# The Cauchy transform

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*To Prof. Harold S. Shapiro on the occasion of his seventy-fifth birthday.*

## 1. Motivation

In this expository paper, we wish to survey both past and current work on the space of Cauchy transforms on the unit circle. By this we mean the collection $\mathcal{K}$ of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that take the form

$$(K\mu)(z) := \int \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}, \tag{1.1}$$

where $\mu$ is a finite, complex, Borel measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. Our motivation for writing this paper stems not only from the inherent beauty of the subject, but from its connections with various areas of measure theory, functional analysis, operator theory, and mathematical physics. We will provide a survey of some classical results, some of them dating back to the beginnings of complex analysis, in order to prepare a reader who wishes to study some recent and important work on perturbation theory of unitary and compact operators. We also gather up these results, which are often scattered throughout the mathematical literature, to provide a solid bibliography, both for historical preservation and for further study.

Why is the Cauchy transform important? Besides its obvious use, via Cauchy's formula, in providing an integral representation of analytic functions, the Cauchy transform (with $d\mu = dm$ - normalized Lebesgue measure on the unit circle) is the Riesz projection operator $f \to f_+$ on $L^2 := L^2(\mathbb{T}, m)$, where the Fourier expansions of $f$ and $f_+$ are given by

$$f \sim \sum_{n=-\infty}^{\infty} a_n \zeta^n \text{ and } f_+ \sim \sum_{n=0}^{\infty} a_n \zeta^n.$$

Questions such as which classes of functions on the circle are preserved (continuously) by the Riesz projection were studied by Riesz, Kolmogorov, Privalov, Stein, and Zygmund. For example, the $L^p (1 < p < \infty)$ and Lipschitz classes are preserved under the Riesz projection operator while $L^1$, $L^\infty$, and the space of continuous functions are not. With the Riesz projection operator, we can define the Toeplitz operators $f \to T_\phi(f) := (\phi f)_+$, where $\phi \in L^\infty$. Questions about continuity, compactness, etc., depend not only on properties of the symbol $\phi$ but on how the

Cauchy transform acts on the underlying space of functions $f$. The Cauchy transform is also related to the classical conjugation operator $Q\mu = 2\Im(K\mu)$, at least for real measures $\mu$, and similar preservation and continuity questions arise.

Though this classical material is certainly both elegant and important, our real inspiration for wanting to write this survey is the relatively recent work beginning with a seminal paper of Clark [18] which relates the Cauchy transform to perturbation theory. Due to recent advances of Aleksandrov [4] and Poltoratski [65, 66, 67], this remains an active area of research rife with many interesting problems connecting Cauchy transforms to a variety of ideas in classical and modern analysis.

Let us take a few moments to describe the basics of Clark's results. According to Beurling's theorem [25, p. 114], the subspaces of the classical Hardy space $H^2$ invariant under the unilateral shift $Sf = zf$ have the form $\vartheta H^2$, where $\vartheta$ is an inner function. Consequently, the backward shift operator

$$S^* f = \frac{f - f(0)}{z}$$

has invariant subspaces $(\vartheta H^2)^{\perp}$. A description of $(\vartheta H^2)^{\perp}$, involving the concept of a 'pseudocontinuation', can be found in [16, 24, 75]. Clark studied the compression

$$S_{\vartheta} = P_{\vartheta} S | (\vartheta H^2)^{\perp}$$

of the shift $S$ to the subspace $(\vartheta H^2)^{\perp}$, where $P_{\vartheta}$ is the orthogonal projection of $H^2$ onto $(\vartheta H^2)^{\perp}$ (see [60, p. 18]), and determined that all possible rank-one unitary perturbations of $S_{\vartheta}$ (in the case where $\vartheta(0) = 0$) are given by

$$U_{\alpha} f := S_{\vartheta} f + \langle f, \frac{\vartheta}{z} \rangle \alpha, \quad \alpha \in \mathbb{T}.$$

Furthermore, $U_{\alpha}$ is unitarily equivalent to the operator 'multiplication by $z$', $g \to zg$, on the space $L^2(\sigma_{\alpha})$, where $\sigma_{\alpha}$ is a certain singular measure on $\mathbb{T}$ naturally associated with the inner function $\vartheta$. This equivalence is realized by the unitary operator

$$\mathcal{F}_{\alpha} : (\vartheta H^2)^{\perp} \to L^2(\sigma_{\alpha}),$$

which maps the reproducing kernel

$$k_{\lambda}^{\vartheta}(z) = \frac{1 - \overline{\vartheta(\lambda)} \vartheta(z)}{1 - \overline{\lambda} z}$$

for $(\vartheta H^2)^{\perp}$ to the function

$$\zeta \to \frac{1 - \overline{\vartheta(\lambda)} \alpha}{1 - \overline{\lambda} \zeta}$$

in $L^2(\sigma_{\alpha})$ and extends by linearity and continuity. This measure $\sigma_{\alpha}$ arises as follows: For each $\alpha \in \mathbb{T}$ the analytic function

$$\frac{\alpha + \vartheta(z)}{\alpha - \vartheta(z)} \tag{1.2}$$

on the unit disk has positive real part, which, by Herglotz's theorem [25, p. 3], takes the form

$$\Re\left(\frac{\alpha+\vartheta(z)}{\alpha-\vartheta(z)}\right)=\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}\,d\sigma_\alpha(\zeta),$$

where the right-hand side of the above equation is the Poisson integral $(P\sigma_\alpha)(z)$ of a positive measure $\sigma_\alpha$. Without too much difficulty, one can show that the measure $\sigma_\alpha$ is carried by the set $\{\zeta\in\mathbb{T}:\vartheta(\zeta)=\alpha\}$ and hence singular with respect to Lebesgue measure on the circle. Moreover, the measures $(\sigma_\alpha)_{\alpha\in\mathbb{T}}$ are pairwise singular.

This idea extends via eq.(1.2) beyond inner functions $\vartheta$ to any $\phi$ in the unit ball of $H^\infty$ to create a family of positive measures $(\mu_\alpha)_{\alpha\in\mathbb{T}}$ associated with $\phi$. On the other hand, for a given positive measure $\mu$, the Poisson integral $P\mu$ of $\mu$ is a positive harmonic function on the disk and so

$$P\mu=\Re\left(\frac{1+\phi}{1-\phi}\right) \tag{1.3}$$

for some $\phi\in\operatorname{ball}(H^\infty)$. That is to say, every positive measure $\mu$ is the Clark measure $\mu_1$ for some $\phi$ in the unit ball of $H^\infty$. It is worth mentioning that $\phi$ is an inner function if and only if the boundary function for $(1+\phi)/(1-\phi)$ is purely imaginary. Using eq.(1.3) as well as Fatou's theorem (which says that the boundary function for $P\mu_1$ is $d\mu_1/dm$ almost everywhere [25, p. 4]), we conclude that $\phi$ is inner if and only if $\mu_1\perp m$.

The family $(\mu_\alpha)_{\alpha\in\mathbb{T}}$ of Clark measures for some function $\phi\in\operatorname{ball}(H^\infty)$ also provide a disintegration of normalized Lebesgue measure $m$ on the circle. A beautiful theorem of Aleksandrov [4] says that

$$\int_{\mathbb{T}}\mu_\alpha\,dm(\alpha)=m,$$

where the integral is interpreted in the weak-$*$ sense, that is,

$$\int_{\mathbb{T}}\left[\int_{\mathbb{T}}f(\zeta)\,d\mu_\alpha(\zeta)\right]\,dm(\alpha)=\int_{\mathbb{T}}f(\zeta)\,dm(\zeta)$$

for all continuous functions $f$ on $\mathbb{T}$. Moreover, if $\Sigma=\{\,\zeta\in\mathbb{T}\colon|\phi(\zeta)|=1\,\}$,

$$\int_{\mathbb{T}}\mu_\alpha^s\,dm(\alpha)=\chi_\Sigma\cdot m\quad\text{and}\quad\int_{\mathbb{T}}\mu_\alpha^{ac}\,dm(\alpha)=(1-\chi_\Sigma)\cdot m,$$

where $\mu_\alpha^s$ is the singular part and $\mu_\alpha^{ac}$ is the absolutely continuous part of $\mu_\alpha$ (with respect to Lebesgue measure $m$), and again, the above equation is interpreted in the weak-$*$ sense.

One can show that the Cauchy transform of $\sigma_\alpha$ (a Clark measure for the inner function $\vartheta$)

$$(K\sigma_\alpha)(z)=\int\frac{1}{1-\bar{\zeta}z}d\sigma_\alpha(\zeta)$$

is equal to the function

$$\frac{1}{1-\overline{\alpha}\vartheta(z)}.$$

Furthermore, one can use this to produce the following formula for $\mathcal{F}_{\alpha}^{*} : L^2(\sigma_\alpha) \to (\vartheta H^2)^{\perp}$ in terms of the 'normalized' Cauchy transform

$$\mathcal{F}_{\alpha}^{*} f = \frac{K(f\, d\sigma_\alpha)}{K(\sigma_\alpha)}.$$

Poltoratski [65] showed that some striking things happen here. The first is that for $\sigma_\alpha$-almost every $\zeta \in \mathbb{T}$, the non-tangential limit of the above normalized Cauchy transform exists and is equal to $f(\zeta)$. On the other hand, for $g \in (\vartheta H^2)^{\perp}$, the non-tangential limits certainly exist almost everywhere with respect to Lebesgue measure on the circle (since $(\vartheta H^2)^{\perp} \subset H^2$). But in fact, for $\sigma_\alpha$-almost every $\zeta$, the non-tangential limit of $g$ exists and is equal to $(\mathcal{F}_\alpha g)(\zeta)$. Remarkable here is the significance of the role Cauchy transforms play not only in the above perturbation problem involving the rank-one perturbations of the model operator $S_\vartheta$ but in other perturbation problems as well (see [67] and the references therein).

After a brief historical introduction in §2, we begin in §3 with a description of some of the classical function theoretic properties of Cauchy transforms, viewed as functions on the unit disk. This will include their boundary and mapping properties, as well as their relationships with the classical Hardy spaces $H^p$. In §4 we explore the general question: Which analytic functions on the disk can be represented as Cauchy transforms of measures on the circle? Though the answer to this question is still incomplete, there is a related result of Aleksandrov, extending work of Tumarkin, which characterizes the analytic functions on $\widehat{\mathbb{C}}\setminus\mathbb{T}$ which are Cauchy transforms of measures on the circle.

