The center of the wreath product of symmetric group algebras*

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ABSTRACT. We consider the wreath product of two symmetric groups as a group of blocks permutations and we study its conjugacy classes. We give a polynomiality property for the structure coefficients of the center of the wreath product of symmetric group algebras. This allows us to recover an old result of Farahat and Higman about the polynomiality of the structure coefficients of the center of the symmetric group algebra and to generalize our recent result about the polynomiality property of the structure coefficients of the center of the hyperoctahedral group algebra.

1. Introduction

The conjugacy classes of the symmetric group $\mathcal{S}_n$ can be indexed by partitions of n. The conjugacy class associated to a partition λ is the set of all permutations with cycle-type λ. The center of the symmetric group algebra is the algebra over $\mathbb{C}$ generated by the conjugacy classes of the symmetric group. Its structure coefficients have nice combinatorial properties. In [1], Farahat and Higman, gave a polynomiality property for the structure coefficients of the center of the symmetric group algebra. By introducing partial permutations in [3], Ivanov and Kerov, gave a combinatorial proof to this result.

We introduce in this paper the group $\mathcal{B}_{kn}^k$ which permutes n blocks of k elements each. The permutation of the k elements in each block is allowed. This group is the symmetric group $\mathcal{S}_n$ if $k = 1$ and in case $k = 2$ it is the hyperoctahedral group $\mathcal{H}_n$ on $2n$ elements. In general, we show that the group $\mathcal{B}_{kn}^k$ is isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ of the symmetric group $\mathcal{S}_k$ by the symmetric group $\mathcal{S}_n$. It is well known that the conjugacy classes of $\mathcal{H}_n$ are indexed by pairs of partitions (λ, δ) verifying $|\lambda| + |\delta| = n$, see [2] and [7]. We show that in general, for any fixed integer k, the conjugacy classes of the group $\mathcal{B}_{kn}^k$ are indexed by families of partitions $\lambda = (\lambda(\rho))_{\rho \vdash k}$ indexed by the set of partitions of k such that the sum of the sizes of all $\lambda(\rho)$ equals n. This comes with no surprise since it agrees with the description of the conjugacy classes of the wreath product $G \wr \mathcal{S}_n$, where G is a finite group, given by Specht in [6], see also [4] and [5]. However, our way to prove this fact for $\mathcal{S}_k \wr \mathcal{S}_n$ seems to be easier.

Recently, in [10], we developed a framework in which the polynomiality property for double-class algebras, and subsequently centers of groups algebra, holds. In particular, we showed that our framework contains the sequence of the symmetric groups and that of the hyperoctahedral groups. Thus we obtained again the result of Farahat and Higman for the structure coefficients of the center of the symmetric group algebra. In addition we gave a polynomiality property for the structure coefficients of the center of the hyperoctahedral group algebra in [10, Section 6.2].

In this paper we show that the general framework we gave in [10] contains the sequence of groups $(\mathcal{B}_{kn}^k)_n$ when k is a fixed integer. Thus, we will be able to give a polynomiality property for the structure coefficients of the center of the group $\mathcal{B}_{kn}^k$ algebra. This result can be obtained as an application of Theorem 2.13 given by Wang in [12] which shows that the structure coeffecients of the center of $G \wr \mathcal{S}_n$ are polynomials in n for any finite group G. However, the paper of Wang uses the Farahat-Higman approach while the way taken in this paper is different and seems to be easier to arrive to the result in the particular case when $G = \mathcal{S}_k$. A particular attention to the cases $k = 2$ (hyperoctahedral group) and $k = 3$ is given. In a separate paper [11] we generalise the concept of partial permutation of Ivanov and Kerov to k-partial permutation and we give a more combinatorial proof to this result.

The paper is organized as follows. In Section 2, we review all necessary definitions of partitions and we describe the conjugacy classes of the symmetric group. Then, we define explicitly the group $\mathcal{B}_{kn}^k$ in Section 3 and we study in details its conjugacy classes. After this, we show that it is

isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$. In Sections 4.1 and 4.2 a special treatment is given for the cases $k = 2$ and $k = 3$ respectively. The last section contains our main result, that is a polynomiality property for the structure coefficients of the center of the group $\mathcal{B}_{kn}^k$ algebra. In addition some examples are given for the cases $k = 2$ and $k = 3$.

2. Partitions and conjugacy classes of the symmetric group

If n is a positive integer, we denote by $\mathcal{S}_n$ the symmetric group of permutations on the set $[n] := \{1, 2, \ldots, n\}$. A *partition* λ is a list of integers $(\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_l \geqslant 1$. The λ_i are called the *parts* of λ; the *size* of λ, denoted by $|\lambda|$, is the sum of all of its parts. If $|\lambda| = n$, we say that λ is a partition of n and we write $\lambda \vdash n$. The number of parts of λ is denoted by $l(\lambda)$. We will also use the exponential notation $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots)$, where $m_i(\lambda)$ is the number of parts equal to i in the partition λ. In case there is no confusion, we will omit λ from $m_i(\lambda)$ to simplify our notation. If $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots, n^{m_n(\lambda)})$ is a partition of n then $\sum_{i=1}^n i m_i(\lambda) = n$. We will dismiss $i^{m_i(\lambda)}$ from λ when $m_i(\lambda) = 0$. For example, we will write $\lambda = (1^2, 3, 6^2)$ instead of $\lambda = (1^2, 2^0, 3, 4^0, 5^0, 6^2, 7^0)$. If λ and δ are two partitions we define the *union* $\lambda \cup \delta$ and subtraction $\lambda \setminus \delta$ (if exists) as the following partitions:

$$\lambda \cup \delta = (1^{m_1(\lambda)+m_1(\delta)}, 2^{m_2(\lambda)+m_2(\delta)}, 3^{m_3(\lambda)+m_3(\delta)}, \ldots),$$

$$\lambda \setminus \delta = (1^{m_1(\lambda)-m_1(\delta)}, 2^{m_2(\lambda)-m_2(\delta)}, \ldots) \text{ if } m_i(\lambda) \geqslant m_i(\delta) \text{ for any } i.$$

A partition is called *proper* if it does not have any part equal to 1. The proper partition associated to a partition λ is the partition $\bar{\lambda} := \lambda \setminus (1^{m_1(\lambda)}) = (2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots)$.

The *cycle-type* of a permutation of $\mathcal{S}_n$ is the partition of n obtained from the lengths of the cycles that appear in its decomposition into product of disjoint cycles. For example, the permutation

$$(2, 4, 1, 6)(3, 8, 10, 12)(5)(7, 9, 11)$$

of $\mathcal{S}_{12}$ has cycle-type $(1, 3, 4^2)$. In this paper we will denote the cycle-type of a permutation ω by $\mathrm{ct}(\omega)$. It is well known that two permutations of $\mathcal{S}_n$ belong to the same conjugacy class if and only if they have the same cycle-type. Thus the conjugacy classes of the symmetric group $\mathcal{S}_n$ can be indexed by partitions of n. If $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, 3^{m_3(\lambda)}, \ldots, n^{m_n(\lambda)})$ is a

partition of n, we will denote by C_λ the conjugacy class of $\mathcal{S}_n$ associated to λ:

$$C_\lambda := \{\sigma \in \mathcal{S}_n \mid \mathrm{ct}(\sigma) = \lambda\}.$$

The cardinal of C_λ is given by:

$$|C_\lambda| = \frac{n!}{z_\lambda},$$

where

$$z_\lambda := 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \ldots n^{m_n(\lambda)} m_n(\lambda)!.$$

3. The conjugacy classes of the group $\mathcal{B}_{kn}^k$

In this section we define the group $\mathcal{B}_{kn}^k$ then we study its conjugacy classes. We show in Proposition 1 that these latter are indexed by families of partitions indexed by all partitions of k. Then in Proposition 2 we give an explicit formula for the size of any of its conjugacy classes. The group $\mathcal{B}_{kn}^k$ is isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ as will be shown in Proposition 3. However, we decided to work with this copy of the wreath product in this paper since it seems very natural to present our main result.

