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The Cauchy–Riemann equations on product domains

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Abstract We establish the L^2 theory for the Cauchy–Riemann equations on product domains provided that the Cauchy–Riemann operator has closed range on each factor. We deduce regularity of the canonical solution on $(p, 1)$-forms in special Sobolev spaces represented as tensor products of Sobolev spaces on the factors of the product. This leads to regularity results for smooth data.

1 Introduction

In this paper we study the existence and regularity for the solution of the inhomogeneous Cauchy–Riemann equations, or the $\overline{\partial}$-equation on product domains. When the product domain is a polydisc in $\mathbb{C}^n$, the solution to the $\overline{\partial}$-equation can be obtained by an inductive process from the solution in one variable given by the Cauchy integral formula for the disc. This is known as the Dolbeault–Grothendieck Lemma (see [10, Theorem 2.1.6]; For other approaches, see [27,28]) which is the analog for the $\overline{\partial}$-operator of the Poincaré lemma for the exterior derivative d.

We are interested here in the $\overline{\partial}$-problem in the L^2 setting. For a bounded pseudoconvex domain in $\mathbb{C}^n$, or more generally in a Stein manifold, L^2 existence theorems have been established in Hörmander [18]. We prove L^2 existence on a product, i.e., we show that $\overline{\partial}$ has closed range on a product provided that $\overline{\partial}$ has closed range on each factor domain.

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Theorem 1.1 *For $j = 1, \dots, N$, let Ω_j be a relatively compact domain with Lipschitz boundary in a complex hermitian manifold M_j. Let $\Omega \subset M_1 \times \cdots \times M_N$ be the product domain $\Omega = \Omega_1 \times \cdots \times \Omega_N$. Suppose the $\overline{\partial}$-operator has closed range in $L^2(\Omega_j)$ for all degrees for each j, then the $\overline{\partial}$ operator has closed range for all degrees in $L^2(\Omega)$. Furthermore, the Künneth formula holds for the L^2 cohomology*:

$$H^*_{L^2}(\Omega) = H^*_{L^2}(\Omega_1)\widehat{\otimes} \cdots \widehat{\otimes} H^*_{L^2}(\Omega_N),$$

where $\widehat{\otimes}$ denotes the Hilbert space tensor product.

If the L^2 space on the domain Ω_j is defined with respect to a weight function ϕ_j, i.e., if on Ω_j we use the norm $\int_{\Omega_j} |f|^2 e^{-\phi_j} dV$, the same statement holds if $\sum_{j=1}^N \phi_j$ is used as a weight on the product Ω.

A classical approach to the study of partial differential equations on product domains is by separation of variables and spectral representation. This method can be applied to the $\Box$-operator (the complex Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$): see [17, pp. 103ff] for the case of compact complex manifolds and [15] for the case of the polydisc. A proof of a general version of Theorem 1.1 (without the assumption of relative compactness or boundary regularity of the Ω_j) may be given using separation of variables and spectral theory (see [9].) However, it is difficult to use this method to draw conclusions about the regularity of the solution of the $\overline{\partial}$-equation. There is a different approach, using a direct construction of a solution operator on a product domain, used first in [34] for the de Rham complex, and we use this approach to prove Theorem 1.1. Using this method we not only prove Theorem 1.1, but also obtain regularity results for the canonical solution of $\overline{\partial}$.

The closed-range property given by Theorem 1.1 has numerous applications. First it immediately gives that the Hodge decomposition holds for the product domain Ω. Notice that it is not assumed that the domains Ω_j are non-compact: the theorem can be applied to the case when the product domain is a product $D \times M$ of a bounded pseudoconvex domain D in $\mathbb{C}^n$ and a compact complex manifold M. Though the proof for Theorem 1.1 is not difficult, it has not been stated explicitly in the literature.

We obtain boundary regularity results for the canonical solution of the $\overline{\partial}$-equation on product domains in $\mathbb{C}^n$ or complex hermitian manifolds. The regularity for the canonical solution of the $\overline{\partial}$-equation and the $\overline{\partial}$-Neumann operator on a polydisc have been studied extensively (see [4, 12–14] and the references in these works.) There is also a considerable amount of work for the $\overline{\partial}$-equation on domains with Lipschitz boundary or piecewise smooth domains (see [25]). Notice that a product domain is only *piecewise* smooth even if each factor domain has smooth boundary. Thus the boundary is only Lipschitz. It is known that on a general Lipschitz domain, the $\overline{\partial}$-Neumann operator or even the Green's operator for the Dirichlet problem (see [3, 30]) is not regular near the singular part of the domain. Thus one cannot expect the $\overline{\partial}$-Neumann operator to be regular near the product of the boundaries of the factor domains. This is confirmed by the explicit computations in [12–14]. One would expect that the canonical solution might also not be regular. An interesting feature is that while the $\overline{\partial}$-Neumann operator on product domains might not be well-behaved, the canonical solution still exhibits regularity on certain Sobolev spaces. Before our results here, only $\mathcal{C}^k$ estimates were known for the special case of the polydisc using an explicit integral formula (see [24].)

In order to state precise regularity results on the canonical solution operator we introduce special Sobolev spaces, called *Partial Sobolev spaces*, denoted by $\widetilde{W}^k(\Omega)$ (for definition of $\widetilde{W}^k(\Omega)$, see Sect. 5.) If $W^k(\Omega)$ denotes the usual Sobolev space of functions having L^2-derivatives of order k on Ω, we have $W^{Nk}(\Omega) \subset \widetilde{W}^k(\Omega) \subset W^k(\Omega)$. We prove the following regularity result for the canonical solution operator in the partial Sobolev spaces on a product pseudoconvex domain.

Theorem 1.2 *Let Ω be the same as in Theorem 1.1. Then the $\overline{\partial}$-Neumann operator N exists for all degrees on (p, q)-forms with L^2 coefficients. Assume further that for each j, the domain Ω_j is smoothly bounded, and the $\overline{\partial}$-Neumann operator on Ω_j preserves the space of forms with coefficients in $W^k(\Omega_j)$ for every integer $k \geq 0$. For any p with $0 \leq p \leq \dim_{\mathbb{C}} \Omega$, let f be a $\overline{\partial}$-closed $(p, 1)$-form on Ω orthogonal to the $(p, 1)$-harmonic forms such that the coefficients of f are in the partial Sobolev space $\widetilde{W}^l(\Omega)$, for some integer $l \geq 0$. Then, the canonical solution $u = \overline{\partial}^* \mathsf{N} f$ of the equation $\overline{\partial} u = f$ also has coefficients in $\widetilde{W}^l(\Omega)$.*

As with Theorem 1.1, if the L^2 space on Ω_j is defined with respect to a weight function ϕ_j, the same conclusion holds if the $\overline{\partial}$-Neumann operator N_{ϕ_j} with weight ϕ_j preserves $W^k(\Omega_j)$ forms, and the canonical solution on the product is taken with respect to the weight $\sum_{j=1}^N \phi_j$. Note that it follows from the inclusions $W^{Nk}(\Omega) \subset \widetilde{W}^k(\Omega) \subset W^k(\Omega)$ that if the $(p, 1)$-form f has coefficients in $W^{Nk}(\Omega)$, then the canonical solution $\overline{\partial}^* \mathsf{N} f$ has coefficients in $W^k(\Omega)$. Of course this loss of smoothness disappears on using the correct space $\widetilde{W}^k(\Omega)$. Also note that unless D is a domain in $\mathbb{C}^n$, the $\overline{\partial}$-equation for (p, q) forms on a domain D in a complex manifold cannot be reduced to the $\overline{\partial}$-equation for $(0, q)$ forms.

