k-\rho_j^k)^2\right)dx\,\le\,C.$$

In view of the assumption (9.3), there exists $C_1 > 0$ depending on γ such that

$$\int_0^1\frac{p'(\rho_j^k+\tau(\rho_-^k-\rho_j^k))}{\rho_j^k+\tau(\rho_-^k-\rho_j^k)}\,(1-\tau)\,d\tau\,\ge\,C_1\,\min\big\{1,(\rho_-^k-\rho_j^k)^{\gamma-2}\big\},$$

which yields

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\left(\rho_-^k\big(\frac{m_-^k}{\rho_-^k}-\frac{m_j^k}{\rho_j^k}\big)^2+|\rho_-^k-\rho_j^k|^\gamma\right)dx\,\le\,C.$$

The Cauchy-Schwarz inequality implies

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\rho_-^k\big|\frac{m_-^k}{\rho_-^k}-\frac{m_j^k}{\rho_j^k}\big|\,dx\,\le\,C\,h^{-1/2},\tag{9.44}$$

and

$$\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}|\rho_-^k-\rho_j^k|\,dx\,\le\,C\,h^{1/\gamma-1}.\tag{9.45}$$

For any bounded set $\Omega\subset\mathbb{R}\times[0,T]$ and for any weak entropy pair (η,q), we deduce from (9.37), (9.38), (9.40)–(9.45), and Propositions 9.4 and 9.5 that, for any $\varphi\in\mathcal{C}_0(\Omega)$,

$$\begin{aligned}
&|M(\varphi)|=0,\\
&|S^h(\varphi)|\,\le\,C\,\|\varphi\|_{\mathcal{C}_0}\int_0^T\sum\{x'(t)[\eta_*]-[q_*]\}\,dt\,\le\,C\,\|\varphi\|_{\mathcal{C}_0(\Omega)},\\
&|L_1^h(\varphi)|\,\le\,C\,\|\varphi\|_{\mathcal{C}_0}\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}dx\int_0^1(\mathbf{u}_-^k-\mathbf{u}_j^k)^\top\nabla^2\eta_*(\mathbf{u}_j^k+\tau(\mathbf{u}_-^k-\mathbf{u}_j^k))(\mathbf{u}_-^k-\mathbf{u}_j^k)\,(1-\tau)\,d\tau\\
&\qquad\qquad\le\,C\,\|\varphi\|_{\mathcal{C}_0(\Omega)}.
\end{aligned}$$

Hence $|(M^h+S^h+L_1^h)(\varphi)|\,\le\,C\,\|\varphi\|_{\mathcal{C}_0}$, which yields a uniform bound in the space of bounded measures $\mathcal{M}(\Omega)$ for $M^h+S^h+L_1^h$, considered as a functional on the space of continuous functions:

$$\|M^h+S^h+L_1^h\|_{\mathcal{M}(\Omega)}\,\le\,C.$$

The embedding theorem $\mathcal{M}(\Omega)\overset{\text{compact}}{\hookrightarrow}W^{-1,q_0}(\Omega)$, $1<q_0<2$, yields that

$$M^h+S^h+L_1^h\quad\text{is a compact sequence in }W^{-1,q_0}(\Omega).\tag{9.46}$$

It remains to treat $L_2^h(\varphi)$. Let $\varphi\in\mathcal{C}_0^\alpha(\Omega)$, $\frac12<\alpha<1$. We distinguish two cases :

(i) For $1<\gamma\le 2$, we deduce from (9.43) that

$$\begin{aligned}
|L_2^h(\varphi)|\,&\le\,h^\alpha\,\|\varphi\|_{\mathcal{C}_0^\alpha}\sum_k\left(\sum_j\int_{x_{j-1}}^{x_{j+1}}|\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k)|^2\,dx\right)^{1/2}\\
&\le\,h^{\alpha-1/2}\|\nabla\eta\|_{L^\infty}\|\varphi\|_{\mathcal{C}_0^\alpha}\left(\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}|\mathbf{u}_-^k-\mathbf{u}_j^k|^2\,dx\right)^{1/2}\\
&\le\,C\,h^{\alpha-1/2}\|\varphi\|_{W_0^{1,p}(\Omega)},\quad\text{for all }p>\frac{2}{1-\alpha}.
\end{aligned}\tag{9.47}$$

(ii) For $\gamma > 2$, the estimates (9.44) and (9.45) yield

$$\begin{aligned} |L_2^h(\varphi)| &\le h^\alpha \, \|\nabla\eta\|_{L^\infty} \|\varphi\|_{C_0^\alpha} \sum_{j,k} \int_{x_{j-1}}^{x_{j+1}} \left(|\rho_-^k - \rho_j^k| + \rho_-^k \Big|\frac{m_-}{\rho_-^k} - \frac{m_j^k}{\rho_j^k}\Big| \right) dx \\ &\le C\, h^{\alpha+1/\gamma-1} \, \|\varphi\|_{C_0^\alpha(\Omega)}. \end{aligned} \tag{9.48}$$

The estimates (9.47) and (9.48) imply

$$\|L_2^h\|_{W^{-1,q_0}(\Omega)} \le C h^{\alpha_0} \longrightarrow 0, \quad \text{when } h \to 0, \;\text{ for } 1 < q_0 < \frac{2}{1+\alpha} < 2, \tag{9.49}$$

where $\alpha_0 = \max\{\alpha - 1/2, \alpha - 1 + 1/\gamma\}$. Finally, we combine (9.46) with (9.49) to obtain that

$$M^h + S^h + L_1^h + L_2^h \quad \text{is compact in} \quad W^{-1,q_0}(\Omega). \tag{9.50}$$

Since $0 \le \rho(x,t) \le C, |m(x,t)/\rho(x,t)| \le C$, we have that

$$M^h + S^h + L_1^h + L_2^h \quad \text{is bounded in} \quad W^{-1,r}(\Omega), r > 2. \tag{9.51}$$

The interpolation lemma in [96] (also see [39]), (9.50), and (9.51) imply that

$$M^h + S^h + L_1^h + L_2^h \quad \text{is compact in} \quad W^{-1,2}(\Omega),$$

which implies that

$$\partial_t \eta(\mathbf{u}^h) + \partial_x q(\mathbf{u}^h) \quad \text{is compact in} \quad W^{-1,2}(\Omega). \tag{9.52}$$

In view of Theorem 9.1 and (9.52), there exists a subsequence $\mathbf{u}^{h_\ell}(x,t)$ converging for almost every (x,t) to a limit function $(\rho, m) \in L^\infty$.

Now we check here that $\mathbf{u}(x,t) = (\rho, m)(x,t)$ is actually an entropy solution of the Cauchy problem (1.14) and (9.1). For any weak entropy pair (η, q) with convex η and for any nonnegative function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, we obtain from (9.37) and (9.38) that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \big(\eta(\mathbf{u}^h)\partial_t\varphi + q(\mathbf{u}^h)\partial_x\varphi\big)\, dxdt + \int_{\mathbb{R}} \eta(\mathbf{u}^h(x,0))\varphi(x,0)\, dx \\ &= S^h(\varphi) + \sum_{j,k} \varphi_j^k \int_{x_{j-1}}^{x_{j+1}} (\eta(\mathbf{u}_-^k) - \eta(\mathbf{u}_j^k))\, dx \\ &\quad + \sum_{j,k} \int_{x_{j-1}}^{x_{j+1}} (\varphi(x,t_k) - \varphi_j^k)\, (\eta(\mathbf{u}_-^k) - \eta(\mathbf{u}_j^k))\, dx. \end{aligned} \tag{9.53}$$

Since η is a convex function, it satisfies the entropy inequality so that

$$S^h(\varphi) \ge 0, \tag{9.54}$$

and

$$\begin{aligned} &\sum_{j,k} \varphi_j^k \int_{x_{j-1}}^{x_{j+1}} (\eta(\mathbf{u}_-^k) - \eta(\mathbf{u}_j^k))\, dx \\ &= \sum_{j,k} \varphi_j^k \int_{x_{j-1}}^{x_{j+1}} \int_0^1 (\mathbf{u}_-^k - \mathbf{u}_j^k)^\top \nabla^2\eta(\mathbf{u}_j^k + \tau(\mathbf{u}_-^k - \mathbf{u}_j^k))(\mathbf{u}_-^k - \mathbf{u}_j^k)\,(1-\tau)\, d\tau dx \;\ge\; 0. \end{aligned} \tag{9.55}$$

Furthermore, for $1 < \gamma \le 2$, one has

$$\begin{aligned} &|\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}(\varphi(x,t_k)-\varphi_j^k)(\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k))\,dx| \\ &\le Ch^{1/2}\|\varphi\|_{\mathcal{C}_0^1}\left(\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}|\mathbf{u}_-^k-\mathbf{u}_j^k|^2\,dx\right)^{1/2}. \end{aligned}$$

Thus, when $h \to 0$,

$$|\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}(\varphi(x,t_k)-\varphi_j^k)(\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k))\,dx| \;\le\; C\,h^{1/2}\;\to 0. \tag{9.56}$$

For $\gamma > 2$, (9.43) and (9.44) imply

$$\begin{aligned} &|\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}(\varphi-\varphi_j^k)(\eta(\mathbf{u}_-^k)-\eta(\mathbf{u}_j^k))\,dx| \\ &\le C\,h\|\varphi\|_{\mathcal{C}_0^1}\sum_{j,k}\int_{x_{j-1}}^{x_{j+1}}\left(|\rho_-^k-\rho_j^k|+\rho_-^k|\frac{m_-^k}{\rho_-^k}-\frac{m_j^k}{\rho_j^k}|\right)dx \le C\,\|\varphi\|_{\mathcal{C}_0^1}\,h^{1/\gamma}\;\to 0. \end{aligned} \tag{9.57}$$

Since $\left|\frac{m^{h_\ell}(x,t)}{\rho^{h_\ell}(x,t)}\right| \le C$ and $(\rho^{h_\ell}, m^{h_\ell})(x,t) \longrightarrow (\rho,m)(x,t)$ for almost every (x,t), we have $0 \le \rho(x,t) \le C$ and $\frac{|m(x,t)|}{\rho(x,t)} \le C$ almost everywhere. We also conclude from (9.53)–(9.57) that $\mathbf{u}(x,t) = (\rho,m)(x,t)$ satisfies the entropy inequality

$$\int_0^\infty\int_{\mathbb{R}}(\eta(\mathbf{u})\partial_t\phi+q(\mathbf{u})\partial_x\phi)\,dxdt+\int_{\mathbb{R}}\eta(\mathbf{u}_0(x))\,\phi(x,0)\,dx\;\ge\;0,$$

for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}\times[0,\infty))$. This completes the proof of Theorem 9.2. □

Remark 9.2. The convergence for the γ-law case was first proved by Ding-Chen-Luo [96, 97] and by Chen [37]. The proof presented above for the general pressure law basically follows [96, 97, 37] with some simplifications and modifications. We refer to Tadmor [296] for further discussions on various approximate solutions of nonlinear conservation laws and related equations.

9.5. **Existence and Compactness of Entropy Solutions.**

Theorem 9.3 (Existence and Compactness)**.** *Assume that the initial data* $(\rho_0, m_0)(x)$ *satisfy* (9.36). *Assume that System* (1.14) *satisfies* (6.22) *and* (9.3). *Then*

(i) *There exists an entropy solution* $(\rho,m)(x,t)$ *of the Cauchy problem* (1.14) *and* (9.1), *in the sense of Definition 9.1, globally defined in time.*

(ii) *The solution operator* $(\rho,m)(\cdot,t) = S_t(\rho_0,m_0)(\cdot)$, *defined in Definition 9.1, is compact in* L^1_{loc} *for* $t > 0$.

Proof. The existence is a direct corollary of Theorem 9.2. Now we prove the compactness.

Consider any (oscillatory) sequence of initial data $(\rho_0^\epsilon, m_0^\epsilon)(x)$, $\epsilon > 0$, satisfying

$$0 \le \rho_0^\epsilon(x) \le C_0, \quad |m_0^\epsilon(x)| \le C_0\,\rho_0^\epsilon(x), \tag{9.58}$$

with $C_0 > 0$ independent of $\epsilon > 0$. Then there exists $C > 0$ independent of $\epsilon > 0$ such that the corresponding sequence $(\rho^\epsilon, m^\epsilon)(x,t)$, determined by Theorem 9.2, satisfies

$$0 \le \rho^\epsilon(x,t) \le C, \quad |m^\epsilon(x,t)| \le C\,\rho^\epsilon(x,t).$$

Since $(\rho^\epsilon, m^\epsilon)(x,t)$ are entropy solutions satisfying

$$\partial_t\eta(\rho^\epsilon,m^\epsilon)+\partial_x q(\rho^\epsilon,m^\epsilon) \le 0$$

in the sense of distributions, for any C^2 convex weak entropy pair (η, q), we deduce from the Murat Lemma (see Murat [237] or [39] for the details) that

$$\partial_t \eta(\rho^\epsilon, m^\epsilon) + \partial_x q(\rho^\epsilon, m^\epsilon) \qquad \text{is compact in} \qquad H^{-1}_{loc}(\mathbb{R}^2_+),$$

for any weak entropy pair (η, q), not necessarily convex. Combining with Theorem 9.1 yields that $(\rho^\epsilon, m^\epsilon)(x,t)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$, which implies our conclusion. □

Remark 9.3. The existence and compactness of entropy solutions for the fluids obeying the γ-law, the case $\gamma = \frac{N+2}{N}, N \geq 5$ odd, was treated by DiPerna [106], the case $1 < \gamma \leq 5/3$ by Ding-Chen-Luo [96] and Chen [37], the case $\gamma \geq 3$ by Lions-Perthame-Tadmor [203], and then $1 < \gamma < 3$ by Lions-Perthame-Souganidis [202]. For the more general pressure law (6.22) and (9.3), Theorem 9.3 is due to Chen-LeFloch [57, 59].

Remark 9.4. Notice that Greenberg and Rascle [144] found an interesting nonlinear system with only C^1 (but not C^2) flux function admitting time-periodic and space-periodic solutions, which indicates that the compactness and asymptotic decay of entropy solutions are sensitive with respect to the smoothness of the flux functions. However, Theorem 9.3 shows that, although the flux-function of System (1.14) is only Lipschitz continuous, the entropy solution operator is still compact in L^1_{loc} for this system.

9.6. **Decay of Periodic Entropy Solutions.** Now we show the large-time decay of periodic entropy solutions in L^∞ in the sense of Definition 9.1, established in Chen-Frid [46].

Theorem 9.4 (Decay)**.** *Consider the Cauchy problem* (1.14) *and* (9.1) *satisfying* (6.22) *and* (9.3). *Let* $(\rho, m) \in L^\infty(\mathbb{R}^2_+)$ *be its periodic entropy solution with period* $[0, a]$. *Then* $(\rho, m)(x,t)$ *asymptotically decays as* $t \to \infty$*:*

$$ess \lim_{t\to\infty} \int_0^a \big(|\rho(x,t) - \bar\rho| + |m(x,t) - \bar m|\big)\, dx = 0,$$

where $(\bar\rho, \bar m) := \frac{1}{a}\int_0^a (\rho_0, m_0)(x) dx$.

Proof. We divide the proof into four steps:
Step 1. Set

$$\mathbf{u}^\epsilon(x,t) = (\rho^\epsilon, m^\epsilon)(x,t) := (\rho, m)(x/\epsilon, t/\epsilon).$$

Then $\mathbf{u}^\epsilon(x,t)$ is a sequence of entropy solutions with oscillatory initial data. Theorem 9.3 implies the compactness of $\mathbf{u}^\epsilon(x,t)$ in $L^1_{loc}(\mathbb{R}^2_+)$. Therefore, there exists a subsequence (still denoted) $\mathbf{u}^\epsilon(x,t)$ converging to some function $\overline{\mathbf{u}}(x,t) \in L^\infty(\mathbb{R}^2_+)$ in $L^1_{loc}(\mathbb{R}^2_+)$. We conclude that $\overline{\mathbf{u}}(x,t) = \overline{\mathbf{u}}(t)$ from the periodicity of $\mathbf{u}^\epsilon(x,t)$.

Now, writing the equation of $\mathbf{u}^\epsilon(x,t)$ in the weak integral form and setting $\epsilon \to 0$, we can check that

$$\partial_t \overline{\mathbf{u}}(t) = 0$$

in the sense of distributions. This implies from the periodicity of $\mathbf{u}_0(x)$ that $\overline{\mathbf{u}}(t) = \overline{\mathbf{u}} := \frac{1}{a}\int_0^a \mathbf{u}_0(x)dx = w^* - lim_{\epsilon\to 0}\, \mathbf{u}_0(x/\epsilon)$.

Since the limit is unique, the whole sequence $\mathbf{u}^\epsilon(x,t)$ strongly converges to $\overline{\mathbf{u}}$ in $L^1_{loc}(\mathbb{R}^2_+)$ when $\epsilon \to 0$. Therefore, we have

$$\int_0^1 \int_{|x|\leq rt} |\mathbf{u}^\epsilon(x,t) - \overline{\mathbf{u}}| dxdt \to 0, \qquad \text{when } \epsilon \to 0. \tag{9.59}$$

Step 2. Define the following quadratic entropy-entropy flux pairs:

$$\begin{cases} \alpha(\mathbf{u}, \bar{\mathbf{u}}) = \eta_*(\mathbf{u}) - \eta_*(\bar{\mathbf{u}}) - \nabla\eta_*(\bar{\mathbf{u}}) \cdot (\mathbf{u} - \bar{\mathbf{u}}), \\ \beta(\mathbf{u}, \bar{\mathbf{u}}) = q_*(\mathbf{u}) - q_*(\bar{\mathbf{u}}) - \nabla\eta_*(\bar{\mathbf{u}}) \cdot (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\bar{\mathbf{u}})), \end{cases} \tag{9.60}$$

where (η_*, q_*) is the special entropy-entropy flux pair with convex η_*, defined in (9.6). Then the periodic entropy solution $\mathbf{u}(x,t)$ satisfies the entropy inequality

$$\partial_t \alpha(\mathbf{u}, \bar{\mathbf{u}}) + \partial_x \beta(\mathbf{u}, \bar{\mathbf{u}}) \le 0$$

in the sense of distributions. It follows that there exists $\mathcal{T} \subset \mathbb{R}_+$ with $meas(\mathcal{T}) = 0$ such that, for any $q \in \mathbb{R}$,

$$\int_q^{q+a} \alpha(\mathbf{u}(x,t_2), \bar{\mathbf{u}})dx \le \int_q^{q+a} \alpha(\mathbf{u}(x,t_1), \bar{\mathbf{u}})dx, \tag{9.62}$$

for all $0 \le t_1 < t_2, t_1, t_2 \notin \mathcal{T}$.