If i and k are two positive integers, we denote by $p_k(i)$ the following set of size k :

$$p_k(i) := \{(i-1)k+1, (i-1)k+2, \ldots, ik\}.$$

The above set $p_k(i)$ will be called a k-tuple in this paper. We define the group $\mathcal{B}_{kn}^k$ to be the subgroup of $\mathcal{S}_{kn}$ formed by permutations that send each set of the form $p_k(i)$ to another set with the same form:

$$\mathcal{B}_{kn}^k := \{w \in \mathcal{S}_{kn}; \ \forall\, 1 \leqslant r \leqslant n, \ \exists\, 1 \leqslant r' \leqslant n \mid w(p_k(r)) = p_k(r')\}.$$

Example 1. $\left(\begin{smallmatrix} 1\,2\,3 \mid 4\,5\,6 \\ 1\,3\,2 \mid 6\,5\,4 \end{smallmatrix}\right) \in \mathcal{B}_6^3$ but $\left(\begin{smallmatrix} 1\,2\,3 \mid 4\,5\,6 \\ 1\,3\,6 \mid 2\,4\,6 \end{smallmatrix}\right) \notin \mathcal{B}_6^3$.

Remark 1. The sign $\mid$ will be added after k elements in the two-lines notation of a permutation to let the reader easily see that we are working in the case k. In the previous example k was 3.

The group $\mathcal{B}_{kn}^k$ as it is defined here appears in [10, Section 7.4]. When $k = 1$, it is clear that $\mathcal{B}_n^1$ is the symmetric group $\mathcal{S}_n$. When $k = 2$, the group $\mathcal{B}_{2n}^2$ is the hyperoctahedral group $\mathcal{H}_n$ on $2n$ elements, see [9] where

the author treats $\mathcal{H}_n$ as being the group $\mathcal{B}_{2n}^2$. It would be clear to see that the order of the group $\mathcal{B}_{kn}^k$ is

$$|\mathcal{B}_{kn}^k| = (k!)^n n!$$

The decomposition of a permutation $\omega \in \mathcal{B}_{kn}^k$ into product of disjoint cycles has a remarkable pattern. To see this, fix a k-tuple $p_k(i)$ and a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_l)$ of k. Suppose now that while writing ω as a product of disjoint cycles we get among the cycles the following pattern:

$$\mathcal{C}_1 = (a_1, \ldots, a_2, \ldots, a_{\rho_1}, \ldots), \tag{1}$$
$$\mathcal{C}_2 = (a_{\rho_1+1}, \ldots, a_{\rho_1+2}, \ldots, a_{\rho_1+\rho_2}, \ldots),$$
$$\vdots$$
$$\mathcal{C}_l = (a_{\rho_1+\cdots+\rho_{l-1}+1}, \ldots, a_{\rho_1+\cdots+\rho_{l-1}+2}, \ldots, a_{\rho_1+\cdots+\rho_{l-1}+\rho_l}, \ldots),$$

where

$$\{a_1, a_2, \ldots, a_{\rho_1+1}, \ldots, a_{\rho_1+\cdots+\rho_{l-1}+\rho_l}\} = p_k(i) \text{ for a certain } i \in [n].$$

We should remark that since $\omega \in \mathcal{B}_{kn}^k$, if we consider $b_j = \omega(a_j)$ for any $1 \leqslant j \leqslant |\rho|$ then there exists $r \in [n]$ such that:

$$p_k(r) = \{b_1, b_2, \ldots, b_{\rho_1+1}, \ldots, b_{\rho_1+\cdots+\rho_{l-1}+\rho_l}\}.$$

This can be redone till we reach a_2 which implies that in cycle $\mathcal{C}_1$, we have the same number of elements between a_i and a_{i+1} for any $1 \leqslant i \leqslant \rho_1 - 1$. Thus the size of the cycle $\mathcal{C}_1$ is a multiple of ρ_1, say $|\mathcal{C}_1| = m\rho_1$. The same can be done for all the other cycles $\mathcal{C}_i$ and in fact for any $1 \leqslant i \leqslant l$, $|\mathcal{C}_i| = m\rho_i$. In addition, if we take the set of all the elements that figure in the cycles $\mathcal{C}_i$ we will get a disjoint union of m k-tuples. That means:

$$\sum_{j=1}^{l} |\mathcal{C}_j| = \sum_{j=1}^{l} m\rho_j = mk.$$

Now construct the partition $\omega(\rho)$ by grouping all the integers m as above.

Remark 2. We should pay the reader's attention to two important remarks after this construction. First of all, for any partition ρ of k there are rk elements involved when adding the part r to $\omega(\rho)$. That means:

$$\sum_{\rho \vdash k} \sum_{r \geqslant 1} krm_r(\omega(\rho)) = kn \text{ which implies } \sum_{\rho \vdash k} \sum_{r \geqslant 1} rm_r(\omega(\rho)) = n.$$

Now let p_ω denotes the blocks permutation of n associated to ω. That is $p_\omega(i) = j$ whenever $\omega(p_k(i)) = p_k(j)$. Adding a part r to one of the partitions $\omega(\rho)$ implies that r is the length of one of the cycles of p_ω. Thus we have:

$$\bigcup_{\rho \vdash k} \omega(\rho) = \text{ct}(p_\omega).$$

In other words, one may first find the cycle-type of p_ω and then distribute all its parts between the partitions $\omega(\rho)$ according to the above construction.

Example 2. Consider the following permutation, written in two-lines notation, ω of $\mathcal{B}_{24}^3$

$$\omega = \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & | & 7 & 8 & 9 & | & 10 & 11 & 12 & | & 13 & 14 & 15 & | & 16 & 17 & 18 & | & 19 & 20 & 21 & | & 22 & 23 & 24 \\ 12 & 10 & 11 & | & 20 & 21 & 19 & | & 8 & 7 & 9 & | & 1 & 2 & 3 & | & 16 & 18 & 17 & | & 15 & 14 & 13 & | & 5 & 4 & 6 & | & 22 & 23 & 24 \end{pmatrix}.$$

Its decomposition into product of disjoint cycles is:

$$\begin{aligned} \omega \; = \; & (1, 12, 3, 11, 2, 10)(4, 20)(5, 21, 6, 19)(7, 8)(9)(13, 16, 15, 17, 14, 18) \\ & (22)(23)(24). \end{aligned}$$

The first cycle $(1, 12, 3, 11, 2, 10)$ contains all the elements of $p_3(1)$ thus it contributes to $\omega(3)$. In it, there are two 3-tuples namely $p_3(1)$ and $p_3(4)$ thus we should add 2 to the partition $\omega(3)$. For a similar reason, the cycle $(13, 16, 15, 17, 14, 18)$ contributes to $\omega(3)$ by 2 also, thus $\omega(3)$ becomes $(2, 2)$. By looking to the cycles $(4, 20)(5, 21, 6, 19)$ we see that 5 and 6 belong to the same cycle while 4 belongs to the other, thus these cycles will contribute to $\omega(2, 1)$. Since they are formed out of the two 3-tuples $p_3(2)$ and $p_3(7)$, we should add the part 2 to the partition $\omega(2, 1)$. In the same way, the cycles $(7, 8)(9)$ contribute to $\omega(2, 1)$ by 1 to become the partition $(2, 1)$. The remaining cycles $(22)(23)(24)$ give $\omega(1, 1, 1) = (1)$. The reader should remark that p_ω is the permutation $(1, 4)(2, 7)(3)(5, 6)(8)$ of 8 and that:

$$\omega(3) \cup \omega(2, 1) \cup \omega(1^3) = \text{ct}(p_\omega) = (1^2, 2^3).$$