The space of Cauchy transforms can be viewed in a natural way as the dual space of the disk algebra, or equivalently, as the quotient space $M/\overline{H_0^1}$. We will describe this duality in §5, including facts about the weak and weak-$*$ topology on this space. In §6 we will describe multiplication and division in the space of Cauchy transforms and examine such questions as: (i) Which bounded analytic functions $\phi$ on the disk satisfy $\phi\mathcal{K} \subseteq \mathcal{K}$ (the multiplier question)? (ii) If $\vartheta$ is inner and divides $K\mu$, that is, $K\mu/\vartheta \in H^p$ for some $p > 0$, does $K\mu/\vartheta$ belong to $\mathcal{K}$ (the divisor question)? In §7 we explore the classical operators on $H^p$ (forward and backward shifts, composition operators, and the Cesàro operator) in the setting of the space of Cauchy transforms. Questions relating to continuity and invariant subspaces will be explored. In §8 we will touch more briefly on such topics as the distribution of boundary values of Cauchy transforms. This will include the Havin-Vinogradov-Tsereteli theorem, and its recent improvement by Poltoratski, as well as Aleksandrov's weak-type characterization using the $A$-integral. We will also discuss the maximal properties of Cauchy transforms arising in the recent work of Poltoratski.

Conspicuously missing from this survey are results about the Cauchy transform of a measure compactly supported in the plane. Certainly this is an important object to study and there are many wonderful ideas here. However, broadening the survey to include those Cauchy transforms opens up such a vast array of topics from so many other fields of analysis such as potential theory and partial differential equations, that our original theme and motivation for writing this survey - a brief overview of the subject with a solid bibliography for further study - would be lost. We feel that focusing on the Cauchy transform of measures on the circle links the classical function theory with more modern applications to perturbation theory. If one is interested in exploring the Cauchy transform of a measure on the plane we suggest that the books [9, 28, 61] are a good place to start.

## 2. Some early history

In a series of papers from the mid 1800's [86], Cauchy developed what is known today as the 'Cauchy integral formula': If $f$ is analytic in $\{|z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$, then

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad |z| < 1.$$

After Cauchy, others, such as Sokhotski, Plemelj, and Privalov [58], examined the 'Cauchy integral'

$$\widehat{\phi}(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\phi(\zeta)}{\zeta - z} d\zeta,$$

where the boundary function $f(\zeta)$ in Cauchy's formula is replaced by a 'suitable' function $\phi$ defined only on the unit circle $\mathbb{T}$. Amongst other things, they explored the relationship between the density function $\phi$ and the limiting values of $\widehat{\phi}$, as $|z| \to 1$, as well as the Cauchy principal-value integral

$$\frac{1}{2\pi i} P.V. \int_{\mathbb{T}} \frac{\phi(\zeta)}{\zeta - e^{i\theta}} d\zeta.$$

In particular, Sokhotski [87] in his 1873 thesis (see also [54, Vol. I, p. 316]) proved the following.

**Theorem 2.1** (Sokhotski, 1873)**.** *Suppose $\phi$ is continuous on $\mathbb{T}$ and, for a particular $\zeta_0 \in \mathbb{T}$, satisfies the condition*

$$|\phi(\zeta) - \phi(\zeta_0)| \leqslant C|\zeta - \zeta_0|^{\alpha}, \ \ \zeta \in \mathbb{T}$$

*for some positive constants $C$ and $\alpha$. Then the limits*

$$\widehat{\phi}^{-}(\zeta_0) := \lim_{r \to 1^-} \widehat{\phi}(r\zeta_0) \quad \text{and} \quad \widehat{\phi}^{+}(\zeta_0) := \lim_{r \to 1^-} \widehat{\phi}(\zeta_0/r)$$

*exist and moreover, $\widehat{\phi}^{-}(\zeta_0) - \widehat{\phi}^{+}(\zeta_0) = \phi(\zeta_0)$. Furthermore, the Cauchy principal-value integral*

$$P.V. \int_{\mathbb{T}} \frac{\phi(\zeta)}{\zeta - \zeta_0} d\zeta := \lim_{\varepsilon \to 0} \int_{|\zeta - \zeta_0| > \varepsilon} \frac{\phi(\zeta)}{\zeta - \zeta_0} d\zeta$$

*exists and*

$$\widehat{\phi}^{+}(\zeta_0) + \widehat{\phi}^{-}(\zeta_0) = \frac{1}{\pi i} P.V. \int_{\mathbb{T}} \frac{\phi(\zeta)}{\zeta - \zeta_0} d\zeta.$$

Though Sokhotski first proved these results in 1873, the above formulas are often called the 'Plemelj formulas' due to reformulations and refinements of them by J. Plemelj [64] in 1908.

I. I. Privalov, in a series of papers and books[1], beginning with his 1919 Saratov doctoral dissertation [70], began to examine the Plemelj formulas for integrals of Cauchy-Stieltjes type

$$\widehat{F}(z) := \frac{1}{2\pi i} \int_{[0,2\pi]} \frac{1}{1 - e^{-i\theta} z} dF(\theta),$$

where $F$ is a function of bounded variation on $[0, 2\pi]$. Privalov, knowing the recently discovered integration theory of Lebesgue and following the lead of his teacher Golubev [30], developed the Sokhotski-Plemelj formulas for these Cauchy-Stieltjes integrals.

**Theorem 2.2** (Privalov, 1919)**.** *Suppose $F$ is a function of bounded variation on $[0, 2\pi]$. Then for almost every $t$, $\widehat{F}^{-}(e^{it})$, the non-tangential limit of $\widehat{F}(z)$ as $z \to e^{it}$ $(|z| < 1)$ and $\widehat{F}^{+}(e^{it})$, the non-tangential limit at $z \to e^{it}$ $(|z| > 1)$, exist and moreover, $\widehat{F}^{-}(e^{it}) - \widehat{F}^{+}(e^{it}) = F'(t)$. Furthermore, for almost every $t$, the Cauchy principal-value integral*

$$P.V. \int_{[0,2\pi]} \frac{1}{1 - e^{-i\theta} e^{it}} dF(\theta) := \lim_{\varepsilon \to 0} \int_{|t-\theta| > \varepsilon} \frac{1}{1 - e^{-i\theta} e^{it}} dF(\theta)$$

*exists and*

$$\widehat{F}^{+}(e^{it}) + \widehat{F}^{-}(e^{it}) = \frac{1}{\pi i} P.V. \int_{[0,2\pi]} \frac{1}{1 - e^{-i\theta} e^{it}} dF(\theta).$$

A well-known theorem from Fatou's 1906 thesis [26] (see also [25, p. 39]) states that $\widehat{F}(re^{it}) - \widehat{F}(e^{it}/r) \to F'(t)$ as $r \to 1^{-}$, whenever $F'(t)$ exists (which is almost everywhere). Privalov's contribution was to prove the existence of the principal-value integral as well as to generalize to the case when the unit circle is replaced with a general rectifiable curve (see [31, 58] for more).

A question explored for some time, and for which there is still no completely satisfactory answer is: Which analytic functions $f$ on $\mathbb{D}$ can be represented as a Cauchy-Stieltjes integral? Perhaps a first step in answering this question would be to determine which functions can be represented as the Cauchy integral of their boundary values. The following theorem of the brothers Riesz [72] (see also [25, p. 41]) gives the answer [2].

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[1]There is Privalov's famous book [71] as well as a nice survey of his work in [53].

[2]A nice survey of these results (and others) can be found in [31, Ch. IX].

**Theorem 2.3** (F. and M. Riesz). *Let $f$ be analytic on $\mathbb{D}$. Then $f$ has an almost everywhere defined, integrable - with respect to Lebesgue measure on $\mathbb{T}$ - non-tangential limit function $f^*$ and*

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f^*(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}$$

*if and only if $f$ belongs to $H^1$.*

Though the question as to which analytic functions on $\mathbb{D}$ can be represented as Cauchy integrals of their *boundary functions* has been answered, there is no complete description of these functions, though there are some nice partial results (see Theorem 4.9 below).

To give the reader some historical perspective, we stated these classical theorems of Sokhotski, Plemelj, Riesz, and Privalov, in terms of Cauchy-Stieltjes integrals of functions of bounded variation. However, for the rest of this survey, we will use the more modern, but equivalent, notation of Cauchy integrals of finite, complex, Borel measures on the circle as in eq.(1.1). This is done by equating a function of bounded variation with a corresponding measure and vice-versa (see [39, p. 331] for further details).

## 3. Some basics about the Cauchy transform

Before moving on, let us set some notation. Throughout this survey, $\mathbb{C}$ will denote the complex numbers, $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere, $\mathbb{D} := \{|z| < 1\}$ the open unit disk, $\mathbb{D}^- := \{|z| \leqslant 1\}$ its closure, $\mathbb{T} := \partial\mathbb{D} = \{|z| = 1\}$ its boundary, and $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$ will denote the (open) extended exterior disk. $M := M(\mathbb{T})$ will denote the complex, finite, Borel measures on $\mathbb{T}$, $M_+$ the positive measures in $M$, $M_a := \{\mu \in M : \mu \ll m\}$ ($dm = |d\zeta|/2\pi$ is normalized Lebesgue measure on the circle), and $M_s := \{\mu \in M : \mu \perp m\}$. The norm on $M$, the total variation norm, will be denoted by $\|\mu\|$. The Lebesgue decomposition theorem says that $M = M_a \oplus M_s$ and that if $\mu = \mu_a + \mu_s$ ($\mu_a \in M_a, \mu_s \in M_s$), then $\|\mu\| = \|\mu_a\| + \|\mu_s\|$.

For $0 < p \leqslant \infty$ we will let $L^p := L^p(m)$ denote the usual Lebesgue spaces, $H^p$ the standard Hardy spaces, and $\|\cdot\|_p$ the norm on these spaces. When $p = \infty$, $H^\infty$ will denote the bounded analytic functions on $\mathbb{D}$. Recall that every function in $H^p$ has finite non-tangential limits almost everywhere and this boundary function belongs to $L^p$. In fact, the $L^p$ norm of this boundary function is the $H^p$ norm. We will use $N$ to denote the Nevanlinna class of the disk and $N^+$ to denote the Smirnov class. We refer the reader to some standard texts about the Hardy spaces [25, 29, 40, 51, 77].

The following theorem of F. and M. Riesz [25, p. 41] plays an important role in the theory of Cauchy transforms and will be mentioned many times in this survey.

**Theorem 3.1** (F. and M. Riesz theorem)**.** *Suppose* $\mu \in M$ *satisfies*

$$\int \zeta^n d\mu(\zeta) = 0 \quad \textit{whenever } n = 0, 1, 2, \cdots.$$

*Then* $d\mu = \phi dm$, *where* $\phi \in H^1_0 = \{f \in H^1 : f(0) = 0\}$.