Step 3. Given $t > 0, t \notin \mathcal{T}$, we take all the rectangles given by $x \in [q, q+a]$, for q integer, and $s \in [[rt]/(2r), t]$, in the interior of the cone $\{|x| \le rs \;:\; 0 \le s \le t\}$ ($[a]$ is the largest integer less than or equal to a). The number of such rectangles is larger than $[rt]$. Using the periodicity of $u(x,\cdot)$ in x and the inequality (9.62) with $t_2 = t$, which holds for a.e. $t_1 = s \in (0,t)$ over the period $[q, q+a]$, we obtain that there exist $c_0 > 0, C > 0$, independent of t, such that

$$\begin{aligned} c_0 \int_0^a \alpha(\mathbf{u}(x,t), \bar{\mathbf{u}})dx &\le \frac{[rt]}{t^2} \int_{\frac{[rt]}{2r}}^t \int_0^a \alpha(\mathbf{u}(x,t), \bar{\mathbf{u}})dxds \\ &\le \frac{[rt]}{t^2} \int_{\frac{[rt]}{2r}}^t \int_0^a \alpha(\mathbf{u}(x,s), \bar{\mathbf{u}})dxds \\ &\le \frac{1}{t^2} \int_0^t \int_{|x| \le rs} \alpha(\mathbf{u}(x,s), \bar{\mathbf{u}})dxds \\ &\le C \int_0^1 \int_{|x| \le rs} |\mathbf{u}^\epsilon(x,s) - \bar{\mathbf{u}}|dxds \to 0, \qquad \epsilon = \frac{1}{t} \to 0. \end{aligned}$$

That is,

$$ess \lim_{t\to\infty} \int_0^a \int_0^1 (1-\tau)(\mathbf{u}(x,t) - \bar{\mathbf{u}})^\top \nabla^2 \eta_*(\bar{\mathbf{u}} + \tau(\mathbf{u}(x,t) - \bar{\mathbf{u}}))(\mathbf{u}(x,t) - \bar{\mathbf{u}})d\tau dx = 0. \tag{9.63}$$

Step 4. We observe the following:

(a). For $1 < \gamma \le 2$, the entropy η_* is uniformly convex, and then (9.63) is equivalent to

$$ess \lim_{t\to\infty} \int_0^a |\mathbf{u}(x,t) - \bar{\mathbf{u}}|^2 dx = 0. \tag{9.64}$$

(b). For $\gamma > 2$, (9.63) implies that

$$ess \lim_{t\to\infty} \int_0^a \left(\rho(x,t)(\frac{m(x,t)}{\rho(x,t)} - \frac{\bar{m}}{\bar{\rho}})^2 + |\rho(x,t) - \bar{\rho}|^\gamma \right) dx = 0. \tag{9.65}$$

Note by Hölder's inequality that

$$|m - \bar{m}|^2 \le C\left(\rho(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}})^2 + (\rho - \bar{\rho})^2 \right), \quad \int_\alpha^\beta |\rho - \bar{\rho}|^2 dx \le C\left(\int_\alpha^\beta |\rho - \bar{\rho}|^\gamma dx \right)^{1/2}. \tag{9.66}$$

We conclude from (9.65), (9.66), and the uniform boundedness of the solution $(\rho, m)(x,t)$ that

$$ess \lim_{t\to\infty} \int_0^a (|\rho(x,t) - \bar{\rho}| + |m(x,t) - \bar{m}|)\, dx = 0. \tag{9.67}$$

Combining (9.64) with (9.67) leads to the completion of the proof. □

Remark 9.5. Theorem 9.4 indicates that periodic entropy solutions asymptotically decay to the unique constant state, determined solely by the initial data.

Remark 9.6. Although the proof above for L^∞ entropy solutions only in the one-dimensional case, the argument applies to any space dimension and may extend to entropy solutions in any L^p for $p \geq 1$. See Chen-Frid [46].

9.7. **Stability of Rarefaction Waves and Vacuum States.** We now consider the global stability of rarefaction waves in a broad class of entropy solutions in L^∞ containing the vacuum states for (1.14). Rarefaction waves are the only case that may produce a vacuum state in later time in the Riemann solutions when the Riemann initial data stay away from the vacuum.

In §6.3, we have discussed the global solvability of the Riemann problem (1.14) and (6.21). Now we show the global stability of rarefaction waves in the following broader class of entropy solutions of (1.14) and (9.1) containing the vacuum states.

Definition 9.2. *A bounded measurable function* $\mathbf{u}(x,t) = (\rho, m)(x,t)$ *is an entropy solution of* (1.14) *and* (9.1) *in* $\mathbb{R}^2_+$ *if* $\mathbf{u}(x,t)$ *satisfies the following:*

(i) *There exists a constant* $C > 0$ *such that*

$$0 \leq \rho(x,t) \leq C, \qquad |m(x,t)/\rho(x,t)| \leq C;$$

(ii) $\mathbf{u}(x,t)$ *satisfies the equations in* (1.14) *and one physical entropy inequality in the sense of distributions in* $\mathbb{R}^2_+$*, i.e., for any nonnegative function* $\phi \in C^1_0(\mathbb{R}^2_+)$,

$$\int_0^\infty \int_{-\infty}^\infty (\eta(\mathbf{u})\partial_t \phi + q(\mathbf{u})\partial_x \phi)\, dx\, dt + \int_{-\infty}^\infty \eta(\mathbf{u}_0)(x)\phi(x,0)dx \geq 0, \tag{9.68}$$

for $(\eta, q) = \pm(\rho, m), \pm(m, \frac{m^2}{2\rho} + p(\rho)), (\eta_*, q_*)$, *where* (η_*, q_*) *is the mechanical energy-energy flux pair defined in* (9.6).

Remark 9.7. In Definition 9.2, we require that the entropy solutions in L^∞ satisfy solely one physical entropy inequality, besides the equations (1.14), thus admitting a broader class than the usual class of entropy solutions in L^∞ that satisfy all weak Lax entropy inequalities (compare with Definition 9.1).

Remark 9.8. For the Cauchy problem (1.14), (9.1), (6.22), and (9.3), there exists a global entropy solution satisfying all weak Lax entropy inequalities (see Theorem 9.3).

The following theorem is taken from Chen [42].

Theorem 9.5. *Let* $\mathbf{R}(x/t)$ *be the Riemann solution of* (1.14), (6.21), (6.22), *and* (9.2), *consisting of one or two rarefaction waves, constant states, and possible vacuum states, as constructed in §6.3. Let* $\mathbf{u}(x,t)$ *be any entropy solution of* (1.14), (6.22), (9.1), *and* (9.2) *in* $\mathbb{R}^2_+$ *in the sense of Definition 9.2. Then, for any* $L > 0$,

$$\int_{|x|\leq L} \alpha(\mathbf{u}, \mathbf{R})(x,t)\, dx \leq \int_{|x|\leq L+Nt} \alpha(\mathbf{u}_0, \mathbf{R}_0)(x)\, dx, \tag{9.69}$$

where $N > 0$ *depends only on* $C > 0$ *in Definition 9.2 and is independent of* t, *and*

$$\alpha(\mathbf{u}, \mathbf{R}) \equiv (\mathbf{u} - \mathbf{R})^\top \left(\int_0^1 \nabla^2 \eta_*(\mathbf{R} + \tau(\mathbf{u} - \mathbf{R}))d\tau \right) (\mathbf{u} - \mathbf{R}) > 0,$$

if $\mathbf{u} \neq \mathbf{R}$ *and both stay away from the vacuum.*

In particular, if $\mathbf{u}_0(x) = \mathbf{R}_0(x)$ *a.e., then* $\mathbf{u}(x,t) = \mathbf{R}(x/t)$ *a.e..*

Proof. Without loss of generality, we prove the assertion only for the Riemann solution (6.29) which consists of two rarefaction waves with vacuum states as intermediate states. The other cases can be proved similarly. The proof is based on normal traces and the generalized Gauss-Green theorem for divergence-measure vector fields in L^∞, established in Chen-Frid [50, 51], and the techniques developed in [47, 49, 104] for strictly hyperbolic systems. One of the new difficulties here is that strict hyperbolicity fails at the vacuum, yielding singular derivatives of the mechanical energy at the vacuum, which is absent in the strictly hyperbolic case. Another difficulty is that the entropy solutions are only in L^∞.

Step 1. Denote $\mathbf{u} = (\rho, m)$ and $\mathbf{R} = (\bar\rho, \bar m)$. First we renormalize the mechanical energy-energy flux pair in (9.6) as in (9.60) and consider

$$\mu = \partial_t \alpha(\mathbf{u}(x,t), \mathbf{R}(x/t)) + \partial_x \beta(\mathbf{u}(x,t), \mathbf{R}(x/t)), \;\; d = \partial_t \eta_*(\mathbf{u}(x,t)) + \partial_x q_*(\mathbf{u}(x,t)).$$

Since $\mathbf{u}(x,t)$ is an entropy solution, $\mu \le 0$ in any region in which $\mathbf{R}$ is constant and $\mu \le 0$, in the sense of distributions. Then μ and d are Radon measures, and $(q_*(\mathbf{u}), \eta_*(\mathbf{u}))(x,t)$ and $(\beta(\mathbf{u}(x,t), \mathbf{R}(x/t)), \alpha(\mathbf{u}(x,t), \mathbf{R}(x/t)))$ are divergence-measure vector fields on $\mathbb{R}^2_+$.

Step 2. Let

$$\Omega_1 := \{\, (x,t) \;:\; \lambda_1(\mathbf{u}_-) < x/t < v_{c_1},\; t>0 \,\},\; \Omega_2 := \{\, (x,t) \;:\; v_{c_2} < x/t < \lambda_2(\mathbf{u}_+),\; t>0 \,\},$$

the rarefaction wave regions of the Riemann solution, and

$$\Omega_0 := \{(x,t) \;:\; v_{c_1} < x/t < v_{c_2},\, t > 0\,\}$$

the vacuum region.

Over the regions $\Omega_j, j=1,2$,

$$\mu = \partial_t \alpha(\mathbf{u}, \mathbf{R}) + \partial_x \beta(\mathbf{u}, \mathbf{R}) = d - (\partial_x \mathbf{R})^\top \nabla^2 \eta_*(\mathbf{R}) Q\mathbf{f}(\mathbf{u}, \mathbf{R}), \tag{9.71}$$

where $Q\mathbf{f}(\mathbf{u}, \mathbf{R}) = \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{R}) - \nabla \mathbf{f}(\mathbf{R})(\mathbf{u} - \mathbf{R})$, and we used the fact that $\nabla^2 \eta_* \nabla \mathbf{f}$ is symmetric. Recall that, for $(x,t) \in \Omega_j$,

$$\partial_x \mathbf{R}(x/t) = \frac{1}{t} \mathbf{r}_j(\mathbf{R}(x/t)), \qquad j = 1, 2. \tag{9.72}$$

Then, by (9.71) and (9.72), for any Borel set $E \subset \Omega_j$, $j=1,2$, we have

$$\mu(E) = d(E) - \int_E \frac{1}{t} \mathbf{r}_j(\mathbf{R})^\top \nabla^2 \eta_*(\mathbf{R}) Q\mathbf{f}(\mathbf{u}, \mathbf{R})(x,t)\, dxdt. \tag{9.73}$$

Over the vacuum region $\Omega_0, \bar\rho(x,t) = 0$, we may choose the velocity

$$\bar v(x,t) = x/t, \quad v_{c_1} < x/t < v_{c_2}.$$

Then a careful calculation yields

$$\mu = d - \frac{1}{t}\left(\rho \left(\bar v - \frac{m}{\rho}\right)^2 + p(\rho)\right),$$

which implies that, for any Borel set $E \subset \Omega_0$,

$$\mu(E) = d(E) - \int_E \frac{1}{t}\left(\rho \left(\bar v - \frac{m}{\rho}\right)^2 + p(\rho)\right)(x,t) dxdt. \tag{9.74}$$

Step 3. For any $\delta > 0$, denote

$$\ell_1^\delta(t) = \{x/s = \lambda_1(\mathbf{u}_L),\; \delta < s < t\}, \qquad \ell_2^\delta(t) = \{x/s = v_{c_1},\; \delta < s < t\},$$
$$\ell_3^\delta(t) = \{x/s = v_{c_2},\; \delta < s < t\}, \qquad \ell_4^\delta(t) = \{x/s = \lambda_2(\mathbf{u}_R),\; \delta < s < t\}.$$

Then

$$\mu\{\ell_j^\delta(t)\} = d\{\ell_j^\delta\} \le 0, \qquad j = 1,2,3,4. \tag{9.75}$$

Step 4. For any $L > 0$, let $\Pi^\delta_{L,t}$ denote the region $\{\, (x,s) \;:\; |x| < L + M(t-s),\, 0 < \delta < s < t\}$ and $\Omega_j^\delta(t) = \Omega_j \cap \Pi^\delta_{L,t}$, $\Omega_j(t) = \Omega_j \cap \{(x,s) \;:\; 0 < s < t\}, j = 0,1,2$, where

$$M \ge M_0 := \|\beta(\mathbf{u}, \mathbf{R})/\alpha(\mathbf{u}, \mathbf{R})\|_{L^\infty(\mathbb{R}^2_+)}.$$

First, by the entropy inequality (9.68), the Gauss-Green formula for divergence-measure vector fields in [50], and the convexity of $\eta_*(\mathbf{u})$ in $\mathbf{u}$, it is standard (cf. [63]) to deduce that any entropy solution defined in Definition 9.2 assumes its initial data $\mathbf{u}_0(x)$ strongly in L^1_{loc}:

$$\lim_{t\to 0}\int_{|x|\le K}|\mathbf{u}(x,t)-\mathbf{u}_0(x)|dx=0,\quad \text{for any } K>0. \tag{9.76}$$

Furthermore, we apply normal traces and the Gauss-Green formula for divergence-measure vector fields in [50] to conclude again

$$\mu\{\Pi^\delta_{t,L}\}=\int_{|x|\le L}\alpha(\mathbf{u}(x,t),\mathbf{R}(x/t))\,dx-\int_{|x|\le L+M(t-\delta)}\alpha(\mathbf{u}(x,\delta),\mathbf{R}(x/\delta))\,dx$$
$$+\int_{\partial\Pi^\delta_{t,L}}(\beta,\alpha)\cdot\nu\,d\sigma,$$

where ν is the unit outward normal field and σ is the boundary measure. Then we can choose $M\ge M_0$ such that $\int_{\partial\Pi^\delta_{t,L}}(\beta,\alpha)\cdot\nu\,d\sigma\ge 0$. Therefore, we have

$$\mu\{\Pi^\delta_{t,L}\}\ge\int_{|x|\le L}\alpha(\mathbf{u}(x,t),\mathbf{R}(x/t))\,dx-\int_{|x|\le L+M(t-\delta)}\alpha(\mathbf{u}(x,\delta),\mathbf{R}(x/\delta))\,dx. \tag{9.77}$$

On the other hand, since $\mathbf{R}(x/t)$ is constant in each component of $\Pi_t-\cup_{j=0}^2\Omega^\delta_j(t)-\cup_{j=1}^4\ell^\delta_j(t)$ and $d\le 0$, we have

$$\mu\{\Pi^\delta_{t,L}\}\le-\sum_{j=1}^2\int_{\Omega^\delta_j(t)}\frac{1}{s}\mathbf{r}_j(\mathbf{R})^\top\nabla^2\eta_*(\mathbf{R})Q\mathbf{f}(\mathbf{u},\mathbf{R})(x,s)\,dxds, \tag{9.78}$$

from (9.73)–(9.75).

Step 5. A careful direct calculation yields

$$\begin{aligned} h_j(x,s) &:= \mathbf{r}_j(\mathbf{R})^\top\nabla^2\eta_*(\mathbf{R})Q\mathbf{f}(\mathbf{u},\mathbf{R})(x,s)\\ &= \frac{2p'(\bar\rho)}{\bar\rho p''(\bar\rho)+2p'(\bar\rho)}\left(\rho\left(\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right)^2+p(\rho)-p(\bar\rho)-p'(\bar\rho)(\rho-\bar\rho)\right)(x,s)\ge 0,\end{aligned}$$

for $j=1,2$, since $p(\rho)$ is convex in $\rho\ge 0$. Also, from (6.22) and (9.2), we can see that $h_j(x,s), j=1,2$, are uniformly bounded everywhere, even near the vacuum, which means that $h_j(x,s), j=1,2$, are integrable in $\Omega_1(t)\cup\Omega_2(t)$ as $s>0$. This fact in combination with (9.77) and (9.78) yields

$$\int_{|x|\le L}\alpha(\mathbf{u}(x,t),\mathbf{R}(x/t))\,dx\le\int_{|x|\le L+M(t-\delta)}\alpha(\mathbf{u}(x,\delta),\mathbf{R}(x/\delta))\,dx. \tag{9.79}$$

Then (9.76) and (9.79) imply (9.69) as $\delta\to 0$. This completes the proof.

In the previous proof, the values of the divergence-measure field $(\beta(\mathbf{u},\mathbf{R}),\alpha(\mathbf{u},\mathbf{R}))(x,t)$ on the line segments in the (x,t)-plane should be understood in the sense of normal traces. If one wishes to forego the normalization of our solution through normal traces, then (9.69) should be considered to hold for almost all $t\in[0,\infty)$.

As a corollary, the following theorem in Chen [42] holds.