Definition 1. If $\omega \in \mathcal{B}_{kn}^k$, define **type**$(\omega)$ to be the following family of partitions indexed by partitions of k

$$\mathbf{type}(\omega) := (\omega(\rho))_{\rho \vdash k}.$$

Proposition 1. *Two permutations α and β of $\mathcal{B}_{kn}^k$ are in the same conjugacy class if and only if they both have the same type.*

Proof. Suppose $\alpha = \gamma\beta\gamma^{-1}$ for some $\gamma \in \mathcal{B}_{kn}^k$ and fix a partition ρ of k. Suppose that the cycles $\mathcal{C}_1, \ldots, \mathcal{C}_l$ of Equation (1) contribute to $\beta(\rho)$ then the cycles

$$\mathcal{C}_1' = (\gamma(a_1), \ldots, \gamma(a_2), \ldots, \gamma(a_{\rho_1}), \ldots),$$

$$\mathcal{C}_2' = (\gamma(a_{\rho_1+1}), \ldots, \gamma(a_{\rho_1+2}), \ldots, \gamma(a_{\rho_1+\rho_2}), \ldots),$$

$$\vdots$$

$$\mathcal{C}_l' = (\gamma(a_{\rho_1+\cdots+\rho_{l-1}+1}), \ldots, \gamma(a_{\rho_1+\cdots+\rho_{l-1}+2}), \ldots, \gamma(a_{\rho_1+\cdots+\rho_{l-1}+\rho_l}), \ldots),$$

will contribute to $\alpha(\rho)$ and they have respectively the same lengths as $\mathcal{C}_1, \ldots, \mathcal{C}_2$. This proves the first implication.

Conversely, $\mathbf{type}(\alpha) = \mathbf{type}(\beta)$ means that $\alpha(\rho) = \beta(\rho)$ for any partition ρ of k. In order to simplify, we will look at the elements of two fixed k-tuples, say $p_k(1) = \{1, 2, \ldots, k\}$ for α and $p_k(2) = \{k+1, k+2, \ldots, 2k\}$ for β, such that the distribution of the elements of $p_k(1)$ among the cycle decomposition in α is similar to that of the elements of $p_k(2)$ in β. In other words, the cycle decompositions of α and β are as follows:

$$\alpha = (1, \ldots, 2, \ldots, \rho_1, \ldots)(\rho_1+1, \ldots, \rho_1+2, \ldots, \rho_1+\rho_2, \ldots)$$
$$\cdots (\rho_1 + \cdots + \rho_{l-1} + 1, \ldots, \rho_1 + \cdots + \rho_{l-1} + 2,$$
$$\ldots, \rho_1 + \cdots + \rho_{l-1} + \rho_l, \ldots)$$

and

$$\beta = (k+1, \ldots, k+2, \ldots, k+\rho_1, \ldots)$$
$$(k + \rho_1 + 1, \ldots, k + \rho_1 + 2, \ldots, k + \rho_1 + \rho_2, \ldots)$$
$$\cdots (k + \rho_1 + \cdots + \rho_{l-1} + 1, \ldots, k + \rho_1 + \cdots + \rho_{l-1} + 2,$$
$$\ldots, k + \rho_1 + \cdots + \rho_{l-1} + \rho_l, \ldots)$$

Construct γ to be the permutation that orderly takes each element of the above cycles of α to each element with the same order in β, that is:

$$\gamma(1) = k+1, \gamma(\alpha(1)) = \beta(k+1), \ldots \gamma(\rho_1) = k + \rho_1, \ldots$$

It is then clear that $\alpha = \gamma^{-1}\beta\gamma$ and $\gamma \in \mathcal{B}_{kn}^k$. $\qquad\square$

Proposition 2. *If ω is a permutation of $\mathcal{B}_{kn}^k$ then the size $\mathbf{conj}(\omega)$ of the conjugacy class of ω is given by*

$$\mathbf{conj}(\omega) = \frac{n!(k!)^n}{\displaystyle\prod_{\rho \vdash k} z_{\omega(\rho)} z_\rho^{l(\omega(\rho))}}.$$

Proof. Suppose that $\mathrm{ct}(p_\omega) = \lambda$, where λ is a partition of n. We will explain how to obtain all the elements $\gamma \in \mathcal{B}_{kn}^k$ that have the same type as ω. First of all, p_γ must be decomposed in cycles with sizes equivalent to those of p_ω. In other words, p_γ should have the same cycle-type as p_ω and there are

$$\frac{n!}{z_\lambda}$$

choices to make p_γ.

Now suppose that $m_r(\lambda) \neq 0$ for some $r \in [n]$. That means that there are exactly $m_r(\lambda)$ cycles of length r distributed in the family $(\omega(\rho))_{\rho \vdash k}$. Suppose they all appear in the partitions $\omega(\rho_1), \ldots, \omega(\rho_m)$. In order to have $\mathbf{type}(\gamma) = \mathbf{type}(\omega)$, we should make $\gamma(\rho_1) = \omega(\rho_1)$, $\gamma(\rho_2) = \omega(\rho_2)$ and so on. That is $\gamma(\rho_1)$ should have parts equal to r as many as $\omega(\rho_1)$ has and so on. The total number of γ verifying these conditions is

$$\frac{m_r(\lambda)!}{\displaystyle\prod_{1 \leqslant j \leqslant m} m_r(\omega(\rho_j))!} = \frac{m_r(\lambda)!}{\displaystyle\prod_{\rho \vdash k} m_r(\omega(\rho))!}$$

It remains to see how many $\gamma \in \mathcal{B}_{kn}^k$ one can make when he nows all $\gamma(\rho)$. For this fix a partition ρ_1 of k such that $\gamma(\rho_1) \neq \varnothing$ and suppose m_1 is a part of $\gamma(\rho_1)$. There will be m_1 k-tuples involved in the construction of γ now. Choose one of them then distribute its elements according to ρ_1 in $\frac{k!}{z_{\rho_1}}$ ways. To complete γ, there will be $(k!)^{m_1-1}$ choices for the elements between any two consecutive elements of the fixed k-tuples in any chosen cycle.

By bringing together all the above arguments, the size $\mathbf{conj}(\omega)$ of the conjugacy class of ω is:

$$\mathbf{conj}(\omega) = \frac{n!}{z_\lambda} \prod_{r, m_r(\lambda) \neq 0} \frac{m_r(\lambda)!}{\displaystyle\prod_{\rho \vdash k} m_r(\omega(\rho))!} \frac{(k!)^n}{z_\rho^{l(\omega(\rho))}} = \frac{n!(k!)^n}{\displaystyle\prod_{\rho \vdash k} z_{\omega(\rho)} z_\rho^{l(\omega(\rho))}}.$$

This equality is due to the fact that

$$l(\lambda) = \sum_{r \geqslant 1} m_r(\lambda) = \sum_{r \geqslant 1} \sum_{\rho \vdash k} m_r(\omega(\rho)). \qquad \square$$

We turn now to show that the group $\mathcal{B}_{kn}^k$ is isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$. The wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ is the group with underlying set $\mathcal{S}_k^n \times \mathcal{S}_n$ and product defined as follows:

$$((\sigma_1, \ldots, \sigma_n); p).((\epsilon_1, \ldots, \epsilon_n); q) = ((\sigma_1 \epsilon_{p^{-1}(1)}, \ldots, \sigma_n \epsilon_{p^{-1}(n)}); pq),$$

for any $((\sigma_1, \ldots, \sigma_n); p), ((\epsilon_1, \ldots, \epsilon_n); q) \in \mathcal{S}_k^n \times \mathcal{S}_n$. The identity in this group is $(1; 1) := ((1_k, 1_k, \ldots, 1_k); 1_n)$, where 1_i denotes the identity function of $\mathcal{S}_i$. The inverse of an element $((\sigma_1, \sigma_2, \ldots, \sigma_n); p) \in \mathcal{S}_k \wr \mathcal{S}_n$ is given by

$$((\sigma_1, \sigma_2, \ldots, \sigma_n); p)^{-1} := ((\sigma_{p(1)}^{-1}, \sigma_{p(2)}^{-1}, \ldots, \sigma_{p(n)}^{-1}); p^{-1}).$$

For each permutation $\omega \in \mathcal{B}_{kn}^k$ and each integer $i \in [n]$, define ω_i to be the normalized restriction of ω on the block $p_\omega^{-1}(i)$. That is:

$$
\begin{aligned}
\omega_i \quad : \quad [k] \quad &\to \quad [k] \\
b \quad &\mapsto \quad \omega_i(b) := \omega\big(k(p_\omega^{-1}(i) - 1) + b\big)\%k,
\end{aligned}
$$

where % means that the integer is taken modulo k (for a multiple of k we use k instead of 0).