For $\mu \in M$, define the analytic function on $\widehat{\mathbb{C}} \setminus \mathbb{T}$

$$\widehat{\mu}(z) := \int \frac{d\mu(\zeta)}{1 - \overline{\zeta} z}. \tag{3.2}$$

The function $\widehat{\mu}$ is called the *Cauchy transform* of $\mu$. We let

$$K\mu := \widehat{\mu}|\mathbb{D}$$

and set

$$\mathcal{K} := \{K\mu : \mu \in M\}$$

to be the *space of Cauchy transforms*. A few facts are immediate from the definition: (i) $\widehat{\mu}$ has an analytic continuation to $\widehat{\mathbb{C}} \setminus \operatorname{supp}(\mu)$, (ii) $\widehat{\mu}(\infty) = 0$, (iii) $\widehat{\mu}$ has the following series expansions:

$$(K\mu)(z) = \sum_{n=0}^{\infty} \mu_n z^n, \ z \in \mathbb{D}, \qquad \widehat{\mu}(z) = -\sum_{n=1}^{\infty} \frac{\mu_{-n}}{z^n}, \ z \in \mathbb{D}_e, \tag{3.3}$$

where

$$\mu_n := \int e^{-in\theta} d\mu(e^{i\theta}), \ \ n \in \mathbb{Z}$$

are the Fourier coefficients of $\mu$, (iv) $\widehat{\mu}$ satisfies the growth condition

$$|\widehat{\mu}(z)| \leqslant \frac{\|\mu\|}{|1 - |z||}, \ \ |z| \neq 1. \tag{3.4}$$

For a given $f \in \mathcal{K}$, there are a variety of measures $\mu \in M$ such that $f = K\mu$. For example, by eq.(3.3), $K(\overline{\phi} dm) = 0$ whenever $\phi \in H^1_0$. By the F. and M. Riesz theorem however, these are the only measures for which $K\mu = 0$. For $f \in \mathcal{K}$, let

$$M_f := \{\mu \in M : f = K\mu\}$$

be the set of 'representing measures' for $f$. For $\mu \in M$ decomposed (uniquely) as $\mu = \mu_a + \mu_s$ ($\mu_a \in M_a$ and $\mu_s \in M_s$) we can apply the F. and M. Riesz theorem to show that all measures in $M_f$ have the same singular part $\mu_s$. For $\mu_1, \mu_2 \in M_f$, $d\mu_1 - d\mu_2 = \overline{\phi}\, dm$ for some $\phi \in H^1_0$. Thus $M_f$ can be identified with a coset in the quotient space $M/\overline{H^1_0}$. We will explore $M_f$ further in §5.

A routine argument using Lebesgue's dominated convergence theorem shows that

$$\lim_{r \to 1^-} (1 - r)(K\mu)(r\zeta) = \mu(\{\zeta\})$$

and so $K\mu$ is poorly behaved at the points $\zeta$ where $\mu(\{\zeta\}) \neq 0$. This can indeed be a dense subset of $\mathbb{T}$. Despite this seemingly poor behavior of $K\mu$ near $\mathbb{T}$, there is some regularity in the boundary behavior of a Cauchy transform. For $0 < p \leqslant \infty$, let $H^p(\widehat{\mathbb{C}} \setminus \mathbb{T})$ denote the class of analytic functions $f$ on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ for which

$$\|f\|^p_{H^p(\widehat{\mathbb{C}}\setminus\mathbb{T})} := \sup_{r \neq 1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty.$$

A well-known theorem of Smirnov [85] (see also [25, p. 39]) is the following.

**Theorem 3.5** (Smirnov). *If $\mu \in M$, then*

$$\widehat{\mu} \in \bigcap_{0<p<1} H^p(\widehat{\mathbb{C}} \setminus \mathbb{T})$$

*and moreover, $\|\widehat{\mu}\|_{H^p(\widehat{\mathbb{C}}\setminus\mathbb{T})} \leqslant c_p \|\mu\|$, where $c_p = O((1-p)^{-1})$.*

The above theorem says that for fixed $0 < p < 1$, the operator $\mu \to K\mu$ is a continuous linear operator from $M$ to $H^p$. What is the norm of this operator? Equivalently, what is the best constant $A_p$ in the estimate $\|K\mu\|_p \leqslant A_p\|\mu\|$? We do not know the answer to this. However, we can say that

$$\sup\{\|K\mu\|_p : \mu \in M_+, \|\mu\| = 1\} = \|\frac{1}{1-z}\|_p.$$

To see this note that $K\mu$ is subordinate to $\phi(z) = (1-z)^{-1}$. Now use Littlewood's subordination theorem [25, p. 10]. For any complex measure $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4), \mu_j \geqslant 0$, one can use a slight variation of the above argument four times to prove Smirnov's theorem: $\|K\mu\|_p \leqslant A_p\|\mu\|$. However, the best constant $A_p$ is unknown for general complex measures.

The containment

$$\mathcal{K} \subsetneq \bigcap_{0<p<1} H^p$$

is strict since $f(z) = (1-z)^{-1}\log(1-z) \in H^p$ for all $0 < p < 1$ but does not satisfy the growth condition in eq.(3.4). Also worth pointing out here again is Theorem 2.3 which says that every $f \in H^p$ $(p \geqslant 1)$ can be written as the Cauchy integral of its boundary function. Thus

$$\bigcup_{p\geqslant 1} H^p \subsetneq \mathcal{K} \subsetneq \bigcap_{0<p<1} H^p.$$

The first containment above is strict since $(1-z)^{-1} = K\delta_1$ but does not belong to $H^1$. For $0 < p < 1$, the boundary functions for $H^p$ functions certainly belong to $L^p$. However, they may not be integrable on the circle and so their Cauchy integral does not always make sense. There is a remedy for this in the theory of '$A$-integrals' that will be described in §4, which says that some, but not all, Cauchy transforms can be written as the Cauchy $A$-integral of their boundary functions.

Smirnov's theorem has been refined in a variety of ways. The first refinement due to M. Riesz [73] (see also [25, p. 54]) deals with the case when the measure $\mu$

takes the form $d\mu = f dm$ for $f \in L^p$ $(1 < p < \infty)$. Since the notation $K(f dm)$ can be a bit cumbersome, we use $f_+$ to denote $K(f dm)$ for $f \in L^1$.

**Theorem 3.6** (M. Riesz). *If $1 < p < \infty$ then $f_+ \in H^p$ whenever $f \in L^p$. Moreover, the map $f \to f_+$ from $L^p$ to $H^p$ is continuous and onto.*

For $f \in L^p$ $(1 < p < \infty)$ with Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} f_n \zeta^n, \quad f_n := \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi},$$

it is not too difficult to see that the $L^p$ boundary function for $f_+$ has Fourier series

$$f_+ \sim \sum_{n=0}^{\infty} f_n \zeta^n.$$

Thus we can think of the Cauchy transform as projection operator $P : L^p \to H^p$, $Pf = f_+$. Hollenbeck and Verbitsky [41] compute the norm of the Riesz projection as

$$\sup\{\|f_+\|_p : \|f\|_{L_p} = 1\} = \frac{1}{\sin(\pi/p)}, \quad 1 < p < \infty.$$

The endpoint cases $p = 1$ and $p = \infty$ are more complicated. For example [25, pp. 63 - 64], the $f \in L^1$ whose Fourier series is

$$\sum_{n=2}^{\infty} \frac{\cos n\theta}{\log n}$$

has Cauchy transform equal to

$$f_+ = \sum_{n=2}^{\infty} \frac{z^n}{\log n}$$

which does not belong to $H^1$ [25, p. 48][3]. It is worth remarking here that not only does the Riesz projection $f \to f_+$ fail to be continuous from $L^1$ onto $H^1$, but there is no other continuous projection of $L^1$ onto $H^1$ [59]. If the function is slightly better than $L^1$, there is the following theorem of Zygmund [100] (see also [25, p. 58]): If $|f| \log^+ |f| \in L^1$, then $f_+ \in H^1$.

Despite these pathologies with $p = 1$, there is a well-known, and often revisited, theorem about the Cauchy transforms of $L^1$ functions due to Kolmogorov [50] (see also [51, p. 92]).

**Theorem 3.7** (Kolmogorov).
*For $f \in L^1$,*

$$m(|f_+| > \lambda) \leqslant A \frac{\|f\|_1}{\lambda}, \quad \lambda > 0.$$

[3]If $f = \sum_n a_n z^n \in H^1$, then $\sum_n |a_n|/(n+1) \leqslant \pi \|f\|_1$.

We use the notation $m(|f_+| > \lambda)$ as a shorthand for $m(\{e^{i\theta} : |f_+(e^{i\theta})| > \lambda\})$. Kolmogorov's theorem can be use to give an alternate proof of Theorem 3.5 (see [51, p. 98]). It has also been generalized in several directions. First, the estimate

$$m(|K\mu| > \lambda) = O(\frac{1}{\lambda})$$

holds for any $\mu \in M$, not just the absolutely continuous ones. This result is somewhat folklore by now and a proof can be obtained by making minor changes to the proof of Kolmogorov's theorem in [51, p. 92]. Secondly, for $\mu \ll m$, Kolmogorov's estimate can be improved to

$$m(|K\mu| > \lambda) = o(\frac{1}{\lambda}). \tag{3.8}$$

Indeed, this inequality is true when $d\mu = p\,dm$, where $p$ is a trigonometric polynomial. Now approximate any $L^1$ function with trigonometric polynomials and use Theorem 3.7. More surprising [93] is that the converse of this is true, namely

$$m(|K\mu| > \lambda) = o(\frac{1}{\lambda}) \Leftrightarrow \mu \ll m.$$

We will see a much stronger version of this in §8.

The $f \in L^\infty$ whose Fourier series is

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n}$$

has Cauchy transform

$$f_+ = \sum_{n=1}^{\infty} \frac{z^n}{n} = \log(\frac{1}{1-z})$$

which is not bounded. As was true in the $L^1$ case, not only is the Riesz projection $f \to f_+$ not a continuous projection of $L^\infty$ onto $H^\infty$, there is no other continuous projection of $L^\infty$ onto $H^\infty$ [40, p. 155]. There is, however, a theorem of Spanne [88] and Stein [90] which characterizes the Cauchy transforms of $L^\infty$ functions.

**Theorem 3.9** (Spanne, Stein)**.** *For $f \in L^\infty$, the Cauchy transform $f_+$ belongs to $BMOA$, the analytic functions of bounded mean oscillation. Moreover, the map $f \to f_+$ is continuous from $L^\infty$ onto $BMOA$.*

See [29, Chapter 6] for a definition of $BMO$ and $BMOA$ and their basic properties. For now, note that $H^\infty \subsetneq BMOA$.