Theorem 9.6. *Let* $\mathbf{R}(x/t)$ *be the Riemann solution of* (1.14), (6.21), (6.22), *and* (9.2), *consisting of one or two rarefaction waves, constant states, and possibly vacuum states, as constructed above. Let* $\mathbf{u}(x,t)$ *be any entropy solution of* (1.14), (6.22), (9.1), *and* (9.2) *with initial data*

$$\mathbf{u}|_{t=0}=\mathbf{R}_0(x)+\mathbf{P}_0(x),$$

in the sense of Definition 9.2. Then $\mathbf{R}(x/t)$ *is asymptotically stable under the initial perturbation* $P_0(x) \in L^1 \cap L^\infty$ *in the sense of*

$$\lim_{t\to\infty} \int_{|\xi|\le L} \alpha(\mathbf{u}(\xi t, t), \mathbf{R}(\xi))d\xi = 0, \quad \text{for any } L > 0.$$

Remark 9.9. The analysis above has also been extended in Chen [42] to the system for non-isentropic fluids, which is more complicated. This has been achieved by identifying a good Lyapunov functional and making an appropriate choice of entropy functions. Also see Chen-Frid [51] for more recent results.

9.8. **Other Results.** Further results include the following.

Equations of Elasticity: Consider the equations in (1.15) with $p(\tau) = -\sigma(\tau)$, $\sigma'(\tau) > 0$. In elasticity, genuine nonlinearity is typically precluded by the fact that the medium in question can sustain discontinuities in both the compressive and expansive phases of the motion. In the simplest model for common rubber, one postulates that the stress σ, as a function of the strain τ, switches from concave in the compressive mode $\tau < 0$ to convex in the expansive mode $\tau > 0$, i.e.

$$sgn(\tau\sigma''(\tau)) > 0, \qquad \text{if} \quad \tau \neq 0. \tag{9.80}$$

In [105], DiPerna proved the existence of global entropy solutions in L^∞ of System (1.15) and (9.80). Also see Shearer [278], Lin [200], and Gripenberg [146]. As a corollary, the compactness and decay of global entropy solutions follows with the aid of the approach in Chen-Frid [46].

Euler Equations for Non-Isentropic Fluids: Consider the Euler equations for non-isentropic fluids in (1.13). Selecting (τ, v, S) as the state vector, we have the constitutive relations

$$(e, p, \theta) = (\hat{e}(\tau, S), \hat{p}(\tau, S), \hat{\theta}(\tau, S)) \tag{9.81}$$

satisfying the conditions

$$p = -\hat{e}_\tau, \qquad \theta = \hat{e}_S.$$

Under the standard assumptions $\hat{p}_v < 0$ and $\hat{\theta} > 0$, System (1.13) is strictly hyperbolic.

Consider the following class of constitutive relations

$$p = h(\tau - \alpha S), \quad e = \beta S - \int^w h(y)dy, \quad \theta = \alpha h(\tau - \alpha S) + \beta > 0, \tag{9.82}$$

where $\alpha, \beta, w = \tau - \alpha S$, and $h(w)$ is a smooth function with $h'(w) < 0$ satisfying

$$h''(w) - 4\frac{\alpha h'(w)^2}{\alpha h(w) + \beta} \begin{cases} > 0, & \text{if } w < \hat{w}, \\ < 0, & \text{if } w > \hat{w}. \end{cases} \tag{9.83}$$

The model (9.82) can be regarded as a "first-order correction" to the general constitutive relations (see [43]).

The existence and compactness of distributional entropy solutions for the Cauchy problem of (1.13) and (9.81)–(9.83) was established in Chen-Dafermos [43], and the decay of periodic solutions was established in Chen-Frid [46]. In particular, although the periodic solutions do not decay because of linear degeneracy of the system, several important physical quantities, including the velocity, the pressure, and the temperature, do asymptotically decay.

As for the Euler equation with more general constitutive relations, including those for polytropic gases with $p = (\gamma - 1)\rho e$, the problems of existence, compactness, and decay of entropy solutions with arbitrarily large initial data, beyond the BV theory, are still open.

10. Global Discontinuous Solutions V: the Multidmensional Case

In this section we discuss global discontinuous solutions for the multidimensional Euler equations for compressible fluids.

10.1. Multidimensional Euler Equations with Geometric Structure. We first discuss global solutions with geometric structure for the multidimensional Euler equations for isentropic gas dynamics in (1.9) and (1.10).

Consider spherically symmetric solutions outside a solid core:

$$\rho(\mathbf{x},t)=\rho(r,t), \quad \mathbf{m}(\mathbf{x},t)=m(r,t)\frac{\mathbf{x}}{r}, \quad r=|\mathbf{x}|\geq 1. \tag{10.1}$$

Then $(\rho,m)(r,t)$ is determined by the equations:

$$\begin{cases} \partial_t\rho+\partial_r m=-\frac{A'(r)}{A(r)}m, \\ \partial_t m+\partial_r(\frac{m^2}{\rho}+p(\rho))=-\frac{A'(r)}{A(r)}\frac{m^2}{\rho}, \end{cases} \tag{10.2}$$

subject to the Cauchy data:

$$(\rho,m)|_{t=0}=(\rho_0,m_0)(r), \quad r>1, \tag{10.3}$$

with homogeneous boundary condition:

$$m|_{r=1}=0, \tag{10.4}$$

where $A(r)=\frac{2\pi^{d/2}}{\Gamma(d/2)}r^{d-1}$ is the surface area of d-dimensional sphere.

Although System (10.2) is here presented in the context of spherically symmetric flow, the same system also describes many flows, important in physics, such as the transonic nozzle flow with variable cross-sectional area $A(r)\geq c_0>0$.

The eigenvalues of (10.2) are $\lambda_\pm=\frac{m}{\rho}\pm c=c(M\pm 1)$, where $c=\sqrt{p'(\rho)}$ is the sound speed and $M=\frac{m}{\rho c}$ is the Mach number. We notice that $\lambda_+-\lambda_-=2c(\rho)=2\rho^{\frac{\gamma-1}{2}}\to 0$ as $\rho\to 0$. On the other hand, the geometric source speed is zero, and the eigenvalues $\lambda_\pm$ are also zero near $M\approx\pm 1$, which indicates that there is also nonlinear resonance between the geometric source term and the characteristic modes.

The natural issues associated with this problem are: (a) whether the solution has the same geometric structure globally; (b) whether the solution blows up to infinity in a finite time, especially the density. These issues are not easily resolved through physical experiments or numerical simulations, especially the second one, due to the limited capacity of available instruments and computers. The central difficulty of this problem in the unbounded domain lies in the reflection of waves from infinity and their amplification as they move radially inwards. Another difficulty is that the associated steady-state equations change type from elliptic to hyperbolic at the sonic point; such steady-state solutions are fundamental building blocks in our approach.

Consider the steady-state solutions:

$$\begin{cases} m_r=-\frac{A'(r)}{A(r)}m, \\ (\frac{m^2}{\rho}+p(\rho))_r=-\frac{A'(r)}{A(r)}\frac{m^2}{\rho}, \\ (\rho,m)|_{r=r_0}=(\rho_0,m_0). \end{cases} \tag{10.5}$$

The first equation can be integrated directly to get

$$A(r)m=A(r_0)m_0. \tag{10.6}$$

The second equation can be rewritten as

$$(A(r)\frac{m^2}{\rho})_r + A(r)p(\rho)_r = 0.$$

Hence, using (10.6) and $\theta = \frac{\gamma-1}{2}$, we have

$$\rho^{2\theta}(\theta M^2 + 1) = \rho_0^{2\theta}(\theta M_0^2 + 1). \tag{10.7}$$

Then (10.6) and (10.7) become

$$\left(\frac{\rho}{\rho_0}\right)^{\theta+1} = \frac{A(r_0)M_0}{A(r)M}, \qquad \left(\frac{\rho}{\rho_0}\right)^{2\theta} = \frac{\theta M_0^2 + 1}{\theta M^2 + 1}. \tag{10.8}$$

Eliminating ρ in (10.8) yields

$$F(M) = \frac{A(r_0)}{A(r)} F(M_0), \tag{10.9}$$

where

$$F(M) = M\left(\frac{1+\theta}{1+\theta M^2}\right)^{\frac{\theta+1}{2\theta}}$$

satisfies

$$\begin{cases} F(0) = 0, F(1) = 1; \quad F(M) \to 0, \text{ when } M \to \infty; \\ F'(M)(1-M) > 0, \text{ when } M \in [0,\infty); \\ F'(M)(1+M) > 0, \text{ when } M \in (-\infty, 0]. \end{cases}$$

Thus we see that, if $A(r) < A(r_0)|F(M_0)|$, no smooth solution exists because the right-hand side of (10.9) exceeds the maximum values of $|F|$. If $A(r) > A(r_0)|F(M_0)|$, there are two solutions of (10.9), one with $|M| > 1$ and the other with $|M| < 1$, since the line $F = \frac{A(r_0)}{A(r)}F(M_0)$ intersects the graph of $F(M)$ at two points.

For $A'(r) = 0$, the system becomes the one-dimensional isentropic Euler equations, which have been discussed in Section 9.

For $A'(r) \neq 0$, the existence of global solutions for the transonic nozzle flow problem was obtained in Liu [207] by first incorporating the steady-state building blocks into the random choice method [130], provided that the initial data have small total variation and are bounded away from both sonic and vacuum states. A generalized random choice method was introduced to compute transient gas flows in a Laval nozzle in [129, 135]. A global entropy solution with spherical symmetry was constructed in [224] for $\gamma = 1$, and the local existence of such an entropy solution for $1 < \gamma \leq \frac{5}{3}$ was also discussed in [225]. Also see Liu [206, 207, 208], Glaz-Liu [129], Glimm-Marshall-Plohr [135], Embid-Goodman-Majda [109], and Fok [119].

In Chen-Glimm [53], a numerical shock capturing scheme was developed and applied for constructing global solutions of (1.9) and (1.10) with geometric structure and large initial data in L^∞ for $1 < \gamma \leq 5/3$, including both spherically symmetric flows and transonic nozzle flows. The case $\gamma \geq 5/3$ was treated in [66]. It was proved that the solutions do not blow up to infinity in a finite time. More precisely, the following theorem due to Chen-Glimm [53] holds:

Theorem 10.1. *There exists a family of approximate solutions* $(\rho^\epsilon, m^\epsilon)(r,t)$ *of* (10.2) *such that*

(i) $0 \leq \rho^\epsilon(r,t) \leq C$, $|\frac{m^\epsilon(r,t)}{\rho^\epsilon(r,t)}| \leq C$;

(ii) $\partial_t\eta(\rho^\epsilon, m^\epsilon) + \partial_r q(\rho^\epsilon, m^\epsilon)$ *is compact in* $H^{-1}_{\rm loc}(\Omega)$ *for any weak entropy pair* (η, q), *where* $\Omega \subset \mathbb{R}^2_+$ *or* $\Omega \subset (1,\infty)\times\mathbb{R}_+$.

Furthermore, there is a convergent subsequence $(\rho^{\epsilon_\ell}, m^{\epsilon_\ell})(r,t)$ *of approximate solutions* $(\rho^\epsilon, m^\epsilon)(r,t)$ *such that*

$$(\rho^{\epsilon_\ell}, m^{\epsilon_\ell})(r,t) \to (\rho, m)(r,t), \quad a.e.,$$

and the limit function $(\rho, m)(r,t)$ *is a global entropy solution of* (10.2) *with the assigned initial data in* L^∞ *and satisfies*

$$0 \le \rho(r,t) \le C, \quad |\frac{m(r,t)}{\rho(r,t)}| \le C.$$

Moreover, $(\rho, \mathbf{m})(\mathbf{x},t)$*, defined in* (10.1) *through* $(\rho, m)(r,t)$ *of* (10.2)–(10.4)*, is a global entropy solution of* (1.9) *and* (1.10) *with spherical symmetry outside the solid core for the initial data in* L^∞.

The approach in Chen-Glimm [53] for constructing the family of approximate solutions in Theorem 10.1 is to merge shock capturing ideas with the fractional-step techniques in order to develop first-order Godunov shock capturing schemes, replacing the usual piecewise constant building blocks by piecewise smooth ones. The main point is to use the steady-state solutions, which incorporate the main geometric source terms, in order to modify the wave strengths in the Riemann solutions. This construction yields better approximate solutions and permits a uniform L^∞ bound. There are two technical difficulties to achieve this, both due to transonic phenomena. The first one is that no smooth steady-state solution exists in each cell in general. This problem was solved by introducing a standing shock. The other is that the constructed steady-state solution in each cell must satisfy the following requirements:

(a). The oscillation of the steady-state solution around the Godunov value must be of the same order as the cell length so as to obtain the L^∞ estimate for the convergence arguments;

(b). The difference between the average of the steady-state solution over each cell and the Godunov value must be higher than first-order in the cell length in order to ensure the consistency of the corresponding approximate solutions with the Euler equations. That is,

$$\frac{1}{\Delta r}\int_{(j-\frac{1}{2})\Delta r}^{(j+\frac{1}{2})\Delta r} \mathbf{u}(r, k\Delta t - 0)dr = \mathbf{u}_j^k(1 + O(|\Delta r|^{1+\delta})), \quad \delta > 0.$$

These requirements are naturally satisfied by smooth steady-state solutions that stay away from the sonic state in the cell. The general case must include the transonic steady-state solutions. The sonic difficulty was overcome, as in experimental physics, by introducing an additional standing shock with continuous mass and by adjusting its left state and right state in the density and its location to control the growth of the density. These requirements can yield the H^{-1} compactness estimates for entropy dissipation measures

$$\partial_t \eta(\rho^\epsilon, m^\epsilon) + \partial_r q(\rho^\epsilon, m^\epsilon)$$

and the strong compactness of approximate solutions $(\rho^\epsilon, m^\epsilon)(r,t)$ with the aid of the compactness framework discussed in Section 9.3.

Remark 10.1. The above method has been applied to studying the Euler-Poisson equations for compressible fluids, which describe the dynamic behavior of many flows of physical importance including the propagation of electrons in submicron semiconductor devices and the biological transport of ions for channel proteins. See Chen-Wang [65] and the references cited therein.

Remark 10.2. Some related results for entropy solutions with symmetric structure can be found in [40, 76, 289, 322, 336, 337].

Remark 10.3. In the spherically symmetric problem, one of the main difficulties is from infinity because of the reflection of waves from infinity and their amplification as they move radially inwards; this difficulty has been overcome above. Another difficulty is the singularity of entropy solutions at the origin; it would be interesting to study the existence and behavior of spherically symmetric entropy solutions near the origin.

10.2. **The Multidimensional Riemann Problem.** The multidimensional Riemann problem is very important, since it serves as a building block and standard test model of mathematical theories and numerical methods for solving nonlinear systems of conservation laws, especially the Euler equations for compressible fluids, in any space dimension; and its solutions may also determine the large-time behavior of general entropy solutions. See Glimm-Majda [134], Chern-Glimm-McBryan-Plohr-Yaniv [69], Glimm-Klingenberg-McBryan-Plohr-Sharp [132], and Chen-Frid [47, 48, 49].

The elementary waves are the building blocks out of which a Riemann solution is constructed. The Riemann solution is characterized by invariance under scale transformations:

$$\mathbf{x} \to \alpha\mathbf{x}, \qquad t \to \alpha t, \qquad \alpha > 0,$$

while the elementary wave is invariant under additional symmetry: it moves as a travelling wave with a fixed velocity. The elementary waves for the Euler equations for a polytropic fluid were classified in the following theorem by Glimm-Klingenberg-McBryan-Plohr-Sharp [132].

Theorem 10.2. *Generally, the elementary waves for the Euler equations are one of the following simple types: cross, overtake, Mach triple point, diffraction, and transmission.*

Two-dimensional Riemann problems arise when one-dimensional waves cross or overtake one another or when these waves reflect from or interact with walls or boundaries. Generally, an interaction will arise when two waves meet or a single wave meets a boundary; it is such simple and generic problems which are fundamental. The following two problems have been studied extensively on the level of experiment and computation:

(a) the shock-wedge problem of reflection of a shock wave by a wedge in a shock tube (e.g. [323, 93, 134]);

(b) the shock diffraction problem of reflection and transmission of a shock wave by a contact surface (e.g. [1, 134]).

There are a series of topologically distinct patterns for the various reflected, transmitted and incident waves. Similar issues apply to the interior interaction of waves. Moreover, a two-dimensional Riemann problem can also be generated by the self-interactions of a single two-dimensional elementary wave. See Glimm [131] for more detailed discussions.

In Chang-Chen-Yang [31, 32], Kurganov-Tadmor [173], Lax-Liu [184], Schulz-Collins-Glaz [270], and Zhang-Zheng [335], the two-dimensional Riemann problem with the following form was analyzed for gas dynamics: The initial Riemann values are constant states in each quadrant of the (x, y)-plane, and the four initial constant states satisfy that each jump in the initial data away from the origin produces exactly one of planar forward shocks, backward shocks, forward centered rarefaction waves, backward centered rarefaction waves, or slip surfaces. It was shown that all possible wave combinations can be clarified into nineteen genuine different cases, and there may be some subcases in each case. For each case, numerical solutions of each subcase were illustrated by using various shock capturing methods, and the corresponding theoretical analyses were given by the method of characteristics.

In particular, in the case of the interaction of rarefaction waves propagating in the opposite direction, the numerical solutions clearly show that two compressive waves, even shock waves, appear in the solutions. This phenomenon can be explained as the effect of compression of the flow characteristics. The observation of the essential difference of two types of contact discontinuities distinguished by the sign of the vorticity yields two genuinely different cases for the interaction of four contact discontinuities. For one case, the four contact discontinuities roll up and generate a vortex, and the density monotonically decreases along the stream curves. For the other, two shock waves are formed; and, in the subsonic region between two shock waves, a new kind of nonlinear hyperbolic waves appears, called smoothed Delta-shock waves, in the compressible Euler flow, which were first observed by Chang-Chen-Yang [31, 32]. The formation of Delta-shocks and the phenomena of concentration and cavitation in the vanishing pressure limit have been rigorously analyzed in Chen-Liu [54].

In general, the solution structures of the Riemann problem are extremely complicated. The following four numerical examples show the complexity of the density contour curves for different interactions of elementary waves in the Riemann problem. Figs. 10.1 and 10.2 were taken from Lax-Liu [184]; and Figs. 10.3 and 10.4 from Kurganov-Tadmor [173], respectively.