To use this result, we need to understand the regularity of the $\overline{\partial}$-Neumann operator on the factor domains. There is a vast literature on the regularity of the $\overline{\partial}$-Neumann operator on smooth and pseudoconvex domains. In particular, regularity is known when the boundary is strongly pseudoconvex (see [22]) or finite type (see [8]), or if the boundary has a plurisubharmonic defining function (see [6]) or if the boundary has transverse symmetry (see [1].) However, for each $s > 0$, there exists a pseudoconvex domain with smooth boundary such that the $\overline{\partial}$-Neumann operator or the canonical solution is not regular in the Sobolev space W^s (see [2]). Even in this case, we can obtain regularity in a weighted Sobolev space (see [23].) Using these results, one can draw many corollaries from Theorem 1.2 regarding the regularity of the solution of the $\overline{\partial}$-problem in Sobolev spaces or spaces of smooth forms (see Corollary 6.1 below.) One example is the following:

Corollary 1.3 *Suppose that the smoothly bounded pseudoconvex domains $\Omega_1, \ldots, \Omega_N$ in hermitian manifolds of dimension $n_1, \ldots, n_j$ respectively are such that for each j, and every $0 \leq p \leq n_j$, the canonical solution operator on Ω_j maps the space $\mathcal{C}^\infty_{p,1}(\overline{\Omega_j})$ of $(p, 1)$ forms smooth up to the boundary to $\mathcal{C}^\infty_{p,0}(\overline{\Omega_j})$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_N$. For $0 \leq \boldsymbol{p} \leq \sum_{j=1}^N n_j$, let f be a $\overline{\partial}$-closed $(\boldsymbol{p}, 1)$ form with $\mathcal{C}^\infty(\overline{\Omega})$ coefficients. Then $\overline{\partial}^* \mathsf{N} f$ also has coefficients in $\mathcal{C}^\infty(\overline{\Omega})$, where N is the $\overline{\partial}$-Neumann* *operator on Ω. Further, the Bergman projection on the space of functions $L^2(\Omega)$ preserves the space $\mathcal{C}^\infty(\overline{\Omega})$.*

Note that if f is orthogonal to the harmonic forms, $\overline{\partial}^*\mathsf{N}f$ is the canonical solution to $\overline{\partial}u = f$. Also, Corollary 1.3 applies to the product of domains which are strongly pseudoconvex or more generally of finite type. In contrast, we note that when the domain is the intersection of two balls, the Bergman projection is not regular near the nongeneric points of the boundary (see [3]).

Notice that on a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, if the canonical solution $\overline{\partial}^*\mathsf{N}$ is regular, it follows that the $\overline{\partial}$-Neumann operator N, and the adjoint of the canonical solution operator $\overline{\partial}\mathsf{N}$ are all exact regular on Sobolev spaces (see [10].) However, the same method cannot be applied to the adjoint of the canonical solution or to the $\overline{\partial}$-Neumann operator on a product domain. On a product domain, the canonical solution is regular, but neither the $\overline{\partial}$-Neumann operator N nor the operator $\overline{\partial}\mathsf{N}$ is regular near the boundary.

The plan of this paper is as follows: in Sects. 2 and 3 we establish terminology and notation regarding the L^2 $\overline{\partial}$-problem and tensor products of forms, and discuss some basic properties of the objects involved. We note here that although our results have been stated for general manifolds, in these sections, for simplicity of exposition and notation, we give the definitions for domains in Euclidean space $\mathbb{C}^n$. The generalization to manifolds is easy and left to the reader. Also, due to the nature of the proof, we need to consider spaces of forms of arbitrary degrees. For example, we denote by $L^2_*(D)$ the space of forms with square integrable coefficients on a domain D. The key observation in Sect. 2 is that closure of the range of the $\overline{\partial}$-operator is a necessary as well as sufficient condition for representation of cohomology classes by harmonic forms (see Lemma 2.2.) The next Sect. 4 represents the central argument of the paper. Starting from the canonical solution operator and the harmonic projection on the factor domains, we write down a formula (18) defining a solution operator S on the product, which coincides with the canonical solution operator $\overline{\partial}^*\mathsf{N}$ on (0, 1)-forms. Using S we give a simple proof of Theorem 1.1. In Sect. 5 we consider the tensor products of Sobolev spaces, which gives rise to the partial Sobolev spaces referred to above. This is used in the last Sect. 6 to prove regularity results.

2 The L^2 setting for the $\overline{\partial}$ problem

2.1 Spaces of forms on domains

We recall the definition and notation used in the L^2 theory of the $\overline{\partial}$-operator. Let D be a bounded domain in $\mathbb{C}^n$, and let ϕ be a continuous function on $\overline{D}$. We denote by $L^2(D)$ the space of square integrable functions on D with respect to weight ϕ, which has the norm

$$\|f\| = \int_D |f|^2 e^{-\phi} dV,$$

where dV is the volume form on $\mathbb{C}^n$ induced by the standard hermitian metric. (Note that we have suppressed ϕ from the notation.)

We denote by $L^2_*(D)$ the space of differential forms with coefficients in $L^2(D)$. More generally, for any space of functions $\mathcal{F}(D)$ on D, we will let $\mathcal{F}_*(D)$ denote the space of forms with coefficients in $\mathcal{F}$. Then $\mathcal{F}_*(D)$ can be thought of as a vector space direct sum

$$\mathcal{F}_*(D) = \bigoplus_{\substack{0\le p\le n\\ 0\le q\le n}} \mathcal{F}_{p,q}(D) \tag{1}$$

of the spaces of forms of bidegree (p, q).

Often the space $\mathcal{F}(D)$ will be a Hilbert space. Then we can give $\mathcal{F}_*(D)$ a Hilbert space structure in the following way: first, we declare that forms of different bidegrees are orthogonal, so that the sum in (1) is now an orthogonal direct sum of Hilbert Subspaces. Any form $f \in \mathcal{F}_{p,q}(D)$ can be uniquely represented as

$$f = \sum_{I,J}{}' f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

where $I = (i_1, \dots, i_p) \in \mathbb{N}^p$, and $J = (j_1, \dots, j_q) \in \mathbb{N}^q$ are multi-indices, $f_{I,J} \in \mathcal{F}(D)$, $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, and the notation $\sum'$ means that the summation is over strictly increasing multi-indices only, i.e. $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. We define the norm of f as

$$\|f\|^2_{\mathcal{F}_*(D)} = \sum_{I,J}{}' \|f_{I,J}\|^2_{\mathcal{F}(D)}. \tag{2}$$

In this paper, the Hilbert space $\mathcal{F}$ will be either a usual L^2 space (possibly with weight), a Sobolev space, or a partial Sobolev space on product domains (to be defined in Sect. 5.) These notions easily extend to spaces of forms on domains in hermitian manifolds (see [10, Chapter 5]) using the natural pointwise inner-product on forms induced by the hermitian structure.

2.2 The L^2 Dolbeault complex

We now recall the definition of the the $\overline{\partial}$-operator on the Hilbert space $L^2_*(D)$ of forms with square integrable coefficients on D. The $\overline{\partial}$-operator is the closed, densely defined unbounded operator from $L^2_*(D)$ to itself which coincides with the usual $\overline{\partial}$ operator from $\mathcal{C}^\infty_*(\overline{D})$ to $\mathcal{C}^\infty_*(\overline{D})$, and which has been extended as a distributional operator to the dense domain of definition

$$\operatorname{dom}(\overline{\partial}) = \{f \in L^2_*(D) \colon \overline{\partial} f \in L^2_*(D)\}.$$

In the terminology of [7], the operator $\overline{\partial}$ is the differential map of a *Hilbert Complex*, i.e., a cochain complex, in which the cochain space $\mathrm{dom}(\overline{\partial})$ is a dense subspace of a graded Hilbert Space, and the differential is a closed, densely defined unbounded linear map of the graded Hilbert space into itself. Note that the map $\overline{\partial}$ has bidegree $(0, 1)$, i.e., it maps (p, q) forms to $(p, q + 1)$ forms.

We denote by $\overline{\partial}^*$ the Hilbert space adjoint of $\overline{\partial}$. This is again a closed, densely defined operator on $L^2_*(\Omega)$. Its domain $\mathrm{dom}(\overline{\partial}^*)$ is in general very different from $\mathrm{dom}(\overline{\partial})$, because of the natural boundary conditions that the Hilbert space adjoint must satisfy. The map $\overline{\partial}^*$ is of bidegree $(0, -1)$.

A form $f \in L^2_*(D)$ is said to be *harmonic*, if $\overline{\partial} f = \overline{\partial}^* f = 0$. The harmonic forms $\mathcal{H}_*(D)$ form a closed subspace of $L^2_*(D)$. The orthogonal projection $P : L^2_*(D) \to \mathcal{H}_*(D)$ is called the *harmonic projection*, which is of course a map of bidegree $(0, 0)$. Note that since $\overline{\partial}^*$ vanishes on $L^2_{0,0}(D) \equiv L^2(D)$, the space $\mathcal{H}_{0,0}(D)$ can be identified with the space $L^2(D) \cap \mathcal{O}(D)$ of square integrable holomorphic functions, the *Bergman space* associated to D. The operator $P_{0,0}$ is the *Bergman projection* onto square integrable holomorphic functions.

2.3 The closed range property and its consequences

Let $g \in L^2_*(D)$ be such that $\overline{\partial} g = 0$. In order to solve the equation $\overline{\partial} u = g$ in the L^2 sense first we need to show that the L^2 $\overline{\partial}$-operator has closed range. In general, the closed-range property is not easy to establish, even with a smooth boundary. Subtle holomorphically invariant convexity properties of the boundary of D control whether $\overline{\partial}$ has closed range on D (see the example on p. 76 of [16].) Note that, in contrast, for the L^2 d-complex on a Riemannian manifold the operator d always has closed range when the boundary is $\mathcal{C}^2$ or even Lipschitz [26,29].