Example 3. Consider the following permutation α of $\mathcal{B}_{18}^3$:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & | & 4\ 5\ 6 & | & 7\ 8\ 9 & | & 10\ 11\ 12 & | & 13\ 14\ 15 & | & 16\ 17\ 18 \\ 12 & 10 & 11 & | & 5\ 6\ 4 & | & 8\ 7\ 9 & | & 15\ 13\ 14 & | & 16\ 18\ 17 & | & 3 & 2 & 1 \end{pmatrix}.$$

The blocks permutation associated to α is $p_\alpha = (1, 4, 5, 6)(2)(3)$. In addition, we have:

$$\alpha_1 = (1, 3), \quad \alpha_2 = (1, 2, 3), \quad \alpha_3 = (1, 2), \quad \alpha_4 = (1, 3, 2), \quad \alpha_5 = (1, 3, 2)$$

and

$$\alpha_6 = (2, 3).$$

Proposition 3. *The map*

$$
\begin{aligned}
\psi \quad : \quad \mathcal{B}_{kn}^k \quad &\to \quad \mathcal{S}_k \wr \mathcal{S}_n \\
\omega \quad &\mapsto \quad \psi(\omega) := ((\omega_1, \ldots, \omega_n); p_\omega),
\end{aligned}
$$

is a group isomorphism.

Proof. ψ is clearly a bijection with inverse given by:

$$
\begin{aligned}
\phi \quad : \quad \mathcal{S}_k \wr \mathcal{S}_n \quad &\to \quad \mathcal{B}_{kn}^k \\
((\sigma_1, \sigma_2, \ldots, \sigma_n); p) \quad &\mapsto \quad \sigma,
\end{aligned}
$$

where $\sigma\big(k(a - 1) + b\big) = k(p(a) - 1) + \sigma_{p(a)}(b)$ for any $a \in [n]$ and any $b \in [k]$. It remains to show that if $x = ((\sigma_1, \sigma_2, \ldots, \sigma_n); p)$ and

$y = ((\epsilon_1, \epsilon_2, \ldots, \epsilon_n); q)$ are two elements of $\mathcal{S}_k \wr \mathcal{S}_n$ then $\phi(x.y) = \phi(x)\phi(y)$. To prove this let $a \in [n]$ and $b \in [k]$. On the right hand we have:

$$\phi(x)\phi(y)\big(k(a-1)+b\big) = \phi(x)\big(k(q(a)-1)+\epsilon_{q(a)}(b)\big)$$

$$= k(p(q(a))-1) + \sigma_{p(q(a))}(\epsilon_{q(a)}(b))$$

and on the left hand we have:

$$\phi(xy)\big(k(a-1)+b\big) = k((pq)(a)-1) + \sigma_{(pq)(a)}\epsilon_{p^{-1}((pq)(a))}(b).$$

This shows the equality $\phi(x.y) = \phi(x)\phi(y)$ and finishes the proof. □

Example 4. Recall the permutation $\alpha \in \mathcal{B}_{18}^3$ of Example 3 and consider the following permutation β of the same group:

$$\beta = \begin{pmatrix} 1\ 2\ 3\ |\ 4\ \ 5\ \ 6\ \ |\ 7\ 8\ 9\ |\ 10\ 11\ 12\ |\ 13\ 14\ 15\ |\ 16\ 17\ 18 \\ 4\ 5\ 6\ |\ 18\ 17\ 16\ |\ 8\ 9\ 7\ |\ \ 1\ \ \ 2\ \ \ 3\ \ |\ 12\ 11\ 10\ |\ 15\ 14\ 13 \end{pmatrix}.$$

We have:

$$\alpha\beta = \begin{pmatrix} 1\ 2\ 3\ |\ 4\ 5\ 6\ |\ 7\ 8\ 9\ |\ 10\ 11\ 12\ |\ 13\ 14\ 15\ |\ 16\ 17\ 18 \\ 5\ 6\ 4\ |\ 1\ 2\ 3\ |\ 7\ 9\ 8\ |\ 12\ 10\ 11\ |\ 14\ 13\ 15\ |\ 17\ 18\ 16 \end{pmatrix},$$

$$\psi(\alpha) = \Big(\big((1,3),(1,2,3),(1,2),(1,3,2),(1,3,2),(2,3)\big); (1,4,5,6)(2)(3)\Big),$$

$$\psi(\beta) = \Big(\big(1,1,(1,2,3),(1,3),(1,3),(1,3)\big); (1,2,6,5,4)(3)\Big)$$

and

$$\psi(\alpha\beta) = \Big(\big(1,(1,2,3),(2,3),(1,3,2),(1,2),(1,2,3)\big); (1,2)(3)(4)(5)(6)\Big).$$

Now we can easily verify that $\psi(\alpha\beta) = \psi(\alpha).\psi(\beta)$.

4. Special cases

In this section we treat two special cases, the case of $k = 2$ and that of $k = 3$. We see that when $k = 2$, the group $\mathcal{B}_{2n}^2$ is the hyperoctahedral group on $2n$ elements. There are many papers in the literature, see [7] and [2] for examples, that study the representations of the hyperoctahedral group. Especially we are interested here in studying its conjugacy classes. We recover some of its nice properties using our approach.

4.1.   Case $k = 2$: The hyperoctahedral group

When $k = 2$ there are only two types of partitions of 2, mainly $\lambda_1 = (1^2)$ and $\lambda_2 = (2)$. Now we are going to see how things go in the context of $\mathcal{B}_{2n}^2$. We use the same arguments given in [10, Section 6.2]. If $a \in p_2(i)$, we shall denote by $\bar{a}$ the element of the set $p_2(i) \setminus \{a\}$. Therefore, we have, $\bar{\bar{a}} = a$ for any $a \in [2n]$. As seen in our general construction, the cycle decomposition of a permutation of $\mathcal{B}_{2n}^2$ will have two types of cycles. To see this, suppose that ω is a permutation of $\mathcal{B}_{2n}^2$ and take the following cycle $\mathcal{C}$ of its decomposition:

$$\mathcal{C} = (a_1, \ldots, a_l).$$

We distinguish two cases :

    1) $\overline{a_1}$ appears in the cycle $\mathcal{C}$, for example $a_j = \overline{a_1}$. Since $\omega \in \mathcal{B}_{2n}^2$ and $\omega(a_1) = a_2$, we have $\omega(\overline{a_1}) = \overline{a_2} = \omega(a_j)$. Likewise, since $\omega(a_{j-1}) = \overline{a_1}$, we have $\omega(\overline{a_{j-1}}) = a_1$ which means that $a_l = \overline{a_{j-1}}$. Therefore,

$$\mathcal{C} = (a_1, \ldots a_{j-1}, \overline{a_1}, \ldots, \overline{a_{j-1}})$$

    and $l = 2(j-1)$ is even. We will denote such a cycle by $(\mathcal{O}, \overline{\mathcal{O}})$.