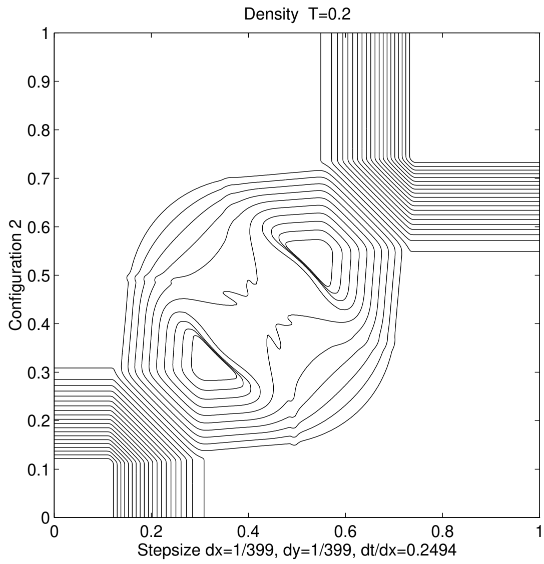

Figure 10.1: Interaction of Four Rarefaction Waves

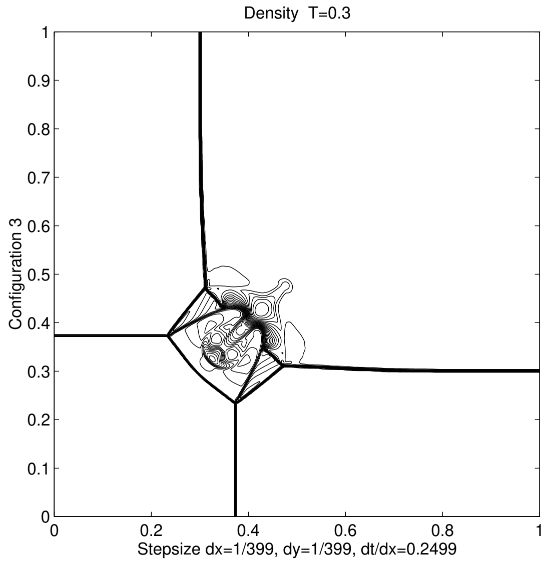

Figure 10.2: Interaction of Four Shock Waves

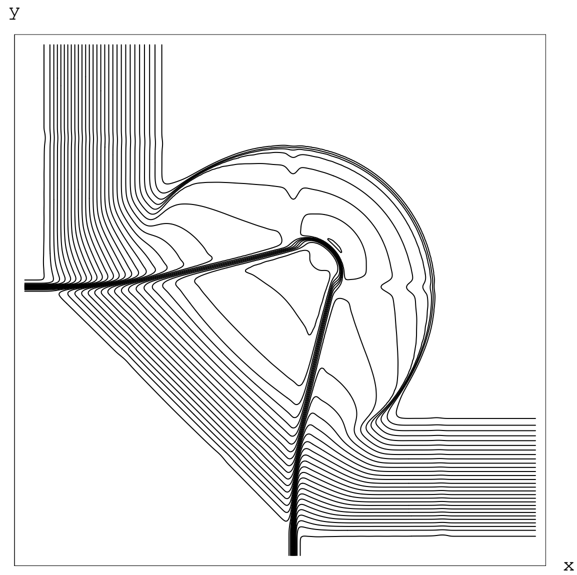

Figure 10.3: Interaction of Two Rarefaction Waves and Two Contact Discontinuities

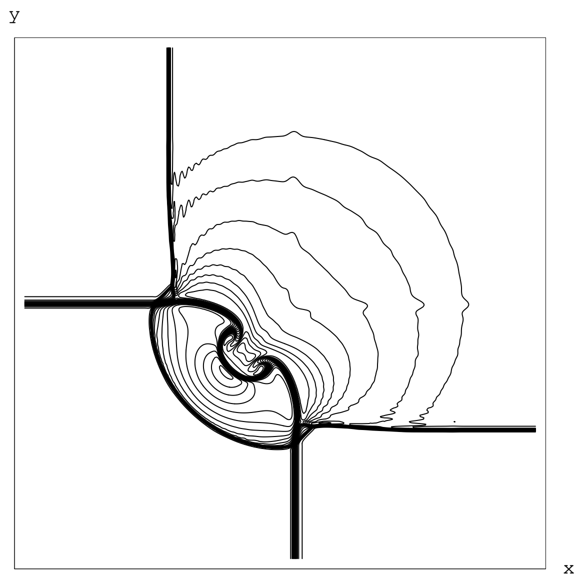

Figure 10.4: Interaction of Two Shock Waves and Two Contact Discontinuities

More analytical and numerical results about the Riemann problem and shock reflection problems for the isentropic or non-isentropic Euler equations can be found in

AbdElFattah-Henderson-Lozzi [1], Chang-Chen [30], Chang-Hsiao [33], Deschambault-Glass [93], Gamba-Rosales-Tabak [127], Glimm-Majda [134], Harabetian [151], Hunter-Brio [156], Hunter-Keller [157], Keller-Blank [168], Li-Zhang-Yang [192], Lighthill [196], Serre [274, 275], Tabak-Rosales [295], Tesdall-Hunter [305], Woodward-Colella [323], Zakharian-Brio-Hunter-Webb [332], Zheng [338], and the references cited therein.

For potential compressible fluid flows, recent mathematical efforts have been made to establish the existence and behavior of solutions. See Chen [67, 68], Chen-Feldman [45], Gamba-Morawetz [126], Gu [147], Li [193], Lien-Liu [199], Morawetz [230, 232], Zheng [333], and the references cited therein. A related model, called the unsteady transonic small disturbance (UTSD) equations, has been analyzed in Canic-Keyfitz-Lieberman [27] and Canic-Keyfitz-Kim [28].

For the Euler equations for pressureless, isentropic fluids, global Riemann solutions are now well-understood. We refer the reader to Bouchut-James [15], Chen-Liu [54], Ding-Wang [98], Grenier [145], LeFloch [186], Poupaud-Rascle [260], Sheng-Zhang [280], Tan-Zhang [298], Yang-Huang [326], and the references cited therein. Also see E-Khanin-Mazel-Sinai [107] and E-Rykov-Sinai [108] for the effects of random initial data and stochastic forcing.

For the multidimensional Riemann problem for Hamilton-Jacobi equations, we refer the reader to Glimm-Kranzer-Tan-Tangerman [136], Bardi-Osher [8], and the references cited therein.

11. Euler Equations for Compressible Fluids with Source Terms

In §2 – §10, we have discussed the Cauchy problem for the Euler equations for equilibrium, compressible fluids. In this section, we discuss two of the most important examples for the Euler equations for compressible fluids with source terms: Relaxation and combustion.

11.1. **Euler Equations with Relaxation.** The Euler equations with relaxation in (1.1), (1.21), and (1.22) fit into a general setting of hyperbolic systems of conservation laws in the form:

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}(\mathbf{U}) + \frac{1}{\epsilon}\mathbf{S}(\mathbf{U}) = 0, \quad x \in \mathbb{R}^d, \tag{11.1}$$

where $\mathbf{U} = \mathbf{U}(\mathbf{x}, t) \in \mathbb{R}^N$ represents the density vector of basic physical variables, and ϵ is the relaxation time, which is very short. It is assumed that the system is hyperbolic, and the relaxation term $\mathbf{S}(\mathbf{U})$ is endowed with an $n \times N$ constant matrix $\mathbf{Q}$ with rank $n < N$ such that $\mathbf{QS}(\mathbf{U}) = 0$. This yields n independent conserved quantities $\mathbf{u} = \mathbf{QU}$. In addition, it is assumed that each $\mathbf{u}$ uniquely determines a local equilibrium value $\mathbf{U} = \mathcal{E}(\mathbf{u})$ satisfying $\mathbf{S}(\mathcal{E}(\mathbf{u})) = 0$ and such that $\mathbf{Q}\mathcal{E}(\mathbf{u}) = \mathbf{u}$, for all $\mathbf{u}$.

For the system in (1.1), (1.21), and (1.22), $N = d+3, n = d+2, \mathbf{U} = (\rho, \mathbf{m}, E, \rho q)^\top, \mathbf{u} = (\rho, \mathbf{m}, E)^\top$, $\mathcal{E}(\mathbf{u}) = (\rho, \mathbf{m}, E, \rho Q(\rho, e))^\top$, and

$$\mathbf{Q} = \begin{pmatrix} \mathbf{I}_{(d+2)\times(d+2)} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\mathbf{I}_{(d+2)\times(d+2)}$ is the $(d+2)\times(d+2)$ identity matrix.

The simplest models of (11.1) are 2×2 systems:

$$\begin{cases} \partial_t u + \partial_x f(u, v) = 0, \\ \partial_t v + \partial_x g(u, v) + \frac{1}{\epsilon} h(u, v) = 0, \end{cases} \tag{11.2}$$

where $h(u, v) = a(u, v)(v - e(u)), a(u, e(u)) \neq 0$. For such systems, $d = 1, N = 2, n = 1, U = (u, v)^\top, \mathcal{E}(u) = (u, e(u))^\top$, and $\mathbf{Q} = (1, 0)$. In particular, the p-system is a special case of (11.2):

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p(u) + \frac{1}{\epsilon}(v - f(u)) = 0, \end{cases} \tag{11.3}$$

with $\Lambda_1 = -\sqrt{p'(u)} < \Lambda_2 = \sqrt{p'(u)}$.

One of the most important issues is the relaxation limit of hyperbolic systems of conservation laws with stiff relaxation terms to the corresponding local systems. This may model how the dynamic limit from the continuum and kinetic nonequilibrium processes to the equilibrium processes is attained, as the relaxation time tends to zero. Typical examples for such a process include gas flow near thermo-equilibrium, viscoelasticity with vanishing memory, kinetic theory with small Knudsen number, and phase transition with short transition time.

The local equilibrium limit turns out to be highly singular because of shock and initial layers and to involve many challenging problems in nonlinear analysis and applied sciences. Roughly speaking, the relaxation time measures how far the nonequilibrium states are away from the corresponding equilibrium states; understanding its limit behavior is equivalent to understanding the stability of the equilibrium states. It connects nonlinear integral partial differential equations with nonlinear partial differential equations. This limit also involves the singular limit problem from nonlinear strictly hyperbolic systems to mixed hyperbolic-elliptic ones, or in some cases even purely elliptic (see [61]). The basic issue for such a limit is stability.

Consider System (11.3). If $f(u) = \lambda u, p(u) = \Lambda^2 u$, then u satisfies

$$\partial_t u + \lambda \partial_x u + \epsilon(\partial_{tt} u - \Lambda^2 \partial_{xx} u) = 0. \tag{11.4}$$

The limit $\epsilon \to 0$ is stable if and only if the characteristic speeds satisfy $-\Lambda < \lambda < \Lambda$ (cf. [322]).

To understand the stability of the zero relaxation limit for the nonlinear case, we first analyze the p-system in (11.3). Notice that $v^\epsilon = f(u^\epsilon) - \epsilon(\partial_t v^\epsilon + \partial_x p(u^\epsilon))$. If one can show

$$(u^\epsilon, v^\epsilon)(x,t) \to (u,v)(x,t), \text{ a.e.,}$$

then the zero relaxation limit of $(u^\epsilon, v^\epsilon)(x,t)$ is a weak solution of the local equilibrium:

$$\begin{cases} v = f(u), \\ \partial_t u + \partial_x f(u) = 0. \end{cases} \tag{11.5}$$

Consider a formal expansion of $v^\epsilon(x,t)$ in the form:

$$v^\epsilon \approx f(u^\epsilon) + \epsilon v_1(u^\epsilon) + \epsilon^2 v_2(u^\epsilon) + \cdots . \tag{11.6}$$

Then, in the ϵ^0-level, one has

$$\begin{aligned} &\partial_t u^\epsilon + \partial_x f(u^\epsilon) \approx 0, \\ &\partial_t f(u^\epsilon) + \partial_x p(u^\epsilon) + v_1(u^\epsilon) \approx 0, \end{aligned} \tag{11.7}$$

which implies

$$v_1(u^\epsilon) \approx -\left(p'(u^\epsilon) - f'(u^\epsilon)^2\right) \partial_x u^\epsilon. \tag{11.8}$$

Dropping all the higher-order terms in the expansion leads to a first-order correction to the local equilibrium approximation in the form:

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) \approx \epsilon \partial_x \left((p'(u^\epsilon) - f'(u^\epsilon)^2) \partial_x u^\epsilon \right). \tag{11.9}$$

This evolution equation will be dissipative provided the following stability criterion holds:

$$\Lambda_1 < \lambda < \Lambda_2,$$

where $\lambda = f'(u^\epsilon), \Lambda_j = (-1)^j \sqrt{p'(u^\epsilon)}$.

For the general system (11.1), similar arguments yield that the first correction is

$$\begin{cases} \mathbf{U} = \mathcal{E}(\mathbf{u}) - \epsilon(\nabla_{\mathbf{U}} \mathbf{S}(\mathcal{E}(\mathbf{u})))^{-1}(\mathbf{I} - \mathbf{P}(\mathbf{u}))\nabla_x \cdot \mathbf{F}(\mathcal{E}(\mathbf{u})), \\ \partial_t \mathbf{u} + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = \epsilon \nabla_x \cdot \{\mathbf{Q} \nabla_{\mathbf{U}} \mathbf{F}(\mathcal{E}(\mathbf{u}))(\nabla_{\mathbf{U}} \mathbf{S}(\mathcal{E}(\mathbf{u})))^{-1}(\mathbf{I} - \mathbf{P}(\mathbf{u}))\nabla_x \cdot \mathbf{F}(\mathcal{E}(\mathbf{u}))\}, \end{cases} \tag{11.10}$$

where $\mathbf{P}(\mathbf{u}) = \nabla_{\mathbf{u}} \mathcal{E}(\mathbf{u}) \mathbf{Q}$ is a projection ($\mathbf{P}^2 = \mathbf{P}$) onto the tangent space of the image of $\mathcal{E}(\mathbf{u})$.

Definition 11.1. *A twice-differential function $\Phi(\mathbf{U})$ is called an entropy for System* (11.1) *provided that*

(i) $\nabla^2\Phi(\mathbf{U})\nabla\mathbf{F}(\mathbf{U})\cdot\omega$ *is symmetric, for any* $\omega\in S^{d-1}$*;*

(ii) $\nabla\Phi(\mathbf{U})\mathbf{S}(\mathbf{U})\geq 0$*;*

(iii) The following are equivalent:

(a) $\mathbf{S}(\mathbf{U})=0$*,*

(b) $\nabla\Phi(\mathbf{U})\mathbf{S}(\mathbf{U})=0$*,*

(c) $\nabla\Phi(\mathbf{U})=\nu^\top\mathbf{Q}$*, for some* $\nu\in\mathbb{R}^n$*.*

An entropy Φ *is called convex if*

$$\nabla^2\Phi(\mathbf{U})\geq 0. \tag{11.11}$$

If the inequality (11.11) *is strict, the entropy* $\Phi(\mathbf{U})$ *is called strictly convex.*

Such a strictly convex entropy exists for many physical systems. For example, under certain conditions, Coquel-Perthame [26] showed that the system in (1.1), (1.21), and (1.22) has a globally defined, strictly convex entropy. In Chen-Levermore-Liu [61], the following theorem was proved.

Theorem 11.1. *Suppose that System* (11.1) *is endowed with a strictly convex entropy pair* $(\Phi,\mathbf{\Psi})$*. Then*

(i) The local equilibrium approximation

$$\partial_t\mathbf{u}+\nabla_x\cdot\mathbf{f}(\mathbf{u})=0 \tag{11.12}$$

is hyperbolic with strictly convex entropy pair:

$$(\eta(\mathbf{u}),\mathbf{q}(\mathbf{u}))=(\Phi,\mathbf{\Psi})|_{\mathbf{U}=\mathcal{E}(\mathbf{u})}. \tag{11.13}$$

(ii) The characteristic speeds of (11.12) *associated with any wave number* $\omega\in\mathbb{R}^d$ *are determined as the critical values of the restricted Rayleigh quotient:*

$$\mathbf{w}\to\frac{\mathbf{W}^\top\nabla^2_{\mathbf{U}}\Phi(\mathcal{E}(\mathbf{u}))\nabla_{\mathbf{U}}\mathbf{F}(\mathcal{E}(\mathbf{u}))\cdot\omega\,\mathbf{W}}{\mathbf{W}^\top\nabla^2_{\mathbf{U}}\Phi(\mathcal{E}(\mathbf{u}))\mathbf{W}}, \tag{11.14}$$

where $\mathbf{W}=\nabla_{\mathbf{u}}\mathcal{E}(\mathbf{u})\mathbf{w}$ *for* $\mathbf{w}\in\mathbb{R}^n$*. The characteristic speeds of* (11.12) *are interlaced with the characteristic speeds of* (11.1)*. That is, given a wave number* $\omega\in\mathbb{R}^d$*, for each* $\mathbf{u}\in\mathbb{R}^n$*, if the characteristic speeds* $\Lambda_k=\Lambda_k(\mathcal{E}(\mathbf{u}))$ *of* (11.1) *satisfy*

$$\Lambda_1\leq\cdots\leq\Lambda_k\leq\Lambda_{k+1}\leq\cdots\leq\Lambda_N,$$

while the characteristic speeds $\lambda_j=\lambda_j(\mathbf{u})$ *of* (11.12) *satisfy*

$$\lambda_1\leq\cdots\leq\lambda_k\leq\lambda_{k+1}\leq\cdots\leq\lambda_n,$$

then

$$\lambda_j\in[\Lambda_j,\Lambda_{j+N-n}].$$

(iii) The first correction (11.10) *is locally dissipative with respect to the entropy* $\eta(\mathbf{u})$*. For a* 2×2 *system in* (11.2)*, this implies the subcharacteristic stability condition:*

$$\Lambda_1<\lambda<\Lambda_2,\quad on\quad v=e(u). \tag{11.15}$$

(iv) For the 2×2 *system in* (11.2) *satisfying the strictly subcharacteristic stability condition* (11.15)*, the existence of a strictly convex entropy pair* (η,q) *for the local equilibrium equation implies the existence of a strictly convex entropy pair* (Φ,Ψ) *for System* (11.2) *over an open set* $\mathbf{O}_\eta$ *containing the local equilibrium curve* $v=e(u)$*, along which* (11.13) *is satisfied.*

Theorem 11.1 indicates that a strictly convex entropy function always exists for 2×2 systems endowed with the strictly subcharacteristic condition (11.15) in the regions which are close to the local equilibrium curves. It would be interesting to explore an approach to construct such an entropy for hyperbolic systems of conservation laws with relaxation. Generally, the convexity of entropy could fail at the nonequailibrium states which are far away from the local equilibrium manifolds. In [310], Tzavaras considered the criteria to have such an entropy, as dictated from compatibility with the second law of thermodynamics in the form of the Clausius-Duhem inequality, and found that, roughly speaking, the existence of the entropy is equivalent to the requirement of the relaxation model to be compatible with the second law.

