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THE CAUCHY PROBLEM FOR THE SEMILINEAR QUINTIC SCHRÖDINGER EQUATION IN ONE DIMENSION

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Abstract. We show that the Cauchy problem for the quintic NLS on $\mathbf{R}$ is globally well posed in H^s for $4/9 < s \leq 1/2$. Since we work below the energy space we cannot immediately use the energy. Instead we use the "I-method" introduced by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. This method allows us to define a modification of the energy functional that is "almost conserved" and thus can be used to iterate the local result.

1. Introduction

We consider the semilinear Schrödinger initial-value problem (IVP)

$$iu_t + u_{xx} - |u|^4 u = 0, \quad u(x,0) = u_0(x) \in H^s(\mathbf{R}),\ t \in \mathbf{R}. \tag{1.1}$$

This equation is proposed as a modification of the Gross-Pitaevski (GP) approximation in low-dimensional Bose Liquids [8]. The GP approximation is a long-wavelength theory widely used to describe a variety of properties of dilute Bose condensates. However, in low dimensions ($d \leq 2$), an essential modification of the GP theory is necessary. In this context equation (1.1) is the modification of the GP equations in the important one-dimensional case where the deviations from the GP theory are largest. See in particular equations (4) and (10) in [8]. In addition to the physical motivation, we note that (1.1) is the one-dimensional, L^2-critical nonlinear Schrödinger equation and there is therefore purely mathematical motivation for its detailed study. The Cauchy problem for equation (1.1) is known to be locally well posed for $s > 0$. This result was proved by T. Cazenave and F. B. Weissler [3]. A local result also exists for s=0 but the time of existence depends on the profile of the data as well as the norm. The equation satisfies the following two conservation laws, among others that we will not use in this paper.

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Mass conservation:

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

And energy conservation:

$$E(u)(t) = \frac{1}{2}\int |u_x(t)|^2 dx + \frac{1}{6}\int |u(t)|^6 dx = E(u_0).$$

Since we are in the defocussing case we can iterate to get a global solution for $s \geq 1$ using the energy conservation. Below the energy space the best global result is due to J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [4],[5]. They proved that (1.1) is globally well posed for $s > 1/2$. To do so, they introduced the "modified energy" functional and they proved that it is almost conserved. The precise meaning of the term almost conserved will be apparent after some definitions. In this article we extend this result and prove global well posedness, in other words we prove that a global solution exists for all time for $u_0 \in H^s$, $4/9 < s \leq 1/2$.

Remark. Note that in the focussing case (where in the front of the nonlinearity we have the plus instead of the minus sigh) we can also prove global well posedness for $4/9 < s \leq 1/2$ following step by step the proofs below, but with the crucial assumption that $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q is the unique positive solution (up to translations) of

$$Q_{xx} - Q + |Q|^4 u = 0.$$

For a comprehensive review of the most important properties of the semilinear NLS the reader should consult [2].

2. Notation

We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C. If there exist constants C and D such that $DB \leq A \leq CB$ we say that $A \sim B$ and $A \gg B$ if there does not exist a constant C such that $A \leq CB$. If $\|f\|_{H^s} = \|\langle\xi\rangle^s \hat{f}(\xi)\|_{L^2_\xi}$ is the inhomogeneous Sobolev norm then we can define the $X_{s,b}$ spaces as the set of tempered distributions such that

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s \langle\tau - |\xi|^2\rangle^b \hat{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi} < \infty,$$

where

$$\hat{u}(\xi,\tau) = \int\int e^{-i(x\xi + t\tau)} u(x,t)\, dt\, dx$$

is the space-time Fourier transform of u and $\langle\xi\rangle = 1 + |\xi|$. We also define the restricted $X_{s,b}(I \times R)$ spaces by

$$\|u\|_{X_{s,b}(I\times R)} = \inf\{\|U\|_{s,b} : U|_{I\times R} = u\}.$$

With D we define the operator with symbol $|\xi|$ and J stands for the Bessel potential of order 1, or in other terms $J = \langle D \rangle$. We now give some useful notation for multilinear expressions. If $n \geq 2$ is an even integer we define a spatial multiplier of order n to be the function $M_n(\xi_1, \xi_2, \dots, \xi_n)$ on the hyperplane $\Gamma_n = \{(\xi_1, \xi_2, \dots, \xi_n) \in R^n : \xi_1 + \xi_2 + \ldots + \xi_n = 0\}$ which we endow with the standard measure $\delta(\xi_1 + \xi_2 + \ldots + \xi_n)$. If M_n is a multiplier of order n, $1 \leq j \leq n$ is an index, and $k \geq 1$ is an even integer, we define the elongation $X_j^k(M_n)$ of M_n to be the multiplier of order $n+k$ given by

$$X_j^k(M_n)(\xi_1, \xi_2, \dots, \xi_{n+k}) = M_n(\xi_1, \dots, \xi_{j-1}, \xi_j + \ldots + \xi_{j+k}, \xi_{j+k+1}, \dots, \xi_{n+k}).$$

In addition if M_n is a multiplier of order n and $f_1, f_2, ..., f_n$ are functions on $\mathbf{R}$ we define

$$\Lambda_n(M_n; f_1, f_2, ..., f_n) = \int_{\Gamma_n} M_n(\xi_1, \xi_2, \dots, \xi_n) \prod_{i=1}^{n} \hat{f}_j(\xi_j),$$

where we adopt the notation $\Lambda_n(M_n; f) = \Lambda_n(M_n; f, \bar{f}, ..., f, \bar{f})$. Observe that $\Lambda_n(M_n; f)$ is invariant under permutations of the even ξ_j indices, or of the odd ξ_j indices.