An important consequence of the closed range property on D is the existence of the *Canonical-* or *Kohn's solution operator* K, which is a bounded map from $L^2_*(D)$ to itself of bidegree $(0, -1)$, and is a right-inverse of the operator $\overline{\partial}$. For every $f \in \mathrm{img}(\overline{\partial})$, we define Kf to be the unique solution to $\overline{\partial} u = f$ which is orthogonal to $\ker(\overline{\partial})$. We then extend K to all of $L^2_*(D)$ by setting $K \equiv 0$ on $(\mathrm{img}(\overline{\partial}))^\perp$ and extending linearly. The map K is bounded by the closed graph theorem, and is represented in terms of the $\overline{\partial}$-Neumann operator N on D as $K = \overline{\partial}^* \mathsf{N}$. We further have the following:

Lemma 2.1 *If $\overline{\partial}$ has closed range, and K is the canonical solution operator, then on* $\mathrm{dom}(\overline{\partial})$ *we have*

$$I - P = \overline{\partial} K + K \overline{\partial}. \tag{3}$$

Further, the ranges of the three operators $\overline{\partial} K$, $K \overline{\partial}$ and P are orthogonal.

Proof Since $\mathrm{img}(\overline{\partial})$ is closed in $L^2_*(D)$, we have the Strong Hodge decomposition:

$$L^2_*(D) = (\mathrm{img}(\overline{\partial}^*)) \oplus (\mathrm{img}(\overline{\partial})) \oplus \mathcal{H}_*(D),$$

where $\mathcal{H}_*(D)$ is the Hilbert space of harmonic forms, and $\oplus$ means that the summands are orthogonal (see [10].) Let Q and R be the orthogonal projections from $L^2_*(D)$ onto the closed subspaces $\mathrm{img}(\overline{\partial})$ and $\mathrm{img}(\overline{\partial}^*)$ respectively. Then on $L^2_*(D)$, we have $I = P + Q + R$. From the definition of K, we have $\overline{\partial} K = Q$. Also, on $\mathrm{dom}(\overline{\partial})$ we have $K\overline{\partial} = R$ by noting that the left hand side is the identity on $\left(\ker(\overline{\partial})\right)^\perp = \mathrm{img}(\overline{\partial}^*)$ and zero on the orthogonal complement $\ker\overline{\partial}$. Equation (3) now follows. The last statement follows from the method of proof. ◻

For any domain D, the L^2 *Dolbeault Cohomology* space is the graded vector space

$$H^*_{L^2}(D) = \frac{\ker(\overline{\partial})}{\mathrm{img}(\overline{\partial})}.$$

Note that in the quotient topology, this is a Hilbert space if $\mathrm{img}(\overline{\partial})$ is closed (and not even Hausdorff if $\mathrm{img}(\overline{\partial})$ is not closed, see [32, Chapter 1, Sect. 2.3].) If $\mathrm{img}(\overline{\partial})$ is closed, we have

$$\begin{aligned} H^*_{L^2}(D) &\cong (\ker(\overline{\partial})) \cap \left(\mathrm{img}(\overline{\partial})\right)^\perp \\ &= (\ker(\overline{\partial})) \cap (\ker(\overline{\partial}^*)) \\ &= \mathcal{H}_*(D), \end{aligned}$$

so that the cohomology space is naturally isomorphic to the space of harmonic forms. We will now recall the less well-known converse to this statement, due to Kodaira (see [11, p. 165].) Let

$$[.] : \ker(\overline{\partial}) \to H^*_{L^2}(D)$$

denote the natural projection onto the quotient space. We have the following:

Lemma 2.2 *Let η be the linear map from the vector space of harmonic forms $\mathcal{H}_*(D)$ to the cohomology vector space $H^*_{L^2}(D)$ given by $\eta(f) = [f]$. Then*

(i) *η is injective.*
(ii) *If η is also surjective, then the range of $\overline{\partial}$ is closed.*

Proof (i) For $(0,0)$-forms, i.e. functions, the space $\mathcal{H}_{0,0}(D)$ coincides by definition with the cohomology space $H^{0,0}_{L^2}(D)$. For forms of higher degree, a harmonic form in $\ker(\eta)$ is of the form $\overline{\partial} g$ with $\overline{\partial}^*(\overline{\partial} g) = 0$ so that

$$\begin{aligned} 0 &= (\overline{\partial}^*(\overline{\partial} g), g) \\ &= \left\|\overline{\partial} g\right\|^2. \end{aligned}$$

(ii) Since η is an isomorphism, we can identify $\mathcal{H}_*(D)$ with the cohomology space $H^*_{L^2}(D)$. Since $\mathcal{H}_*(D)$ is a closed subspace of the Hilbert Space $L^2_*(D)$, the space $H^*_{L^2}(D)$ becomes a Hilbert space in the natural way. Then the map $[\cdot]$

can be thought of as an operator from the Hilbert space $\ker(\overline{\partial}) \subset L^2_*(D)$ to the Hilbert space $H^*_{L^2}(\Omega)$. Since η is surjective, every element of $\ker(\overline{\partial})$ can be written as $f + \overline{\partial} g$, where $f \in \mathcal{H}^*(D)$. Then $[(f + \overline{\partial} g)] = f$, using the identification of $\mathcal{H}_*(D)$ and $H^*_{L^2}(D)$. Since $\left\| f + \overline{\partial} g \right\|^2 = \|f\|^2 + \left\| \overline{\partial} g \right\|^2 \geq \|f\|^2$, so that $\left\| [f + \overline{\partial} g] \right\| \leq \left\| f + \overline{\partial} g \right\|$, it follows that $[\cdot]$ is actually a bounded map. Therefore, $\ker[\cdot] = \operatorname{img}(\overline{\partial})$ is closed. □

3 Differential forms on product domains

3.1 Algebraic tensor product of spaces of forms

Let H_1 and H_2 be $\mathbb{C}$-vector spaces. We denote by $\mathsf{H}_1 \otimes \mathsf{H}_2$ the *algebraic* tensor product (over $\mathbb{C}$) of H_1 and H_2 : then $\mathsf{H}_1 \otimes \mathsf{H}_2$ can be thought of as the space of finite sums of elements of the type $x \otimes y$, where $x \in \mathsf{H}_1$ and $y \in \mathsf{H}_2$, where $\otimes : \mathsf{H}_1 \times \mathsf{H}_2 \to \mathsf{H}_1 \otimes \mathsf{H}_2$ is the canonical bilinear map (see e.g. [33, Sect. 3.4] for the purely algebraic definition.) Similarly $\mathsf{H} = \mathsf{H}_1 \otimes \mathsf{H}_2 \otimes \cdots \otimes \mathsf{H}_N$ denotes the algebraic tensor product of N vector spaces $\mathsf{H}_1, \cdots \mathsf{H}_N$. We call an element of H of the form $x_1 \otimes \cdots \otimes x_N$ a *simple tensor.*

When $\mathsf{H}_1, \ldots, \mathsf{H}_N$ are realized as spaces of forms on domains (or manifolds) $\Omega_1, \ldots, \Omega_N$, there is a concrete realization of the algebraic tensor product H as a space of forms on the product domain $\Omega = \Omega_1 \times \cdots \times \Omega_N$. For $j = 1, \ldots, N$, let $f_j \in \mathsf{H}_j$, so that f_j is a form on the domain Ω_j and let $\pi_j : \Omega \to \Omega_j$ denote the projection onto the jth factor Ω_j from the product Ω. We define a form on Ω, the *tensor product* of the forms $f_1, \ldots, f_N$, by setting

$$f_1 \otimes \cdots \otimes f_N = \pi_1^* f_1 \wedge \cdots \wedge \pi_N^* f_N, \tag{4}$$

which we will call a *simple decomposable* form. Then $\mathsf{H}_1 \otimes \cdots \otimes \mathsf{H}_N$ is the linear span of the simple decomposable forms. It is easy to verify that this construction gives rise to a vector space isomorphic to the usual algebraic definition of a tensor product by the universal property.

3.2 Hilbert tensor products

We now specialize to the case where the factors H_j are Hilbert spaces. For ease of exposition, we assume that $N = 2$, and the general case should be obvious. We can define an inner product on the algebraic tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ defined above by setting

$$(x \otimes y, z \otimes w) = (x, z)_{\mathsf{H}_1} (y, w)_{\mathsf{H}_2},$$

and extending bilinearly. This is well-defined thanks to the bilinearity of $\otimes$. This makes $\mathsf{H}_1 \otimes \mathsf{H}_2$ into a pre-Hilbert space, and its completion is a Hilbert space denoted by $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$, the *Hilbert tensor product* of the spaces H_1 and H_2. The algebraic tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ sits inside $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ as a dense subspace. We will refer to any element of $\mathsf{H}_1 \otimes \mathsf{H}_2$ (thought of as a subspace of $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$) as a *decomposable form*. For further details on Hilbert tensor products, see [33, Sect. 3.4], or for a more intrinsic approach [21, Sect. 2.6, vol. 1].

