

**THE CAUCHY PROPERTY OF THE GENERALIZED
APPROXIMATELY CONTINUOUS
PERRON INTEGRAL**

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1. Introduction. We say an integral has the Cauchy property, if it satisfies the following condition (C).

(C) If $f(x)$ is defined in $[a, b]$ and is integrable in each interval $[a + \varepsilon, b - \eta]$, where $a < a + \varepsilon < b - \eta < b$ and

$$\lim_{\varepsilon, \eta \rightarrow 0} \int_{a+\varepsilon}^{b-\eta} f(t) dt \quad (*)$$

exists, then $f(x)$ is integrable in $[a, b]$ and the integral over $[a, b]$ is equal to the above limit.

Both the special and the general Denjoy integrals have this property. M. E. Grimshaw [1] proved that the approximately continuous Perron integral defined by J. C. Burkill [2] satisfies the condition (C) with the approximate limit instead of the ordinary limit in (*).

By the use of a similar method we will show that the corresponding property is possessed by the generalized approximately continuous Perron integral defined by G. Sunouchi and M. Utagawa [3].

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2. Generalized approximately continuous Perron integral.

DEFINITION 2. 1. $U(x)$ [$L(x)$] is termed upper [lower] function of a measurable $f(x)$ in $[a, b]$, provided that

- (i) $U(a) = 0$ [$L(a) = 0$],
- (ii) $\underline{AD} U(x) > -\infty$ [$\overline{AD} L(x) < +\infty$] at each point x ,
- (iii) $\underline{AD} U(x) \geq f(x)$ [$\overline{AD} L(x) \leq f(x)$] at each point x .

DEFINITION 2. 2. If $f(x)$ has upper and lower functions in $[a, b]$ and

$$\text{l. u. b. } L(b) = \text{g. l. b. } U(b),$$

then $f(x)$ is termed integrable in AP-sense or AP-integrable. The common value of the two bounds is called the definite AP-integral of $f(x)$ and

denoted by $(AP) \int_a^b f(t)dt$.

G. Sunouchi and M. Utagawa [3] have proved the following results.

THEOREM 2. 1. *The function $U(x) - L(x)$ is non-decreasing and non-negative.*

THEOREM 2. 2. *If $f(x)$ is AP-integrable in $[a, b]$, then $f(x)$ is so in every interval $[a, x]$ for $a \leq x \leq b$.*

THEOREM 2. 3. *The indefinite integral $F(x) \equiv (AP) \int_a^x f(t)dt$ is approximately continuous.*

THEOREM 2. 4. *The function $F(x)$ is approximately derivable almost everywhere and*

$$AD F(x) = f(x), \text{ a. e.}$$

3. Cauchy property of AP-integral. We shall prove the Cauchy property of the AP-integral in the following form.

THEOREM 3. 1. *If $f(x)$ is AP-integrable in $[a, \beta]$, where $a \leq \beta < b$ and has the integral $F(x)$ in the interval $a \leq x < b$, and if*

$$\text{ap} \lim_{x \rightarrow b} F(x) = 1,$$

then $f(x)$ is AP-integrable in $[a, b]$ and

$$(AP) \int_a^b f(t)dt = 1.$$

PROOF. We put $F(b) = 1$. Then $F(x)$ is approximately continuous at b since $\text{ap} \lim_{x \rightarrow b} F(x) = 1$. Hence, there exists a certain set S which includes the point b and has unit density on the left at b , and on which

$$F(x) \rightarrow 1 \text{ as } x \rightarrow b.$$

Let $\{b_n\}$ ($n \geq 1$) be an increasing sequence of S converging to b , and put $b_0 = a$.

