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THE CENTRAL LIMIT THEOREM FOR LOCAL LINEAR STATISTICS IN CLASSICAL COMPACT GROUPS AND RELATED COMBINATORIAL IDENTITIES[1]

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We discuss the CLT for the global and local linear statistics of random matrices from classical compact groups. The main parts of our proofs are certain combinatorial identities, much in the spirit of works by M. Kac and H. Spohn.

1. Introduction. Let M be a unitary matrix chosen at random with respect to the Haar measure on the unitary group $U(n)$. We denote the eigenvalues of M by $\{\exp(i\cdot\theta_j)\}_{j=1}^n$, where $-\pi \le \theta_1, \theta_2, \dots, \theta_n < \pi$. The joint distribution of the eigenvalues (called the Weyl measure) is absolutely continuous with respect to the Lebegue measure $\prod_{j=1}^n d\theta_j$ on the n-dimensional tori, and its density is given by

$$P_{U(n)}(\theta_1, \dots, \theta_n) = \frac{1}{(2\pi)^n \cdot n!} \prod_{1\le j<k\le n} |\exp(i\cdot\theta_j) - \exp(i\cdot\theta_k)|^2 \tag{1.1}$$

(see [36]). Throughout the paper we will be interested in the global and local linear statistics

$$S_n(f) = \sum_{j=1}^n f(\theta_j), \tag{1.2}$$

$$S_n\big(g(L_n\cdot)\big) = \sum_{j=1}^n g(L_n\cdot\theta_j), \tag{1.3}$$

$$L_n \to \infty, \qquad \frac{L_n}{n} \to 0.$$

The optimal conditions on f, g for our purposes are

$$\sum_{k=-\infty}^{\infty} |\hat f(k)|^2 \cdot |k| < \infty, \tag{1.4}$$

$$\int_{-\infty}^{\infty} |\hat g(t)|^2 \cdot |t|\, dt < \infty, \tag{1.5}$$

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where

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(t) e^{itx}\, dt.$$

However, in order to simplify the exposition we will always assume that f has a continuous derivative on a unit circle [$f \in C^1(S^1)$] and g is a Schwartz function [$g \in J(\mathbb{R}^1)$].

Let us denote by E_n the mathematical expectation with respect to Haar measure. We start with the formulation of the result which is essentially due to C. Andréief [2]; for a modern day reference see [35] (and also [14]).

PROPOSITION.

$$E_n \exp\big(tS_n(f)\big) = \det\big(Id + (e^{tf} - 1)K_n\big) = \det\big(Id + (e^{tf} - 1)Q_n\big), \tag{1.6}$$

where $(e^{tf} - 1)$ *is a multiplication operator and* $K_n, Q_n\colon L^2(S^1) \to L^2(S^1)$ *are the integral operators with the kernels*

$$K_n(x, y) = \frac{1}{2\pi} \frac{\sin\big((n/2)(x-y)\big)}{\sin\big((x-y)/2\big)}, \tag{1.7}$$

$$Q_n(x, y) = \sum_{j=0}^{n-1} \frac{1}{\sqrt{2\pi}} e^{ijx} \frac{1}{\sqrt{2\pi}} e^{-ijy}. \tag{1.8}$$

REMARK 1. K_n, Q_n are unitary equivalent and are the operators of a finite rank. In particular, Q_n is just a projection operator on the first n harmonic functions of the unit circle.

One of the ingredients of the proof of the proposition is the following chain of equalities:

$$\begin{aligned} p_{U(n)}(\theta_1, \dots, \theta_n) &= \frac{1}{n!}(2\pi)^{-n} \det\big(\exp(i \cdot (j-1) \cdot \theta_k)\big)_{1 \le j, k \le n} \\ &\quad \cdot \det\big(\exp(-i \cdot (j-1) \cdot \theta_k)\big)_{1 \le j, k \le n} \\ &= \frac{1}{n!} \det\big(Q_n(\theta_j, \theta_k)\big)_{1 \le j, k \le n} \\ &= \frac{1}{n!} \det\big(K_n(\theta_j, \theta_k)\big)_{1 \le j, k \le n}. \end{aligned} \tag{1.9}$$

Remark 1 allows us to rewrite the Fredholm determinants in (1.6) as the Toeplitz determinant with the symbol $\exp(t \cdot f(\cdot))$:

$$\begin{aligned} E_n \exp\Big(t \sum_{j=1}^{n} f(\theta_j)\Big) &= D_{n-1}\big(\exp(t \cdot f)\big) \\ &= \det\Big(\frac{1}{2\pi} \int_0^{2\pi} \exp(tf(x)) \exp(i(j-k)x)\, dx\Big)_{1 \le j, k \le n}. \end{aligned} \tag{1.10}$$