Now let $\mathcal{F}(\Omega_1)$ and $\mathcal{G}(\Omega_2)$ be Hilbert Spaces of functions on Ω_1 and Ω_2 respectively, and let $\mathcal{F}_*(\Omega_1)$ and $\mathcal{G}_*(\Omega_2)$ be the Hilbert Spaces of forms with coefficients in $\mathcal{F}(\Omega_1)$ and $\mathcal{G}(\Omega_2)$ respectively, with the norm given by (2). It is easily verified from the definitions that there is an isometric equality of Hilbert spaces:

$$\mathcal{F}_*(\Omega_1) \widehat{\otimes} \mathcal{G}_*(\Omega_2) = (\mathcal{F} \widehat{\otimes} \mathcal{G})_*(\Omega_1 \times \Omega_2), \tag{5}$$

with an obvious extension to the case of more than two factor domains.

3.3 Forms with square integrable coefficients

We now recall the most important case of the above constructions. Another example will be considered in Sect. 5.

Recall the following classical fact, which we will use repeatedly:

Lemma 3.1 ([20, p. 369]) *Let Ω_1, Ω_2 be domains in Euclidean spaces (or manifolds), and let $\Omega = \Omega_1 \times \Omega_2$. Then every function in $\mathcal{C}_0^\infty(\Omega)$ can be approximated in the $\mathcal{C}^k$ norm (where $0 \le k \le \infty$) by functions in the algebraic tensor product $\mathcal{C}_0^\infty(\Omega_1) \otimes \mathcal{C}_0^\infty(\Omega_2)$.*

Now, $\mathcal{C}_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, and the decomposable compactly supported smooth functions $\mathcal{C}_0^\infty(\Omega_1) \otimes \mathcal{C}_0^\infty(\Omega_2)$ are dense in the uniform norm (and therefore in the L^2 norm) in the space $\mathcal{C}_0^\infty(\Omega)$. It follows that:

$$L^2(\Omega_1) \widehat{\otimes} L^2(\Omega_2) = L^2(\Omega_1 \times \Omega_2).$$

Combining with (5) we have:

$$L^2_*(\Omega_1) \widehat{\otimes} L^2_*(\Omega_2) = L^2_*(\Omega_1 \times \Omega_2). \tag{6}$$

3.4 Tensor products of operators

Again, for clarity we confine ourselves to the case $N = 2$. Let $\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}'_1, \mathsf{H}'_2$ be Hilbert Spaces. Given bounded linear operators $T_1 : \mathsf{H}_1 \to \mathsf{H}'_1$ and $T_2 : \mathsf{H}_2 \to \mathsf{H}'_2$, we can define an algebraic tensor product $T_1 \otimes T_2$ which maps the algebraic tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ into $\mathsf{H}'_1 \otimes \mathsf{H}'_2$: on decomposable tensors it is given by $(T_1 \otimes T_2)(x \otimes y) = T_1 x \otimes T_2 y$ and extended linearly. Then $T_1 \otimes T_2$ is bounded on the dense subspace $\mathsf{H}_1 \otimes \mathsf{H}_2$ and therefore extends to a bounded linear operator $T_1 \widehat{\otimes} T_2$ from $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ to $\mathsf{H}'_1 \widehat{\otimes} \mathsf{H}'_2$.

This construction can be extended to densely defined unbounded linear operators, provided they are *closed.* (see [21, Sect. 11.2, vol. 2].) Given closed (or even closable) operators $T_1 : \mathrm{dom}(T_1) \to \mathsf{H}'_1$ and $T_1 : \mathrm{dom}(T_2) \to \mathsf{H}'_2$, where $\mathrm{dom}(T_1)$ and

$\mathrm{dom}(T_2)$ are dense subspaces of the Hilbert spaces H_1 and H_2, the algebraic tensor product (which is densely defined on $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ with domain $\mathrm{dom}(T_1) \otimes \mathrm{dom}(T_2)$) is closable (see [21, Proposition 11.2.7 (vol. 2)].) Its closure, denoted by $T_1 \widehat{\otimes} T_2$ is a closed densely defined operator from $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ to $\mathsf{H}'_1 \widehat{\otimes} \mathsf{H}'_2$. Note that this definition agrees with the previous one, when both T_1 and T_2 are bounded.

4 $\overline{\partial}$ on a product domain in the L^2 sense

In this section we construct a solution operator to the $\overline{\partial}$-problem on a product domain in terms of the canonical solution operators on the factor domains, and show that the operator constructed in fact gives the canonical solution on $\ker \overline{\partial}_{p,1}$, the $\overline{\partial}$-closed $(p, 1)$-forms. We use the following notation: for $j = 1, \ldots, N$, let Ω_j be a bounded domain in Euclidean space $\mathbb{C}^{n_j}$. All our arguments and results will have easy generalizations to relatively compact domains in hermitian manifolds, which we leave for the reader. We will assume that the boundary of each domain Ω_j is Lipschitz, i.e., it can be represented locally in holomorphic coordinates as the graph of a Lipschitz function. For each j, we also fix a weight function ϕ_j continuous on $\overline{\Omega_j}$. We use the L^2 space of forms $L^2_*(\Omega_j)$ on the domain Ω_j with the weight ϕ_j, i.e., the norm of a function f is given by $\|f\|^2 = \int_{\Omega_j} |f|^2 e^{-\phi_j} dV$, where dV is the volume form induced by the hermitian metric. (If we want spaces without weights, we simply take $\phi_j \equiv 0$.)

The product domain $\Omega = \Omega_1 \times \cdots \times \Omega_N$ also has Lipschitz boundary. We will consider $L^2_*(\Omega)$ with the weight $\phi = \phi_1 + \cdots + \phi_N$ (and with the product hermitian metric.) The analog of formula (6) holds with this choice of metric and weight:

$$L^2_*(\Omega) = L^2_*(\Omega_1)\widehat{\otimes} \cdots \widehat{\otimes} L^2_*(\Omega_N).$$

Our fundamental assumption will be the following: *For each j, the L^2 $\overline{\partial}$-operator (with weight ϕ_j) on Ω_j has closed range in each degree.*

We remark that the closed range property is independent of the weight function ϕ_j as long as it is continuous to the boundary since the L^2 spaces are the same. We will show that the $\overline{\partial}$ operator on Ω (with weight ϕ) also has closed range and deduce a formula for the canonical solution on $\ker(\overline{\partial})$.

4.1 Construction of solution operator on smooth decomposable forms

For simplicity of exposition, we from now on consider the case $N = 2$, that is we have two domains Ω_1 and Ω_2 and we are trying to solve the L^2 $\overline{\partial}$-problem on the product $\Omega = \Omega_1 \times \Omega_2$. In this section we write down some algebraic formulas which hold for smooth decomposable forms on Ω.

We first note that if $f \in \mathcal{C}^\infty_*(\overline{\Omega_1})$ and $g \in \mathcal{C}^\infty_*(\overline{\Omega_2})$, then we have

$$\overline{\partial}(f \otimes g) = \overline{\partial}_1 f \otimes g + \sigma_1 f \otimes \overline{\partial}_2 g, \tag{7}$$

where $\overline{\partial}_1, \overline{\partial}_2, \overline{\partial}$ denote the $\overline{\partial}$ operator on the domains $\Omega_1, \Omega_2, \Omega$ respectively, and σ_1 is the map on $\mathcal{C}^\infty_*(\overline{\Omega_1})$ which is multiplication by $(-1)^{p+q}$ on $\mathcal{C}^\infty_{p,q}(\overline{\Omega_1})$. Note that if

T be any linear map of odd degree on the space $\mathcal{C}^\infty_*(\overline{\Omega_1})$ (i.e. the degrees of Tf and f differ by an odd integer) then we obviously have

$$\sigma_1 T = -T\sigma_1. \tag{8}$$

Extending (7) bilinearly to $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$, we obtain the *Leibnitz formula* for smooth decomposable forms:

$$\overline{\partial} = \overline{\partial}_1 \otimes I_2 + \sigma_1 \otimes \overline{\partial}_2. \tag{9}$$

Let K_1, K_2 be the canonical solution operators on Ω_1, Ω_2 (see Sect. 2.3.) We define an operator S from $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$ into $L^2_*(\Omega_1) \otimes L^2_*(\Omega_2)$ by the formula

$$S = K_1 \otimes I_2 + \sigma_1 P_1 \otimes K_2, \tag{10}$$

where P_j denotes the harmonic projection on the domain Ω_j (see Sect. 2.2.) It will be proved in the next section that S extends to $L^2_*(\Omega)$, and coincides on (0,1)-forms with the canonical solution operator on the product Ω. In this section, we take a first step in this direction by proving the following homotopy formula:

Lemma 4.1 *On the space of smooth decomposable forms* $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$, *we have*

$$\overline{\partial} S + S\overline{\partial} = I - P_1 \otimes P_2, \tag{11}$$

where I is the identity map.