    2) $\overline{a_1}$ does not appear in the cycle $\mathcal{C}$. Take the cycle $\overline{\mathcal{C}}$ which contains $\overline{a_1}$. Since $\omega(a_1) = a_2$ and $\omega \in \mathcal{B}_{2n}^2$, we have $\omega(\overline{a_1}) = \overline{a_2}$ and so on. That means that the cycle $\overline{\mathcal{C}}$ has the following form,

$$\overline{\mathcal{C}} = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_l})$$

    and that $\mathcal{C}$ and $\overline{\mathcal{C}}$ appear in the cycle decomposition of ω.

The cycles of the first case will contribute to $\omega(\lambda_2)$ while those of the second will contribute to $\omega(\lambda_1)$. Suppose now that the cycle decomposition of a permutation ω of $\mathcal{B}_{2n}^2$ is as follows:

$$\omega = \mathcal{C}_1\overline{\mathcal{C}_1}\mathcal{C}_2\overline{\mathcal{C}_2}\ldots\mathcal{C}_k\overline{\mathcal{C}_k}(\mathcal{O}^1, \overline{\mathcal{O}^1})(\mathcal{O}^2, \overline{\mathcal{O}^2})\ldots(\mathcal{O}^l, \overline{\mathcal{O}^l}),$$

where the cycles $\mathcal{C}_i$ (resp. $(\mathcal{O}^j, \overline{\mathcal{O}^j})$) are written decreasingly according to their sizes. From this decomposition, we obtain that the parts of the partition $\omega(\lambda_1)$ are the sizes of the sets $\mathcal{C}_j$, while the parts of the partition $\omega(\lambda_2)$ are the sizes of the sets $\mathcal{O}^i$:

$$\omega(\lambda_1) = (|\mathcal{C}_1|, \ldots, |\mathcal{C}_k|), \quad \omega(\lambda_2) = (|\mathcal{O}^1|, \ldots, |\mathcal{O}^l|)$$

and

$$|\omega(\lambda_1)| + |\omega(\lambda_2)| = n.$$

Example 5. Consider the following permutation ω of $\mathcal{B}_{16}^2$

$$\omega = \begin{pmatrix} 1 & 2 & | & 3 & 4 & | & 5 & 6 & | & 7 & 8 & | & 9 & 10 & | & 11 & 12 & | & 13 & 14 & | & 15 & 16 \\ 14 & 13 & | & 1 & 2 & | & 16 & 15 & | & 7 & 8 & | & 12 & 11 & | & 10 & 9 & | & 4 & 3 & | & 5 & 6 \end{pmatrix}.$$

Its decomposition into product of disjoint cycles is as follows:

$$\omega = (1, 14, 3)(2, 13, 4)(7)(8)(9, 12)(10, 11)(5, 16, 6, 15).$$

Then $\omega(\lambda_1) = (3, 2, 1)$ and $\omega(\lambda_2) = (2)$.

Apply Proposition 2 to obtain the following result.

Corollary 1. *The size of the conjugacy class of a permutation $\omega \in \mathcal{B}_{2n}^2$ is:*

$$\mathbf{conj}(\omega) = \frac{2^n n!}{2^{l(\omega(\lambda_1)) + l(\omega(\lambda_2))} z_{\omega(\lambda_1)} z_{\omega(\lambda_2)}}.$$

The above Corollary 1 is a well known formula for the sizes of the conjugacy classes of the hyperoctahedral group, see [7] and [2].

4.2.   Case $k = 3$

In a way similar to that of case $k = 2$, the fact that there are only three partitions of 3 suggests that there are three types of cycles in the decomposition into product of disjoint cycles of a permutation $\omega \in \mathcal{B}_{3n}^3$.

Let $\mathcal{C}$ be a cycle of $\omega \in \mathcal{B}_{3n}^3$, we distinguish the following three cases:
1) first case: all three elements of a certain $p_3(s)$ belong to $\mathcal{C}$. For simplicity, suppose $s = 1$ and

$$\mathcal{C} = (a_1 = 1, a_2, a_3, \ldots, a_j = 2, a_{j+1}, \ldots, a_l = 3, a_{l+1}, \ldots a_k).$$

Since $\omega \in \mathcal{B}_{3n}^3$, the sets

$$\{a_2, a_{j+1}, a_{l+1}\}, \{a_3, a_{j+2}, a_{l+2}\}, \ldots, \{a_{j-1}, a_{l-1}, a_k\}$$

all have the form $p_3(m)$ and thus $\mathcal{C}$ is a cycle of length $3(j - 1)$ that contains a union of sets of the form $p_3(r)$.
2) second case: two and only two elements of a certain $p_3(s)$ belong to $\mathcal{C}$. Say,

$$\mathcal{C} = (a_1 = 1, a_2, a_3, \ldots, a_j = 2, a_{j+1}, \ldots, a_k).$$

Since $\omega \in \mathcal{B}_{3n}^3$, there exists integers b_i, $1 \leqslant i \leqslant j - 1$, such that

$$\{a_1, a_j\} = p_3(b_1) \setminus \{c_1\},$$
$$\{a_2, a_{j+1}\} = p_3(b_2) \setminus \{c_2\},$$
$$\vdots$$
$$\{a_{j-1}, a_k\} = p_3(b_{j-1}) \setminus \{c_{j-1}\},$$

and another cycle $(c_1, c_2, \ldots, c_{j-1})$ should thus appear in the decomposition of ω.

3) third case: all three elements of a certain $p_3(s)$ belong to different cycles. In this case, all three cycles will have the same lengths and each one of them will contain elements belonging to different triplets.

Now for any permutation $\omega \in \mathcal{B}_{3n}^3$, define γ_ω to be the partition obtained from the lengths divided by three of the cycles of the first case, β_ω the partition obtained from the lengths of the cycles of the second case divided by two and α_ω the partition obtained from the lengths of the cycles of the third case. It would be clear that

$$\omega(\lambda_1) = \alpha_\omega, \quad \omega(\lambda_2) = \beta_\omega, \quad \omega(\lambda_3) = \gamma_\omega \text{ and } |\gamma_\omega| + |\beta_\omega| + |\alpha_\omega| = n.$$

For example, for the permutation ω of Example 2, we have:

$$\gamma_\omega = (2, 2), \quad \beta_\omega = (2, 1) \text{ and } \alpha_\omega = (1).$$

Using Proposition 2, we obtain the following result.

Corollary 2. *The size* **conj**(ω) *of the conjugacy class of* $\omega \in \mathcal{B}_{3n}^3$ *is:*

$$\frac{(3!)^n n!}{(3!)^{l(\alpha_\omega)} z_{\alpha_\omega} 2^{l(\beta_\omega)} z_{\beta_\omega} 3^{l(\gamma_\omega)} z_{\gamma_\omega}} = 2^{n - l(\alpha_\omega) - l(\beta_\omega)} 3^{n - l(\alpha_\omega) - l(\gamma_\omega)} \cdot \frac{n!}{z_{\alpha_\omega} z_{\beta_\omega} z_{\gamma_\omega}}.$$

5.   The center of the group $\mathcal{B}_{kn}^k$ algebra

In this section, we present in Theorem 1 a polynomiality property for the structure coefficients of the center of the group $\mathcal{B}_{kn}^k$ algebra. This can be seen as a generalisation of the Farahat and Higman result in [1] and our result in [10] that gave polynomiality properties for the structure coefficients of the center of the symmetric group and the hyperoctahedral group algebras respectively. A special treatment for the cases $k = 2$ and $k = 3$ is given.

5.1. The algebra $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$

The center of the group $\mathcal{B}_{kn}^k$ algebra will be denoted $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$. It is the algebra over $\mathbb{C}$ spanned by the "formal sum of elements of the" conjugacy classes of $\mathcal{B}_{kn}^k$. According to Proposition 1, these are indexed by families of partitions $x = (x(\lambda))_{\lambda \vdash k}$ satisfying the property:

$$|x| := \sum_{\lambda \vdash k} |x(\lambda)| = n \tag{2}$$

and for each such a family its associated conjugacy class C_x is

$$C_x = \{t \in \mathcal{B}_{kn}^k \text{ such that } \mathbf{type}(t) = x\}$$

while its formal sum of elements is

$$\mathbf{C}_x := \sum_{t \in C_x} t.$$

From now on and unless stated otherwise, x is a family of partition would mean $x = (x(\lambda))_{\lambda \vdash k}$ with the condition 2.