For spaces of smooth functions, there are results about the action of the Cauchy transform. Some examples: (i) $C_+ = VMOA$ (where $C$ denotes the continuous functions on $\mathbb{T}$, $C_+ = \{f_+ : f \in C\}$, and $VMOA$ are the analytic functions of vanishing mean oscillation) [80] (ii) If $\Lambda^n_\alpha$ ($n = 0, 1, 2, \ldots, 0 < \alpha < 1$) denotes the Lipschitz classes and $\Lambda^n_*$ ($n = 0, 1, 2, \ldots$) denotes the Zygmund classes on $\mathbb{T}$, then $(\Lambda^n_\alpha)_+ \subseteq \Lambda^n_\alpha$ and $(\Lambda^n_*)_+ \subseteq \Lambda^n_*$ [69] (see also [51, p. 110] and [101]) (iii) One can check just by looking at Fourier and power series coefficients that if $f \in C^{\infty}(\mathbb{T})$, then every derivative of $f_+$ has a continuous extension to the closed disk $\mathbb{D}^-$.

What about Cauchy transforms of functions from weighted $L^p$ spaces? The question here is the following: Given $1 < p < \infty$, what are the conditions on a measure $\mu \in M_+$ (the non-negative measures in $M$) such that

$$\int |g_+|^p d\mu \leqslant C \int |g|^p d\mu$$

for all trigonometric polynomials $g$? Though there has been earlier work on this problem [32, 38] (for example $d\mu$ must take the form $w dm$), the definitive result here is a celebrated theorem of Hunt, Muckenhoupt, and Wheeden [45] (a precise condition on the weight $w$ called the '$A_p$ condition').

In looking at the references for the above results, one notices that they all deal with 'conjugate functions' (*fonctions conjuguées*). The relationship between this conjugation operator and the Cauchy transform is as follows. A computation reveals that

$$\frac{1}{1-\overline{\zeta}z} = \frac{1}{2}\left[\, 1 + P_z(\zeta) + iQ_z(\zeta)\,\right],$$

where

$$P_z(\zeta) = \frac{1-|z|^2}{|\zeta - z|^2} \qquad Q_z(\zeta) = \frac{2\Im(\overline{\zeta}z)}{|\zeta - z|^2}, \;\; \zeta \in \mathbb{T}, z \in \mathbb{D}, \tag{3.10}$$

are the Poisson and conjugate Poisson kernels. For $f \in L^1$,

$$f_+(z) = \frac{1}{2}\left[\, \int_{\mathbb{T}} f(\zeta)dm(\zeta) + \int_{\mathbb{T}} P_z(\zeta)f(\zeta)dm(\zeta) + i\int_{\mathbb{T}} Q_z(\zeta)f(\zeta)dm(\zeta)\,\right].$$

By Fatou's theorem [25, p. 5],

$$\lim_{\measuredangle z \to e^{i\theta}} \int_{\mathbb{T}} P_z(\zeta)f(\zeta)dm(\zeta) = f(e^{i\theta}) \text{ a.e.}$$

and [101, Vol. I, p. 131] the following limit

$$\widetilde{f}(e^{i\theta}) := \lim_{\measuredangle z \to e^{i\theta}} \int_{\mathbb{T}} Q_z(\zeta)f(\zeta)dm(\zeta)$$

exists for almost all $e^{i\theta}$. Here $\measuredangle$ denotes the non-tangential limit. Moreover,

$$\widetilde{f}(e^{i\theta}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{|\theta - t| > \varepsilon} \cot\left(\, \frac{\theta - t}{2}\,\right) f(e^{it})\, dt \text{ a.e.}$$

One can also define, for $\mu \in M$,

$$(Q\mu)(z) = \int Q_z(\zeta)d\mu(\zeta)$$

and note that the above function has non-tangential limits almost everywhere denoted by $\widetilde{\mu}(e^{i\theta})$. The operator $f \to \widetilde{f}$ (or $\mu \to \widetilde{\mu}$) is called the *conjugation operator*. Thus, for example,

$$f_+(e^{i\theta}) = \frac{1}{2}\ \Big[ \int_{\mathbb{T}} f(\zeta) dm(\zeta) + f(e^{i\theta}) + i\widetilde{f}(e^{i\theta}) \Big] \quad \text{a.e.}$$

Thus questions about the continuity of the operator $f \to f_+$ (on spaces of functions on $\mathbb{T}$) are equivalent to questions about the continuity of the conjugation operator $f \to \widetilde{f}$. The norm of the operator $f \to f_+$ is known when $1 < p < \infty$ (see the remarks following Theorem 3.6) and not quite understood when $0 < p < 1$. For the conjugation operator $f \to \widetilde{f}$ (as an operator from $L^p$ to $L^p$ when $p > 1$ and an operator from $L^1$ to $L^p$ for $0 < p < 1$), the norm has been computed by Pichorides [63] as: $\tan(\pi/2p)$ if $1 < p \leqslant 2$; $\cot(\pi/2p)$ if $p > 2$; $(\cos(p\pi/2))^{-1/p}\|f\|_1 \leqslant \|\widetilde{f}\|_p \leqslant 2^{1/p-1}(\cos(p\pi/2))^{-1/p}\|f\|_1$ for $0 < p < 1$.

A variant of Kolmogorov's theorem (Theorem 3.7) says that for fixed $0 < p < 1$ and $\mu \in M$, $\widetilde{\mu}$ belongs to $L^p$ and

$$\|\widetilde{\mu}\|_{L^p} \leqslant c_p \|\mu\|.$$

B. Davis [19, 20, 21] computes the best constant $c_p$, at least for real measures $M_{\mathbb{R}}$ as

$$\sup\{\|\widetilde{\mu}\|_{L^p} : \mu \in M_{\mathbb{R}}, \|\mu\| = 1\} = \|\widetilde{\nu}\|_{L^p},$$

where $\nu$ is a measure given by $\nu(\{1\}) = 1/2, \nu(\{-1\}) = -1/2$ and $|\nu|(\mathbb{T}\backslash\{-1, 1\}) = 0$.

Kolmogorov's theorem says that

$$m(|K\mu| > \lambda) \leqslant C\frac{\|\mu\|}{\lambda}, \quad \mu \in M.$$

The best constant $C$ is unknown. However, there is information about the best constant in the related inequality

$$m(|\widetilde{\mu}| > \lambda) \leqslant C\frac{\|\mu\|}{\lambda}.$$

When $\mu \ll dm$, the best constant $C$ is $\Theta^{-1}$, where $\Theta = (1 - 3^{-2} + 5^{-2} - \cdots)/(1 + 3^{-2} + 5^{-2} + \cdots)$. For $\mu \in M_+$, the best constant $C$ is one.

Here is a good place to mention that the conjugation operator $Q$ is closely related to the *Hilbert transform*

$$(Hf)(x) := P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t} dt,$$

defined almost everywhere for $f \in L^1(\mathbb{R}, dx)$. Results for the conjugation operator frequently have direct analogs for the Hilbert transform in that they are both singular integral operators [29, 91].

## 4. Which analytic functions are Cauchy transforms?

For an analytic $f$ on $\mathbb{D}$, when is $f = K\mu$? Gathering up the observations from previous sections of this survey, there are the following necessary conditions.

**Proposition 4.1.** *Suppose $f = K\mu$ for some $\mu \in M$. Then*

1. *$f$ satisfies the growth condition $|f(z)| \leqslant C_\mu(1 - |z|)^{-1}$.*
2. *$f$ has finite non-tangential limits $m$-almost everywhere on $\mathbb{T}$ and*

$$m(|f| > \lambda) \leqslant C_f/\lambda, \quad \lambda > 0.$$

3. *$f \in H^p$ for all $0 < p < 1$ and $\|f\|_p = O((1 - p)^{-1})$.*
4. *If $f = \sum_{n \geqslant 0} a_n z^n$, then $(a_n)_{n \geqslant 1}$ is a bounded sequence of complex numbers.*

Known necessary and sufficient conditions for an analytic function on the disk to be a Cauchy transform are difficult to apply and in a way, the very question is unfair. For example, suppose that $f$ is analytic on $\mathbb{D}$ with power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

and we want to know if $f = K\mu$ on $\mathbb{D}$ for some $\mu \in M$. Since, from eq.(3.3), $K\mu$ can be written as

$$(K\mu)(z) = \mu_0 + \mu_1 z + \mu_2 z^2 + \cdots ,$$

we would be trying to determine (comparing $a_n$ with $\mu_n$) the measure $\mu$ from only 'half' its Fourier coefficients (the non-negative ones).

If one can settle for a functional analysis condition, there is characterization of $\mathcal{K}$ [33], albeit difficult to apply. The proof follows directly by a duality argument (see §5 below).

**Theorem 4.2** (Havin)**.** *Suppose $f = \sum_{k=0}^{\infty} a_k z^k$ is analytic on $\mathbb{D}$. Then the following statements are equivalent.*

1. *There is some constant $C > 0$, depending only on $f$, such that*

$$\Big| \sum_{k=0}^{p} \lambda_k a_k \Big| \leqslant C \max \Big\{ \Big| \sum_{k=0}^{p} \frac{\lambda_k}{z^{k+1}} \Big| : z \in \mathbb{T} \Big\}$$

*for any complex numbers $\lambda_0, \ldots, \lambda_p$.*

2. *$f = K\mu$ for some $\mu \in M$.*

Instead of asking whether or not an analytic function defined only on $\mathbb{D}$ is a Cauchy transform, suppose we were to ask if an analytic function $f$ on $\widehat{\mathbb{C}}\backslash\mathbb{T}$ is a Cauchy transform $\widehat{\mu}$ on $\widehat{\mathbb{C}}\backslash\mathbb{T}$ (recall the definition of $\widehat{\mu}$ from eq.(3.2)). This is a more tractable question since we would be comparing the Laurent series of these two functions which would involve knowing *all* of the Fourier series coefficients of $\mu$ and not just the non-negative ones as before (see eq.(3.3)). An early result which answers this question is one of Tumarkin [95] (see [55] for a generalization).

**Theorem 4.3** (Tumarkin)**.** *Let $f$ be analytic on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ with $f(\infty) = 0$ and set $f_1 = f|\mathbb{D}$ and $f_2 = f|\mathbb{D}_e$. Then $f = \widehat{\mu}$ for some $\mu \in M$ if and only if*

$$\sup_{0<r<1} \int_{\mathbb{T}} |f_1(r\zeta) - f_2(\zeta/r)| dm(\zeta) < \infty. \tag{4.4}$$

The Havin and Tumarkin results are easily seen to be equivalent using the technique of dual extremal problems as pioneered by S. Ja. Khavinson [36, 37] and W. Rogosinski and H. S. Shapiro [74].