The next issue is how the strong convergence of the zero relaxation limit to the local equilibrium equations can be achieved for systems with a strictly convex entropy. For this purpose, we consider a 2×2 system in (11.2).

Assume that $\mathbf{U}^\epsilon(x,t) = (u^\epsilon, v^\epsilon)(x,t) \subset K$, bounded open convex set, are solutions of (11.2), which satisfy the following entropy condition: For any convex entropy pair (Φ, Ψ),

$$\partial_t \Phi(\mathbf{U}^\epsilon) + \partial_x \Psi(\mathbf{U}^\epsilon) + \frac{1}{\epsilon}\Phi_v(\mathbf{U}^\epsilon) h(\mathbf{U}^\epsilon) \le 0, \qquad \text{for all} \quad \nabla^2_{\mathbf{U}}\Phi(\mathbf{U}) \ge 0,$$

in the sense of distributions. For simplicity, it is assumed that there exist two convex and dissipative entropy pairs $(\Phi_i, \Psi_i), i = 1, 2$, on K such that

$$\phi_2(u) - \phi_1(u) = cf(u), \qquad c \neq 0,$$

where $\phi(u) = \Phi_i|_{v=e(u)}, f(u) = f_1(u, e(u))$. The existence of such entropy functions is related to the stability theory (Theorem 11.1) (see [60, 61, 241, 46]).

Theorem 11.2. *Assume that there is no interval in which $f(u)$ is linear. Let the Cauchy data $(u_0^\epsilon, v_0^\epsilon)(x)$ satisfy*

$$\|(u_0^\epsilon - \overline{u}, v_0^\epsilon - \overline{v})\|_{L^2} \le C < \infty,$$

with $\overline{v} = e(\overline{u})$. Then $\mathbf{U}^\epsilon(x,t)$ strongly converges almost everywhere:

$$\mathbf{U}^\epsilon(x,t) \to \mathbf{U}(x,t), \quad a.e.$$

The limit function $\mathbf{U}(x,t) = (u,v)(x,t)$ satisfies that

(i) $v(x,t) = e(u(x,t))$ almost everywhere for $t > 0$;

(ii) $u(x,t)$ is the unique entropy solution of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u|_{t=0} = w^* - \lim u_0^\epsilon(x). \end{cases}$$

Theorem 11.2 proved in [60, 63] was obtained by combining the compactness theorem in [62] with the uniqueness theorem in [63], with the aid of Theorem 11.1. This limit is of compressible Euler type. Theorem 11.2 shows that, when the stability condition is satisfied, the solutions of the relaxation system indeed tend to the solutions of the local relaxation approximation, which are inviscid conservation laws, The main difficulty here is that the solutions of the full system are only measurable functions with certain boundedness. The following remarks are in order:

(a). Notice that the initial data may even be far from equilibrium. The convergence result indicates that the limit function $(u,v)(x,t)$ indeed goes into the local equilibrium instantaneously as t becomes positive. This shows that the limit is highly singular. In fact, this limit consists of two processes simultaneously: one is the initial layer limit, and the other is the shock layer limit.

(b). The compactness of the zero relaxation limit indicates that the sequence $\mathbf{U}^\epsilon(x,t)$ is compact no matter how oscillatory the initial data are. Note that the relaxation systems are allowed to be linearly degenerate; and the initial oscillations can propagate along the linearly degenerate fields for the homogeneous systems (cf. [38]). This shows that the relaxation mechanism coupled with the nonlinearity of the equilibrium equations can kill the initial oscillations, just as the nonlinearity of the homogeneous system can kill the initial oscillations.

(c). The above discussions are based on the L^∞ a priori estimate. In many physical systems, such estimates can be derived. Examples include the p-system and the models in viscoelasticity, chromatography, and combustion (see [46, 60, 61, 241, 264, 309, 322]), which possess natural invariant regions.

The technique based on the extensions of entropies has been further pursued by Serre [276] for semilinear and kinetic relaxations of systems of conservation laws.

Another technique based on some strong dissipation estimates on derivatives, which are available for several semilinear systems, has been used by Tzavaras [310], Gosse-Tzavaras [140], and the references cited therein.

For some special models, even uniform BV bounds of relaxation solutions $(u^\epsilon, v^\epsilon)(x,t)$ can be obtained, which ensures convergence to the zero relaxation limit. see Natalini [241], Tveito-Winther [309], Shen-Tveito-Winther [279], and the references cited therein.

We are now concerned with the weakly nonlinear limit for (11.2). Let

$$u^\epsilon = \overline{u} + \epsilon w^\epsilon, \qquad v^\epsilon = \overline{v} + \epsilon z^\epsilon, \tag{11.16}$$

where $(\overline{u}, \overline{v}) = (\overline{u}, e(\overline{u}))$ is an equilibrium state.

Upon rescaling the time variable t and translating the space variable x, as the slow time variable ϵt and the moving space variable $x - \lambda(\overline{u})t$, respectively,

$$(x,t) \to (x - \lambda(\overline{u})t, t),$$

the flux function in System (11.2) with the stability condition satisfies

$$\lambda(\overline{u}) = 0, \qquad \Lambda_1(\overline{u})\Lambda_2(\overline{u}) < 0.$$

The limit process as $\epsilon \to 0$ is a weakly nonlinear limit corresponding to the limit from the Boltzmann equation to the Navier-Stokes equations for incompressible fluids. The main observation is that the linearization of the local relaxation approximation about an equilibrium reduces to a simple advection dynamics with the equilibrium characteristic speed. This can be understood in a formal fashion. If one applies the same asymptotic scaling to the first correction to the local equilibrium approximation, one again arrives at the weakly nonlinear approximation. This shows that the weakly nonlinear limit is a distinguished limit of the local equilibrium limit and makes clear why it inherits the good features of the former. The advantage of the weakly nonlinear limit is that the solutions of the Burgers equation are smooth even for the case that the initial data are not smooth. Thus the solutions remain globally consistent with all the assumptions that were used to derive the weakly nonlinear approximation.

In [61], the weakly nonlinear approximation was justified by using the stability theory and the energy estimate techniques. The linearized version of the limit is well understood and is related to "random walk" in Brownian motion (cf. [116, 174, 256]). From (11.2) and (11.16), $(w^\epsilon, z^\epsilon)(x,t)$ satisfy

$$\begin{cases} \epsilon^2 \partial_t w^\epsilon + \partial_x f(\overline{u} + \epsilon w^\epsilon, \overline{v} + \epsilon z^\epsilon) = 0, \\ \epsilon^2 \partial_t z^\epsilon + \partial_x g(\overline{u} + \epsilon w^\epsilon, \overline{v} + \epsilon z^\epsilon) + \frac{1}{\epsilon} h(\overline{u} + \epsilon w^\epsilon, \overline{v} + \epsilon z^\epsilon) = 0, \\ (w^\epsilon, z^\epsilon)|_{t=0} = (w_0^\epsilon, z_0^\epsilon)(x). \end{cases} \tag{11.17}$$

Theorem 11.3. *There exist* $\epsilon_0 > 0$ *and* $C_0 > 0$ *such that, if* $0 < \epsilon \le \epsilon_0$, *and*

$$\|(w_0^\epsilon, z_0^\epsilon)\|_{H^3} \le C_0, \qquad \|z_0^\epsilon - \frac{e(\overline{u} + \epsilon w_0^\epsilon) - e(\overline{u})}{\epsilon}\|_{L^2} \le C_0\epsilon, \tag{11.18}$$

then there exists a unique global solution $(w^\epsilon, z^\epsilon) \in H^3$ *of* (11.17) *such that*

$$\begin{cases} (w^\epsilon, z^\epsilon)(x,t) \to (w,z)(x,t) \in L^2, \qquad \epsilon \to 0, \\ z(x,t) = e'(\overline{u})w(x,t), \\ \partial_t w + \lambda'(\overline{u})\partial_x(\frac{w^2}{2}) + \dfrac{\Lambda_1(\overline{u})\Lambda_2(\overline{u})}{h_v(\overline{u}, e(\overline{u}))} h_v(\overline{u}, e(\overline{u}))\partial_{xx} w = 0. \end{cases} \tag{11.19}$$

Outline of the Proof. Since the proof of Theorem 11.3 is technical, we list only the main steps below.
Step 1. We replace (Φ, Ψ) by (Φ_*, Ψ_*), where

$$\Phi_*(\mathbf{U}) := \Phi(\mathbf{U}) - \Phi(\overline{\mathbf{U}}) - \nabla\Phi(\overline{\mathbf{U}})(\mathbf{U} - \overline{\mathbf{U}}),$$
$$\Psi_*(\mathbf{U}) := \Psi(\mathbf{U}) - \Psi(\overline{\mathbf{U}}) - \nabla\Psi(\overline{\mathbf{U}})(\mathbf{F}(\mathbf{U}) - \mathbf{F}(\overline{\mathbf{U}})).$$

Then $\Phi_{*v}(u, e(u)) \ge c_0(v - e(u))^2$ for some $c_0 > 0$. From $\nabla^2\Phi_*(\mathbf{U}) > 0$, one has

$$\begin{aligned} &\epsilon\int_{-\infty}^{\infty} \Phi_*(u^\epsilon(x,t), v^\epsilon(x,t))dx + c_0\int_0^t\int_{-\infty}^{\infty} \frac{(v^\epsilon - e(u^\epsilon))^2}{\epsilon} dxd\tau \\ \le &\epsilon\int_{-\infty}^{\infty} \Phi_*(u_0^\epsilon(x), v_0^\epsilon(x))dx \le C\epsilon^3\int_{-\infty}^{\infty}(w_0^\epsilon(x)^2 + z_0^\epsilon(x)^2)dx. \end{aligned}$$

Therefore,

$$\|z^\epsilon - \frac{e(\overline{u} + \epsilon w^\epsilon) - e(\overline{u})}{\epsilon}\|_{L^2} \le C\epsilon. \tag{11.20}$$

Step 2. Eliminating $z^\epsilon(x,t)$ leads to

$$\begin{aligned} &\partial_t w^\epsilon + \frac{\lambda'(\overline{u})}{2}\partial_x(w^\epsilon)^2 + \frac{\Lambda_1(\overline{u})\Lambda_2(\overline{u})}{h_v(\overline{u}, e(\overline{u}))}\partial_{xx}w^\epsilon + \frac{\epsilon^2}{h_v(\overline{u}, v(\overline{u}))}\partial_{tt}w^\epsilon \\ &= E^\epsilon(x, t, D^2w^\epsilon, D^2z^\epsilon). \end{aligned} \tag{11.21}$$

Using the energy estimates yields

$$\sum_{\substack{i,j=1 \\ i+j\le 3}} \epsilon^{2(i-1)}\int_0^t\int_{-\infty}^{\infty} |\partial_\tau^i\partial_x^j(w^\epsilon, \epsilon^i z^\epsilon)|^2(x,\tau)dxd\tau \le C, \tag{11.22}$$

where C is a constant independent of ϵ.

Step 3. Then we prove $E^\epsilon(x, t, D^2w^\epsilon, D^2z^\epsilon) \to 0$, when $\epsilon \to 0$.

Step 4. Since $\|w^\epsilon\|_{H^1} \le C$, the Sobolev embedding theorem yields that there exists a subsequence (still denoted) $w^\epsilon(x,t)$ converging strongly in L^2. That is,

$$w^\epsilon(x,t) \to w(x,t).$$

Estimate (11.20) implies that $z^\epsilon(x,t)$ strongly converges in L^2:

$$z^\epsilon(x,t) \to e'(\overline{u})w(x,t).$$

Then Theorem 10.3 follows.

More details of the proof can be found in Chen-Levermore-Liu [61].

Some further results for hyperbolic systems of conservation laws with relaxation can be found in [39, 46, 159, 160, 241, 276, 309, 310] and the references cited therein.

Some recent ideas and approaches in attacking hyperbolic conservation laws with memory can be found in Dafermos [85], Nohel-Rogers-Tzavaras [247], and Chen-Dafermos [44] with the aid of the compensated compactness methods. For special memory kernels, these conservation laws reduce to hyperbolic systems of conservation laws with relaxation.

11.2. **Euler Equations for Exothermically Reacting Fluids.** We now consider the Euler equations in (1.12) and (1.23) ($d = 1$), which governs the behavior of plane detonation waves. In a detonation wave, the effect of pressure gradient, which supports the shock wave, and the conversion of chemical energy to mechanical energy is far greater than the diffusive effect of viscosity, heat conduction, and diffusion of chemical species. This justifies the use of the Euler equations in (1.12) and (1.23), rather than the Navier-Stokes equations, in this context. The shock wave solutions in this model are jump discontinuities. This is a very good representation of the shock waves one observes experimentally, which have a width of several molecular mean free paths. The reaction zone of a detonation wave, by way of contrast, is generally hundreds of mean free paths wide.

The main interest in this system of equations lies in a new type of behavior exhibited by solutions. Whereas non-reacting shock waves are known to be stable under reasonable assumptions [221], linearized stability analysis, as well as numerical and physical experiments, have shown that certain steady detonation waves are unstable [16, 113, 117, 185, 251]. One particular kind of instability that takes place within the context of one space dimension produces pulsating detonation waves. In certain parameter regimes, steady planar detonation waves are unstable and evolve into oscillating waves. These oscillating waves generate a steady stream of waves which propagate behind the wave [64]. This implies that the exothermic reaction can increase the total variation in a number of ways. For example, in the formation of a detonation wave, a chemical reaction behind a shock wave can increase the strength of that shock wave. More subtle phenomena are also possible. In a nearly constant, unreacted state, a very small variation in temperature can cause the gas in one region to react prior to the gas in nearby regions, resulting in a large increase in total variation. Moreover, the hot spot created by such an event would generate waves, some of which would be shock or rarefaction waves. These waves could propagate away from the hot spot before the remaining reactant ignites.

The theorem we discuss here from Chen-Wagner [64] is a first-step in dealing with these difficulties. It is assumed that the initial data are such that the reaction rate function $\phi(\theta)$ never vanishes. In a sense, this is a very realistic condition. Typically $\phi(\theta)$ has the Arrhenius form (1.23):

$$\phi(\theta) = Ke^{-\theta_0/\theta},$$

which vanishes only at absolute zero temperature. However, in a typical unburned state, $\phi(\theta)$ is very small. This assumption is made in order to obtain uniform decay of the reactant to zero. Thus, although the total variation of the solution may very well increase while the reaction is active, the reaction must eventually die out. Consequently, the increase in total variation can be estimated rigorously.

Consider a one-parameter family of functions $e(\tau, S, \epsilon)$, $\tau = 1/\rho, \epsilon \geq 0$, which is C^5 and satisfies (1.5). For a polytropic gas, $\epsilon = \gamma - 1$. It is assumed that, when $\epsilon = 0$, the equation of state is that of an isothermal gas:

$$e(\tau, S, 0) = -\ln \tau + \frac{S}{R}. \tag{11.24}$$

For a polytropic gas,

$$e(\tau, S, \epsilon) = \frac{1}{\epsilon}\left((\tau \exp(-S/R))^{-\epsilon} - 1\right). \tag{11.25}$$

One may easily check that this function is C^∞ and that, as $\epsilon \to 0+$, all partial derivatives converge uniformly on compact sets in $\tau > 0$ to the corresponding derivatives of $e(\tau, S, 0)$. In particular, one may use L'Hôpital's rule to calculate

$$\partial_\epsilon e(\tau, S, 0) = \frac{1}{2}\left(-\ln\tau + \frac{S}{R}\right)^2, \tag{11.26}$$

and that $\partial_\epsilon e(\tau, S, \epsilon)$ is continuous at $\epsilon = 0$, $\tau > 0$. The value $\epsilon = 0$ is mathematically special because, at this value, System (1.12) and (1.23), in Lagrangian coordinates, has a complete set of Riemann invariants:

$$(r, s, S, Z) = (v - \ln(p), v + \ln(p), S, Z). \tag{11.27}$$

Moreover, all shock, rarefaction, and contact discontinuity curves in the (r, s, S, Z)-space are invariant under translation of the base point. We also use (r, s, S, Z) as the coordinates for the analysis in $\epsilon \geq 0$. Note that, since $p = -\partial_\tau e(\tau, S, \epsilon)$, and $e(\tau, S, \epsilon)$ is C^5, the transformation between (τ, v, S) and (r, s, S) is C^4 and is a diffeomorphism (e.g. [302]).

Theorem 11.4. *Let $K \subset \{(\tau, v, S, Z) : \tau > 0\} \subset \mathbb{R}_+ \times \mathbb{R}^2 \times [0,1]$ be a compact set, and let $N \geq 1$ be any positive constant. Then there exists a constant $C_0 = C_0(K, N) > 0$, independent of $\epsilon > 0$, such that, for every initial data $(\tau_0, v_0, S_0, Z_0)(x) \in K$ with $\mathrm{TV}_{\mathbb{R}}(\tau_0, v_0, S_0, Z_0) \leq N$, when*

$$\epsilon\, \mathrm{TV}_{\mathbb{R}}(\tau_0, v_0, S_0, Z_0) \leq C_0, \tag{11.28}$$

the Cauchy problem (1.12) *and* (1.23) *in Lagrangian coordinates, with the initial data determined by $(r_0, s_0, S_0, Z_0)(x)$, has a global BV entropy solution $\mathbf{U}(x,t) = (\tau, v, e + \frac{v^2}{2}, Z)(x,t)$.*

There is a trade-off between the size of ϵ (or $\gamma - 1$) and the size of the initial data allowed. When ϵ is close to 0, the initial data are allowed to be of large total variation. There is also a trade–off between the minimum reaction rate and the size of the initial data allowed. If the minimum reaction rate is slow, the increase in total variation due to the reaction is potentially large so that the initial data are only allowed to be of small total variation. If, however, the minimum reaction rate is large, then larger initial data are somewhat allowed.