Let x and y be two family of partitions. In the algebra $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$, the product $\mathbf{C}_x \mathbf{C}_y$ can be written as a linear combination as following

$$\mathbf{C}_x \mathbf{C}_y = \sum_z c_{xy}^z \mathbf{C}_z \tag{3}$$

where z runs through all the families of partitions. The coefficients c_{xy}^z that appear in this equation are called the structure coefficients of the center of the group $\mathcal{B}_{kn}^k$ algebra.

5.2. Polynomiality of the structure coefficients of $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$

When $k = 1$, Farahat and Higman were the first to give a polynomiality property for the structure coefficients of the center of the symmetric group algebra in [1]. In the case of the center of the hyperoctahedral group algebra, we gave a polynomiality property for its structure coefficients in [10]. The goal of this section is to generalize these results and show that the structure coefficients of $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$ have a polynomiality property for any fixed k.

The most natural way to see a permutation $\omega \in \mathcal{B}_{kn}^k$ as an element of $\omega \in \mathcal{B}_{k(n+1)}^k$ is by extending it by identity. By doing so, the new permutation will have the same type as ω except that the partition $\omega(1^k)$ will become $\omega(1^k) \cup (1)$.

Definition 2. A family of partitions $x = (x(\lambda))_{\lambda \vdash k}$ is said to be proper if and only if the partition $x(1^k)$ is proper. If $x = (x(\lambda))_{\lambda \vdash k}$ is a proper family of partitions such that $|x| < n$, we define $C_x(n)$ to be the set of elements $t \in \mathcal{B}_{kn}^k$ that have type equals to x except that $x(1^k)$ is replaced by $x(1^k) \cup (1^{n-|x|})$.

If $x = (x(\lambda))_{\lambda \vdash k}$ is a proper family of partitions such that $|x| = n_0$ then for any $n > n_0$ we have by Proposition 2 the following result:

$$
\begin{aligned}
|C_x(n)| &= \frac{n!(k!)^n}{z_{x(1^k) \cup (1^{n-n_0})}(k!)^{l(x(1^k))+n-n_0} \displaystyle\prod_{\lambda \vdash k, \lambda \neq (1^k)} z_{x(\lambda)} z_\lambda^{l(x(\lambda))}} \\
&= \frac{n!(k!)^{n_0-l(x(1^k))}}{z_{x(1^k)}(n-n_0)! \displaystyle\prod_{\lambda \vdash k, \lambda \neq (1^k)} z_{x(\lambda)} z_\lambda^{l(x(\lambda))}}.
\end{aligned}
$$

Take x and y to be two proper families of partitions. For any integer $n > |x|, |y|$ we have the following equation in $Z(\mathbb{C}[\mathcal{B}_{kn}^k])$

$$
\mathbf{C}_x(n)\mathbf{C}_y(n) = \sum_z c_{xy}^h(n)\mathbf{C}_z(n), \tag{4}
$$

where h runs through all proper families of partitions verifying $|z| \leqslant n$.

In [10], under some conditions, a formula describing the form of the structure coefficients of centers of finite group algebras is given. We show below that the sequence $(\mathcal{B}_{kn}^k)_n$ satisfies these conditions. This will allow us to use [10, Corollary 6.3] in order to give a polynomiality property for the structure coefficients $c_{xy}^h(n)$ described in Equation (4).

We will show below the conditions required in [10] for our sequence of groups $(\mathcal{B}_{kn}^k)_n$. To avoid repetitions and confusing notations and since the integer k will be fixed, we will use simply G_n to denote the group $\mathcal{B}_{kn}^k$.

Hypothesis 1: For any integer $1 \leqslant r \leqslant n$, there exists a group G_n^r isomorphic to G_{n-r}. Set

$$
G_n^r := \{\omega \in G_n \text{ such that } \omega(i) = i \text{ for any } 1 \leqslant i \leqslant kr\}.
$$

for this reason.

Hypothesis 2: The elements of G_n^r and G_r commute between each other which is normal since the permutations in these groups act on disjoint sets.

Hypothesis 3: $G_{n+1}^r \cap G_n = G_n^r$ which is obvious.

Hypothesis 4: For any $z \in G_n$, $\mathrm{k}(G_n^{r_1} z G_n^{r_2}) := \min\{s|G_n^{r_1} z G_n^{r_2} \cap G_s \neq \varnothing\} \leqslant r_1 + r_2$. To prove this, remark first that the size of the set $\{1, \ldots, kr_1\} \cap \{z(1), \ldots, z(kr_2)\}$ is a multiple of k since $z \in G_n$, say it is km. Suppose that

$$\{h_1, \ldots, h_{kr_1 - km}\} = \{1, \ldots, kr_1\} \setminus \{z(1), \ldots, z(kr_2)\}.$$

We can find a permutation of the following form

$$\begin{pmatrix} 1 & | & 2 & | & \ldots & | & kr_2 & | & kr_2+1 & | & \ldots & | & kr_1+kr_2-km & | & kr_1+kr_2-km+1 & | & \ldots & | & kn \\ z(1) & | & z(2) & | & \ldots & | & z(kr_2) & | & h_1 & | & \ldots & | & h_{kr_1-km} & | & * & & & | & \ldots & | & * \end{pmatrix}$$

in $z G_n^{r_2}$ since it contains permutations that fixes the first kr_2 images of z. The stars are used to say that the images may not be fixed. Since the multiplication by an element of $G_n^{r_1}$ to the left permutes the elements greater than kr_1 in the second line defining this permutation, the set $G_n^{r_1} z G_n^{r_2}$ contains thus a permutation of the following form

$$\begin{pmatrix} 1 & | & 2 & | & \ldots & | & kr_2 & | & kr_2+1 & | & \ldots & | & kr_1+kr_2-km & | & kr_1+kr_2-km+1 & | & \ldots & | & kn \\ * & | & * & | & \ldots & | & * & | & h_1 & | & \ldots & | & h_{kr_1-km} & | & kr_1+kr_2-km+1 & | & \ldots & | & kn \end{pmatrix}$$

This permutation is also in $G_{r_1+r_2-m}$ which ends the proof.

Hypothesis 5: If $z \in G_n$ then we have $z G_n^{r_1} z^{-1} \cap G_n^{r_2} = G_n^{r(z)}$ where

$$
\begin{aligned}
r(z) &= |\{z(1), z(2), \ldots, z(kr_1), 1, \ldots, kr_2\}| \\
&= kr_1 + kr_2 - |\{z(1), z(2), \ldots, z(kr_1)\} \cap \{1, \ldots, kr_2\}|.
\end{aligned}
$$

To prove this, let $a = zbz^{-1}$ be an element of G_n which fixes the kr_2 first elements while b fixes the kr_1 first elements. Then a also fixes the elements $z(1), \ldots, z(kr_1)$ which proves that $z G_n^{r_1} z^{-1} \cap G_n^{r_2} \subset G_n^{r(z)}$. In the opposite direction, if p is a permutation of n which fixes the elements of the set $\{z(1), z(2), \ldots, z(kr_1), 1, \ldots, kr_2\}$ then p is in $G_n^{r_2}$ and in addition $z^{-1}pz$ is in $G_n^{r_1}$ which implies that $p = zz^{-1}pzz^{-1}$ is in $z G_n^{r_1} z^{-1}$.

Now with all the necessary hypotheses verified we can apply the main result in [10] to get the following theorem.