The asymptotics of (1.10) for large n is given by the strong Szegö limit theorem,

$$D_{n-1}\big(\exp(t\cdot f)\big)=\exp\bigg(tn\hat{f}(0)+\frac{1}{2}t^2\sum_{-\infty}^{+\infty}|k||\hat{f}(k)|^2+\bar{o}(1)\bigg). \tag{1.11}$$

(See [34] and [22, 17, 11, 15, 16, 37, 38, 25, 5, 19, 7, 8, 26, 31, 39] for further developments.)

In probabilistic terms (1.11) claims that $ES_n(f)=(n/2\pi)\int_{-\pi}^{\pi}f(\theta)\,d\theta+\bar{o}(1)$ (actually the remainder term is zero), and the centralized random variable $\sum_{j=1}^n f(\theta_j)-E_n\sum_{j=1}^n f(\theta_j)$ converges in distribution to the normal law $N(0,\sum_{-\infty}^{\infty}|k||\hat{f}(k)|^2)$ (see [13, 19, 20, 21, 12].

Our first goal is to establish a similar result for the local linear statistics.

THEOREM 1. *Let $g\in J(\mathbb{R}^1)$, $L_n\to+\infty$, $L_n/n\to 0$. Then $E_n\sum_{j=1}^n g(L_n\cdot\theta_j)=(n/2\pi\cdot L_n)\int_{-\infty}^{\infty}g(x)\,dx$, and the centralized random variable $\sum_{j=1}^n(g(L_n\cdot\theta_j)-E\sum_{j=1}^n g(L_n\theta_j)$ converges in distribution to the normal law $N(0,(1/2\pi)\times\int_{-\infty}^{+\infty}|\hat{g}(t)|^2|t|\,dt)$.*

We give a combinatorial proof which holds both in the local and global cases. In some sense our approach is close to the heuristic arguments in [18]. We start with a lemma.

LEMMA 1. *Let $C_{l,n}(f)$ be the lth cumulant of $S_n(f)$. Then for $l=1$, $C_{l,n}(f)=\hat{f}(0)n$ and for $l>1$,*

$$\begin{aligned}\bigg|C_{l,n}(f)-&\sum_{k_1+\cdots+k_l=0}\hat{f}(k_1)\cdot\,\cdots\,\cdot\hat{f}(k_l)\sum_{m=1}^{l}\frac{(-1)^{m-1}}{m}\\ &\times\sum_{\substack{l_1+\cdots+l_m=l,\\ l_1\ge 1,\ldots,l_m\ge 1}}\frac{l!}{l_1!\cdot\,\cdots\,\cdot l_m!}\bigg(n-\max\bigg(0,\sum_{i=1}^{l_1}k_i,\sum_{i=1}^{l_1+l_2}k_i,\ldots,\sum_{i=1}^{l_1+\cdots+l_{m-1}}k_i\bigg)\\ &\qquad-\max\bigg(0,\sum_{i=1}^{l_1}(-k_i),\sum_{i=1}^{l_1+l_2}(-k_i),\ldots,\sum_{i-1}^{l_1+\cdots+l_{m-1}}(-k_i)\bigg)\bigg)\bigg|\\ &\le \mathrm{const}_l\sum_{\substack{k_1+\cdots+k_l=0\\ |k_1|+\cdots+|k_l|>n}}|k_1||\hat{f}(k_1)|\cdot\,\cdots\,\cdot|\hat{f}(k_l)|.\end{aligned} \tag{1.12}$$

REMARK 2. One can see that for sufficiently smooth f the r.h.s. of (1.12) goes to zero as $n\to\infty$.

REMARK 3. An analogous result to Lemma 1 was established in [33] for the determinantal random point field with the sine kernel (see also Remark 4 below).

The proof of Lemma 1 will be given in Section 2. At this moment we observe that it implies the following lemma.