There is an interesting common thread connecting the results with previous ones concerning balance laws (cf. [89, 97, 219, 329, 330]). While earlier results had in view lower-order terms that exerted a damping effect, or otherwise reduced total variation, the result in Theorem 11.4 requires the decay of the lower-order term, even though total variation may increase in the process. Thus, in either case, decay of some kind seems essential.

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References

[1] A. M. AbdElFattah, L. F. Henderson, and A. Lozzi, *Pressure shock waves at a slow-fast gas interface*, J. Fluid Mech. **76** (1976), 157-176.

[2] S. Alinhac, *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux*, Invent. Math. **111** (1993), 627-670.

[3] S. Alinhac, *Blowup for Nonlinear Hyperbolic Equations*, Birkhauser: Boston, 1995.

[4] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) **135** (1983), 293–318.

[5] P. Baiti, P. G. LeFloch, and B. Piccoli, *Uniqueness of classical and nonclassical solutions for nonlinear hyperbolic systems,* J. Diff. Eqs. **172** (2001), 59–82.

[6] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337-403.

[7] J. M. Ball, *A version of the fundamental theorem for Young measures*, In: PDEs and Continuum Models of Phase Transitions (Nice, 1988), Lecture Notes in Phys. **344** (1989), 207-215, Springer-Verlag: Berlin-New York.

[8] M. Bardi and S. Osher, *The nonconvex multidimensional Riemann problem for Hamilton-Jacobi equations*, SIAM J. Math. Anal. **22** (1991), 344–351.

[9] C. Bardos, F. Golse, and C. D. Levermore, *Fluid dynamic limits of discrete velocity kinetic equations*, In: Advances in Kinetic Theory and Continuum Mechanics, R. Gatignol and Soubbaramayer, eds., Springer-Verlag: Berlin-New York (1991), 57-91.

[10] F. Bereux, E. Bonnetier, and P. G. LeFloch, *Gas dynamics system: two special cases*, SIAM J. Math. Anal. **28** (1997), 499-515.

[11] A. V. Betsadze, *Equations of the Mixed Type*, Macmillan Company: New York, 1964.

[12] S. Bianchini and A. Bressan, *A center manifold technique for tracing viscous waves*, Preprint, 2001.

[13] S. Bianchini and A. Bressan, In preparation, 2002.

[14] A. Blokhin, and Y. Trakhinin, *Stability of strong discontinuities in fluids and MHD*, In this volume.

[15] F. Bouchut and F. James, *Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness*, Commun. Partial Diff. Eqs. **24** (1999), 2173–2189.

[16] A. Bourlioux, A. Majda, and V. Roytburd, *Theoretical and numerical structure for unstable one–dimensional detonations*, SIAM J. Appl. Math. **51** (1991), 303–343.

[17] U. Brauer, *Breakdown of smooth solutions of the three-dimensional Euler-Poisson system*, J. Math. Phys. **39** (1998), 1050-1074.

[18] A. Bressan, *Global solutions of systems of conservation laws by wave-front tracking*, J. Math. Anal. Appl. **170** (1992), 414-432.

[19] A. Bressan, *The unique limit of the Glimm scheme*, Arch. Rational Mech. Anal. **130** (1995), 205-230.

[20] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One Dimensional Cauchy Problem*, Oxford University Press: Oxford, 2000.

[21] A. Bressan, G. Crasta, and B. Piccoli, *Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, Mem. Amer. Math. Soc. **146**, AMS: Providence, 2000.

[22] A. Bressan and P. Goatin, *Oleinik type estimates and uniqueness for $n \times n$ conservation laws*, J. Diff. Eqs. **156** (1999), 26-49.

[23] A. Bressan and P. LeFloch, *Uniqueness of weak solutions to systems of conservation laws*, Arch. Rational Mech. Anal. **140** (1970), 301-317.

[24] A. Bressan and M. Lewicka, *A uniqueness condition for hyperbolic systems of conservation laws*, Discrete Contin. Dynam. Systems, **6** (2000), 673-682.

[25] A. Bressan, T.-P. Liu, and T. Yang, *L^1 stability estimates for $n \times n$ conservation laws*, Arch. Rational Mech. Anal. **149** (1999), 1-22.

[26] R. Caflisch and G. C. Papanicolaou, *The fluid dynamic limit of a nonlinear model Boltzmann equations*, Comm. Pure Appl. Appl. **32** (1979), 521–554.

[27] S. Canic, B. Keyfitz, and G. M. Lieberman, *A proof of existence of perturbed steady transonic shocks via a free boundary problem*, Comm. Pure Appl. Math. **53** (2000), 484-511.

[28] S. Canic, B. Keyfitz, and E. H. Kim, *A free boundary problem for a quasilinear degenerate elliptic equation: regular reflection of weak shocks*, Preprint, 2000.

[29] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer-Verlag: New York, 1988.

[30] T. Chang and G.-Q. Chen, *Diffraction of planar shock along the compressive corner*, Acta Math. Sci. **6**, 241–257 (1986).

[31] T. Chang, G.-Q. Chen, and S. Yang, 2-*D Riemann problem in gas dynamics and formation of spiral*, In: Nonlinear Problems in Engineering and Science–Numerical and Analytical Approach (Beijing, 1991), 167-179, Science Press: Beijing, 1992.

[32] T. Chang, G.-Q. Chen, and S. Yang, *On the Riemann Problem for two-dimensional Euler equations I: Interaction of shocks and rarefaction waves; II: Interaction of contact discontinuities*, Discrete Contin. Dynam. Systems, **1** (1995), 555-584; **6** (2000), 419-430.

[33] T. Chang and L. Hsiao, *The Riemann Problem and Interaction of Waves in Gas Dynamics,* Longman Scientific & Technical, Harlow; and John Wiley & Sons, Inc.: New York, 1989.

[34] J. S. Chapman and T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press: Cambridge, 1970.

[35] J. -Y. Chemin, *Dynamique des gaz à masse totale finie*, Asymptotic Anal. **3** (1990), 215-220.

[36] J. Y. Chemin, *Remarques sur l'apparition de singularités dans les ecoulements Euleriens compressibles*, Commun. Math. Phys. **133** (1990), 323-329.

[37] G.-Q. Chen, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III)*, Acta Math. Sci. **6** (1986), 75-120 (in English); **8** (1988), 243-276 (in Chinese).

[38] G.-Q. Chen, *The method of quasidecouping for discontinuous solutions to conservation laws*, Arch. Rational Mech. Anal. **121** (1992), 131-185.

[39] G.-Q. Chen, *Entropy, Compactness, and Conservation Laws*, Lecture notes, Northwestern University, 2001.

[40] G.-Q. Chen, *Remarks on global solutions to the compressible Euler equations with spherical symmetry*, Proc. Royal Soc. Edinburgh, **127A** (1997), 243-259.

[41] G.-Q. Chen, *Hyperbolic systems of conservation laws with a symmetry*, Commun. Partial Diff. Eqs. **16** (1991), 1461-1487.

[42] G.-Q. Chen, *Vacuum states and stability of rarefaction waves for compressible flow*, Meth. Appl. Anal. **7** (2000), 337-362.

[43] G.-Q. Chen and C. M. Dafermos, *The vanishing viscosity method in one-dimensional thermoelasticity*, Trans. Amer. Math. Soc. **347** (1995), 531-541.

[44] G.-Q. Chen and C. M. Dafermos, *Global solutions in L^∞ for a system of conservation laws of viscoelastic materials with memory*, J. Partial Diff. Eqs. **10** (1997), 369-383.

[45] G.-Q. Chen and M. Feldman, *Multidimensional transonic shocks and free boundary problems for nonlinear equations of mixed type*, Preprint, Northwestern University, August 2001.

[46] G.-Q. Chen and H. Frid, *Decay of entropy solutions of nonlinear conservation laws*, Arch. Rational Mech. Anal. **146** (1999), 95-127.

[47] G.-Q. Chen and H. Frid, *Large-time behavior of entropy solutions of conservation laws*, J. Diff. Eqs. **152** (1999), 308-357.

[48] G.-Q. Chen and H. Frid, *Uniqueness and asymptotic stability of Riemann solutions for the compressible Euler equations*, Trans. Amer. Math. Soc. **353** (2000), 1103-1117.

[49] G.-Q. Chen and H. Frid, *Large-time behavior of entropy solutions in L^∞ for multidimensional conservation laws*, In: Advances in Nonlinear Partial Differential Equations and Related Areas, pp. 28-44, G.-Q. Chen et al (eds.), World Scientific, Singapore, 1998.

[50] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Rational Mech. Anal. **147** (1999), 89-118.

[51] G.-Q. Chen and H. Frid, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Preprint, Northwestern University, August 2001.

[52] G.-Q. Chen, H. Frid, and Y. Li, *Uniqueness and stability of Riemann solutions with large oscillation in gas dynamics,* Commun. Math. Phys. 2001 (to appear).

[53] G.-Q. Chen and J. Glimm, *Global solution to the compressible Euler equations with geometrical structure*, Commun. Math. Phys. **179** (1996), 153-193.

[54] G.-Q. Chen and H. Liu, *Concentration and cavitation in the vanishing pressure limit in solutions of the Euler equations for compressible fluids*, Preprint, Northwestern University, August 28, 2001.

[55] G.-Q. Chen, J. W. Jerome, C.-W. Shu, and D. Wang, *Two carrier semiconductor device models with geometric structure and symmetry*, In: Modeling and Computation for Applications in Mathematics, Science, and Engineering, pp. 103-140, Oxford University Press: Oxford, 1998.

[56] G.-Q. Chen and P.-T. Kan, *Hyperbolic conservation laws with umbilic degeneracy*, I: Arch. Rational Mech. Anal. **130** (1995), 231-276, II: Arch. Rational Mech. Anal. (2001) (to appear).

[57] G.-Q. Chen and P. LeFloch, *Compressible Euler equations with general pressure law*, Arch. Rational Mech. Anal. **153** (2000), 221-259.

[58] G.-Q. Chen and P. LeFloch, *Entropies and flux-splittings for the isentropic Euler equations*, Chinese Annals Math. Ser. B, **22** (2001), 145-158.

[59] G.-Q. Chen and P. LeFloch, *Existence theory for the isentropic Euler equations*, Preprint, Northwestern University, 2001 (with P. LeFloch).

[60] G.-Q. Chen and T.-P. Liu, *Zero relaxation and dissipation limits for hyperbolic conservation laws*, Comm. Pure. Appl. Math. **46** (1993), 255-281.

[61] G.-Q. Chen, C. D. Levermore, and T.-P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1994), 787-830.

[62] G.-Q. Chen and Y.-G. Lu, *A study to applications of the theory of compensated compactness*, Chinese Science Bulletin, **34** (1989), 15-19.

[63] G.-Q. Chen and M. Rascle, *Initial layers and uniqueness of weak entropy solutions to hyperbolic conservation laws*, Arch. Rational Mech. Anal. **153** (2000), 205-220.

[64] G.-Q. Chen and D. Wagner, *Global entropy solutions to exothermically reacting, compressible Euler equations*, J. Diff. Eqs. 2001 (to appear).

[65] G.-Q. Chen and D. Wang, *Convergence of shock capturing schemes for the compressible Euler-Poisson equations*, Commun. Math. Phys. **179** (1996), 333-364.

[66] G.-Q. Chen and D. Wang, *Shock capturing approximations to the compressible Euler equations with geometric structure and related equations*, Z. Angew. Math. Phys. **49** (1998), 341-362.

[67] S.-X. Chen, *Asymptotic behavior of supersonic flow past a convex combined wedge*, Chinese Annals Math. **19B** (1998), 255–264.

[68] S.-X. Chen, *Supersonic flow past a concave double wedge,* Sci. China, **41A** (1998), 39–47.

[69] I.-L. Chern, J. Glimm, O. McBryan, B. Plohr, and S. Yaniv, *Front tracking for gas dynamics*, J. Comput. Phys. **62** (1986), 83–110.

[70] A. J. Chorin and J. E. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag: Berlin-Heidelberg-New York, 1979.

[71] K, Chueh, C. Conley, and J. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J. **26** (1977), 373-392.

[72] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. **72** (1993), 247–286.

[73] P. Constantin, *Some mathematical problems of fluid mechanics*, Proceedings of the International Congress of Mathematicians, Vols. 1-2 (Zürich, 1994), 1086-1095, Birkhäuser, Basel, 1995.

[74] E. Conway and J. Smoller, *Global solutions of the Cauchy problem for quasilinear first-order equations in several space variables*, Comm. Pure Appl. Math. **19** (1966), 95-105.

[75] F. Coquel and B. Perthame, *Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics*, In: SIAM J. Numer. Anal. **35** (1998).

[76] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Springer-Verlag: New York, 1962.

[77] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience Publishers, Inc.: New York, 1953.

[78] M. G. Crandall and P. E. Souganidis, *Convergence of difference approximations of quasilinear evolution equations*, Nonlinear Anal. **10** (1986), 425–445.

[79] B. Dacorogna, *Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functionals*, Lecture Notes in Math. **922**, Springer-Verlag: New York, 1982.

[80] C. M. Dafermos, *Polygonal approximations of solutions of the initial-value problem for a conservation law*, J. Math. Anal. Appl. **38** (1972), 33-41.

[81] C. M. Dafermos, *Generalized characteristics and the structure of solutions of hyperbolic conservation laws*, Indiana Univ. Math. J. **26** (1977), 1097-1119.

[82] C. M. Dafermos, *Can dissipation prevent the breaking of waves ?*, In: Transactions of the Twenty-Sixth Conference of Army Mathematicians, 187-198, ARO Rep. **81**, 1, U. S. Army Res. Office, Research Triangle Park, N.C., 1981.

[83] C. M. Dafermos, *Development of singularities in the motion of materials with fading memory*, Arch. Rational Mech. Anal. **91** (1985), 193-205.

[84] C. M. Dafermos, *Generalized characteristics in hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **107** (1989), 127–155.

[85] C. M. Dafermos, *Solutions in L^∞ for a conservation law with memory*, Analyse Mathématique et Applications, pp. 117-128, Gauthier-Villars: Paris, 1988.

[86] C. M. Dafermos, *Large time behavior of periodic solutions of hyperbolic systems of conservation laws*, J. Diff. Eqs. **121** (1995), 183–202.

[87] C. M. Dafermos, *Entropy and the stability of classical solutions of hyperbolic systems of conservation laws*, In: Recent Mathematical Methods in Nonlinear Wave Propagation (Montecatini Terme, 1994), pp. 48–69, Lecture Notes in Math. **1640**, Springer-Verlag: Berlin, 1996.

[88] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: Berlin, 2000.

[89] C. M. Dafermos and L. Hsiao, *Hyperbolic systems of balance laws with inhomogeneity and dissipation* Indiana Univ. Math. J. **31** (1982), 471-491.

[90] C. M. Dafermos and L. Hsiao, *Development of singularities in solutions of the equations of nonlinear thermoelasticity*, Q. Appl. Math. **44** (1986), 463-474.

[91] C. M. Dafermos and J. A. Nohel, *Energy methods for nonlinear hyperbolic Volterra integrodifferential equations*, Commun. Partial Diff. Eqs. **4** (1979), 219-278.

[92] G. Darboux, *Legons Sur La Theorie Generale des Surfaces*, T. II, Paris, 1914-1915.

[93] R. L. Deschambault, I. I. Glass, *An update on nonstationary oblique shock-wave reflections, actual isopicnics and numerical experiments*, J. Fluid Mech. **131** (1983), 27-57.

[94] X. Ding, *Theory of conservation laws in China*, In: Hyperbolic Problems, Theory, Numerics, Applications, pp. 110-119, J. Glimm et al (eds.), World Scientific: Singapore, 1996.

[95] X. Ding, T. Zhang, C.-H. Wang, L. Hsiao, and T.-C. Li, *A study of the global solutions for quasilinear hyperbolic systems of conservation laws*, Scientica Sinica, **16** (1973), 317-335.

[96] X. Ding, G.-Q. Chen, and P. Luo, *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (I)-(II)*, Acta Math. Sci. **5** (1985), 483-500, 501-540 (in English); **7** (1987), 467-480, **8** (1988), 61-94 (in Chinese).

[97] X. Ding, G.-Q. Chen, and P. Luo, *Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for isentropic gas dynamics*, Commun. Math. Phys. **121** (1989), 63-84.

[98] X. Ding and Z. Wang, *Existence and uniqueness of discontinuous solutions defined by Lebesgue-Stieltjes integral*, Sci. China, **39A** (1996), 807–819.

[99] R. DiPerna, *Existence in the large for quasilinear hyperbolic conservation laws*, Arch. Rational Mech. Anal. **52** (1973), 244-257.

[100] R. DiPerna, *Singularities of solutions of nonlinear hyperbolic systems of conservation laws*, Arch. Rational Mech. Anal. **60** (1975), 75-100.

[101] R. DiPerna, *Decay and asymptotic behavior of solutions to nonlinear hyperbolic systems of conservation laws*, Indiana Univ. Math. J. **24** (1975), 1047-1071.

[102] R. DiPerna, *Global existence of solutions to nonlinear hyperbolic systems of conservation laws*, J. Diff. Eqs. **20** (1976), 187-212.

[103] R. DiPerna, *Decay of solutions of hyperbolic systems of conservation laws with a convex extension*, Arch. Rational Mech. Anal. **64** (1977), 1-46.

[104] R. DiPerna, *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137-188.

[105] R. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Rational Mech. Anal. **88** (1985), 223-270.