3. The "I-method" and the basic theorem

The "I-method" for the Schrödinger equation in one dimension was developed in [4],[5]. Traditionally, to prove global well posedness in H^1 we use the energy given by

$$E(u) = \frac{1}{2} \int_{\mathbf{R}} |\partial_x u|^2 dx + \frac{1}{6} \int_{\mathbf{R}} |u|^6 dx$$

which can be written using the multilinear notation as

$$E(u) = -\frac{1}{2} \Lambda_2(\xi_1 \xi_2; u) + \frac{1}{6} \Lambda_6(1; u).$$

In our case since we work in H^s with $s < 1$ we cannot use the energy $E(u)$. So we are looking for a substitute notion of "energy" that can be defined for a less regular solution and that has a very slow increment in time, with respect to a large parameter N. This will be enough to establish our global result. To do so we consider in the frequency space a C^∞ monotone multiplier $m(\xi)$ taking values in [0,1] such that

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| < N \\ (\frac{|\xi|}{N})^{s-1} & \text{if } |\xi| > 2N, \end{cases}$$

where $N \gg 1$ is a fixed large number which we shall choose later. Next we define the multiplier operator $I : H^s \to H^1$ such that

$$\widehat{(Iu)}(\xi) = m(\xi)\hat{u}(\xi).$$

This operator is smoothing of order $1-s$. Indeed we have:

$$\|u\|_{s_0,b_0} \lesssim \|Iu\|_{s_0+1-s,b_0} \lesssim N^{1-s}\|u\|_{s_0,b_0} \tag{3.1}$$

for any $s_0, b_0 \in \mathbf{R}$. Our substitute energy will be defined by $E^1(u) = E(Iu)$. Obviously this energy makes sense even if u is only in H^s. Thus

$$E(Iu) = \frac{1}{2}\int_{\mathbf{R}} |\partial_x Iu|^2 dx + \frac{1}{6}\int_{\mathbf{R}} |Iu|^6 dx = -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{6}\Lambda_6(m_1 ... m_6),$$

where $m_j = m(\xi_j)$. We also define the second energy

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{6}\Lambda_6(M_6(\xi_1, \xi_2, ..., \xi_6)),$$

where $M_6(\xi_1, \xi_2, ..., \xi_6)$ is a multiplier to be chosen later. The reason that we will use the second energy rather than the first is that, as we shall prove shortly, the second energy has better decay-with-N properties. For an example of an equation where the first energy is used, see [4]. Using equation (1.1) we can write

$$u_t = iu_{xx} - iu\bar{u}u\bar{u}u$$

and

$$\bar{u}_t = i\bar{u}_{xx} - i\bar{u}u\bar{u}u\bar{u}.$$

Now taking the Fourier transform of these identities we have the following simple calculation

$$\begin{aligned}
&\frac{d}{dt}\Lambda(M_2; u(t)) = \frac{d}{dt}\int_{\Gamma_2} M_2(\xi_1, \xi_2)\hat{u}_1(\xi_1)\hat{\bar{u}}_2(\xi_2) \\
&= \int_{\Gamma_2} M_2(\xi_1, \xi_2)(\frac{d}{dt}\hat{u}_1(\xi_1))\hat{\bar{u}}_2(\xi_2) + \int_{\Gamma_2} M_2(\xi_1, \xi_2)(\frac{d}{dt}\hat{\bar{u}}_2(\xi_2))\hat{u}_1(\xi_1) \\
&= \int_{\Gamma_2} M_2(\xi_1, \xi_2)\{-i\xi_1^2\hat{u}_1(\xi_1) - i(u_1\widehat{\bar{u}_1 u_1 \bar{u}_1} u_1(\xi_1))\}\hat{\bar{u}}_2(\xi_2) \\
&+ \int_{\Gamma_2} M_2(\xi_1, \xi_2)\{i\xi_2^2\hat{\bar{u}}_2(\xi_2) + i(\bar{u}_2\widehat{u_2\bar{u}_2 u_2}\bar{u}_2(\xi_2))\}\hat{u}_1(\xi_1) \\
&= i\Lambda_2\Big(M_2\sum_{j=1}^{2}(-1)^j\xi_j^2; u(t)\Big) - i\Lambda_6\Big(\sum_{j=1}^{2}(-1)^j X_j^4(M_2); u(t)\Big).
\end{aligned}$$