Proof First note that

$$\begin{aligned}\overline{\partial} S &= (\overline{\partial}_1 \otimes I_2 + \sigma_1 \otimes \overline{\partial}_2)(K_1 \otimes I_2 + \sigma_1 P_1 \otimes K_2)\\ &= \overline{\partial}_1 K_1 \otimes I_2 + \sigma_1 K_1 \otimes \overline{\partial}_2 + P_1 \otimes \overline{\partial}_2 K_2,\end{aligned}$$

where one term vanishes because $\overline{\partial}_1 P_1 = 0$. Similarly, since by the Hodge decomposition, $P_1\overline{\partial}_1 = 0$, we have,

$$\begin{aligned}S\overline{\partial} &= (K_1 \otimes I_2 + \sigma_1 P_1 \otimes K_2)(\overline{\partial}_1 \otimes I_2 + \sigma_1 \otimes \overline{\partial}_2)\\ &= K_1\overline{\partial}_1 \otimes I_2 + K_1\sigma_1 \otimes \overline{\partial}_2 + P_1 \otimes K_2\overline{\partial}_2\\ &= K_1\overline{\partial}_1 \otimes I_2 - \sigma_1 K_1 \otimes \overline{\partial}_2 + P_1 \otimes K_2\overline{\partial}_2,\end{aligned}$$

where we have used (8) in the last line along with the fact that K_1 has degree -1. Combining the two expressions and canceling the middle terms we have

$$\begin{aligned}\overline{\partial} S + S\overline{\partial} &= (\overline{\partial}_1 K_1 + K_1\overline{\partial}_1) \otimes I_2 + P_1 \otimes (\overline{\partial}_2 K_2 + K_2\overline{\partial}_2)\\ &= (I_1 - P_1) \otimes I_2 + P_1 \otimes (I_2 - P_2)\\ &= I_1 \otimes I_2 - P_1 \otimes P_2,\end{aligned}$$

where we have used the homotopy formula (3) in each factor. The result follows. □

4.2 Density results: extension to $\operatorname{dom}(\bar\partial)$

In this section we use a density argument to extend the formulas of the last section. We first recall the following:

Lemma 4.2 ([10, Lemma 4.3.2, part (i)]) *If D is a Lipschitz domain, then the space $\mathcal{C}^\infty_*(\overline{D})$ of forms with $\mathcal{C}^\infty(\overline{D})$ coefficients is dense in the graph-norm in the domain* $\operatorname{dom}(\bar\partial)$ *of the L^2 $\bar\partial$ operator on D.*

Since D is Lipschitz, it is locally star-shaped. This is a special case of Friedrichs' Lemma and follows from smoothing by convolution with a mollifier; see Section 1.2 in Chapter I in Hörmander [18] or Part (i) of proof of the Density Lemma 4.3.2 in [10]. The following is now easy:

Lemma 4.3 *$\mathcal{C}^\infty(\overline{\Omega_1}) \otimes \mathcal{C}^\infty(\overline{\Omega_2})$ is dense in the domain of $\bar\partial$ in the graph norm of the $\bar\partial$-operator on $\Omega = \Omega_1 \times \Omega_2$.*

Proof Given a form $f \in \operatorname{dom}(\bar\partial)$ on Ω, by the Lemma 4.2, we can approximate it in the graph norm by a form $\tilde f \in \mathcal{C}^\infty_*(\overline{\Omega})$. Note that it easily follows from Lemma 3.1 that every form in $\mathcal{C}^\infty_*(\overline{\Omega})$ can be approximated in the $\mathcal{C}^k$ norm (where $0 \le k \le \infty$) by forms in the algebraic tensor product $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$. Therefore, approximating $\tilde f$ by a form in $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$ in the $\mathcal{C}^1$ norm (which dominates the graph norm) our result follows. □

We now extend the formulas of the previous section from the space $\mathcal{C}^\infty_*(\overline{\Omega_1}) \otimes \mathcal{C}^\infty_*(\overline{\Omega_2})$ of smooth decomposable forms (which is dense in the graph norm of $\bar\partial$) to $\operatorname{dom}(\bar\partial)$.

Lemma 4.4 *On the dense subspace* $\operatorname{dom}(\bar\partial) \subset L^2_*(\Omega)$ *we have*:

$$\bar\partial = \bar\partial_1 \widehat{\otimes} I_2 + \sigma_1 \widehat{\otimes}\, \bar\partial_2. \tag{12}$$

The operator S defined in (10) can be extended to $L^2_*(\Omega)$ by the formula

$$S = K_1 \widehat{\otimes} I_2 + \sigma_1 P_1 \widehat{\otimes} K_2, \tag{13}$$

and on $\operatorname{dom}(\bar\partial)$ *the following homotopy formula holds*:

$$\bar\partial S + S\bar\partial = I - P_1 \widehat{\otimes} P_2. \tag{14}$$

Proof All three formulas follow from the corresponding formulas for decomposable forms by taking limits, using Lemma 4.3 for (12) and (14). □

4.3 Consequences

Using the homotopy formula (14), we can now prove:

Theorem 4.5 *Let Ω_1 and Ω_2 be two bounded domains in complex hermitian manifolds with Lipschitz boundaries. Suppose that the $\overline{\partial}$ operator has closed range in $L^2(\Omega_j)$ for all degrees, where $j = 1, 2$. Then $\overline{\partial}$ has closed range in $L^2(\Omega)$ for the product domain $\Omega = \Omega_1 \times \Omega_2$.*

Proof We recall the result established in Lemma 2.2: if the map $\eta(f) = [f]$ is surjective, then $\overline{\partial}$ has closed range. In other words, we need to show that for every cohomology class $\alpha \in H^*_{L^2}(\Omega)$ there is a *harmonic form* $h \in \mathcal{H}_*(\Omega)$ such that $\alpha = [h]$. We will actually do better. We will show that there is such a h in the tensor product $\mathcal{H}_*(\Omega_1) \widehat{\otimes} \mathcal{H}_*(\Omega_2) \subset \mathcal{H}_*(\Omega)$. Note that this will also show that

$$\mathcal{H}_*(\Omega_1) \widehat{\otimes} \mathcal{H}_*(\Omega_2) = \mathcal{H}_*(\Omega). \tag{15}$$

Indeed, let $f \in \ker(\overline{\partial})$ be a form representing the cohomology class α, i.e. $\alpha = [f]$. Then, from the homotopy formula (14), we have

$$f - \overline{\partial}(Kf) = (P_1 \widehat{\otimes} P_2) f.$$

Therefore, the form $(P_1 \widehat{\otimes} P_2) f \in \mathcal{H}_*(\Omega_1) \widehat{\otimes} \mathcal{H}_*(\Omega_2)$ also represents the same cohomology class α, i.e. $[(P_1 \widehat{\otimes} P_2) f] = \alpha$. Therefore every cohomology class in $H^*_{L^2}(\Omega)$ can be represented by a harmonic form in $\mathcal{H}_*(\Omega)$ (indeed by a harmonic form in the possibly smaller subspace $\mathcal{H}_*(\Omega_1) \widehat{\otimes} \mathcal{H}_*(\Omega_2)$.) This shows that the map η of Lemma 2.2 is surjective. The equality (15) now follows from the fact that η is injective. □

We now note a few important consequences of the above result:

Corollary 4.6 (i) *The L^2 Künneth formula holds for the Dolbeault cohomology with L^2 coefficients:*

$$H^*_{L^2}(\Omega) = H^*_{L^2}(\Omega_1) \widehat{\otimes} H^*_{L^2}(\Omega_2) \tag{16}$$

(ii) *The harmonic projections satisfy $P = P_1 \widehat{\otimes} P_2$*

Proof Part (i) follows from the natural isomorphisms $H^*_{L^2}(\Omega) \cong \mathcal{H}_*(\Omega)$, $H^*_{L^2}(\Omega_1) \cong \mathcal{H}_*(\Omega_1)$ and $H^*_{L^2}(\Omega_2) \cong \mathcal{H}_*(\Omega_2)$ (note that the range of $\overline{\partial}$ is closed in each case.) Part (ii) follows from comparing the homotopy formulas (14) and (3), or directly from (15). □

We now come to the most significant consequence:

Theorem 4.7 *For $0 \le p \le n$, the restriction of the map S defined in (13) to the $\overline{\partial}$-closed $(p, 1)$-forms coincides with the restriction of the canonical solution operator $\overline{\partial}^* \mathsf{N}$ to the same space.*

Proof From the Hodge decomposition, we have for (p, q) forms that $\ker(\overline{\partial}_{p,q}) = \operatorname{img}(\overline{\partial}_{p,q-1}) \oplus \mathcal{H}_{p,q}(\Omega)$. If $q = 0$, it follows that

$$\ker(\overline{\partial}_{p,0}) = \mathcal{H}_{p,0}(\Omega) = \bigoplus_{j+k=p} \mathcal{H}_{j,0}(\Omega_1)\widehat{\otimes}\mathcal{H}_{k,0}(\Omega_2),$$

by (15).