[106] R. DiPerna, *Convergence of viscosity method for isentropic gas dynamics*, Commun. Math. Phys. **91** (1983), 1-30.

[107] W. E, K. Khanin, A. Mazel, and Y. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. Math. (2) **151** (2000), 877–960.

[108] W. E, Y. Rykov, Y. Sinai, *Generalized variational principles, global weak solutions, and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Commun. Math. Phys. **177** (1996), 349–380.

[109] P. Embid, J. Goodman, and A. Majda, *Multiple steady state for 1-D transonic flow*, SIAM J. Sci. Stat. Comp. **5** (1984), 21-41.

[110] S. Engelberg, H. Liu, and E. Tadmor, *Critical thresholds in Euler-Poisson equations*, Indiana Univ. Math. J. **50** (2001), 109-157.

[111] B. Engquist and W. E, *Large time behavior and homogenization of solutions of two-dimensional conservation laws*, Comm. Pure Appl. Math. **46** (1993), 1–26.

[112] B. Engquist, M. Luskin, and A. Majda, *Computational Fluid Dynamics and Reacting Gas Flows*, IMA Vols. in Math. and its Appl. **12**, Springer-Verlag: New York, 1988.

[113] J. J. Erpenbeck. *Stability of idealized one-reaction detonations*, Phys. Fluids, **7** (1964), 684–696.

[114] L. Euler, *Institutiones Calculi Integralis*, III, Petropoli, 1790.

[115] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, CBMS **72**, AMS: Providence, Rhode Island, 1990.

[116] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear PDE*, Proc. Royal Soc. Edinburgh, **111A** (1989), 141-172.

[117] W. Fickett and W. W. Wood. *Flow calculation for pulsating one-dimensional detonation*, Phys. Fluids, **9** (1966), 903–916.

[118] A. F. Filippov, *Differential equations with discontinuous right-hand side*, Mat. Sb. (N.S.), **51** (1960), 99-128.

[119] S. K. Fok, *Extensions of Glimm's method to the problem of gas flow in a duct of variable cross-section*, Ph. D. Thesis, Department of Mathematics, University of California at Berkeley, 1981.

[120] J. Francheteau and G. Métivier, *Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels* (in French), Astérisque, **268**, SMF: Paris, 2000.

[121] K. O. Friedrichs, *Nonlinear hyperbolic differential equations for functions of two independent variables*, Amer. J. Math. **70** (1948), 555-589.

[122] K. O. Friedrichs, *Symmetric hyperbolic linear differential equations*, Comm. Pure Appl. Math. **7** (1954), 345-392.

[123] S. Friedlands, J. W. Robbin, and J. Sylvester, *On the crossing rule*, Comm. Pure Appl. Math. **37** (1984), 19-38.

[124] K. Friedrichs and P. Lax, *Systems of conservation laws with a convex extension*, Proc. Nat. Acad. Sci., U.S.A. **68** (1971), 1686-1688.

[125] I. Gamba, *Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductors*, Commun. Partial Diff. Eqs. **17** (1992), 553-577.

[126] I. M. Gamba and C. S. Morawetz, *A viscous approximation for a 2-D steady semiconductor or transonic gas dynamic flow: existence theorem for potential flow*, Comm. Pure Appl. Math. **49** (1996), 999–1049.

[127] I. M. Gamba, R. Rosales, and E. Tabak, *Constraints on possible singularities for the unsteady transonic small disturbance (UTSD) equations*, Comm. Pure Appl. Math. **52** (1999), 763–779.

[128] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), 1794-1797.

[129] H. Glaz and T.-P. Liu, *The asymptotic analysis of wave interactions and numerical calculation of transonic nozzle flow*, Adv. Appl. Math. **5** (1984), 111-146.

[130] J. Glimm, *Solutions in the large for nonlinear hyperbolic system of equations*, Comm. Pure Appl. Math. **18** (1965), 95-105.

[131] J. Glimm, *The interaction of nonlinear hyperbolic waves*, Comm. Pure Appl. Math. **41** (1988), 569-590.

[132] J. Glimm, C. Klingenberg, O. McBryan, B. Plohr, D. Sharp, and S. Yaniv, *Front tracking and two-dimensional Riemann problems,* Adv. Appl. Math. **6** (1985), 259–290.

[133] J. Glimm and P. D. Lax, *Decay of Solutions of Nonlinear Hyperbolic Conservation Laws*, Mem. Amer. Math. Soc. **101**, AMS: Providence, 1970.

[134] J. Glimm and A. Majda, *Multidimensional Hyperbolic Problems and Computations*, IMA Volumes in Mathematics and its Applications, **29**, Springer-Verlag: New York, 1991.

[135] J. Glimm, G. Marshall, and B. Plohr, *A generalized Riemann problem for quasi-one dimensional gas flow*, Adv. Appl. Math. **5** (1984), 1-30.

[136] J. Glimm, H. C. Kranzer, D. Tan, and F. M. Tangerman, *Wave fronts for Hamilton-Jacobi equations: the general theory for Riemann solutions in* $\mathbf{R}^n$, Commun. Math. Phys. **187** (1997), 647–677.

[137] P. Goatin and P. G. LeFloch, *Sharp* L^1 *continuous dependence of solutions of bounded variation for hyperbolic systems of conservation laws,* Arch. Rational Mech. Anal. **157** (2001), 35–73.

[138] E. Godlewski and P.-A. Raviart, *Numerical Approximation of Hyperbolic Systems of Conservation Laws*, Springer-Verlag: New York, 1996.

[139] S. Godunov, *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb. **47** (89) (1959), 271-360.

[140] L. Gosse and A. Tzavaras, *Convergence of relaxation schemes to the equations of elastodynamics*, Math. Comp. **70** (2001), 555–577.

[141] M. Grassin, *Global smooth solutions to Euler equations for a perfect gas*, Indiana Univ. Math. J. **47** (1998), 1397-1432.

[142] M. Grassin and D. Serre, *Existence de solutions globales et régulières aux équations d'Euler pour un gaz parfait isentropique*, C. R. Acad. Sci. Paris Ser. I, **325** (1997), 721-726.

[143] J. Greenberg, *Estimates for fully developed shock solutions to the equation* $\partial u/\partial t - \partial v/\partial x = 0$ *and* $\partial v/\partial t - \partial\sigma(u)/\partial x = 0$, Indiana Univ. Math. J. **22** (1973), 989-1003.

[144] J. Greenberg and M. Rascle, *Time periodic solutions of conservation laws*, Arch. Rational Mech. Anal. **115** (1991), 395-407.

[145] E. Grenier, *Existence globale pour le système des gaz sans pression*, C. R. Acad. Sci. Paris Sr. I Math. **321** (1995), 171–174.

[146] G. Gripenberg, *Compensated compactness and one-dimensional elastodynamics*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), 227–240.

[147] C.-H. Gu, *A method for solving the supersonic flow past a curved wedge (in Chinese)*, Fudan Univ. J. **7** (1962), 11-14.

[148] C. Gu, D. Li, W. Yu, and Z. Hou, *Discontinuous initial-value problem for hyperbolic systems of quasilinear equations (I)-(III)*, Acta Math. Sinica, **11** (1961), 314-323, 324-327; **12** (1962), 132-143 (in Chinese).

[149] Y. Guo, *Smooth irrotational flows in the large to the Euler-Poisson system in* $\mathbf{R}^{3+1}$, Comm. Math. Phys. 195 (1998), 249-265.
[150] Y. Guo and A. S. Tahvildar-Zadeh, *Formation of singularities in relativistic fluid dynamics and in spherically symmetric plasma dynamics*, In: Nonlinear Partial Differential Equations, G.-Q. Chen and E. DiBenedetto (Eds.), Contemp. Math. **238**, pp. 151-161, AMS: Providence, 1999.
[151] E. Harabetian, *Diffraction of a weak shock by a wedge*, Comm. Pure Appl. Math. **40** (1987), 849–863.
[152] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag: Berlin, 1997.
[153] W. J. Hrusa and S. A. Messaoudi, *On formation of singularities in one-dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal. **111** (1990), 135-151.
[154] L. Hsiao, *Quasilinear Hyperbolic Systems and Dissipative Mechanisms*, World Scientific Publishing: Singapore, 1997.
[155] J. Hu and P. G. LeFloch, L^1 *continuous dependence property for systems conservation laws*, Arch. Rational Mech. Anal. **151** (2000), 45-93.
[156] J. Hunter and M. Brio, *Weak shock reflection*, J. Fluid Mech. **410** (2000), 235–261.
[157] J. Hunter and J. Keller, *Weak shock diffraction*, Wave Motion, **6** (1984),79–89; 321.
[158] H. K. Jenssen, *Blowup for systems of conservation laws*, SIAM J. Math. Anal. 31 (2000), 894-908.
[159] S. Jin and C. Levermore, *Numerical schemes for hyperbolic conservation laws with stiff relaxation terms*, J. Comput. Phys. **126** (1996), 449–467.
[160] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235-277.
[161] F. John, *Formation of singularities in one-dimensional nonlinear wave propagation*, Comm. Pure Appl. Math. **27** (1974), 377-405.
[162] F. John, *Blow-up for quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29-51.
[163] T. Kato, *Nonstationary flows of viscous and ideal fluids in* $\mathbb{R}^2$, J. Functional Anal. **9** (1972), 296-305.
[164] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **58** (1975), 181-205.
[165] T. Kato, *Quasi-linear equations of evolution with applications to partial differential equations*, Lecture Notes in Math. **448**, pp. 25-70, Springer-Verlag: Berlin, 1975.
[166] S. Kawashima, A. Matsumura, and T. Nishida, *On the fluid dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equations*, Commun. Math. Phys. **70** (1979), 97-124.
[167] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes*, Springer-Verlag: New York, 1987.
[168] J. B. Keller and A. Blank, *Diffraction and reflection of pulses by wedges and corners*, Comm. Pure Appl. Math. **4** (1951), 75–94.
[169] J. B. Keller and L. Ting, *Periodic vibrations of systems governed by nonlinear partial differential equations*, Comm. Pure Appl. Math. **19** (1966), 371-420.
[170] S. Klainerman and A. Majda, *Formation of singularities for wave equations including the nonlinear vibrating string*, Comm. Pure Appl. Math. **33** (1980), 241-263.
[171] W. Kosinski, *Gradient catastrophe in the solutions of nonconservative hyperbolic systems*, J. Math. Anal. Appl. **61** (1977), 672-688.
[172] S. Kruzkov, *First-order quasilinear equations with several space variables*, Mat. Sb. **123** (1970), 228-255.
[173] A. Kurganov and E. Tadmor, *Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers*, Preprint, UCLA, 2001.
[174] T. G. Kurtz, *Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics*, Trans. Amer. Math. Soc. **186** (1974), 259-272.
[175] P. D. Lax, *Nonlinear hyperbolic equations*, Comm. Pure Appl. Math. **6** (1953), 231-258.
[176] P. D. Lax, *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math. **7** (1954), 159-193.
[177] P. D. Lax, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), 537-566.
[178] P. D. Lax, *Development of singularities in solutions of nonlinear hyperbolic partial differential equations*, J. Math. Phys. **5** (1964), 611-613.
[179] P. D. Lax, *Shock waves and entropy*, In: Contributions to Nonlinear Functional Analysis, E. A. Zarantonello (ed.), pp. 603-634, Academic Press: New York, 1971.
[180] P. D. Lax, *The formation and decay of shock waves*, Amer. Math. Monthly, **79** (1972), 227-241.
[181] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM: Philadelphia, 1973.
[182] P. D. Lax, *The multiplicity of eigenvalues*, Bull. Amer. Math. Soc. (N.S.), **6** (1982), 213–214.

[183] P. D. Lax, *Problems solved and unsolved concerning linear and nonlinear partial differential equations*, Proceedings of the International Congress of Mathematicians, Vols. 1-2 (Warsaw, 1983), pp. 119–137, PWN: Warsaw, 1984.

[184] P. D. Lax and X.-D. Liu, *Solutions of two dimensional Riemann problem of gas dynamics by positive schemes*, SIAM J. Sci. Comput. **19** (1998), 319-340.

[185] H. I. Lee and D. S. Stewart, *Calculation of linear detonation instability: one dimensional instability of plane detonation*, J. Fluid Mech. **216** (1990), 103–132.

[186] P. G. LeFloch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, In: Nonlinear Evolution Equations that Change Type, 126–138, IMA Vol. Math. Appl. **27**, Springer-Verlag: New York, 1990.

[187] P. G. LeFloch, *Hyperbolic Systems of Conservation Laws : The theory of classical and nonclassical shock waves,* ETH Lecture Notes Series, Birkhäuser, 2002.

[188] C. D. Levermore, *Moment closure hierarchies for kinetic theories*, J. Stat. Phys. **83** (1996), 1021-1065.

[189] R. J. LeVeque, *Numerical Methods for Conservation Laws*, Second edition, Birkhäuser-Verlag: Basel, 1992.

[190] M. Lewicka, *L^1 stability of patterns of non-interacting large shock waves*, Indiana Math. Univ. J. **49** (2000), 1515–1537.

[191] M. Lewicka and K. Trivisa, *On the L^1 well-posedness of systems of conservation laws near solutions containing two large shocks*, J. Diff. Eqs. 2001 (to appear).

[192] J. Li, T. Zhang, and S. Yang, *The Two-Dimensional Riemann Problem in Gas Dynamics*, Longman (Pitman Monographs 98): Essex, 1998.

[193] T.-T. Li, *On a free boundary problem*, Chinese Ann. Math. **1** (1980), 351-358.

[194] T.-T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*, John Wiley & Sons, Ltd.: Chichester, 1994.

[195] T.-T. Li and W.-C. Yu, *Boundary Value Problems for Quasilinear Hyperbolic Systems*, Duke University Press: Durham, 1985.

[196] M. J. Lighthill, *The diffraction of a blast I*, Proc. Royal Soc. London, **198** (1949), 454-470.

[197] L. W. Lin, *On the vacuum state for the equations of isentropic gas dynamics*, J. Math. Anal. Appl. **121** (1987), 406-425.

[198] L. W. Lin, *A note on development of singularities of solutions of nonlinear hyperbolic partial differential equations*, Advances in nonlinear partial differential equations and related areas (Beijing, 1997), G.-Q. Chen et al (eds.), World Sci. Publishing, River Edge, NJ, 1998, 241–249.

[199] W.-C. Lien and T.-P. Liu, *Nonlinear stability of a self-similar 3-dimensional gas flow*, Commun. Math. Phys. **204** (1999), 525–549.

[200] P. X. Lin, *Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics*, Trans. Amer. Math. Soc. **329** (1992), 377–413.

[201] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Vols. 1-2, Oxford University Press: Oxford, 1998.

[202] P.-L. Lions, B. Perthame, and P. Souganidis, *Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Comm. Pure Appl. Math. **49** (1996), 599-638.

[203] P.-L. Lions, B. Perthame, and E. Tadmor, *Kinetic formulation of the isentropic gas dynamics and p-systems*, Commun. Math. Phys. **163** (1994), 169-172.

[204] T.-P. Liu, *Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **30** (1977), 585-610.

[205] T.-P. Liu, *Linear and nonlinear large-time behavior of solutions of hyperbolic conservation laws*, Comm. Pure Appl. Math. **30** (1977), 767-796.

[206] T.-P. Liu, *The development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations*, J. Diff. Eqs. **33** (1979), 92-111.

[207] T.-P. Liu, *Quasilinear hyperbolic systems*, Commun. Math. Phys. **68** (1979), 141-172.

[208] T.-P. Liu, *Nonlinear stability and instability of transonic gas flow through a nozzle*, Commun. Math. Phys. **83** (1983), 243-260.

[209] T.-P. Liu, *Nonlinear resonance for quasilinear hyperbolic equation*, J. Math. Phys. **28** (1987), 2593-2602.

[210] T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Commun. Math. Phys. **108** (1987), 153-175.

[211] T.-P. Liu, *The deterministic version of the Glimm scheme*, Commun. Math. Phys. **57** (1977), 135–148.

[212] T.-P. Liu, *Solutions in the large for the equations of nonisentropic gas dynamics*, Indiana Univ. Math. J. **26** (1977), 147-177.

[213] T.-P. Liu, *Initial-boundary value problems for gas dynamics*, Arch. Rational Mech. Anal. **64** (1977), 137–168.

[214] T.-P. Liu, *Hyperbolic and Viscous Conservation Laws*, CBMS-NSF Regional Conf. Series in Appl. Math. **72**, SIAM: Philadelphia, 2000.

[215] T.-P. Liu and T. Yang, *L^1 stability for systems of hyperbolic conservation laws, In, Nonlinear Partial Differential Equations and Applications*, G.-Q. Chen and E. DiBenedetto (eds.), Contemp. Math. **238**, pp. 183-192, AMS: Providence, 1999.

[216] T.-P. Liu and T. Yang, *L^1 stability of weak solutions for 2×2 systems of hyperbolic conservation laws*, J. Amer. Math. Soc. **12** (1999), 729-774.

[217] T.-P. Liu and T. Yang, *Well-posedness theory for hyperbolic conservation laws*, Comm. Pure Appl. Math. **52** (1999), 1553-1586.

[218] T. Luo and D. Serre, *Linear stability of shock profiles for a rate-type viscoelastic system with relaxation*, Quart. Appl. Math. **56** (1998), 569–586.

[219] M. Luskin and B. Temple, *The existence of a global weak solution to the nonlinear waterhammer problem*, Commun. Pure Appl. Math. **34** (1982), 697-735.

[220] R. C. MacCamy, *An integro-differential equation with application in heat flow*, Quart. Appl. Math. **35** (1977/78), 1-19.

[221] A. Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc. **275**, AMS: Providence, 1983.

[222] A. Majda, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc. **281**, AMS: Providence, 1983.

[223] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Applied Mathematical Sciences **53**, Springer-Verlag: New York, 1984.

[224] T. Makino, K. Mizohata, and S. Ukai, *Global weak solutions of the compressible Euler equations with spherical symmetry (I)-(II)*, Japan J. Indust. Appl. Math. **9** (1992), 431-449; **11** (1994), 417-426.

[225] T. Makino and S. Takeno, *Initial boundary value problem for the spherically symmetric motion of isentropic gas*, Japan J. Indust. Appl. Math. **11** (1994), 171-183.