Thus, in general we can prove the following differentiation law for the multilinear forms:

$$\partial_t\Lambda_n(M_n) = i\Lambda_n(M_n\sum_{j=1}^{n}(-1)^j\xi_j^2) - i\Lambda_{n+4}(\sum_{j=1}^{n}(-1)^jX_j^4(M_n)). \tag{3.2}$$

Our idea is to consider the second energy and use the above differentiation law. Then we get multilinear forms of type Λ_2, Λ_6, Λ_{10}. If we chose M_6 properly we can simplify the expression for the derivative of the second energy. Thus by differentiating

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1\xi_1m_2\xi_2) + \frac{1}{6}\Lambda_6(M_6(\xi_1,\xi_2,...,\xi_6))$$

we have

$$\begin{aligned}\frac{d}{dt}E^2(u) &= -\frac{1}{2}\frac{d\Lambda_2}{dt}(m_1\xi_1m_2\xi_2;u) + \frac{1}{6}\frac{d\Lambda_6}{dt}(M_6;u)\\ &= \frac{-i}{2}\Lambda_2\Big(m_1\xi_1m_2\xi_2\sum_{j=1}^{2}(-1)^j\xi_j^2;u(t)\Big)\\ &\quad+\frac{i}{2}\Lambda_6\Big(\sum_{j=1}^{2}(-1)^jX_j^4(m_1\xi_1m_2\xi_2);u(t)\Big)\\ &\quad+\frac{i}{6}\Lambda_6\Big(M_6(\xi_1,\dots,\xi_6)\sum_{j=1}^{6}(-1)^j\xi_j^2;u(t)\Big)\\ &\quad-\frac{i}{6}\Lambda_{10}\Big(\sum_{j=1}^{6}(-1)^jX_j^4(M_6(\xi_1,\dots,\xi_6));u(t)\Big).\end{aligned}$$

Since on Γ_2 we have $\xi_1+\xi_2=0$ the first term is zero. By symmetrizing and picking

$$M_6(\xi_1,\xi_2,...,\xi_6) = \frac{m_1^2\xi_1^2 - m_2^2\xi_2^2 + m_3^2\xi_3^2 - m_4^2\xi_4^2 + m_5^2\xi_5^2 - m_6^2\xi_6^2}{\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2}$$

we can force Λ_6 to be zero. We will show shortly, in Proposition 1, that M_6 is well defined and bounded. The only multilinear form is now Λ_{10} and its multiplier is a sum of elongations of M_6. As we show below this multiplier is bounded, which is the main ingredient in exploiting the decay of the derivative of the second energy. For calculations of this type, although in a different context, see [5] where a Schrödinger equation with derivative nonlinearity is treated. Finally, using the fundamental theorem of calculus we have the following lemma.

Lemma 1. *Let u be an H^1 solution to* (1.1)*. Then for any $T \in \mathbf{R}$ and $\delta > 0$ we have*

$$E^2(u(T+\delta)) - E^2(u(T)) = \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t))dt,$$

where $M_{10} = -\frac{i}{5!6!}\sum\{M_6(\xi_{abcde}, \xi_f, \xi_g, \xi_h, \xi_i, \xi_j) - M_6(\xi_a, \xi_{bcdef}, \xi_g, \xi_h, \xi_i, \xi_j) + M_6(\xi_a, \xi_b, \xi_{cdefg}, \xi_h, \xi_i, \xi_j) - M_6(\xi_a, \xi_b, \xi_c, \xi_{defgh}, \xi_i, \xi_j) + M_6(\xi_a, \xi_b, \xi_c, \xi_d, \xi_{efghj}, \xi_j) - M_6(\xi_a, \xi_b, \xi_c, \xi_d, \xi_e, \xi_{fghij})\}$*, where the summation runs over all permutations* $\{a, c, e, g, i\} = \{1, 3, 5, 7, 9\}$ *and* $\{b, d, f, h, j\} = \{2, 4, 6, 8, 10\}$*. Furthermore if $|\xi_j| \ll N$ for all j then the multiplier M_{10} vanishes.*

Proof. Only the last statement needs a comment. Notice that if all $|\xi_j| \ll N$ then $m_i = 1$ and consequently $M_6 = 1$ and $M_{10} = 0$.

To iterate the global result by standard limiting arguments we just need an a priori bound for our solutions in H^s. This bound comes from the next theorem.

Theorem 1. *Let u be a global H^1 solution to* (1.1)*. Then for any $T > 0$ and $s > 4/9$ we have that*

$$\sup_{0 \le t \le T} \|u(t)\|_{H^s} \lesssim C_{(\|u_0\|_{H^s}, T)},$$

where the right-hand side does not depend on the H^1 norm of u.

To prove this theorem we need 4 propositions.

Proposition 1. *Assume M_6 is the multiplier given by*

$$M_6(\xi_1, \xi_2, ..., \xi_6) = \frac{m_1^2\xi_1^2 - m_2^2\xi_2^2 + m_3^2\xi_3^2 - m_4^2\xi_4^2 + m_5^2\xi_5^2 - m_6^2\xi_6^2}{\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2}.$$

Then $|M_6| \lesssim C$ on Γ_6.