We claim that the range of $S_{p,1}$ is orthogonal to the space $\ker(\overline{\partial}_{p,0})$. By the computation above, it is sufficient to show that the range of $S_{p,1}$ is orthogonal to every form of the type $g_1 \otimes g_2$, where g_1 and g_2 are harmonic forms of degrees $(j, 0)$ and $(p - j, 0)$, where $0 \le j \le p$. Let f_1, f_2 be L^2 forms such that $f_1 \otimes f_2$ is of bidegree $(p, 1)$. Then,

$$\begin{aligned}(S(f_1 \otimes f_2), g_1 \otimes g_2) &= (K_1 f_1 \otimes f_2 + \sigma_1 P_1 f_1 \otimes K_2 f_2, g_1 \otimes g_2)\\ &= (K_1 f_1, g_1)(f_2, g_2) + (\sigma_1 P_1 f_1, g_1)(K_2 f_2, g_2)\\ &= 0 \cdot (f_2, g_2) + (\sigma_1 P_1 f_1, g_1) \cdot 0\\ &= 0,\end{aligned}$$

where we have used the fact that K_1, K_2 being canonical solutions, have ranges orthogonal to $\overline{\partial}$-closed forms.

If f is a $\overline{\partial}$-closed $(p, 1)$ form orthogonal to the harmonic forms, it follows from formula (14) that $\overline{\partial}(Sf) = f$. Since Sf is orthogonal to $\ker(\overline{\partial})$, it follows that $Sf = \overline{\partial}^* \mathsf{N} f$.

To complete the proof, we need to show that S vanishes on the space of $(p, 1)$ harmonic forms. By formula (15), it follows that we only need to verify this on a harmonic form of the type $f \otimes g$, where f, g are also harmonic forms. We have,

$$\begin{aligned}S(f \otimes g) &= K_1 f \otimes g + \sigma_1 P_1 f \otimes K_2 g\\ &= 0 \otimes g + f \otimes 0\\ &= 0,\end{aligned}$$

since K_1 and K_2 are the canonical solutions on the domains Ω_1 and Ω_2. □

Remark For arbitrary degrees, the operator S is not equal to the canonical solution operator $K = \overline{\partial}^* \mathsf{N}$. In fact, an examination of the proof of Lemma 2.1 shows that for the canonical solution K on a domain, the ranges of the operators $\overline{\partial} K$ and $K \overline{\partial}$ are orthogonal. On the other hand, using the computations used in the proof of Lemma 4.1, we can check that

$$\left(\overline{\partial} S(f \otimes g), S\overline{\partial}(f \otimes g)\right) = -\|K_1 f\|^2 \left\|\overline{\partial}_2 g\right\|^2,$$

so that S is not the canonical solution on the product.

Using a simple induction argument, we can extend the results of this section to N factors. Further, as remarked above, all the arguments generalize to relatively compact domains in hermitian manifolds:

Theorem 4.8 *For $j = 1, \dots, N$, let M_j be a hermitian manifold and let $\Omega_j \Subset M_j$ be a Lipschitz domain. Suppose that the L^2 $\overline{\partial}$-operator on Ω_j (with weight ϕ_j) has closed range for each $1 \le j \le N$. Then we have the following*:

- *the* $\overline{\partial}$*-operator (with weight* $\sum_{j=1}^{N}\phi_j$*) has closed range on* Ω*.*
- *the* L^2 *Künneth formula holds*:

$$H^*_{L^2}(\Omega) = H^*_{L^2}(\Omega_1)\widehat{\otimes}\cdots\widehat{\otimes}H^*_{L^2}(\Omega_N)$$

- *the harmonic projection on* Ω *is given by*

$$P = P_1\widehat{\otimes}\cdots\widehat{\otimes}P_N. \tag{17}$$

- *a solution operator for* $\overline{\partial}$ *on* Ω *is given by*

$$S = \sum_{j=0}^{N-1} T_{N,j}, \tag{18}$$

with

$$T_{N,j} = \tau_j \mathsf{Q}_j \,\widehat{\otimes}\, K_{j+1}\,\widehat{\otimes}\,\mathsf{I}_j,$$

where
  - Q_j *is the harmonic projection on the domain* $U_j = \Omega_1\times\cdots\times\Omega_j$*, (the product of the first* j *factors),*
  - τ_j *is the map on* $L^2_*(U_j)$ *which multiplies forms of degree* d *by* $(-1)^d$*,*
  - I_j *is the identity map on forms on* $\Omega_{j+2}\times\cdots\times\Omega_N$*, and*
  - *it is understood that* $T_{N,0} = K_1\widehat{\otimes}\mathsf{I}_0$ *and* $T_{N,N-1} = \tau_{N-1}\mathsf{Q}_{N-1}\widehat{\otimes}K_N$*.*
- *let* $0 \le p \le \sum_{j=1}^{N}\dim_{\mathbb{C}} M_j$*; on the space of* $\overline{\partial}$*-closed* $(p,1)$ *forms on* Ω*, the solution operator* S *coincides with the canonical solution operator* $\overline{\partial}^*\mathsf{N}$ *of the* $\overline{\partial}$*-equation.*

In particular, this proves Theorem 1.1.

5 Partial Sobolev spaces

5.1 Definitions

Recall that for a Lipschitz domain D in $\mathbb{R}^n$, and an integer $k\ge 0$, the Sobolev space $W^k(D)$ is the Hilbert space obtained by completion of $\mathcal{C}^\infty(\overline{D})$ under the norm given by

$$\|f\|^2_{W^k(D)} = \sum_{[\alpha]\le k}\left\|\mathsf{D}^\alpha f\right\|^2_{L^2(D)},$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index, $[\alpha] = \alpha_1+\cdots+\alpha_n$ is the length of multi-index, and D^α is the partial derivative operator of order α:

$$\mathsf{D}^\alpha = \frac{\partial^{[\alpha]}}{\partial^{\alpha_1}x_1\ldots\partial^{\alpha_n}x_n}.$$

We will obtain regularity estimates for the canonical solution on product domains in a generalized type of Sobolev space suited to the product structure of the domain. We will call these spaces *partial Sobolev spaces*. Such spaces are characterized by the fact that there are some values of the integer l such that the norm controls only *some* distinguished partial derivatives of order l. For the usual Sobolev space $W^k(D)$, the norm controls either all or no derivatives of order l, depending on whether $l \leq k$ or $l > k$.

For convenience of exposition, first consider a product domain $D \Subset \mathbb{R}^n$ represented as $D = D_1 \times D_2$, where $D_1 \Subset \mathbb{R}^{n_1}$ and $D_2 \Subset \mathbb{R}^{n_2}$ are Lipschitz domains, with $n = n_1 + n_2$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with n components. We can write $\alpha = \alpha(1) + \alpha(2)$, where

$$\alpha(1) = (\alpha_1, \ldots, \alpha_{n_1}, \underbrace{0, \ldots, 0}_{n_2}),$$

and

$$\alpha(2) = (\underbrace{0, \ldots, 0}_{n_1}, \alpha_{n_1+1}, \ldots, \alpha_n).$$

Then $\mathsf{D}^{\alpha(1)}$ acts only on the variables which come from D_1 and $\mathsf{D}^{\alpha(2)}$ acts only on the variables that come from D_2 in the product D, and we have $\mathsf{D}^\alpha = \mathsf{D}^{\alpha(1)}\mathsf{D}^{\alpha(2)}$.

The $\widetilde{W}^k$-norm of a function $f \in \mathcal{C}^\infty(\overline{D})$ is defined to be

$$\|f\|_{\widetilde{W}^k(D)} = \sum_{\substack{[\alpha(1)] \leq k \\ [\alpha(2)] \leq k}} \left\|\mathsf{D}^\alpha f\right\|^2_{L^2(D)} \tag{19}$$

Note that the $\widetilde{W}^k$-norm dominates the ordinary W^k-norm on D, and is in turn dominated by the W^{2k}-norm.

We now define the space $\widetilde{W}^k(D)$ to be the completion of $\mathcal{C}^\infty(\overline{D})$ under the norm (19). It is clear how to extend this definition to more than two factors: if $D = D_1 \times \cdots \times D_N$, then the $\widetilde{W}^k$-norm on D is defined as

$$\|f\|^2_{\widetilde{W}^k}(D) = \sum_{\substack{[\alpha(j)] \leq k \\ 1 \leq j \leq N}} \left\|\mathsf{D}^\alpha f\right\|^2_{L^2(D)},$$

where $\alpha(j)$ is the part of the multi-index α corresponding to the factor D_j, defined in analogy with the case $N = 2$ considered above.