There is a refinement of Tumarkin's due to Aleksandrov [2, Thm. 5.3] which even identifies the type of representing measure $\mu$ for $f$. From the previous section, a Cauchy transform $f = \widehat{\mu}$ on $\widehat{\mathbb{C}}\setminus\mathbb{T}$ satisfies the four conditions

$$f(\infty) = 0, \tag{4.5}$$

$$f \in \bigcap_{0<p<1} H^p(\widehat{\mathbb{C}} \setminus \mathbb{T}), \tag{4.6}$$

$$\|f\|_{H^p(\widehat{\mathbb{C}}\setminus\mathbb{T})} = O(\frac{1}{1-p}), \tag{4.7}$$

$$Jf \in L^1, \quad (Jf)(\zeta) := \lim_{r\to 1^-} \left[ \, f(r\zeta) - f(\zeta/r) \, \right] . \tag{4.8}$$

**Theorem 4.9** (Aleksandrov)**.** *Let $f$ be an analytic function on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ satisfying the conditions in eq.(4.5) through eq.(4.8) above. Then $f = \widehat{\mu}$ for some $\mu \in M$. Moreover, if the conditions in eq.(4.5) and eq.(4.6) are satisfied, then*

1. *$f = \widehat{\mu}$ for some $\mu \ll dm$ if and only if*

$$\varliminf_{p\to 1^-} \|f\|_{H^p(\widehat{\mathbb{C}}\setminus\mathbb{T})}(1-p) = 0$$

   *and $Jf \in L^1$.*

2. *$f = \widehat{\mu}$ for some $\mu \perp dm$ if and only if*

$$\varliminf_{p\to 1^-} \|f\|_{H^p(\widehat{\mathbb{C}}\setminus\mathbb{T})}(1-p) < \infty$$

   *and $Jf = 0$ $m$-almost everywhere.*

There is even a further refinement. Let $X$ be a class of analytic functions on $\mathbb{D}$ and $E$ be a closed subset of $\mathbb{T}$. Let $\mathcal{F}(X, E)$ denote the functions $f \in X$ such that $f = K\mu$, where $\mu \in M$ and has support in $E$. Under what conditions on $E$ is $\mathcal{F}(X, E) \neq (0)$? When $0 < p < 1$, notice that $\mathcal{F}(H^p, E) \neq (0)$ for every non-empty set $E$ (Theorem 3.5). When $p \geqslant 1$, Havin proves that $\mathcal{F}(H^p, E) \neq (0)$ if and only $m(E) > 0$ [34]. Though the analysis is more complicated, Hruščev [42] answers this question for the disk algebra and various other spaces of functions that are smooth up to the boundary.

We close this section by mentioning the following generalization of the Cauchy integral formula involving the theory of $A$-integrals as studied by Denjoy, Titchmarsh, Kolmogorov, Ul'yanov, and Aleksandrov. A measurable function $g : \mathbb{T} \to \widehat{\mathbb{C}}$ is *$A$-integrable* if

$$m(|g| > t) = o(1/t) \tag{4.10}$$

and

$$(A) \int g(\zeta) dm(\zeta) := \lim_{t \to \infty} \int_{|g|<t} g(\zeta) dm(\zeta)$$

exists. For $\mu \in M$ with $\mu \ll m$, $f = K\mu$ has non-tangential boundary values $m$-almost everywhere and, by eq.(3.8), $f$ satisfies the condition in eq.(4.10). However, this Cauchy transform may not belong to $H^1$ and so cannot be recovered from its boundary function via the Cauchy integral formula. A theorem of Ul'yanov [96] is the substitute 'Cauchy $A$-integral formula'.

**Theorem 4.11** (Ul'yanov). *For $\mu \in M$ with $\mu \ll m$, the function $f = K\mu$ is $A$-integrable and*

$$f(z) = \frac{1}{2\pi i} (A) \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}. \tag{4.12}$$

A theorem of Aleksandrov [1] says that if $f \in N^+$ (the Smirnov class) satisfies $m(|f| > t) = o(1/t)$ as $t \to \infty$ then eq.(4.12) holds. The condition in eq.(4.10) cannot be weakened since the function $i(1+z)/(1-z)$ cannot be written as the Cauchy $A$-integral of its boundary values. More about this can be found in the references in [79]. There is even a version of the conjugation operator using $A$-integrals in [8] and moreover, this book also contains a nice historical treatment of $A$-integrals.

## 5. Topology on the space of Cauchy transforms

In this section we consider the space $\mathcal{K}$ of Cauchy transforms as a Banach space. We will show how it arises naturally as the dual space to the disk algebra $A$ and indicate its relationship to the space of measures $M$.

Let $C = C(\mathbb{T})$ denote the Banach space of complex-valued continuous functions on the unit circle endowed with the supremum norm $\|f\|_\infty$. Every $\mu \in M$ determines a bounded linear functional $\ell_\mu$ on $C$ by

$$\ell_\mu(f) := \int_{\mathbb{T}} f \, \overline{d\mu}$$

with norm equal to the total variation norm $\|\mu\|$ of $\mu$. Conversely, the Riesz representation theorem [77] guarantees that every bounded linear functional on $C$ has such a representation, and in fact the map $\mu \to \ell_\mu$ is an isometric isomorphism from $M$ onto $C^*$ (the dual space of $C$). Thus, by tradition, we identify $C^*$ with $M$.

The *disk algebra* $A$ is the closure of the analytic polynomials in $C$ or equivalently, the space of functions analytic on $\mathbb{D}$ that have continuous extensions to $\mathbb{D}^-$. Its annihilator $A^\perp$ is a closed subspace of $C^* \simeq M$ which we identify with the set of measures $\mu \in M$ for which

$$\int g\overline{d\mu} = 0 \text{ for all } g \in A.$$

By the F. and M. Riesz theorem (Theorem 3.1), such annihilating measures $\mu$ take the form $d\mu = \overline{f}dm$, where $f \in H^1_0$ and so we identify $A^\perp$ and $\overline{H^1_0}$. We can identify $C^*/A^\perp$ with $A^*$ via the mapping $\ell_\mu + A^\perp \to \ell_\mu|A$ and furthermore, endowing $C^*/A^\perp$ with the usual quotient space norm

$$\|\ell_\mu + A^\perp\| = \operatorname{dist}(\ell_\mu, A^\perp),$$

the above mapping is an isometric isomorphism [78, pp. 96 - 97]. Putting this all together, we have

$$A^* \simeq M/\overline{H^1_0},$$

where, as before, $\simeq$ denotes an isometric isomorphism.

Since $K\mu = 0$ if and only if $\mu \in A^\perp \simeq \overline{H^1_0}$, the map $\mu + \overline{H^1_0} \to K\mu$ from $M/\overline{H^1_0}$ to $\mathcal{K}$ is bijective. From here it makes sense to endow $\mathcal{K}$ with the norm of $M/\overline{H^1_0}$, that is

$$\|K\mu\| := \operatorname{dist}(\mu, \overline{H^1_0}) = \inf\{\|d\mu + \overline{f}dm\| : f \in \overline{H^1_0}\}.$$

Hence $\mathcal{K} \simeq M/\overline{H^1_0}$.

Finally, recall that for $f \in \mathcal{K}$ we let $M_f = \{\mu \in M : f = K\mu\}$ be the set of representing measures for $f$. Since $K\mu = K\nu$ if and only if $d\mu - d\nu = \overline{f}dm$ ($f \in H^1_0$), it follows that

$$\|f\| = \inf\{\|\nu\| : \nu \in M_f\}.$$

Gathering up these facts, we have the following summary result.

**Theorem 5.1.** *The norm dual of $A$ can be identified in an isometric and isomorphic way with $\mathcal{K}$ via the sesquilinear pairing*

$$\langle f, K\mu\rangle = \int_{\mathbb{T}} f\,\overline{d\mu}, \tag{5.2}$$

*or equivalently, by a power series computation,*

$$\langle f, K\mu\rangle = \lim_{r\to 1^-} \sum_{n=0}^{\infty} f_n \overline{\mu_n} r^n, \tag{5.3}$$

*where $(f_n)_{n\geqslant 0}$ are the Taylor coefficients of $f$ and $(\mu_n)_{n\geqslant 0}$ are the Fourier coefficients of $\mu$.*

From the Lebesgue decomposition theorem, the space of measures $M$ admits a direct sum decomposition

$$M = L^1 \oplus M_s.$$

where $L^1$ is identified with the absolutely continuous measures $M_a$. Since $\overline{H_0^1} \subset L^1$, $A^*$ satisfies

$$A^* \simeq L^1/\overline{H_0^1} \oplus M_s.$$

Similarly, the space of Cauchy transforms can be decomposed as

$$\mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s,$$

where $\mathcal{K}_a = \{K\mu : \mu \ll m\}$ and $\mathcal{K}_s = \{K\mu : \mu \perp dm\}$. Furthermore, if $\mu = \mu_a + \mu_s$ ($\mu_a \ll m$ and $\mu_s \perp m$), then

$$\|K\mu\| = \|K\mu_a\| + \|K\mu_s\|.$$

We point out a few more items of interest. First,

$$\mu \perp dm \Rightarrow \|K\mu\| = \|\mu\|.$$

Secondly, $\mathcal{K}_a \simeq L^1/\overline{H_0^1}$ and is the norm closure of the analytic polynomials in $\mathcal{K}$. Thirdly, $\mathcal{K}_s \simeq M_s$ and as such is non-separable since $M_s$ is non-separable. Fourth, since, for $f \in \mathcal{K}$,

$$\|f\| = \inf\{\|\nu\| : \nu \in M_f\},$$

there is a sequence of measures $(\nu_n)_{n\geqslant 1}$ in $M_f$ such that $\|\nu_n\| \leqslant \|f\| + 1/n$ for each $n$. If $\nu$ is a weak-$*$ limit point of $(\nu_n)_{n\geqslant 1}$ in $M$, it follows that $\nu \in M_f$ and $\|\mathcal{K}\mu\| = \|\nu\|$. It is known that the measure $\nu$ is *unique* [35, 46]. We denote this unique representing measure $\nu$ by $\nu_f$. Since all measures in $M_f$ have the same singular part, a similar theorem holds for $\mathcal{K}_a$. Indeed, if $f \in \mathcal{K}_a$, then there is a unique $h \in L^1$ such that $h\,dm \in M_f$ and $\|f\| = \|h\|_1$.

There is another topology on $\mathcal{K}$ namely the weak-$*$ topology it inherits from being the norm dual of $A$. The Banach-Alaoglu theorem says that the closed unit ball of $\mathcal{K}$ is compact. In this case, the weak-$*$ topology on bounded sets can be characterized by its action on sequences. To be specific, we say that a sequence $(f_n)_{n\geqslant 1} \subset \mathcal{K}$ converges weak-$*$ to zero if $\langle g, f_n\rangle \to 0$ for every $g \in A$. From the pointwise estimate

$$|f(z)| \leqslant \frac{1}{1-|z|}\|f\| \text{ for all } z \in \mathbb{D}, f \in \mathcal{K}\ ^4 \tag{5.4}$$

one can prove that a sequence $(f_n)_{n\geqslant 1}$ in $\mathcal{K}$ converges to $f$ weak-$*$ if and only if $(f_n)_{n\geqslant 1}$ is uniformly bounded in the norm of $\mathcal{K}$ and $f_n(z) \to f(z)$ for each $z \in \mathbb{D}$ [13, Prop. 2]. Although $\mathcal{K}$ is not separable in the norm topology, it is separable in the weak-$*$ topology. This follows from the same fact about $M$ [5] and the weak-$*$ continuity of the canonical quotient map $\pi : M \to M/\overline{H_0^1} \simeq \mathcal{K}$.