[226] T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'iquations d'Euler compressible*, Japan J. Appl. Math. **3** (1986), 249-257.

[227] P. Marcati, and R. Natalini, *Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation*, Arch. Rational Mech. Anal. **129** (1995), 129-145.

[228] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag: Wien-New York, 1990.

[229] G. Métivier, *Stability of multi-dimensional weak shocks*, Comm. Partial Diff. Eqs. **15** (1990), 983–1028.

[230] C. S. Morawetz, *On a weak solution for a transonic flow problem*, Comm. Pure Appl. Math. **38** (1985), 797-818.

[231] C. S. Morawetz, *Potential theory for regular and Mach reflection of a shock at a wedge*, Comm. Pure Appl. Math. **47** (1994), 593–624.

[232] C. S. Morawetz, *On steady transonic flow by compensated compactness*, Meth. Appl. Anal. **2** (1995), 257-26

[233] C. S. Morawetz, *On the non-existence of continuous transonic flows past profiles I,II,III*, Comm. Pure Appl. Math. **9** (1956), 45-68; **10** (1957), 107-131; **11** (1958), 129-144.

[234] C. B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag: Berlin, 1966.

[235] J. Moser, *A rapidly convergent iteration method and non-linear partial differential equations I*, Ann. Scuola Norm. Sup. Pisa (3), **20** (1966), 265-315.

[236] I. Müller, and Y. Ruggeri, *Extended Thermodynamics*, Springer-Verlag: New York, 1993.

[237] F. Murat, *The injection of the positive cone of H^{-1} in $W^{-1,q}$ is completely continuous for all $q < 2$*, J. Math. Pures Appl. (9), **60** (1981), 309–322.

[238] F. Murat, *Compacite par compensation*, Ann. Suola Norm. Pisa (4), **5** (1978), 489-507.

[239] F. Murat, *Compacite par compensation, condition necessarie et suffisante de continuite faible sous une hypothese de range constant*, Ann. Scuola Norm. Sup. Pisa (4), **8** (1981), 69-102.

[240] F. Murat, *A survey on compensated compactness*, In: Contributions to Modern Calculus of Variations (Bologna, 1985), 145–183, Pitman Res. Notes Math. Ser. **148**, Longman Sci. Tech.: Harlow, 1987.

[241] R. Natalini, *Convergence to equilibrium for the relaxation approximations of conservation laws*, Comm. Pure Appl. Math. **49** (1996), 795–823.

[242] T. Nishida, *Global solution for an initial-boundary value problem of a quasilinear hyperbolic systems*, Proc. Japan Acad. **44** (1968), 642-646.

[243] T. Nishida, *Equations of fluid dynamics-free surface problems*, Comm. Pure Appl. Math. **39** (1986), 221-238.

[244] T. Nishida, *Equations of motion of compressible viscous fluids*, In: Patterns and Waves—Qualitative Analysis of Nonlinear Differential Equations, T. Nishida, M. Mimura, and H. Fujii (eds.), pp. 97-128, North-Holland: Amsterdam-New York, 1986.

[245] T. Nishida and J. Smoller, *Solutions in the large for some nonlinear hyperbolic conservation laws*, Comm. Pure Appl. Math. **26** (1973), 183-200.

[246] T. Nishida and J. Smoller, *Mixed problems for nonlinear conservation laws*, J. Diff. Eqs. **23** (1977), 244-269.

[247] J. A. Nohel, R. C. Rogers, and A. E. Tzavaras, *Weak solutions for a nonlinear system in viscoelasticity*, Commun. Partial Diff. Eqs. **13** (1988), 97-127.

[248] M. Okada and T. Makino, *Free boundary problem for the equation of spherically symmetrical motion of viscous gas*, Japan J. Appl. Math. **10** (1993), 219-235.

[249] O. Oleinik, *Discontinuous solutions of nonlinear differential equations*, Usp. Mat. Nauk. (N.S.) **12** (1957), 3-73.

[250] S. Osher, *Shock modelling in transonic and supersonic flow*, In: Advanced in Computational Transonics, 607-643, Recent Adv. Numer. Methods Fluids, **4**, Pineridge, Swansea, 1985.

[251] A. K. Oppenheim and R. I. Soloukhin, *Experiments in gasdynamics of explosions*, Ann. Rev. Fluid Mech. **5** (1973), 31–58.

[252] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag: New York.

[253] Y.-J. Peng, *Solutions faibles globales pour l'èquation d'Euler d'un fluide compressible avec de grandes données initiales* (in French), Comm. Partial Diff. Eqs. **17** (1992), 161–187.

[254] B. Perthame, *Nonexistence of global solutions to the Euler-Poisson equations for repulsive forces*, Japan J. Appl. Math. **7** (1990), 363-367.

[255] B. Perthame, *Kinetic Formations of Conservation Laws*, Lecture Notes, Ecole Normale Supérieure at Paris, France, 2001.

[256] M. A. Pinsky, *Lectures on Random Evolutions*, World Scientific: Singapore, 1991.

[257] S. D. Poisson, *Memoire sur l'intéegration des equations lineaires aux derivees partielles*, J. L'ecole Polytechnique, **12** (1823), 19.

[258] F. Poupaud, M. Rascle, J.-P. Vila, *Global solutions to the isothermal Euler-Poisson system with arbitrarily large data*, J. Diff. Eqs. **123** (1995), 93-121.

[259] B. Keyfitz Quinn, *Time-decreasing functionals of nonlinear conservation laws*, Ph.D. thesis, New York University, 1970.

[260] F. Poupaud and M. Rascle, *Measure solutions to the linear multi-dimensional transport equation with non-smooth coefficients*, Commun. Partial Diff. Eqs. **22** (1997), 337–358.

[261] M. A. Rammaha, *Formation of singularities in compressible fluids in two-space dimensions*, Proc. Amer. Math. Soc. **107** (1989), 705-714.

[262] M. A. Rammaha, *On the formation of singularities in magnetohydrodynamic waves*, J. Math. Anal. Appl. **188** (1994), 940-955.

[263] A. D. Rendall, *The initial value problem for self-gravitating fluid bodies*, In: Mathematical Physics X (Leipzig, 1991), pp. 470-474, Springer-Verlag: Berlin, 1992.

[264] H. K. Rhee, R. Aris, and N. R. Amundsen, *First-Order Differential Equations, Theory and Application of Hyperbolic Systems of Quasilinear Equations*, Prentice Hall International Series I, 1986; II, 1989.

[265] B. Riemann, *Uber die Fortpflanzung ebener Luftwellen von endlier Schwingungsweite*, Abhandl. Koenig. Gesell. Wiss., Goettingen, **8** (1860), 43.

[266] N.H. Risebro, *A front-tracking alternative to the random choice method*, Proc. Amer. Math. Soc. **117** (1993), 1125-1139.

[267] D. Schaeffer and M. Shearer, *The classification of* 2×2 *systems of nonstrictly hyperbolic conservation laws, with application to oil recovery*, Comm. Pure Appl. Math. **40** (1987), 141-178.

[268] D. Schaeffer and M. Shearer, *Riemann problems for nonstrictly hyperbolic* 2×2 *systems of conservation laws*, Trans. Amer. Math. Soc. **304** (1987), 267-306.

[269] S. Schochet, *Sufficient conditions for local existence via Glimm's scheme for large BV data*, J. Diff. Eqs. **89** (1991), 317-354.

[270] C. W. Schulz-Rinne, J. P. Collins, and H. M. Glaz, *Numerical solution of the Riemann problem for two-dimensional gas dynamics*, SIAM J. Sci. Comput. **14** (1993), 1394-1414.

[271] L. Schwartz, *Théorie des Distributions*, Actualites Scientifiques et Industrielles 1091, 1122, Herman: Paris, 1950-51.

[272] D. Serre, *La compacitè par compensation pour les systemes hyperboliques non linéaires de deux équations une dimension d'espace*, J. Math. Pure Appl. (9) **65** (1986), 423-468.

[273] D. Serre, *Solutions classiques globales des quations d'Euler pour un fluide parfait compressible*, Ann. Inst. Fourier, **47** (1997), 139–153.

[274] D. Serre, *Spiral waves for the two-dimensional Riemann problem for a compressible fluid* (in French), Ann. Fac. Sci. Toulouse Math. (6), **5** (1996), 125–135.

[275] D. Serre, *Perfect fluid flow in two independent space variables: Reflection of a planar shock by a compressive wedge* (in French), Arch. Rational Mech. Anal. **132** (1995), 15–36.

[276] D. Serre, *Relaxations semi-linéaire et cinétique des systèmes de lois de conservation* (in French), Ann. Inst. H. Poincaré Anal. Non Linéaire, **17** (2000), 169–192.

[277] D. Serre, *Systems of Conservation Laws, I, II,* Cambridge University Press: Cambridge, 1999.

[278] J. Shearer, *Global existence and compactness in L^p for the quasi-linear wave equation*, Commun. Partial Diff. Eqs. **19** (1994), 1829–1877.

[279] W. Shen, A. Tveito, and R. Winther, *On the zero relaxation limit for a system modeling the motions of a viscoelastic solid*, SIAM J. Math. Anal. **30** (1999), 1115–1135.

[280] W.-C. Sheng and T. Zhang, *The Riemann Problem for the Transportation Equations in Gas Dynamics*, Mem. Amer. Math. Soc. **654**, AMS: Providence, 1999.

[281] T. C. Sideris, *Global behavior of solutions to nonlinear wave equations in three dimensions*, Commun. Partial Diff. Eqs. **8** (1983), 1291-1323.

[282] T. C. Sideris, *Formation of singularities in solutions to nonlinear hyperbolic equations*, Arch. Rational Mech. Anal. **86** (1984), 369-381.

[283] T. C. Sideris, *Formation of singularities in three-dimensional compressible fluids*, Commun. Math. Phys. **101** (1985), 475-485.

[284] T. C. Sideris, *The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit*, Indiana Univ. Math. J. **40** (1991), 535-550.

[285] T. C. Sideris, *Delayed singularity formation in 2D compressible flow*, Amer. J. Math. **119** (1997), 371-422.

[286] T. C. Sideris and D. Wang, *Multidimensional compressible Euler equations with damping*, preprint, 2001.

[287] M. Slemrod, *Instability of steady shearing flows in a non-linear viscoelastic fluid*, Arch. Rational Mech. Anal. **68** (1978), 211-225.

[288] M. Slemrod, *Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal. **76** (1981), 97-133.

[289] M. Slemrod, *Resolution of the spherical piston problem for compressible isentropic gas dynamics via a self-similar viscous limit,* Proc. Roy. Soc. Edinburgh, **126A** (1996), 1309–1340.

[290] J. Smith, *The Riemann problem in gas dynamics,* Trans. Amer. Math. Soc. **249** (1979), 1-50.

[291] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, 2nd Ed., Springer-Verlag: New York, 1994.

[292] J. Smoller, and B. Temple, *Global solutions of the relativistic Euler equations*, Comm. Math. Phys. **156** (1993), 67–99.

[293] J. J. Stokes, *Water Waves, The Mathematical Theory with Applications*, John Wiley & Sons, Inc.: New York, 1958.

[294] W. A. Strauss, *Nonlinear Wave Equations*, AMS: Providence, 1989.

[295] E. Tabak and R. Rosales, *Focusing of weak shock waves and the von Neumann paradox of oblique shock reflection*, Phys. Fluids, **6** (1994), 1874–1892.

[296] E. Tadmor, *Approximate solutions of nonlinear conservation laws and related equations*, In: Recent Advances in Partial Differential Equations, Venice, 1996, Proc. Sympos. Appl. Math. **54**, AMS: Providence, R. I. 1998.

[297] E. Tadmor, *A minimum entropy principle in the gas dynamics equations*, Appl. Numer. Math. **2** (1986), 211-219.

[298] D. Tan and T. Zhang, *Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws*, J. Diff. Eqs. **111** (1994), 203-283.

[299] L. Tartar, *Compensated compactness and applications to partial differential equations. In, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium IV*, Res. Notes in Math. **39**, pp. 136-212, Pitman: Boston-London, 1979.

[300] L. Tartar, *Une nouvelle methode de resolution d'equations aux derivees partielles nonlineaires*, Lecture Notes in Math. **665**, pp. 228-241, Springer-Verlag: Berlin, 1977.

[301] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, In: Systems of Nonlinear P.D.E., J. M. Ball (eds.), pp. 263–285, NATO Series, C. Reidel publishing Co. 1983.

[302] B. Temple. *Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics*, J. Diff. Eqs. **41** (1981), 96–161.

[303] B. Temple and R. Young, *The large time existence of periodic solutions for the compressible Euler equations*, Mat. Contemp. **11** (1996), 171-190.

[304] B. Temple and R. Young, *The large time stability of sound waves*, Commun. Math. Phys. **179** (1996), 417-466.
[305] A. Tesdall and J. K. Hunter, *Self-similar solutions for weak shock reflection*, Preprint, 2001.
[306] E. F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics, A Practical Introduction*, Second edition, Springer-Verlag: Berlin, 1999.
[307] K. Trivisa, *A priori estimates in hyperbolic systems of conservation laws via generalized characteristics*, Commun. Partial Diff. Eqs. **22** (1997), 235-267.
[308] C. Truesdell, *Rational Thermodynamics*, 2nd Ed., Springer-Verlag: New York, 1984.
[309] A. Tveito and R. Winther, *On the rate of convergence to equilibrium for a system of conservation laws with a relaxation term*, SIAM J. Math. Anal. **28** (1997), 136–161.
[310] A. Tzavaras, *Materials with internal variables and relaxation to conservation laws*, Arch. Rational Mech. Anal. **146** (1999), 129–155.
[311] W. G. Vincenti and C. H. Kruger, Jr., *Introduction to Physical Gasdynamics*, R. E. Krieger Publication Co. 1982.
[312] A. Volpert, *The Space BV and quasilinear equations*, Mat. Sb. **73** (1967), 255-302.
[313] V. Volterra, *Sulle vibrazioni luminose nei mezzi isotropi*, Rend. Accad. Nazl. Lincel, **1** (1892), 161-170.
[314] D. H. Wagner, *Equivalence of Euler and Lagrangian equations of gas dynamics for weak solutions*, J. Diff. Eqs. **68** (1987), 118-136.
[315] D. Wang, *Global solutions to the Euler-Poisson equations of two-carrier types in one dimension*, Z. Angew. Math. Phys. **48** (1997), 680-693.
[316] D. Wang, *On the Cauchy problem of some dissipative flows*, J. Math. Anal. Appl. **218** (1998), 479-494.
[317] D. Wang, *Global solutions and stability for self-gravitating isentropic gases*, J. Math. Anal. Appl. **229** (1999), 530-542.
[318] D. Wang, *Global solutions to the equations of viscous gas flows*, Proc. Roy. Soc. Edinburgh Sect. A. **131** (2001), 437-449.
[319] D. Wang, *Global solutions and relaxation limits of Euler-Poisson equations*, Z. Angew. Math. Phys. **52** (2001), 620-630.
[320] D. Wang and G.-Q. Chen, *Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation*, J. Diff. Eqs. **144** (1998), 44-65.
[321] W.-C. Wang and Z. Xin, *Asymptotic limit of initial boundary value problems for conservation laws with relaxational extensions,* Comm. Pure Appl. Math. **51** (1998), 505-535.
[322] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience: New York, 1973.
[323] P. Woodward and P. Colella, *The numerical simulation of two-dimensional fluid flow with strong shocks*, J. Comp. Phys. **54** (1984), 115-173.
[324] X.-M. Wu, *Equations of Mathematical Physics* (in Chinese), Chinese Advanced Education Publishing Co.: Beijing, 1956.
[325] G.-J. Yang, *The Euler-Poisson-Darboux Equations* (in Chinese), Yuannan University Press: Yuannan, 1989.
[326] X. Yang and F. Huang, *Two-dimensional Riemann problems of simplified Euler equation*, Chinese Sci. Bull. **43** (1998), 441–444.
[327] T. Yang, *A functional integral approach to shock wave solutions of Euler equations with spherical symmetry*, Commun. Math. Phys. **171** (1995), 607-638.
[328] L. Ying and Z. Teng, *Hyperbolic Systems of Conservation Laws and Difference Methods* (in Chinese), Chinese Scientific Publishing Co.: Beijing, 1991 (in Chinese).
[329] L. Ying and C.-H. Wang, *Global solutions of the Cauchy problem for a nonhomogeneous quasilinear hyperbolic system*, Commun. Pure Appl. Math. **33** (1980), 579-597.
[330] L. Ying and C.-H. Wang, *Solution in the large for nonhomogeneous quasilinear hyperbolic systems of equations*, J. Math. Anal. Appl. **78** (1980), 440-454.
[331] L. C. Young, *Lectures on the Calculus of Variations and Optimal Control Theories*, W. B. Sauders: Philadelphia, 1969.
[332] A. R. Zakharian, M. Brio, J. K. Hunter, and G. M. Webb, *The von Neumann paradox in weak shock reflection*, J. Fluid Mech. **422** (2000), 193-205.
[333] Y.-Q. Zhang, *Global existence of steady supersonic potential flow past a curved wedge with a piecewise smooth boundary,* SIAM J. Math. Anal. **31** (1999), 166–183.
[334] T. Zhang and Y.-F. Guo, *A class of initial-value problem for systems of aerodynamic equations*, Acta Math. Sinica, **15** (1965), 386-396.
[335] T. Zhang and Y. Zheng, *Conjecture on the structure of solutions of the Riemann problem for two-dimensional gas dynamics*, *SIAM J. Math. Anal.* **21** (1990), 593-630.

[336] T. Zhang and Y. Zheng, *Axisymmetric solutions of the Euler equations for polytropic gases*, Arch. Rational Mech. Anal. **142** (1998), 253–279.

[337] T. Zhang and Y. Zheng, *Exact spiral solutions of the two-dimensional Euler equations*, Discrete Contin. Dynam. Systems, **3** (1997), 117–133.

[338] Y. Zheng, *Systems of Conservation Laws: Two-Dimensional Riemann Problems*, Birkhäuser: Boston, 2001.

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