Remark. That the multiplier is bounded on Γ_6, can be checked automatically, using for example any mathematics computer program. Nevertheless, since the analysis is interesting we will prove it below, showing all the crucial steps. Before we start we fix some notation. We define $N_i := |\xi_i|$ and we write $N_1^* \ge N_2^* \ge N_3^* \ge N_4^*$ for the highest, second highest, third highest, and fourth highest values of the frequencies $N_1, N_2, ..., N_6$. Since $\xi_1 + \xi_2 + \ldots + \xi_6 = 0$ we must have that $N_1^* \sim N_2^*$. Finally, notice that if $N_1^* < N$, then $M_6 = 1$. Thus, $N_1^* \gtrsim N$.

Proof. By symmetry we can assume that $N_1^* = |\xi_1|$. Obviously, away from the singular set, the multiplier is bounded. So the only interesting case is when

$$|\xi_1|^2 + |\xi_3|^2 + |\xi_5|^2 \sim |\xi_2|^2 + |\xi_4|^2 + |\xi_6|^2. \tag{3.3}$$

Note that if all the frequencies are equal or equivalent, then M_6 is bounded, then

$$|M_6(\xi_1,\xi_2,...,\xi_6)| \sim |\frac{m_1^2\xi_1^2 - m_2^2\xi_2^2}{\xi_1^2-\xi_2^2}| \sim |m_1^2|$$

and thus $|M_6| \lesssim 1$.

Case 1. $N_1^* \sim N_2^* \gg N_3^*$. *a*) Let $N_2^* = |\xi_3| \sim N_1^*$. Since $N_1^* = |\xi_1|$ we have $2N_1^{*2} + |\xi_5|^2 \sim |\xi_2|^2 + |\xi_4|^2 + |\xi_6|^2$. But $|\xi_2|, |\xi_4|, |\xi_5|, |\xi_6| \lesssim N_3^* \ll N_1^*$ and this contradicts the previous relation.

b) Let $N_2^* = |\xi_2|$. Then again $N_1^{*2} + |\xi_3|^2 + |\xi_5|^2 \sim |\xi_4|^2 + |\xi_6|^2$ and we have again a contradiction unless $|\xi_1|$ is very close to $|\xi_2|$ in which case equation (3.3) is reduced to

$$|\xi_3|^2 + |\xi_5|^2 \sim |\xi_4|^2 + |\xi_6|^2$$

and we can continue as before.

Case 2. $N_1^* \sim N_2^* \sim N_3^*$. a) Let $|\xi_3| = N_2^* \sim N_1^*$. Then $2N_1^{*2} + |\xi_5|^2 \sim |\xi_2|^2 + |\xi_4|^2 + |\xi_6|^2$.

If $|\xi_5| = N_3^* \sim N_1^*$, we have $N_1^{*2} \sim |\xi_2|^3 + |\xi_4|^2 + |\xi_6|^2$ and either $|\xi_2| \sim |\xi_4| \sim |\xi_6| \sim N_1^*$, in which case all frequencies are equivalent, or at least one of $|\xi_2|, |\xi_4|, |\xi_6|$ is equivalent to zero. For example if $|\xi_2| \sim 0$, then

$$|M_6(\xi_1,\xi_2,...,\xi_6)| \sim |\frac{m_1^2\xi_1^2 + m_3^2\xi_3^2 - m_4^2\xi_4^2 + m_5^2\xi_5^2 - m_6^2\xi_6^2}{\xi_1^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2}|$$
$$\sim |\frac{m_1^2\xi_1^2 - m_6^2\xi_6^2}{\xi_1^2-\xi_6^2}| \sim C$$

since $|\xi_1| \sim |\xi_6|$.

If $|\xi_2| = N_3^* \sim N_1^*$, we have $N_1^{*2} + |\xi_5|^2 \sim |\xi_4|^2 + |\xi_6|^2$ where at least one of $|\xi_2|, |\xi_4|, |\xi_6|$ is equivalent to N_1^*.

If $|\xi_5| \sim N_1^*$, then $N_1^{*2} \sim |\xi_4|^2 + |\xi_6|^4$ and the same analysis as above shows that M_6 is bounded.

If $|\xi_4| \sim N_1^*$, then $N_1^{*2} \sim |\xi_6|^2 - |\xi_5|^2$ and if both frequencies are equivalent to N_1^* we are done; so assume that $|\xi_5| \ll N_1^*$.

But $m^2(\xi)\xi^2$ is an increasing function and thus

$$|M_6(\xi_1,\xi_2,...,\xi_6)| \lesssim |\frac{2m_1^2\xi_1^2 - m_2^2\xi_2^2 + m_3^2\xi_3^2 - m_4^2\xi_4^2 - m_6^2\xi_6^2}{\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2-\xi_6^2}|$$
$$\sim |\frac{m_1^2\xi_1^2 - m_6^2\xi_6^2}{\xi_1^2-\xi_6^2}| \sim C$$

b) Let $|\xi_2| = N_2^* \sim N_1^*$. Then $N_1^{*2} + |\xi_3|^2 + |\xi_5|^2 \sim |\xi_4|^2 + |\xi_6|^2$. If $|\xi_5| = N_3^* \sim N_1^*$, then $N_1^{*2} + |\xi_5|^2 \sim |\xi_4|^2 + |\xi_6|^2$ and the same analysis as before gives that $|M_6| \lesssim 1$.