[4] $\|f\|$ is the norm in $\mathcal{K}$. We know from eq.(3.4) that $|f(z)| \leqslant \|\mu\|(1-|z|)^{-1}$, where $\mu \in M_f$. When $\mu = \mu_f$, then $\|\mu\| = \|f\|$.

[5] In $M$, the linear span of the unit point masses $\{\delta_\zeta : \zeta = e^{i\theta}, \theta \in \mathbb{Q}\}$ form a weak-$*$ dense set.

Since $\mathcal{K}$, with the norm topology, is a Banach space, it can be endowed with a weak topology, though the dual space of $\mathcal{K}$ is not a readily identifiable space of analytic functions. There are two interesting theorems regarding this weak topology on $\mathcal{K}$. The first deals with weak completeness. A Banach space is weakly complete (weak Cauchy nets convergence in the space) if and only if it is reflexive (a consequence of Goldstine's theorem [23, p. 13]) and so $\mathcal{K}$ is not weakly complete. However, $\mathcal{K}$ is weakly sequentially complete. In general, a sequence $(x_n)_{n\geqslant 1}$ in a Banach space $X$ is weak Cauchy if the numerical sequence $(\ell(x_n))_{n\geqslant 1}$ converges for each $\ell \in X^*$. Thus $X$ is weakly sequentially complete if every weak Cauchy sequence in $X$ converges weakly to some element of $X$. Mooney's Theorem [57] asserts that $\mathcal{K}$ is weakly sequentially complete and is a consequence of the following result: Let $(\phi_n)_{n\geqslant 1}$ be a sequence in $L^1$ such that the limit

$$\lim_{n\to\infty} \int_{\mathbb{T}} \phi_n f \, dm := L(f)$$

exists for every $f \in H^\infty$. Then there is a function $\phi \in L^1$ such that

$$L(f) = \int_{\mathbb{T}} \phi f \, dm \quad \text{for all } f \in H^\infty.$$

A proof of this can be found in [29, pp. 206–209]. How does this imply Mooney's theorem? Since $\overline{H_0^1}$ is the annihilator of $H^\infty$ in $L^1$, it follows that $H^\infty$ can be regarded as the dual space of $L^1/\overline{H_0^1}$. The above theorem is then just the statement that $L^1/\overline{H_0^1}$ is weakly sequentially complete. Now notice that $\mathcal{K} \simeq L^1/\overline{H_0^1} \oplus M_s$ and so $\mathcal{K}^* \simeq H^\infty \oplus M_s^*$. Suppose $(\phi_n)_{n\geqslant 1}$ is a weak Cauchy sequence in $\mathcal{K}$. Each $\phi_n$ has a unique decomposition as $\phi_n = \psi_n + \nu_n$, where $\psi_n \in \mathcal{K}_a$ and $\nu_n \in \mathcal{K}_s$. Because of the direct sum decomposition of $\mathcal{K}^*$, it is evident that each of the sequences $(\psi_n)_{n\geqslant 1}$ and $(\nu_n)_{n\geqslant 1}$ is a weak Cauchy sequence. By the result quoted above, the first sequence converges weakly to some $\psi \in \mathcal{K}_a$. On the other hand, a theorem of Kakutani says that $M_s$ is isometrically isomorphic to $L^1(\Omega, \Sigma, \mu)$ for some abstract measure space $(\Omega, \Sigma, \mu)$ [47]. It follows that $M_s$ is weakly sequentially complete since every such space $L^1(\Omega, \Sigma, \mu)$ is weakly sequentially complete [23].

The second theorem we will present on the weak topology in $\mathcal{K}$ is a deep result due independently to Delbaen [22] and Kisliakov [49]. For each $f \in \mathcal{K}$, recall that $\mu_f$ is the unique measure such that $f = K\mu_f$ and $\|f\| = \|\mu_f\|$.

**Theorem 5.5.** *Let $W$ be a weakly compact set in $\mathcal{K}$, and let $\widehat{W} = \{\, \mu_f : f \in W \,\}$. Then $\widehat{W}$ is relatively weakly compact.*

A thorough discussion of this theorem can be found in [62, Ch. 7] or [99]. This theorem derives its significance from the Dunford-Pettis Theorem characterizing weakly compact subsets in $M$. This characterization says that a set $W$ in $M$ is weakly compact if and only if it is bounded and uniformly absolutely continuous (cf. [23]).

We now talk about a basis for $\mathcal{K}$. A sequence $(x_n)_{n\geqslant 1}$ in a Banach space $X$ is a *Schauder basis* for $X$ if every $x \in X$ can be written uniquely as

$$x = \sum_{n=1}^{\infty} c_n x_n,$$

where $c_n$ are complex numbers and the $=$ sign means convergence in the norm of $X$. A sequence $(\ell_n)_{n\geqslant 1} \subset X^*$ is called a *weak-$*$ Schauder basis* if every $\ell \in X^*$ can be written uniquely as

$$\ell = \sum_{n=1}^{\infty} d_n \ell_n,$$

where $d_n$ are complex numbers and $=$ means weak-* convergence.

For a Schauder basis $(x_n)_{n\geqslant 1}$, there is a natural sequence $(x_n^*)_{n\geqslant 1}$ of continuous linear functionals defined by

$$x_n^*(f) = c_n, \quad \text{where} \quad f = \sum_{n=1}^{\infty} c_n x_n$$

(remember that the expansion of $f$ is unique and so $x_n^*$ is well-defined). One can prove the following: If $(x_n)_{n\geqslant 1}$ is a Schauder basis for $X$, then $(x_n^*)_{n\geqslant 1}$ is a weak-* Schauder basis for $X^*$. If $(x_n)_{n\geqslant 1}$ is a Schauder basis for $X$, then $(x_n^*)_{n\geqslant 1}$ is a Schauder basis for its closed linear span [23, p. 52].

Let us apply the above to $X = A$ and $X^* \simeq \mathcal{K}$ to identify a weak-* Schauder basis for $\mathcal{K}$ and a Schauder basis for $\mathcal{K}_a \simeq L^1/\overline{H_0^1}$. The result which drives this is one of Bočkarev [10] which produces a Schauder basis $(b_n)_{n\geqslant 1}$ for the disk algebra $A$. Moreover, by its actual construction, $\{b_n\}_{n\geqslant 1}$ is an orthonormal set in $L^2$. Let $B_n := (b_n)_+$. We do this to not confuse $b_n$, the element of the disk algebra $A$, with $B_n$, the element of $\mathcal{K}$. The result here is the following.

**Proposition 5.6.** *$(B_n)_{n\geqslant 1}$ is a weak-$*$ Schauder basis for $\mathcal{K}$ and a Schauder basis for $\mathcal{K}_a$.*

## 6. Multipliers of the space of Cauchy transforms

For a linear space of analytic functions $X$ on the unit disk, an important collection of functions are those $\phi$, analytic on $\mathbb{D}$, for which $\phi f \in X$ whenever $f \in X$. Such $\phi$ are called *multipliers* of $X$ and we denote this class by $M(X)$. For the Hardy spaces $H^p$ $(0 < p \leqslant \infty)$ it is routine to check that $M(H^p) = H^\infty$. For other classes of functions such as the classical Dirichlet space or the Besov spaces, characterizing the multipliers is more difficult [56, 89]. For the space of Cauchy transforms, the multipliers are not completely understood but they do have several interesting properties.

For example, there is a nice relationship between multipliers and Toeplitz operators. If $\phi \in H^\infty$ and $T_{\overline{\phi}}(f) = P(\overline{\phi} f)$ is the co-analytic Toeplitz operator, then the Riesz theorem (Theorem 3.6) says that $P$, and hence $T_{\overline{\phi}}$, is a bounded operator on $H^p$ for $1 < p < \infty$. When $p = \infty$, the Riesz projection $P$ is unbounded. However, this following theorem of Vinogradov [97] determines when $T_{\overline{\phi}}$ is bounded on $H^\infty$.

**Proposition 6.1.** *For $\phi \in H^\infty$, the following are equivalent.*

1. $\phi \in M(\mathcal{K})$.
2. $\phi \in M(\mathcal{K}_a)$.
3. $T_{\overline{\phi}} : A \to A$ *is bounded.*
4. $T_{\overline{\phi}} : H^\infty \to H^\infty$ *is bounded.*

*Moreover, $\|T_{\overline{\phi}} : A \to A\|$ is equal to the multiplier norm, see below, of $\phi$.*

If $\phi$ is a multiplier of $\mathcal{K}$, the closed graph theorem, says that the multiplication operator $f \to \phi f$ on $\mathcal{K}$ is continuous. The norm of this operator is denoted by $\|\phi\|$. The weak-$*$ density of the convex hull of $\{\delta_\zeta : \zeta \in \mathbb{T}\}$ in the unit ball of $M$, together with the identity $(K\delta_\zeta)(z) = (1 - \overline{\zeta} z)^{-1}$ yields [98]

$$\|\phi\| = \sup \left\{ \left\| \frac{\phi}{1 - \overline{\zeta} z} \right\|_{\mathcal{K}} : \zeta \in \mathbb{T} \right\} .$$

Since the constant functions belong to $\mathcal{K}$, any multiplier belongs to $\mathcal{K}$. In fact, one can show that any multiplier $\phi$ is a bounded function on $\mathbb{D}$ and satisfies $\|\phi\|_\infty \leqslant \|\phi\|$. This standard fact is true for the multipliers for just about any Banach space of analytic functions. Although a usable characterization of the multipliers of $\mathcal{K}$ is unknown, there are some sufficient conditions one can quickly check to see if a given $\phi \in H^\infty$ is a multiplier of $\mathcal{K}$. See [98] for details.

**Theorem 6.2** (Goluzina-Havin-Vinogradov)**.** *Let $\phi$ be an analytic function on $\mathbb{D}$. Then any one of the following conditions imply that $\phi$ is a multiplier of $\mathcal{K}$.*

1. $\phi \in H^\infty$ *and*
$$\sup \left\{ \int_{\mathbb{T}} \left| \frac{\phi(\zeta) - \phi(\xi)}{\zeta - \xi} \right| dm(\xi) : \zeta \in \mathbb{T} \right\} < \infty.$$
2. *$\phi$ extends to be continuous on $\mathbb{D}^-$ and*
$$\int_0 \frac{\omega(t)}{t} dt < \infty,$$
*where $\omega$ is the modulus of continuity of $\phi$.*
3. *The quantity*
$$\sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|}{n!} \log(n+2)$$
*is finite.*

This next theorem says that the multipliers of $\mathcal{K}$ are reasonably regular near the boundary of $\mathbb{D}$. Again, see [98] for details.