Theorem 1. *Let x, y and h be three proper families of partitions. For any $n > |x|, |y|, |h|$, the coefficients $c_{xy}^h(n)$ defined in Equation 4 are polynomials in n with non-negative rational coefficients. In addition,*

$$\deg(c_{xy}^h(n)) < |x| + |y| - |h|.$$

Proof. By [10, Corollary 6.3], if $n > |x|, |y|, |h|$ then

$$c_{xy}^h(n) = \frac{|C_x(n)||C_y(n)||\mathcal{B}_{k(n-|x|)}^k||\mathcal{B}_{k(n-|y|)}^k|}{|\mathcal{B}_{kn}^k||C_h(n)|} \sum_{|h| \leqslant r \leqslant |x|+|y|} \frac{a_{xy}^h(r)}{|\mathcal{B}_{k(n-r)}^k|}$$

where the $a_{xy}^h(r)$ are positive, rational and independent numbers of n. Since all the cardinals involved in this formula are known, we get after simplification the following formula for $c_{xy}^h(n)$

$$c_{xy}^h(n) = (k!)^{l(h(1^k))-l(x(1^k))-l(y(1^k))} \frac{z_{h(1^k)}c_h}{z_{x(1^k)}z_{y(1^k)}c_x c_y} \times$$

$$\sum_{|h| \leqslant r \leqslant |x|+|y|} \frac{(k!)^{r-|h|}a_{xy}^h(r)(n-|h|)!}{(n-r)!},$$

where c_x denotes $\prod_{\lambda \vdash k, \lambda \neq (1^k)} z_{x(\lambda)}z_\lambda^{l(x(\lambda))}$. The result follows. $\qquad \square$

In [12, Theorem 2.13], Wang uses the Farahat-Higman approach to prove that the structure coefficients of the center of the wreath product $G \wr \mathcal{S}_n$ are polynomials in n. His result recovers our Theorem 1 with the special case when $G = \mathcal{S}_k$. However, the way we prove Theorem 1 seems to be easier and highlights the importance of the general framework given in [10].

5.3. Special cases

In this section we revisit the two already known results of polynomiality for the structure coefficients of the center of the symmetric group ($k = 1$) algebra and the center of the hyperoctahedral ($k = 2$) group algebra. In addition, as an application of our main theorem, we give a polynomiality property in the case $k = 3$. In these three cases, we give explicit expressions of products of conjugacy classes in the associated center algebra in order to see our results.

k=1: The symmetric group. As seen in Section 2, the conjugacy classes of the symmetric group $\mathcal{S}_n$ are indexed by partitions of n. If λ is a partition of n the size of its associated conjugacy class is

$$|C_\lambda| = \frac{n!}{z_\lambda}.$$

If λ is a proper partition with $|\lambda| < n$, we define $\underline{\lambda}_n$ to be the partition $\lambda \cup (1^{n-|\lambda|})$. Now let λ and δ be two proper partitions with $|\lambda|, |\delta| < n$. In the center of the symmetric group algebra we have the following equation:

$$\mathbf{C}_{\underline{\lambda}_n} \mathbf{C}_{\underline{\delta}_n} = \sum_{\gamma} c_{\lambda\delta}^{\gamma}(n) \mathbf{C}_{\underline{\gamma}_n} \tag{5}$$

where the sum runs through all proper partitions γ satisfying $|\gamma| \leqslant |\lambda| + |\delta|$. If we apply Theorem 1, we re-obtain the following result of Farahat and Higman in [1, Theorem 2.2].

Theorem 2. *Let λ, δ and γ be three proper partitions and let $n \geqslant |\lambda|, |\delta|, |\gamma|$ be an integer. The structure coefficient $c_{\lambda\delta}^{\gamma}(n)$ of the center of the symmetric group algebra defined by Equation (5) is a polynomial in n with non-negative coefficients and*

$$\deg(c_{\lambda\delta}^{\gamma}(n)) \leqslant |\lambda| + |\delta| - |\gamma|.$$

Example 6. The following two complete expressions in $Z(\mathbb{C}[\mathcal{S}_n])$ appear in [8]. For any $n \geqslant 4$

$$\mathbf{C}_{(1^{n-2},2)} \mathbf{C}_{(1^{n-2},2)} = \frac{n(n-1)}{2} \mathbf{C}_{(1^n)} + 3\mathbf{C}_{(1^{n-3},3)} + 2\mathbf{C}_{(1^{n-4},2^2)}$$

and for any $n \geqslant 5$,

$$\mathbf{C}_{(1^{n-2},2)} \mathbf{C}_{(1^{n-3},3)} = 2(n-2)\mathbf{C}_{(1^{n-2},2)} + 4\mathbf{C}_{(1^{n-4},4)} + \mathbf{C}_{(1^{n-5},2,3)}.$$

k=2: The hyperoctahedral group. In Section 4.1, we showed that the conjugacy classes of the hyperoctahedral group are indexed by pairs of partitions (λ, δ) such that $|\lambda| + |\delta| = n$. The partition λ is that associated to the partition $(1, 1)$ of 2 while δ is associated to the partition (2). The size of the class $C_{(\lambda, \delta)}$ is given in Corollary 1

$$|C_{(\lambda, \delta)}| = \frac{2^n n!}{2^{l(\lambda)+l(\delta)} z_\lambda z_\delta}.$$

By Definition 2, the pair (λ, δ) is proper if and only if the partition λ is proper. For a proper pair (λ, δ) of partitions and for any integer $n > |\lambda| + |\delta|$, we define the following pair of partitions:

$$\underline{(\lambda, \delta)}_n := (\lambda \cup (1^{n-|\lambda|-|\delta|}), \delta).$$

Let (λ_1, δ_1) and (λ_2, δ_2) be two proper pairs of partitions. We have the following equation in the center of the hyperoctahedral group algebra for any integer n greater than $|\lambda_1| + |\delta_1|, |\lambda_2| + |\delta_2|$,

$$\mathbf{C}_{\underline{(\lambda_1,\delta_1)}_n} \mathbf{C}_{\underline{(\lambda_2,\delta_2)}_n} = \sum_{(\lambda_3,\delta_3)} c^{(\lambda_3,\delta_3)}_{(\lambda_1,\delta_1)(\lambda_2,\delta_2)}(n) \mathbf{C}_{\underline{(\lambda_3,\delta_3)}_n} \qquad (6)$$

where the sum runs over all the proper pairs of partitions (λ_3, δ_3) satisfying $|\lambda_3| + |\delta_3| \leqslant |\lambda_1| + |\delta_1| + \lambda_2| + |\delta_2|$. As an application of Theorem 1, we re-obtain the following result proved in [10, Corollary 6.11].

Corollary 3. *Let* $(\lambda_1, \delta_1), (\lambda_2, \delta_2)$ *and* (λ_3, δ_3) *be three proper pairs of partitions, then for any* $n \geqslant |\lambda_1| + |\delta_1|, |\lambda_2| + |\delta_2|, |\lambda_3| + |\delta_3|$ *the structure coefficient* $c^{(\lambda_3,\delta_3)}_{(\lambda_1,\delta_1)(\lambda_2,\delta_2)}(n)$ *of the center of the hyperoctahedral group algebra defined in Equation (6) is a polynomial in* n *with non-negative coefficients and we have*

$$\deg(c^{(\lambda_3,\delta_3)}_{(\lambda_1,\delta_1)(\lambda_2,\delta_2)}(n)) \leqslant |\lambda_1| + |\delta_1| + |\lambda_2| + |\delta_2| - |\lambda_3| - |\delta_3|.$$