If $|\xi_4| = N_3^* \sim N_1^*$, then $N_1^{*2} + |\xi_3|^2 + |\xi_5|^2 \sim |\xi_6|^2$. But this implies that $|\xi_6| \sim N_1^*$, and either $|\xi_3|, |\xi_5| \sim N_1^*$, in which case all frequencies are equivalent, or at least one of ξ_3, ξ_5 are close to zero. But this case is treated above. Thus in all cases $|M_6| \lesssim 1$.

Before we state the second proposition let us give the basic estimates that we use throughout this paper. In our arguments we will often use the trivial embedding $\|u\|_{s_1,b_1} \leq \|u\|_{s_2,b_2}$, whenever $s_1 \leq s_2$ and $b_1 \leq b_2$, and the following Strichartz estimates

$$\|u\|_{L_t^6 L_x^6} \leq C\|u\|_{0,1/2+} \tag{3.4}$$

and

$$\|u\|_{L_t^\infty L_x^2} \leq C\|u\|_{0,1/2+}. \tag{3.5}$$

From (3.5) and Sobolev embedding we have that

$$\|u\|_{L_t^\infty L_x^\infty} \leq C\|u\|_{1/2+,1/2+}. \tag{3.6}$$

Finally, interpolation between (3.6) and the trivial estimate

$$\|u\|_{L_t^2 L_x^2} \leq C\|u\|_{0,1/2+}$$

gives us

$$\|u\|_{L_t^p L_x^p} \leq C\|u\|_{\alpha(p),1/2+}, \tag{3.7}$$

where $\alpha(p) = (1/2+)(\frac{p-6}{p})$.

The next step is to prove that the second energy is just a small error plus the first energy $E^1(u)(t) = E(Iu(t))$. This is the result of the following proposition.

Proposition 2. *Assume that $\|Iu\|_{H^1} = O(1)$. Then we have that*

$$E^2(u) \sim E^1(u) + O(1/N)\|Iu\|_{H^1}^6$$

for $N \gg 1$. In particular for $N \gg 1$ we conclude that $\|\partial_x Iu\|_{L^2}^2 \lesssim E^2(u)$.

Proof. Recall that

$$E(Iu) = -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{6}\Lambda_6(m_1 ... m_6)$$

and that for $u \in H^1$,

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 dx + \frac{1}{6}\int_{\mathbb{R}} |u|^6 dx \geq \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 dx \Rightarrow \|\partial_x u\|_{L^2} \lesssim E^{1/2}(u).$$

But then $\|\partial_x Iu\|_{L^2} \lesssim E^1(u)^{1/2}$ which in turn implies

$$\|\partial_x Iu\|_{L^2}^2 \lesssim -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \frac{1}{6}\Lambda_6(m_1...m_6).$$

In addition

$$E^2(u) = -\frac{1}{2}\Lambda_2(m_1\xi_1 m_2\xi_2) + \Lambda_6(M_6) = E^1(u) + \frac{1}{6}\Lambda_6(M_6 - \prod_{i=1}^{6} m_i).$$

Claim:

$$|\Lambda_6(M_6 - \prod_{i=1}^{6} m_i)| \lesssim O(1/N)\|Iu\|_{H^1}^6.$$

Then since $\|\partial_x Iu\|_{L^2}^2 \lesssim E^1(u)$, we are done.

Remarks. 1) We break all the functions into a sum of dyadic constituents ψ_j, each with frequency support $\langle\xi\rangle \sim 2^j, j = 0, ...$ Then we pull the absolute value of the symbols out of the integrals, estimating them pointwise. After bounding the multiplier, the remaining integrals involving the pieces ψ_j are estimated by reversing the Plancherel formula and using duality, Hölder's inequality, and Strichartz's estimates. We can sum over all the frequency pieces ψ_j as long as we keep always a factor $N_{max}^{-\epsilon}$ inside the summation.

2.) Since in all of the estimates that we establish from now on, the right-hand side is in terms of the $X^{s,b}$ norms and the $X^{s,b}$ spaces depend only on the absolute value of the Fourier transform, we can assume without loss of generality that the Fourier transform of all the functions in the estimates are positive and real.