5.2 Basic properties

We now summarize the basic properties of partial Sobolev space $\widetilde{W}^k(D)$, where $D = D_1 \times \cdots \times D_N$. From the definition, $\widetilde{W}^k(D)$ is a Hilbert space in the $\widetilde{W}^k$-norm.

For $k=0$, the space $\widetilde{W}^0(D)$ coincides with $L^2(\Omega)$. In general, for each k, we have continuous inclusions:

$$\mathcal{C}^{Nk}(\overline{D}) \hookrightarrow W^{Nk}(D) \hookrightarrow \widetilde{W}^k(D) \hookrightarrow W^k(D) \hookrightarrow L^2(D). \tag{20}$$

Since $\bigcap_{k\geq 0} W^{Nk}(D) = \bigcap_{k\geq 0} W^k(D) = \mathcal{C}^\infty(\overline{D})$, it follows that

$$\bigcap_{k\geq 0} \widetilde{W}^k(D) = \mathcal{C}^\infty(\overline{D}). \tag{21}$$

The significance of these spaces is explained by:

Lemma 5.1 *For $j=1,\ldots,N$, let $\Omega_j \Subset \mathbb{C}^{n_j}$ be a Lipschitz domain, and denote the product by $\Omega = \Omega_1 \times \cdots \times \Omega_N$. Then we have an isometric equality of Hilbert spaces of forms on Ω:*

$$\widetilde{W}^k_*(\Omega) = W^k_*(\Omega_1)\widehat{\otimes}\cdots\widehat{\otimes}W^k_*(\Omega_N). \tag{22}$$

Proof For simplicity of exposition, we assume $N=2$. Thanks to the comments in Sect. 3.2, in particular equation (5), it follows that we only need to show that

$$\widetilde{W}^k(\Omega) = W^k(\Omega_1)\widehat{\otimes}W^k(\Omega_2).$$

Thanks to Lemma 3.1, it follows easily by $\mathcal{C}^{2k}$ approximation, that $\mathcal{C}^\infty(\overline{\Omega_1}) \otimes \mathcal{C}^\infty(\overline{\Omega_2})$ is dense on each side. Therefore, all it needs to prove isometric equality is to show that the $\widetilde{W}^k$ norm and the tensor product norm coincide on this subspace. A computation shows that we have $\|f\otimes g\|_{\widetilde{W}^k(\Omega_1\times\Omega_2)} = \|f\|_{W^k(\Omega_1)}\|g\|_{W^k(\Omega_2)} = \|f\otimes g\|_{W^k(\Omega_1)\widehat{\otimes}W^k(\Omega_2)}$ □

5.3 Partial Sobolev spaces on manifolds

When for each j, the domain Ω_j is smoothly bounded in a hermitian manifold M_j, we can again define the partial Sobolev space $\widetilde{W}^k(\Omega)$ on the product. The simplest approach is to take (22) to be the definition and deduce the description in terms of distinguished derivatives from there. Alternatively, one can use a partition of unity to define $\widetilde{W}^k(\Omega)$ subordinate to a covering of $\overline{\Omega}$ by coordinate patches.

6 Regularity results

We now prove some results regarding the regularity of the solution of the $\overline{\partial}$-equation on product domains. Our main tool is the operator S defined in Sect. 4.

6.1 Proof of Theorem 1.2

By Theorem 4.7, the solution operator S on the product Ω coincides with the canonical solution operator on $\overline{\partial}$-closed $(p, 1)$-forms. Therefore, it is sufficient to show that S is bounded from $\widetilde{W}^l_{p,1}(\Omega)$ to itself. In fact, it is easy to see that S is bounded from $\widetilde{W}^l_*(\Omega)$ to itself.

The regularity of the $\overline{\partial}$-Neumann operator on $W^k(\Omega_j)$ for each $k \geq 0$ implies that the canonical solution operator as well as the harmonic projection preserves the space of forms with W^k coefficients for each k (see [10, Theorem 6.2.2 and Theorem 6.1.4]; note that in this reference (i) the hypothesis of pseudoconvexity is used only to deduce that the $\overline{\partial}$-Neumann operator is bounded in each Sobolev space, and (ii) although the arguments are stated only for domains in $\mathbb{C}^n$, they generalize easily to relatively compact domains in complex manifolds; for similar results on the Bergman projection, see [5].) Since S is given by (18), in the notation of theorem 4.8, we have

$$\begin{aligned} T_{N,j} &= \tau_j \mathsf{Q}_j \widehat{\otimes} K_{j+1} \widehat{\otimes} \mathsf{I}_j \\ &= \tau_j P_1 \widehat{\otimes} \cdots P_j \widehat{\otimes} K_{j+1} \widehat{\otimes} I_{\Omega_{j+2}} \cdots \widehat{\otimes} I_{\Omega_N}, \end{aligned}$$

where P_ν is the harmonic projection, K_ν is the canonical solution operator and I_{Ω_ν} is the identity map on $L^2_*(\Omega_\nu)$. Therefore, the νth factor in the tensor product representing $T_{N,j}$ is a bounded linear map on $W^k_*(\Omega_\nu)$. It follows (see Sect. 3.4) that $T_{N,j}$ defines a bounded linear map from the tensor product $W^k_*(\Omega_1) \widehat{\otimes} \cdots \widehat{\otimes} W^k_*(\Omega_N)$ to itself, i.e., it is a bounded linear map from $\widetilde{W}^k_*(\Omega)$ to itself. The solution operator S being the sum of the $T_{N,j}$'s is bounded on $\widetilde{W}^k_*(\Omega)$. The proof is complete.

We note here that the hypothesis of Theorem 1.2 are not really necessary. All we need to know to conclude that the canonical solution has coefficients in $\widetilde{W}^l(\Omega)$, if the form f has coefficients in $\widetilde{W}^l(\Omega)$ is the following: for each j, both the canonical solution and the harmonic projection on each factor Ω_j preserves the Sobolev space $W^l(\Omega)$.

6.2 Application to products of weakly pseudoconvex domains

We now consider the $\overline{\partial}$-equation on a product of smoothly bounded pseudoconvex domains:

Corollary 6.1 *For $j = 1, \ldots, N$, let Ω_j be a bounded pseudoconvex domain with smooth boundary in a Euclidean space $\mathbb{C}^{n_j}$. For $n = n_1 + \cdots + n_N$, let $\Omega \subset \mathbb{C}^n$ be the product domain $\Omega = \Omega_1 \times \cdots \times \Omega_N$. Then, for each $k \in \mathbb{N}$, there is an $C_k > 0$ such that, if $t > C_k$, and we use the weight $\phi_t(z) = t\,|z|^2$ on $\mathbb{C}^n$, we have*

- *for $1 \leq q \leq n$, given a $\overline{\partial}$-closed form f in the partial Sobolev space $\widetilde{W}^k_{0,q}(\Omega)$, the form $u = Sf$ is in $\widetilde{W}^k_{0,q-1}(\Omega)$. The form u satisfies $\overline{\partial} u = f$, provided f is orthogonal to the harmonic forms.*
- *if $q = 1$, further we have that u coincides with $\overline{\partial}^*_t \mathsf{N}_t f$, the canonical solution with weight t.*

Proof By the classical solution by Kohn of the weighted $\overline{\partial}$-Neumann problem (see [10, Theorem 6.1.3]), for each $j = 1, \ldots, N$, given an integer $k \geq 0$, there is a $C_k^j > 0$, such that if $t > C_k^j$, the $\overline{\partial}$-Neumann operator is bounded on $W^k(\Omega_j)$ provided the weight is taken to be the function ϕ_t^j on $\mathbb{C}^{n_j}$ given by $\phi_t^j(z) = t\,|z|^2$. The result now follows using the same method as in Theorem 1.2, on taking $C_k = \max_{1\leq j\leq N} C_k^j$ and noting that $\sum_{j=1}^N \phi_t^j = \phi_t$. □

Therefore, it is always possible to solve the $\overline{\partial}$-equation in a product of pseudoconvex domains, with estimates in $\widetilde{W}^k(\Omega)$ using the weight ϕ_t. Using the inclusions (20) and standard results on interpolation, it follows that for each $s \geq 0$, and the operator S maps forms with coefficients in $W^s(\Omega)$ to forms with coefficients in $W^{\frac{s}{N}}(\Omega)$. From this, using a standard "Mittag-Leffler argument" (see [10, pp. 127ff., *Proof of Theorem 6.1.1.*]), one can deduce the following from Corollary 6.1:

Corollary 6.2 *Under the same assumption as in Corollary 6.1, if $f \in \mathcal{C}^\infty_{p,q}(\overline{\Omega})$, is a $\overline{\partial}$-closed form, with $q \neq 0$, then there exists $u \in \mathcal{C}^\infty_{p,q-1}(\overline{\Omega})$ such that $\overline{\partial}u = f$.*

For domains which are the intersection of a finite number of smoothly bounded pseudoconvex domains, such that the boundaries meet transversely at each point of intersection, the existence of a solution to the $\overline{\partial}$-equation smooth up to the boundary has been obtained before [25] using integral kernels. This includes the result of Corollary 6.2, but our method here is simpler and also leads to estimates in Sobolev spaces.