LEMMA 2. *The limit of* $C_{l,n}(f)$, $l > 1$ *exists as* $n \to \infty$ *and is equal to* $\sum_{k_1+\cdots+k_l=0} \hat{f}(k_1)\cdot \cdots \cdot \hat{f}(k_l)\cdot (G(k_1,\dots,k_l)+G(-k_1,\dots,-k_l))$, *where* G *is the piecewise linear continuous function defined by*

$$G(k_1,\dots,k_l) := \sum_{\sigma\in S_l} \sum_{m=1}^{l} \frac{(-1)^m}{m} \sum_{\substack{l_1+\cdots+l_m=l,\\ l_1\geq 1,\dots,l_m\geq 1}} \frac{1}{l_1!\cdot \cdots \cdot l_m!} \times \max\left(0, \sum_{i=1}^{l_1} k_{\sigma(i)}, \sum_{i=1}^{l_1+l_2} k_{\sigma(i)}, \dots, \sum_{i=1}^{l_1+\cdots+l_{m-1}} k_{\sigma(i)}\right). \tag{1.13}$$

PROOF. After opening the brackets in (1.12) we observe that the coefficient in front of n is equal to

$$\sum_{m=1}^{l} \sum_{\substack{l_1+\cdots+l_m=l,\\ l_1\geq 1,\, i=1,\dots,m}} \frac{(-1)^{m-1}}{m} \frac{l!}{l_1!\cdots l_m!} = \begin{cases} 1, & l=1,\\ 0, & l>1. \end{cases} \tag{1.14}$$

Indeed, the generating function of these coefficients is equal to

$$\log\bigl(1+(e^z-1)\bigr) = z. \qquad \square$$

Now the CLT for $\sum_{j=1}^{n} f(\theta_j)$ follows from

MAIN COMBINATORIAL LEMMA. *Let* $k_1,\dots,k_l$ *be arbitrary real numbers such that their sum equals zero. Let* $G(k_1,\dots,k_l)$ *be defined as in* (1.13). *Then*

$$G(k_1,\dots,k_l) = \begin{cases} |k_1| = |k_2|, & \text{if } l=2,\\ 0, & \text{if } l>2. \end{cases}$$

We will prove the lemma in Section 3.

REMARK 4. A similar combinatorial lemma was stated by Spohn in [33]. He studied a time-dependent motion of a system of infinite number of particles governed by the equations

$$d\lambda_j(t) = \sum_{i\neq j} \frac{1}{\lambda_i - \lambda_j} dt + db_j(t),$$

where $\{b_j(t)\}_{j=-\infty}^{+\infty}$-independent standard Brownian motions, and the initial distribution of particles is given by determinantal random point field with the sine kernel $\sin \pi(x-y)/\pi(x-y)$. However, no correct proof of the combinatorial result was given there. For completeness we give a proof of Spohn's lemma independently from the proof of our main combinatorial lemma in Section 3.

Assuming the combinatorial part is done, we can quickly finish the proof of Theorem 1. The formula for the mathematical expectation is trivial. Rewriting

(1.12) for the higher cumulants of $\sum_{j=1}^{n} g(L_n \cdot \theta_j)$ we see that the limit of the lth cumulant is given by

$$(2\pi)^{-1/2} \int \hat{g}(t_1) \cdot \cdots \cdot \hat{g}(t_l) \big(G(t_1, \dots, t_l) + G(-t_1, \dots, -t_l)\big)\, dt_1 \cdots dt_l,$$

where the integral is over the hyperplane $t_1 + \cdots + t_l = 0$.

Theorem 1 is proved. □

REMARK 5. Our method also gives an elementary combinatorial proof of the Szegö theorem (1.11) for $f \in C^1(S^1)$ and sufficiently small complex t. It is different from the one suggested by Kac [22] where the Taylor expansion of $D_n(1 - tg)$ as a function of t was calculated and then a so-called Kac–Spitzer combinatorial lemma was employed to confirm (1.11).

REMARK 6. Results similar to Theorem 1 have been established for other random matrix models in [33, 21, 34, 4, 3, 28, 29, 9].

The rest of this paper is organized as follows. We prove Lemma 1 in Section 2 and the main combinatorial lemma in Section 3. The results analogous to Theorem 1 for orthogonal and symplectic groups are established in Section 4.

2. Proof of Lemma 1. We start with calculating the moments of $S_n(f)$. Recall that the k-point correlation function of the eigenvalues of a random unitary matrix is given by

$$\begin{aligned}\rho_{n,k}(\theta_1, \dots, \theta_k) &= \frac{n!}{(n-k)!} \int_{T^{n-k}} p_{U(n)}(\theta_1, \dots, \theta_n) d\theta_{k+1} \cdots d\theta_n \\ &= \det\big(K_n(\theta_i, \theta_j)\big)_{1 \le i, j \le k} = \det\big(Q_n(\theta_i, \theta_j)\big)_{1 \le i, j \le k}.\end{aligned} \tag{2.1}$$

The Nth moment of $S_n(f)$ is equal to

$$E_n\left(\sum_{i_1=1}^{n} f(\theta_{i_1}) \cdot \cdots \cdot \sum_{i_N=1}^{n} f(\theta_{i_N}) \right),$$

where the indices $i_1, \dots, i_N$ range independently from 1 to n, and in particular can coincide. We need a definition.