Proof of claim. By Proposition 1 we have

$$|\Lambda_6(M_6)| \lesssim |\int \prod_{i=1}^{6} \hat{u}_j| = \int \frac{m_1^* N_1^*}{m_1^* N_1^*} |\prod_{i=1}^{6} \hat{u}_j|.$$

If we use Plancherel, assume that the spacetime Fourier transforms are all real and nonnegative, keeping in mind that $m_1^* N_1^{*-} = \frac{N_1^{*s^-}}{N^{s^-}} N^- \gtrsim N^-$ ($N^- = N^{1-}$), we get

$$\begin{aligned}|\Lambda_6(M_6)| &\lesssim \frac{1}{N^-}\int |JIu_1^*| \cdot |\prod_{1}^{5} u_j| \lesssim \frac{1}{N^-}\|JIu_1^*\|_{L^2}\|\prod_{i=1}^{5} u_j\|_{L^2} \\ &\lesssim \frac{1}{N^-}\|Iu_1^*\|_{H^1}\|u\|_{L^{10}}^5.\end{aligned}$$

But

$$\|u\|_{L^{10}} \lesssim \|u\|_{H^{1/2-1/10}} = \|u\|_{H^{2/5}} \lesssim \|Iu\|_{H^{2/5+1-s}} \le \|Iu\|_{H^1},$$

where we used the Sobolev embedding and (3.1) with $s > 4/9 > 2/5$. Thus

$$|\Lambda_6(M_6)| \lesssim \frac{1}{N^{-}}\|Iu\|^6_{H^1}.$$

Also

$$|\Lambda_6(\prod_{i=1}^{6} m_i)| \lesssim \int \frac{m_1^{*2}N_1^{*2}}{N_1^{*2}}(\prod_{i=1}^{4} m_i)\hat{u}_1...\hat{u}_6$$

$$\lesssim \frac{1}{N^{2-}}\Big(\int |JIu_1^*|\cdot|JIu_2^*|\Big)\prod_{i=1}^{4}\|Iu\|_\infty \lesssim \frac{1}{N^{2-}}\|JIu\|_2^2\|Iu\|_\infty^4 \lesssim \frac{1}{N^{2-}}\|Iu\|^6_{H^1}.$$

The next thing is to prove a local result for the "I-operators". This iteration process is in [4] and the proof that uses techniques as in C. E. Kenig, G. Ponce, and L. Vega [6],[7] can be adapted to our case where $s > 4/9$.

Proposition 3. *Let u be an H^1 solution to* (1.1). *Then if $T \in \mathbf{R}$ is such that $\|Iu(T)\|_{H^1} \leq C$, for some $C > 0$, we have that*

$$\|Iu\|_{X^{1,1/2+}([T,T+\delta]\times\mathbf{R})} \lesssim 1$$

for some $\delta > 0$ that depends on C.

Before we prove the next proposition we state the following Strichartz-type estimate that is due to Bourgain, [1].

Lemma 2. *For any Schwartz functions u, v with Fourier support in $|\xi| \sim R, |\xi| \ll R$, respectively, we have that*

$$\|uv\|_{L_t^2L_x^2} = \|u\bar{v}\|_{L_t^2L_x^2} \lesssim R^{-\frac{1}{2}}\|u\|_{X_{0,1/2+}}\|v\|_{X_{0,1/2+}}.$$

The next proposition shows that the second energy decays. Because the second energy is actually a correction term of the first this is the main step in proving Theorem 1.

Proposition 4. *For any Schwartz function u we have*

$$\Big|\int_T^{T+\delta}\Lambda_{10}(M_{10};u(t))dt\Big| \lesssim N^{-\frac{5}{2}+}\|Iu\|^{10}_{X^{1,1/2+}([T,T+\delta]\times\mathbb{R})}.$$

Proof. In Proposition 1 we proved that M_6 is bounded, so M_{10} is also bounded, since it is a finite sum of elongations of M_6. Thus $|M_{10}| \lesssim C$. Again we perform a Littlewood-Paley decomposition of the ten factors u.

Case A. $N_1^* \sim N_2^* \sim N_3^*$.

$$\Big|\int_T^{T+\delta}\Lambda_{10}(M_{10};u(t))dt\Big| \lesssim \int\int \frac{m_1^{*3}N_1^{*3}}{m_1^{*3}N_1^{*3}}\prod_{i=1}^{10}\hat{u}_j.$$

Observe that since $N_1^* \gtrsim N$

$$m_1^{*3} N_1^{*3-} = (\frac{N_1^{*(3s-3)}}{N^{3s-3}}) N_1^{*3-} \gtrsim (\frac{N_1^*}{N})^{(3s)-} N^{3-} \gtrsim N^{3-}.$$

Thus, $\frac{1}{m_1^{*3} N_1^{*3-}} \lesssim \frac{1}{N^{3-}}$ and

$$\Big| \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t)) dt \Big| \lesssim \frac{1}{N^{3-}} \int\int |JIu| \cdot |JIu| \cdot |JIu| \cdot |\prod_{j=4}^{10} u_j|$$

$$\lesssim \frac{1}{N^{3-}} \|(JIu)(JIu)(JIu)\|_{L_t^2 L_x^2} \| \prod_{j=4}^{10} u_j \|_{L_t^2 L_x^2}$$

$$\lesssim \frac{1}{N^{3-}} \|(JIu)\|^3_{L_t^6 L_x^6} \|u\|^7_{L_t^{14} L_x^{14}},$$

where we applied Hölder's inequality several times. But, by (3.4) and (3.7), we have that

$$\|(JIu)\|_{L_t^6 L_x^6} \lesssim \|(JIu)\|_{X_{0,1/2+}} = \|Iu\|_{X_{1,1/2+}}$$

and

$$\|u\|_{L_t^{14} L_x^{14}} \lesssim \|u\|_{X_{\alpha(14),1/2+}},$$

where $\alpha(14) = (1/2+) \cdot (\frac{8}{14}) = 2/7+$. Thus

$$\|u\|_{L_t^{14} L_x^{14}} \lesssim \|u\|_{X_{2/7+,1/2+}} \lesssim \|Iu\|_{X_{(2/7+)+1-s}} \lesssim \|Iu\|_{X_{1,1/2+}},$$

since $s > 4/9$. So, in this case,

$$\Big| \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t)) dt \Big| \lesssim \frac{1}{N^{3-}} \|Iu\|^{10}_{X^{1,1/2+}([T,T+\delta]\times\mathbf{R})}.$$