6.3 Proof of Corollary 1.3

Fix $1 \leq j \leq N$ and let $0 \leq p \leq n_j$. The canonical solution operator on Ω_j maps the space $\mathcal{C}^\infty_{p,1}(\overline{\Omega})$ of $(p, 1)$-forms smooth up to the boundary to the space $\mathcal{C}^\infty_{p,0}(\overline{\Omega})$. Using formula (3) on $(p, 0)$ forms, we see that the harmonic projection P_j preserves the space $\mathcal{C}^\infty_{p,0}(\overline{\Omega_j})$.

By the Sobolev embedding theorem, the Sobolev norms $\|\cdot\|_{W^k(\Omega_j)}$ form a system of seminorms which define the usual Fréchet space structure on $\mathcal{C}^\infty(\overline{\Omega_j})$. Using a Fréchet space version of the closed graph theorem (see e.g. [20, Theorem 3 on p. 301]), we easily see that the map K_j is continuous from $\mathcal{C}^\infty_{p,1}(\overline{\Omega_j})$ to $\mathcal{C}^\infty_{p,0}(\overline{\Omega_j})$ and P_j is continuous from $\mathcal{C}^\infty_{p,0}(\overline{\Omega_j})$ to itself. Using the characterization of continuous linear maps between Fréchet spaces (see [20, Proposition 2 on p. 97]), we conclude that for each $l \in \mathbb{N}$, there is an $k = k(l, j, p)$ such that K_j maps the Sobolev space $W^k_{p,1}(\Omega_j)$ continuously to the Sobolev space $W^l_{p,0}(\Omega_j)$ and P_j maps the Sobolev space $W^k_{p,0}(\Omega_j)$ to the Sobolev space $W^l_{p,0}(\Omega_j)$. e can assume that for each l, the integer $k_l = k(l, j, p)$ has been chosen to be independent of j and p. Also, since P_j is a projection, it follows that $k_l \geq l$.

Using the formula (18), the argument used in the proof of Theorem 1.2 shows that the operator S maps the Partial Sobolev space $\widetilde{W}^{k_l}_{\boldsymbol{p},1}(\Omega)$ to $\widetilde{W}^l_{\boldsymbol{p},0}(\Omega)$ for each integer l. It follows from (21) that S maps $\mathcal{C}^\infty_{\boldsymbol{p},1}(\overline{\Omega})$ to $\mathcal{C}^\infty_{\boldsymbol{p},0}(\overline{\Omega})$. Using Theorem 4.7 the smoothness up to the boundary of $\overline{\partial}^* \mathsf{N} f$ follows whenever $\overline{\partial} f = 0$ and the $(p, 1)$-form f is smooth up to the boundary. The statement regarding the Bergman projection now follows from the formula $B = I - \overline{\partial}^* \mathsf{N} \overline{\partial} = I - K\overline{\partial}$.

6.4 Some special product domains

We will apply our results to some special cases when the domain is not pseudoconvex or Stein. The first case is the product of an annulus between two pseudoconvex domain and a pseudoconvex domain.

Corollary 6.3 *Let $\Omega_1 = D_2 \backslash \overline{D}_1$ be the annulus between two pseudoconvex domains $D_1 \subset\subset D_2 \Subset \mathbb{C}^n$ with smooth boundary and let Ω_2 be a bounded pseudoconvex domain in $\mathbb{C}^m$ with Lipschitz boundary. Let Ω be the product domain $\Omega = \Omega_1 \times \Omega_2$. Then the $\overline{\partial}$ operator on $L^2(\Omega)$ has closed range. Furthermore, for $0 \leq p \leq n + m$, we have*

$$\dim H^{p,q}_{L^2}(\Omega) = \dim \mathcal{H}_{p,q}(\Omega) = \begin{cases} \infty, & if\ q = 0; \\ 0, & if\ q \neq 0 \quad or\ q \neq n-1; \\ \infty & if\ q = n-1. \end{cases}$$

This corollary follows easily from the fact that $\overline{\partial}$ has closed range on any bounded pseudoconvex domain in $\mathbb{C}^n$ by Hörmander [18] (regardless of the smoothness of the boundary) and for the annulus between smooth pseudoconvex domains (see [29,31]) for all degrees. It also follows from the Hörmander's L^2 existence theorems, the harmonic space $\mathcal{H}_{p,q}(\Omega_2)$ on the pseudoconvex domain Ω_2 vanishes unless $p = q = 0$, when $\mathcal{H}_{0,0}$ is the space of L^2 holomorphic functions. For the annulus, we have that the cohomology $\mathcal{H}_{p,q}(\Omega_1)$ vanishes except for $q = 0$ and $q = n - 1$. Thus the corollary follows from the theorem above.

For the annulus between two concentric balls $\Omega_1 = \{z \in \mathbb{C}^n : 1 < |z| < 2\}$, the nontrivial harmonic spaces $\mathcal{H}_{(p,n-1)}(\Omega_1)$ have been computed explicitly by Hörmander (see Theorem 2.2 and equation (2.3) in [19]). We can apply the corollary to the case when $\Omega = \{z \in \mathbb{C}^n : 1 < |z| < 2\} \times \{z \in \mathbb{C}^m : |z| < 1\}$ in $\mathbb{C}^{n+m}$. In this case, the closed range property for the ball follows from the work of Kohn [22]. For the annulus between two balls, the closure of the range of $\overline{\partial}$ in degree (p, q) follows from [16, pp. 57 ff.] for $q \neq n - 1$ and from [19] if $q = n - 1$. Thus $\overline{\partial}$ has closed range in the product domain Ω. The harmonic space $\mathcal{H}_{(0,0)}$ on Ω is spanned by the monomials in $\mathbb{C}^{m+n}$. The other nontrivial harmonic spaces $\mathcal{H}_{(p,n-1)}(\Omega)$ can be expressed explicitly as the Hilbert tensor products of harmonic forms $\mathcal{H}_{(p,n-1)}(\Omega_1)$ with monomials in $\mathbb{C}^m$. We can therefore obtain a complete description of the harmonic forms in terms of Hilbert tensor products of spaces. Moreover, we have the following existence and regularity results for the $\overline{\partial}$-operator.

Corollary 6.4 *Let $\Omega = \{z \in \mathbb{C}^n : 1 < |z| < 2\} \times \{z \in \mathbb{C}^m : |z| < 1\} = \Omega_1 \times \Omega_2 \Subset \mathbb{C}^{n+m}$, $n \geq 1$ and $m \geq 1$. Then the $\overline{\partial}$-Neumann operator N exists on Ω. For any (p, q)-form f with $\widetilde{W}^k(\Omega)$ (or $C^\infty(\overline{\Omega})$) coefficients, where k is any nonnegative integer and $1 \leq q \leq n + m$, such that $\overline{\partial} f = 0$ and $f \perp \mathcal{H}_{p,q}$, there exists a solution u*

which has $\widetilde{W}^k(\Omega)$ (or $C^\infty(\overline{\Omega})$) coefficients with $\overline{\partial}u = f$ in Ω. If $q = 1$, we can choose $u = \overline{\partial}^*\mathsf{N}f$ to be the canonical solution.

This answers the question posed by X. Chen. Another interesting case is when one of the factors in the product is a compact manifold. In this case, the domain is pseudoconvex in the sense of Levi, but not Stein. Our theorem can also be applied to the following case.

Corollary 6.5 *Let $\Omega = \Omega_1 \times M$ be the product of a bounded pseudoconvex domain Ω_1 in $\mathbb{C}^n$ and let M be a compact complex hermitian manifold. Then the $\overline{\partial}$ operator on $L^2(\Omega)$ has closed range and the Harmonic spaces satisfy the Künneth formula*

$$\mathcal{H}_*(\Omega_1) \otimes \mathcal{H}_*(M) = \mathcal{H}_*(\Omega). \tag{23}$$

In this case, the space $\mathcal{H}_*(M)$ is finite dimensional and the Hilbert Tensor product coincides with the algebraic tensor product. In particular, $\overline{\partial}$ has closed range on the product $\mathbb{D} \times \mathbb{CP}^1$ of the disc and the Riemann sphere, each with its natural metric, thus answering a question raised by J. Cao.

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