**Theorem 6.3.** *If $\phi$ is a multiplier of $\mathcal{K}$, then*

1. *$\phi$ has a finite non-tangential limit for every $\zeta \in \mathbb{T}$.*
2. *The Taylor polynomials are uniformly bounded in $H^\infty$ norm, that is to say,*

$$\sup \Big\{ \Big\| \sum_{k=0}^{n} \frac{\phi^{(k)}(0)}{k!} z^k \Big\|_\infty : n = 0, 1, 2, \dots \Big\} < \infty.$$

3. $(1-z)^2 \phi' \in H^1(\{|z - 1/2| < 1/2\})$.

Statement (1) says that not all Blaschke products are multipliers of $\mathcal{K}$ since there are Blaschke products which do not have radial limits at certain points of the circle. In fact, the Blaschke product with zeros $(a_n)_{n\geqslant 1}$ has a finite non-tangential limit at $\zeta \in \mathbb{T}$ if and only if

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta - a_n|} < \infty.$$

This is an old result of Frostman [27]. Statement (3) says that the singular inner function $\exp((z+1)/(z-1))$ is not a multiplier. If not all inner functions are multipliers, which ones are? This question has a complete answer due to a deep and quite difficult theorem of Hruščev and Vinogradov [44].

**Theorem 6.4** (Hruščev-Vinogradov)**.** *An inner function $\vartheta$ is a multiplier for $\mathcal{K}$ if and only if $\vartheta$ is a Blaschke product whose zeros $(a_n)_{n\geqslant 1}$ - repeated according to multiplicity - satisfy the uniform Frostman condition*

$$\sup \Big\{ \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta - a_n|} : \zeta \in \mathbb{T} \Big\} < \infty.$$

We can also talk about divisors. Suppose that $X$ is a class of analytic functions contained in the Smirnov class $N^+$. We say that $X$ has the *$F$-property* if whenever $f \in X$ and $\vartheta$ is an inner function which 'divides $f$', *i.e.*, $f/\vartheta \in N^+$, then $f/\vartheta \in X$. From the factorization theorem for $H^p$ functions [25, p. 24], $H^p$ certainly has the $F$-property, as do other well-known classes of analytic functions such as $BMOA$, the disk algebra, and the analytic Lipschitz and Besov classes [84]. A theorem of Hruščev and Vinogradov [44, 97] says that the space of Cauchy transforms, as well as their multipliers, also enjoy the $F$-property.

**Theorem 6.5** (Hruščev-Vinogradov)**.**
1. *The space $\mathcal{K}$ has the $F$-property.*
2. *The space of multipliers of $\mathcal{K}$ has the $F$-property.*

## 7. Operators on the space of Cauchy transforms

On the Hardy spaces $H^p$ $(0 < p < \infty)$, there are a variety of operators worthy of study. Several that immediately come to mind are

$$(Sf)(z) = zf(z) \quad \text{(the forward shift)}$$

$$(Bf)(z) = \frac{f - f(0)}{z} \quad \text{(the backward shift)}$$

$$(C_\phi f)(z) = f(\phi(z)) \quad \text{(composition by } \phi : \mathbb{D} \to \mathbb{D})$$

$$(Cf)(z) = \frac{1}{z}\int_0^z \frac{f(t)}{1-t}dt \quad \text{(the Cesàro operator).}$$

We wish to make a few remarks about these operators on the space of Cauchy transforms.

*The forward shift.* It is easy to see that the forward shift $(Sf)(z) = zf(z)$ is a well-defined operator from $\mathcal{K}$ to itself and, by our dual pairing in eq.(5.3), $B^* = S$, where $B$ is the backward shift operator on $A$ (the disk algebra). One quickly sees that for all $f \in A$, $\|Bf\|_\infty = \|f - f(0)\|_\infty \leqslant 2\|f\|_\infty$. Furthermore, if

$$f_r(z) = \frac{z + r}{1 + rz},$$

then $\|f_r - f_r(0)\|_\infty = 1 + r$. It follows that the operator norm of $B : A \to A$ is equal to 2 and so the operator norm of $S : \mathcal{K} \to \mathcal{K}$ is also equal to 2. It also follows from duality that $S$ is continuous on $\mathcal{K}$ when $\mathcal{K}$ is endowed with the weak-$*$ topology.

If $\vartheta$ is an inner function, then $\vartheta H^p$ is a (norm) closed $S$-invariant subspace of $H^p$ for each $0 < p < \infty$ and a theorem of Beurling [25, p. 114] says that all (non-trivial) norm closed $S$-invariant subspaces of $H^p$ take this form. What is the analog of Beurling's theorem for $\mathcal{K}$? Endowed with the norm topology, $\mathcal{K}$ is non-separable and so characterizing its norm closed $S$-invariant subspaces is troublesome. However, $\mathcal{K}$, endowed with the weak-$*$ topology, is separable and so characterizing its weak-$*$ closed $S$-invariant subspaces is a more tractable problem. Since not all inner functions are multipliers of $\mathcal{K}$ (see Theorem 6.4), then $\vartheta\mathcal{K}$ is not always a subset of $\mathcal{K}$ (unlike $\vartheta H^p \subseteq H^p$). However, the subspace

$$\vartheta(\mathcal{K}) := \{f \in \mathcal{K} : f/\vartheta \in \mathcal{K}\}$$

does make sense and is clearly $S$-invariant, although it is not immediately clear that $\vartheta(\mathcal{K})$ is weak-$*$ closed. A theorem of Aleksandrov [3] is our desired 'Beurling's theorem' for $\mathcal{K}$.

**Theorem 7.1** (Aleksandrov)**.** *For each inner function $\vartheta$, $\vartheta(\mathcal{K})$ is a weak-$*$ closed $S$-invariant subspace of $\mathcal{K}$. Furthermore, if $\mathcal{M}$ is a non-zero weak-$*$ closed $S$-invariant subspace of $\mathcal{K}$, then there is an inner function $\vartheta$, such that $\mathcal{M} = \vartheta(\mathcal{K})$.*

*The backward shift.* The backward shift

$$(Bf)(z) = \frac{f - f(0)}{z}$$

is a well-defined operator from $\mathcal{K}$ to itself and is the adjoint (under the pairing eq.(5.3)) of the forward shift $S$ on $A$. Since the operator norm of $S : A \to A$ is equal to one, then the operator norm of $B : \mathcal{K} \to \mathcal{K}$ is also equal to one.

The space $\mathcal{K}_a = \{f_+ : f \in L^1\}$ is the norm closure of the polynomials and hence is separable. A theorem of Aleksandrov [3] (see also [16, p. 99]) characterizes the $B$-invariant subspaces of $\mathcal{K}_a$.

**Theorem 7.2** (Aleksandrov). *If $\mathcal{M}$ is a norm closed $B$-invariant subspace of $\mathcal{K}_a$, then there is an inner function $\vartheta$ such that $f \in \mathcal{M}$ if and only if there is a $G \in N^+(\mathbb{D}_e)$ with $G(\infty) = 0$ and such that*

$$\lim_{r \to 1^-} \frac{f}{\vartheta}(r\zeta) = \lim_{r \to 1^-} G(\zeta/r)$$

*for $m$-almost every $\zeta \in \mathbb{T}$.*

The function $G$ is known in the literature as a 'pseudocontinuation' of the function $f/\vartheta$ (see [75] for more on pseudocontinuations). We compare Aleksandrov's result to the characterization of the backward shift invariant subspaces of $H^2$ which, as mentioned in the introduction, are all of the form $(\vartheta H^2)^{\perp}$. A theorem of Douglas-Shapiro-Shields [24] says that $f \in (\vartheta H^2)^{\perp}$ if and only if there is a function $G \in H^2(\mathbb{D}_e)$, with $G(\infty) = 0$, such that $G$ is a pseudocontinuation of $f/\vartheta$, *i.e.*, $G$ (from the outside of the disk) has the same radial limits as $f/\vartheta$ (from the inside of the disk) almost everywhere.

For a weak-$*$ closed $B$-invariant subspace $\mathcal{N} \subseteq \mathcal{K}$, the dual pairing eq.(5.3) tells us that $\mathcal{N}_{\perp}$ (the pre-annihilator of $\mathcal{N}$) is an $S$-invariant subspace of $A$. Since $A$ is a Banach algebra and polynomials are dense in $A$, then $\mathcal{N}_{\perp}$ is a closed ideal of $A$. A result of Rudin [76] (see also [40, p. 82]) characterizes these ideals by their inner factors and their zero sets on the circle. A result in [15] uses the Rudin characterization to describe the corresponding $\mathcal{N}$ using analytic continuation across certain portions of the circle. This result also has connections to an analytic continuation result of Korenblum [52]. Some partial results on the $B$-invariant subspaces of $\mathcal{K}$, when endowed with the norm topology, can be found in [15].

*Composition operators.* For an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, define, for a Cauchy transform $f$, the function

$$(C_\phi f)(z) = f(\phi(z)).$$

Clearly $C_\phi f$ is an analytic function on $\mathbb{D}$. What is not immediately clear is that $C_\phi f$ is a Cauchy transform. The proof of this follows from the Herglotz theorem [25, p. 2] and the decomposition of any measure as the complex linear sum of four positive measures. Furthermore, we note that if $G$ is simply connected and the Riemann map $g : \mathbb{D} \to G$ belongs to $\mathcal{K}$, then any analytic map $f$ of $\mathbb{D}$ into $G$ belongs to $\mathcal{K}$. Indeed, set $\phi = g^{-1} \circ f$ and notice that $f = C_\phi g \in \mathcal{K}$. Finally, if $f$ is analytic on $\mathbb{D}$ and $\mathbb{C} \setminus f(\mathbb{D})$ contains at least two oppositely oriented half-lines, then $f \in \mathcal{K}$ [12].

Without too much difficulty, one can show that $C_\phi$ has closed graph and so $C_\phi$ is bounded on $\mathcal{K}$. Bourdon and Cima [12] proved that

$$\|C_\phi\| \leqslant \frac{2+2\sqrt{2}}{1-|\phi(0)|}$$

which was improved to

$$\|C_\phi\| \leqslant \frac{1+2|\phi(0)|}{1-|\phi(0)|}$$

by Cima and Matheson [14]. Moreover, equality is attained for certain linear fractional maps $\phi$.