Case B. $N_1^* \sim N_2^* \gg N_3^*$. Since $\frac{1}{m_1^{*2}(N_1^*)^{5/2}} \lesssim \frac{1}{N^{5/2-}}$ we have that

$$\Big| \int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t)) dt \Big| \lesssim \int\int \frac{m_1^{*2} N_1^{*2} (N_1^*)^{1/2}}{m_1^{*2} (N_1^*)^{5/2}} |\prod_{i=1}^{10} \hat{u}_j|$$

$$\lesssim \frac{1}{N^{5/2-}} \int\int |(N_1^*)^{1/2} (JIu_1^*) u_3^* (JIu_2^*)| \prod_{j=1}^{7} u_j|$$

$$\lesssim \frac{1}{N^{5/2-}} \|(JIu_1^*) u_3^*\|_{L_t^2 L_x^2} (N_1^*)^{1/2} \|(\prod_{j=1}^{7} u_j)(JIu_{max})\|_{L_t^2 L_x^2}.$$

The Fourier transform of JIu_1^* is supported in $|\xi| \sim N_1^*$ and the Fourier transform of u_3^* is supported in $|\xi| \sim N_3^* \ll N_1^*$. By Lemma 2, we have that

$$\|(JIu_1^*)u_3^*\|_{L_t^2L_x^2}(N_1^*)^{\frac{1}{2}} \lesssim \|JIu_1^*\|_{X_{0,1/2+}}\|u_3^*\|_{X_{0,1/2+}} = \|Iu_1^*\|_{X_{1,1/2+}}\|u_3^*\|_{X_{0,1/2+}}.$$

In addition,

$$\|\prod_{j=1}^{7} u_j(JIu_{\max})\|_{L_t^2L_x^2} \lesssim \|u_1u_2u_3\|_{L_t^6L_x^6}\|(JIu_{max})\|_{L_t^6L_x^6}\|u_4u_5u_6u_7\|_{L_t^6L_x^6}.$$

By (3.4) again we have

$$\|(JIu_1^*)\|_{L_t^6L_x^6} \lesssim \|Iu_1^*\|_{X_{1,1/2+}}$$

and

$$\|u_1u_2u_3\|_{L_t^6L_x^6} \lesssim \|u\|^3_{L_t^{18}L_x^{18}} \lesssim \|u\|^3_{X_{\alpha(18),1/2+}},$$

where $\alpha(18) = 1/3+ < 4/9$ which implies that $\|u\|^3_{L_t^{18}L_x^{18}} \lesssim \|Iu\|^3_{X_{1,1/2+}}$.

Similarly,

$$\|u_1u_2u_3u_4\|_{L_t^6L_x^6} \lesssim \|u\|^4_{L_t^{24}L_x^{24}} \lesssim \|u\|^4_{X_{\alpha(24),1/2+}},$$

where $\alpha(24) = 3/8+ < 4/9$.

Again, we conclude that $\|u\|^4_{L_t^{24}L_x^{24}} \lesssim \|Iu\|^4_{X_{1,1/2+}}$. So in this case

$$\Big|\int_T^{T+\delta} \Lambda_{10}(M_{10}; u(t))dt\Big| \lesssim \frac{1}{N^{5/2-}}\|Iu\|^{10}_{X^{1,1/2+}([T,T+\delta]\times\mathbb{R})}.$$

Now we are ready to prove Theorem 1.

Proof. Let $\lambda > 0$, to be chosen later. We can easily check that $u(x,t)$ is a solution to (1.1) if and only if $u^\lambda(x,t) = \frac{1}{\lambda^{1/2}}u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is a solution to the same equation with initial data $u_0^\lambda = \frac{1}{\lambda^{1/2}}u(\frac{x}{\lambda})$.

Also we have that $\widehat{u_0^\lambda}(\xi) = \lambda^{-\frac{1}{2}}\hat{u}_0(\lambda\xi)$. In particular

$$\|u_0^\lambda\|^2_{L^2} = \|u_0\|_{L^2}.$$

Since $|m(\xi)| \leq 1$ we have that $\|Iu_0^\lambda\|_2 \leq \|\widehat{u_0^\lambda}\|_2 = \|u_0^\lambda\|_2 = \|u_0\|_2$.