For any positive number ε , we can choose an upper function $U_n(x)$ for $f(x)$ on $[b_n, b_{n+1}]$ ($n \geq 0$), such that

$$0 \leq U_n(x) - [F(x) - F(b_n)] < \frac{\varepsilon}{2^n}, \quad (1)$$

and

$$\underline{AD} U_n(x) > -\infty, \quad \underline{AD} U_n(x) \geq f(x). \quad (2)$$

We define the function $\bar{U}(x)$ for $a \leq x < b$ as follows,

$$\begin{aligned} \bar{U}(x) &= U_0(x) && (a \leq x < b_1) \\ &= \sum_{k=0}^{n-1} U_k(b_{k+1}) + U_n(x) && (b_n \leq x < b_{n+1}). \end{aligned}$$

Then it follows from (1) and (2) that

$$0 \leq \bar{U}(x) - F(x) < 2\varepsilon, \tag{3}$$

and

$$\underline{AD} \bar{U}(x) > -\infty, \underline{AD} \bar{U}(x) \geq f(x) \tag{4}$$

for $a \leq x < b$.

The function $\bar{U}(x) - F(x)$ is non-decreasing for $a \leq x < b$ by Theorem 2. 1 and is bounded in any neighbourhood of b by (3), and so tends to a finite limit as x tends to b from below.

Since $F(x) \rightarrow 1$ as $x \rightarrow b$ on S , $U(x)$ converges to a finite limit as $x \rightarrow b$, x on S .

We define $\bar{U}(b) = \lim_{x \rightarrow b} \bar{U}(x)$ ($x \in S$). Then, we obtain from (3)

$$0 \leq \bar{U}(b) - F(b) \leq 2\varepsilon. \tag{5}$$

Let $\chi(x)$ be a continuous, non-decreasing function in $[a, b]$ such that $\chi(a) = 0$, $\chi(b) = \varepsilon$, $\chi'(b) = +\infty$.

We select the integer p such that oscillation of $\bar{U}(x)$ on $S \cap [b_{p-1}, b]$ is less than ε . This is possible since $\bar{U}(x)$ tends to a finite limit as $x \in S$ tends to b .

Let ω_n be the oscillation of $U(x)$ on $S \cap [b_{n-1}, b]$ for $n \geq p$. We define the function $\varphi(x)$ on $[b_n, b_{n+1}]$ for each $n \geq p$ and at b as follows,

$$\begin{aligned} \varphi(b_n) &= \omega_n, \\ \varphi(b_{n+1}) &= \omega_{n+1}, \\ \varphi(x) &= \text{linear} && (b_n \leq x \leq b_{n+1}), \\ \varphi(b) &= 0. \end{aligned}$$

Finally, we set

$$\begin{aligned} U(x) &= \chi(x) + \bar{U}(x) && (a \leq x < b_p) \\ &= \chi(x) + \bar{U}(x) + \varphi(b_p) - \varphi(x) && (b_p \leq x \leq b). \end{aligned}$$

Then, we obtain from (4)

$$\underline{AD} U(x) \geq f(x), \underline{AD} U(x) > -\infty$$

for $a \leq x < b$, since $\chi(x)$ and $-\varphi(x)$ are non-decreasing functions.

To verify $\underline{AD} U(b)$, we consider $\underline{AD}[\bar{U}(x) + \varphi(b_p) - \varphi(x)]$ at b . By the definition of φ , we obtain

$$\left\{x: \frac{\bar{U}(b) - \bar{U}(x) + \varphi(x)}{b - x} \geq 0\right\} \supset S \cap [b_p, b]$$

and therefore the approximate lower derivate of $\{\bar{U}(x) + \varphi(b_p) - \varphi(x)\}$ at b is not negative. Since $\chi'(b) = \infty$, we have $\underline{AD} U(b) = \infty$. Thus, the function $U(x)$ is an upper function of $f(x)$ on $[a, b]$.

Finally, we have

$$U(b) = \bar{U}(b) + \chi(b) + \varphi(b_p) < \bar{U}(b) + 2\varepsilon$$

and hence by (5)

$$0 \leq U(b) - F(b) < 4\varepsilon.$$

By constructing an upper function for $-f(x)$ in $[a, b]$, we obtain a lower function $L(x)$ such that

$$0 \geq F(b) - L(b) > -4\varepsilon.$$

We have thus proved that $f(x)$ is AP -integrable on $[a, b]$ and that

$$(AP) \int_a^b f(t) dt = 1.$$

REFERENCES

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