Example 7. We give in this example the complete product of the class $C_{((1^{n-2}),(2))}$ by itself whenever $n \geqslant 4$:

$$\mathbf{C}_{((1^{n-2}),(2))}\mathbf{C}_{((1^{n-2}),(2))} = n(n-1)\mathbf{C}_{((1^n),\varnothing)} + 2\mathbf{C}_{((1^{n-4}),(2^2))}$$

$$+2\mathbf{C}_{((1^{n-2}),(1^2))} + 3\mathbf{C}_{((1^{n-3},3),\varnothing)}.$$

$C_{((1^n),\varnothing)}$ is the identity class and since any element in $C_{((1^{n-2}),(2))}$ has its inverse in $C_{((1^{n-2}),(2))}$, the coefficient of $\mathbf{C}_{((1^n),\varnothing)}$ is the size of the conjugacy class $C_{((1^{n-2}),(2))}$ which is $n(n-1)$. The coefficient of $\mathbf{C}_{((1^{n-4}),(2^2))}$ is 2 since if we fix a permutation of $C_{((1^{n-4}),(2^2))}$, say

$$(1,3,2,4)(5,7,6,8)(9)(10)\ldots(2n),$$

then there exists only two pairs $(\alpha; \beta) \in C_{((1^{n-2}),(2))} \times C_{((1^{n-2}),(2))}$ such that

$$\alpha\beta = (1,3,2,4)(5,7,6,8)(9)(10)\ldots(2n).$$

Mainly:

$$(\alpha, \beta) = ((1,3,2,4)(5)\ldots(2n); (1)(2)(3)(4)(5,7,6,8)(9)\ldots(2n))$$

or

$$(\alpha, \beta) = ((1)(2)(3)(4)(5,7,6,8)(9)\ldots(2n); (1,3,2,4)(5)\ldots(2n)).$$

There are only two permutations in $\mathcal{B}_4^2$, namely $\alpha = (1,3,2,4)$ and $\beta = (1,4,2,3)$, such that $\alpha, \beta \in C_{(\varnothing,(2))}$ and $\alpha^2 = \beta^2 = (12)(34)$. Thus the coefficient of $\mathbf{C}_{((1^{n-2}),(1^2))}$ is 2. The last coefficient can be obtained by identifying both sides.

k=3: the group $\mathcal{B}_{3k}^3$ Moving to the case $k = 3$, we showed in Section 4.2 that the conjugacy classes of the group $\mathcal{B}_{3n}^3$ are indexed by triplet of partitions (α, β, γ) such that $|\alpha| + |\beta| + |\gamma| = n$. For a given triplet (α, β, γ), the size of its associated conjugacy class is given in Corollary 2,

$$|C_{(\alpha,\beta,\gamma)}| = 2^{n-l(\alpha)-l(\beta)} 3^{n-l(\alpha)-l(\gamma)} \cdot \frac{n!}{z_\alpha z_\beta z_\gamma}.$$

By Definition 2, the triplet (α, β, γ) is proper if and only if the partition α is proper. Fix three proper triplet of partitions, $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$, the structure coefficient associated to these triplets has the following form according to the proof of Theorem 1,

$$c_{(\alpha_1,\beta_1,\gamma_1)(\alpha_2,\beta_2,\gamma_2)}^{(\alpha_3,\beta_3,\gamma_3)}(n) = \frac{(3!)^{l(\alpha_3)-l(\alpha_1)-l(\alpha_2)} z_{\alpha_3} z_{\beta_3} 2^{l(\beta_3)} z_{\gamma_3} 3^{l(\gamma_3)}}{z_{\alpha_1} z_{\alpha_2} z_{\beta_1} 2^{l(\beta_1)} z_{\gamma_1} 3^{l(\gamma_1)} z_{\beta_2} 2^{l(\beta_2)} z_{\gamma_2} 3^{l(\gamma_2)}} \times$$

$$\sum_r \frac{(3!)^{r-|\alpha_3|-|\beta_3|-|\gamma_3|} a_{(\alpha_1,\beta_1,\gamma_1)(\alpha_2,\beta_2,\gamma_2)}^{(\alpha_3,\beta_3,\gamma_3)}(r)(n - |\alpha_3| - |\beta_3| - |\gamma_3|)!}{(n - r)!}$$

where the sum runs through all integers r with

$$|\alpha_3| + |\beta_3| + |\gamma_3| \leqslant r \leqslant |\alpha_1| + |\beta_1| + |\gamma_1| + |\alpha_2| + |\beta_2| + |\gamma_2|.$$

Corollary 4. *Let $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and $(\alpha_3, \beta_3, \gamma_3)$ be three proper triplets of partitions, then for any $n \geqslant |\alpha_1| + |\beta_1| + |\gamma_1|, |\alpha_2| + |\beta_2| + |\gamma_2|, |\alpha_3| + |\beta_3| + |\gamma_3|$ the coefficient $c_{(\alpha_1,\beta_1,\gamma_1)(\alpha_2,\beta_2,\gamma_2)}^{(\alpha_3,\beta_3,\gamma_3)}(n)$ is a polynomial in n with non-negative coefficients and we have*

$$\deg(c_{(\alpha_1,\beta_1,\gamma_1)(\alpha_2,\beta_2,\gamma_2)}^{(\alpha_3,\beta_3,\gamma_3)}(n)) \leqslant |\alpha_1| + |\beta_1| + |\gamma_1| + |\alpha_2| + |\beta_2| + |\gamma_2|$$
$$- |\alpha_3| - |\beta_3| - |\gamma_3|.$$

Example 8. For $n \geqslant 3$, we leave it to the reader to verify the following expressions in $Z(\mathbb{C}[\mathcal{B}_{3n}^3])$:

$$\mathbf{C}_{((1^{n-2}),(1),(1))} \mathbf{C}_{((1^{n-1}),\varnothing,(1))} = 2\mathbf{C}_{((1^{n-3}),(1),(1^2))} + 2(n - 1)\mathbf{C}_{((1^{n-1}),(1),\varnothing)}$$

$$+ 3\mathbf{C}_{((1^{n-2}),(1),(1))}$$

and

$$\mathbf{C}_{((1^{n-2}),(1),(1))} \mathbf{C}_{((1^{n-1}),(1),\varnothing)} = 2\mathbf{C}_{((1^{n-3}),(1^2),(1))} + 3(n - 1)\mathbf{C}_{((1^{n-1}),\varnothing,(1))}$$

$$+ 4\mathbf{C}_{((1^{n-2}),(1^2),\varnothing)} + 6\mathbf{C}_{((1^{n-2}),\varnothing,(1^2))}.$$

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References

[1] H. Farahat, G. Higman, *The centres of symmetric group rings*, Proc. Roy. Soc. (A) 250 (1959) 212–221.

[2] L. Geissinger, D. Kinch, *Representations of the hyperoctahedral groups*, Journal of Algebra 53(1) (1978) 1–20.

[3] V. Ivanov, S. Kerov, *The algebra of conjugacy classes in symmetric groups, and partial permutations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 256(3)(1999) 95–120.

[4] A. Kerber, *Representations of Permutation Groups I: Representations of Wreath Products and Applications to the Representation Theory of Symmetric and Alternating Groups*, Springer 240 (2006).

[5] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Univ. Press, second edition (1995).

[6] W, Specht, *Eine Verallgemeinerung der symmetrischen Gruppe*, Humboldt-Universität zu Berlin (1932).

[7] J. R. Stembridge, *The projective representations of the hyperoctahedral group*, Journal of Algebra 145(2) (1992) 396–453.

[8] O. Tout, *Polynomiality of the structure coefficients of double-class algebras*, Université de Bordeaux (2014) Theses.

[9] O. Tout, *Structure coefficients of the Hecke algebra of (S_{2n}, B_n)*, The Electronic Journal of Combinatorics 21(4) (2014) 4–35.

[10] O. Tout, *A general framework for the polynomiality property of the structure coefficients of double-class algebras*, Journal of Algebraic Combinatorics 45(4) (2017) 1111–1152.

[11] O. Tout, *k-partial permutations and the center of the wreath product $\mathcal{S}_k \wr \mathcal{S}_n$ algebra*, Journal of Algebraic Combinatorics 53(2) (2021) 389–412.

[12] W. Wang, *The Farahat–Higman ring of wreath products and Hilbert schemes*, Advances in Mathematics 187(2) (2004) 417–446.

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