By definition, $\frac{|m(\xi)|}{|\xi|^{s-1}} \lesssim N^{1-s}$, so we have that

$$\|\partial_x Iu_0^\lambda\|_2 = \|m(\xi)|\xi|\widehat{u_0^\lambda}\|_2 = \|\frac{|m(\xi)|}{|\xi|^{s-1}}|\xi|^s\widehat{u_0^\lambda}\|_2 \leq N^{1-s}\||\xi|^s\widehat{u_0^\lambda}\|_2 \lesssim \frac{N^{1-s}}{\lambda^s}\|u_0\|_{\dot{H}^s}.$$

An application of the Gagliardo-Nirenberg inequality gives

$$\|Iu_0^\lambda\|_6^6 \lesssim \|\partial_x Iu_0^\lambda\|_2^2\|Iu_0^\lambda\|_2^4 \lesssim \frac{N^{2-2s}}{\lambda^{2s}}\|u_0\|^2_{\dot{H}^s}\|u_0\|_2^4.$$

But recall that

$$E(Iu_0^\lambda) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x Iu_0^\lambda|^2 dx + \frac{1}{6}\int_{\mathbb{R}} |Iu_0^\lambda|^6 dx.$$

Thus, $E(Iu_0^\lambda) \lesssim \frac{N^{2-2s}}{\lambda^{2s}}\|u_0\|_{\dot{H}^s}^2$ and for $\lambda \sim N^{\frac{1-s}{s}}$ we have that $E(Iu_o^\lambda) \leq 2/3$, which in turn implies that

$$\|Iu_0^\lambda\|_{H^1}^2 < 1 + \|u_0\|_2^2.$$

So, we succeed in making $\|Iu_0\|_{H^1} \sim 1$ and can apply the previous propositions. The main idea of the proof is that if in each step of the iteration we have the same bound for $\|Iu_0^\lambda\|_{H^1}$ then we can iterate again with the same timestep. Now, by Propositions 2, 3, and 4, we have that

$$E^2(u^\lambda(\delta)) \lesssim E^2(u^\lambda(0)) + CN^{-5/2+} \lesssim E^1(u^\lambda(0)) + \frac{1}{N}\|Iu_0^\lambda\|_{H^1}^6 + CN^{-5/2+} < 1$$

for N large enough. By Proposition 2,

$$\|\partial_x Iu(\delta)^\lambda\|_2^2 \lesssim E^2(u^\lambda(\delta)) < 1$$

and thus

$$\|Iu^\lambda(\delta)\|_{H^1} < 1 + \|u_0\|_2^2$$

so we can continue our argument. We can continue the solution in $[0, M\delta] = [0, T]$ as long as $T \ll N^{5/2-}$. Thus,

$$\|Iu^\lambda(T)\|_{H^1} \lesssim 1 + \|u_0^\lambda\|_2^2 = 1 + \|u_0\|_2^2$$

for all $T \ll N^{5/2-}$. From the definition of I this implies that

$$\|u^\lambda(T)\|_{H^s} \lesssim 1$$

for all $T \ll N^{5/2-}$. Undoing the scaling we have that

$$\|u(T)\|_{H^s} \lesssim C_{N,\lambda}$$

for all $T \ll \frac{N^{5/2-}}{\lambda^2}$. But $\frac{N^{5/2-}}{\lambda^2} \sim N^{\frac{9s-4}{2s}}$ goes to infinity as $N \to \infty$ since $s > 4/9$.

From the above proof, it is evident that, to prove a global well-posedness result for $s > 0$, we need an infinite decay for the "modified energy". So one naturally can think to define the "third energy", or in general higher and higher approximations of the energy, when trying to push down the global result. In our case the symbol, even for the third energy, becomes very singular and it is extremely hard to get an improved decay for the "modified energy". For example we can define the third energy by

$$E^3(u) = E^2(u) + \Lambda(M_{10}').$$

Then by (3.2) we have that

$$\frac{dE^3}{dt}(u) = \frac{dE^2}{dt}(u) + i\Lambda_{10}\Big(M_{10}^{'}\sum_{j=1}^{10}(-1)^j\xi_j^2; u(t)\Big)$$
$$- i\Lambda_{14}\Big(\sum_{j=1}^{10}(-1)^j X_j^4(M_{10}^{'}); u(t)\Big) = \Lambda_{10}\left(M_{10}; u\right)$$
$$+ i\Lambda_{10}\Big(M_{10}^{'}\sum_{j=1}^{10}(-1)^j\xi_j^2; u(t)\Big) - i\Lambda_{14}\Big(\sum_{j=1}^{10}(-1)^j X_j^4(M_{10}^{'}); u(t)\Big),$$

where M_{10} is given by Lemma 1. Thus if we pick $M_{10}^{'} = i\frac{M_{10}}{\sum_{j=1}^{10}(-1)^j\xi_j^2}$, we cancel the Λ_{10} term and thus

$$\frac{dE^3}{dt}(u) = \Lambda_{14}(M_{14}; u),$$

where M_{14} is a finite sum of elongations of $M_{10}^{'}$. But now it is too much to ask for pointwise bounds for $M_{10}^{'}$. So a modification of this method maybe turns out to be what we really need to go down to $s > 0$.

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