The theory of Clark measures, as discussed in § 1, leads us to a characterization of the compact composition operators on $\mathcal{K}$. Following Sarason, for each self map $\phi : \mathbb{D} \to \mathbb{D}$, we define an operator on $M$ as follows: For $\mu$ in $M_+$ the function $(P\mu)(z)$, the Poisson integral of the measure $\mu$, is positive and harmonic in $\mathbb{D}$. Hence, the function $v(z) = (P\mu)(\phi(z))$ is also positive and harmonic. By the Herglotz theorem there is $\nu \in M_+$ with $v(z) = (P\nu)(z)$. Let $S_\phi(\mu) = \nu$, where $\nu$ is the (positive) measure for which $(P\nu)(z) = (P\mu)(\phi(z))$. Extended $S_\phi$ linearly to all of $M$ in the obvious way (writing every measure as a complex linear combination of four positive measures).

**Theorem 7.3** (Sarason [81])**.** *The operator $S_\phi$ is compact if and only if each Clark measure $\mu_\alpha$ associated with $\phi$ satisfies $\mu_\alpha \ll m$ for all $\alpha \in \mathbb{T}$. Moreover, $S_\phi$ maps $L^1$ continuously to $L^1$.*

We comment that Sarason's result is equivalent to J. Shapiro's condition for compactness on $H^2$ [14, 83]. Also worth noting is that if the map $\phi$ is inner, it can not induce a compact operator in this setting, since the Clark measure $\sigma_\alpha$ will be carried by $\{\phi = \alpha\}$, and hence is singular. Using these results, one can show the following [14].

**Theorem 7.4.** *Let $\phi$ be an analytic self map of the disk. Then*

1. *$C_\phi$ is compact on $\mathcal{K}$ if and only if $S_\phi$ is compact on $M$.*
2. *if $C_\phi$ is weakly compact on $\mathcal{K}$, then $C_\phi$ is compact on $\mathcal{K}$.*

*The Cesàro operator.* The Cesàro operator, as it originally appeared in the operator setting, was simply the map defined on the sequence space $(\ell^2)_+$ by

$$(a_n)_{n\geqslant 0} \to (b_N)_{N\geqslant 0},$$

where

$$b_N := \frac{1}{N+1}\big(a_0 + a_1 + \cdots + a_N\big), \quad N = 0, 1, 2, \ldots$$

is the $N$-th Cesàro mean of the sequence $(a_n)_{n\geqslant 0}$. It is easy to see, equating the $\ell^2$ sequence $(a_n)_{n\geqslant 0}$ with the $H^2$ function $f = \sum a_n z^n$, that this operator, denoted by $C$, can be viewed on $H^2$ as the integral operator

$$(Cf)(z) := \frac{1}{z}\int_0^z \frac{f(t)}{(1-t)}dt.$$

Cima and Siskakis [17] prove that the Cesàro operator is continuous from $\mathcal{K}$ to $\mathcal{K}$. An interesting problem to consider is whether or not any of the generalized Cesàro operators on $H^p$ considered in [6, 7] are continuous on $\mathcal{K}$.

## 8. Distribution of boundary values

For a measurable (with respect to normalized Lebesgue measure $m$) function $f$ on $\mathbb{T}$, the distribution function

$$\lambda \to m(|f| > \lambda)$$

is seen throughout analysis. For example, there is Chebyshev's inequality

$$m(|f| > \lambda) \leqslant \frac{1}{\lambda^p}\|f\|_p$$

and the 'layer cake representation' formula

$$\|f\|_p^p = p\int_0^\infty \lambda^{p-1} m(|f| > \lambda)d\lambda.$$

In this brief section, we mention some known results about the distribution functions for $K\mu$ and $Q\mu$ and how they relate in quite surprising ways to $\mu$.

Perhaps the earliest result on the distribution function for $Q\mu$ is one of G. Boole [11] who in 1857 found, in the special case where $\mu$ is a finite positive linear combination of point masses, the exact formula

$$m(Q\mu > \lambda) = m(Q\mu < -\lambda) = \frac{1}{\pi}\arctan\frac{\|\mu\|}{\lambda}.$$

Although, for positive measures with an absolutely continuous component (*i.e.*, $d\mu/dm \not\equiv 0$), Boole's formula is not true, it is 'asymptotically true' as more recent results of Tsereteli [94] and Poltoratski [66] show.

**Theorem 8.1** (Poltoratski)**.** *Let* $\mu \in M_+$ *and* $E_\mu = \{\zeta \in \mathbb{T} : \frac{d\mu}{dm}(\zeta) > 0\}$. *Then*

$$\frac{1}{\pi}\arctan\frac{\|\mu\|}{\lambda} - m(E_\mu) \leqslant m(\{\, Q\mu > \lambda \,\} \setminus E_\mu) \leqslant \frac{1}{\pi}\arctan\frac{\|\mu\|}{\lambda},$$

$$\frac{1}{\pi}\arctan\frac{\|\mu\|}{\lambda} - m(E_\mu) \leqslant m(\{\, Q\mu < -\lambda \,\} \setminus E_\mu) \leqslant \frac{1}{\pi}\arctan\frac{\|\mu\|}{\lambda}.$$

The above theorem is an improvement of [94]. Another fascinating aspect of $Q\mu$ (and hence $K\mu$) is a result of Stein and Weiss [92] (see also [48, p. 71]) which says that if $\chi_U$ is the characteristic function of some set $U \subset \mathbb{T}$, then the distribution function for $Q(\chi_U dm)$ depends only on $m(U)$ and not on the particular geometric structure of $U$.

The next series of results deal with reproducing the singular part of a measure from knowledge of the distribution function for $Q\mu$ (or $K\mu$). Probably one of the earliest of such theorems is this one of Hruščev and Vinogradov [43].

**Theorem 8.2** (Hruščev and Vinogradov). *For any $\mu \in M$,*

$$\lim_{\lambda\to\infty} \pi\lambda\, m(|K\mu| > \lambda) = \|\mu_s\|.$$

Note that if $\mu \ll m$, then we get the well-known estimate (see eq.(3.8))

$$m(|K\mu| > \lambda) = o(1/\lambda).$$

Recent work of Poltoratski [66] says that not only can one recover the total variation norm of $\mu_s$ from the distribution function for $K\mu$, but one can actually recover the measure $\mu_s$. For any measure $\mu \in M$ decomposed as $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$, where $\mu_j \geqslant 0$, let $|\mu|$ be the measure $|\mu| := \sum_{j=1}^4 \mu_j$.

**Theorem 8.3** (Poltoratski). *For $\mu \in M$, the measure $\pi\lambda\chi_{\{|K\mu|>\lambda\}} \cdot m$ converges weak-$*$ to $|\mu_s|$ as $\lambda \to \infty$.*

Though we don't have time to go into the details, we do mention that at the heart of these Poltoratski theorems is the Aleksandrov conditional expectation operator and the relationship between the distribution functions for $Q\mu$ and $K\mu$ and Clark measures (as described in §1).

## 9. The normalized Cauchy transform

If $\sigma_\alpha$ is a Clark measure for the inner function $\vartheta$, the adjoint $\mathcal{F}_\alpha^*$ of the unitary operator $\mathcal{F}_\alpha\colon (\vartheta H^2)^\perp \to L^2(\sigma_\alpha)$ (as described in §1) is

$$\mathcal{F}_\alpha^* f = \frac{K(f\, d\sigma_\alpha)}{K(\sigma_\alpha)}$$

In general, for $\mu \in M_+$ we can consider the *normalized Cauchy transform*

$$V_\mu f = \frac{K(f\, d\mu)}{K(\mu)}, \quad f \in L^1(\mu).$$

This transform has been studied by several authors, beginning with D. Clark [18], with significant contributions by Aleksandrov [4] and Poltoratski [65, 66, 67].

In this section we wish to describe some recent work of Poltoratski on maximal properties of this normalized transform. Poltoratski's motivation for this study is the famous theorem of Hunt, Muckenhoupt, and Wheeden [29, p. 255] [45], which states that the Cauchy transform is bounded on $L^p(w)$, $1 < p < \infty$, if and only if $w$ is an '$A_p$-weight'. Moreover, if $MKf$ denotes the standard nontangential maximal function of the Cauchy transform $Kf$, we have similarly $MKf \in L^p(w)$ for all $f \in L^p(w)$ if and only if $w$ is an $A_p$-weight.

We can ask about the behavior of the normalized Cauchy transform $V_\mu$ and the associated maximal operator $MV_\mu$ on $L^p(\mu)$. We first note that $V_\mu$ is always bounded on $L^2(\mu)$. In fact if $\mu = m$, then $V_\mu(L^2(\mu))$ is just the Hardy space $H^2$, while if $\mu$ is singular $V_\mu(L^2(\mu)) = (\vartheta H^2)^\perp$, where $\vartheta$ is the inner function related to $\mu$ by the formula

$$K\mu = \frac{1}{1-\vartheta}.$$

If $\mu$ is an arbitrary positive measure, $V_\mu(L^2(\mu))$ is the de Branges-Rovnyak space $\mathcal{M}_\vartheta$, where $\vartheta$ comes from the above formula, but is no longer inner [82]. In particular, $V_\mu$ is a bounded operator from $L^2(\mu)$ to $H^2$ for any $\mu \in M_+$. When $p \neq 2$ the situation is described by the following theorem of Aleksandrov [5].

**Theorem 9.1.** *For any $\mu \in M_+$, $V_\mu$ is a bounded operator from $L^p(\mu)$ to $H^p$ for $1 < p \leqslant 2$. In general, it is unbounded for $p > 2$. If $\mu$ is singular and $V_\mu$ is bounded from $L^p(\mu)$ to $H^p$, then $\mu$ is a discrete measure.*

The following theorem of Poltoratski [65] describes the boundary behavior of $V_\mu f$.

**Theorem 9.2.** *Let $\mu \in M_+$ and $f \in L^1(\mu)$. Then the function $V_\mu f$ has finite nontangential boundary values $\mu$-a.e. These values coincide with $f$ $\mu_s$-a.e.*

The above result shows in particular that if $\mu$ is singular, then $V_\mu$ is in fact the identity operator on $L^1(\mu)$, and so is certainly a bounded operator from $L^p(\mu)$ to itself. The situation for arbitrary $\mu$ and for the maximal operator $MV_\mu$ is more involved and was examined by the following theorem of Poltoratski [68].

**Theorem 9.3.** *For any $\mu \in M_+$, $V_\mu$ is a bounded operator from $L^p(\mu)$ to itself for $1 < p \leqslant 2$. In addition, the maximal operator $MV_\mu$ is bounded on $L^p(\mu)$ for $1 < p < 2$, and of weak type $(2,2)$ on $L^2(\mu)$.*

Even for singular $\mu$, the maximal operator $MV_\mu$ may be unbounded. Indeed, Poltoratski provides an example of a singular measure $\mu$ and an $f \in L^\infty(\mu)$ such that $MV_\mu f \notin L^p(\mu)$ for any $p > 2$. Finally, although $MV_\mu$ is always of weak type $(2,2)$, it is not known whether or not it is bounded on $L^2(\mu)$ in general. It is also not known under what conditions it is of weak type $(1,1)$.

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