DEFINITION 1. A partition of a set B is an unordered collection of nonempty disjoint subsets $\mathscr{M} = \{M_1, \dots, M_r\}$, $r = 1, 2, \dots$, of B such that the union of the elements of the partition is the whole B.

Let $\mathscr{M} = \{M_1, \dots, M_r\}$ be a partition of the set $\{1, 2, \dots, N\}$ into subsets determined by coinciding indices among $i_1, \dots, i_N$: $M_1 = \{j_1^{(1)}, \dots, j_{s_1}^{(1)}\}, \dots,$ $M_r = \{j_1^{(r)}, \dots, j_{s_r}^{(r)}\}$, $\bigcup_{i=1}^{r} M_i = \{1, 2, \dots, N\}$, $s_i = |M_i|$, $i = 1, \dots, r$. Then

$$E_n\big(S_n(f)\big)^N = \sum_{\substack{\text{over all} \\ \text{partitions } \mathscr{M}}} E_n \sum_{l_1 \ne l_2 \ne \cdots \ne l_r} f^{s_1}(\theta_{l_1}) \cdot \cdots \cdot f^{s_r}(\theta_{l_r}). \tag{2.2}$$

Let us consider a typical term in (2.2) corresponding to a partition $\mathscr{M}$:

$$
\begin{aligned}
&E_n \sum_{l_1 \neq \cdots \neq l_r} f^{s_1}(\theta_{l_1}) \cdot \cdots \cdot f^{s_r}(\theta_{l_r}) \\
&\quad = \int_{T^r} f^{s_1}(x_1) \cdot \cdots \cdot f^{s_r}(x_r) \cdot \rho_{n,r}(x_1, \cdots, x_r)\, dx_1 \cdots dx_r.
\end{aligned}
\tag{2.3}
$$

By definition of the determinant and (2.1),

$$
\rho_{n,r}(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (-1)^{\sigma} \prod_{i=1}^{r} Q_n(x_i, x_{\sigma(i)}).
$$

Writing the permutation $\sigma \in S_r$ as a product of cyclic permutations we have

$$
\begin{aligned}
&\rho_{n,r}(x_1, \ldots, x_r) \\
&\quad = \sum_{\substack{\text{over partitions} \\ \mathscr{K} \text{ of } \{1,\ldots,r\}}} \left(\prod_{\alpha=1}^{q} \Bigl((-1)^{p_\alpha - 1} \sum_{\substack{\text{over all cyclic} \\ \text{permutations of } K_\alpha}} \prod_{j=1}^{p_\alpha} Q_n\bigl(x_{t_j^{(\alpha)}}, x_{\sigma(t_j^{(\alpha)})}\bigr) \Bigr) \right),
\end{aligned}
\tag{2.4}
$$

where $\{1, \ldots, r\} = \bigcup_1^q K_\alpha$, $K_\alpha = \{t_1^{(\alpha)}, \ldots, t_{p_\alpha}^{(\alpha)}\}$, $\alpha = 1, \ldots, q$, $p_\alpha = |K_\alpha|$.

Substituting (2.4) into (2.3) we arrive at the expression that has the following form:

$$
\sum_{\substack{\text{over partitions} \\ \mathscr{M} = \{M_1, \ldots, M_r\} \text{ of } \{1, \ldots, N\}}} \quad \sum_{\substack{\text{over partitions} \\ \mathscr{K} = \{K_1, \ldots, K_q\} \text{ of } \{1, \ldots, r\}}} \cdots.
$$

To interchange the order of summation we construct a new partition $\mathscr{P} = \{P_1, \ldots, P_q\}$ of $\{1, 2, \ldots, N\}$ as follows: $P_i = \bigcup_{j \in K_i} M_j$, $i = 1, \ldots, q$. Then $\{M_j\}_{j \in K_i}$ gives a partition of P_i that we denote by $\mathscr{P}_i$. We have

$$
\begin{aligned}
&E_n(S_n(f))^N \\
&\quad = \sum_{\substack{\text{over partitions} \\ \mathscr{P} = \{P_1, \ldots, P_q\} \text{ of } \{1, \ldots, N\}}} \\
&\quad\quad \times \left(\prod_{i=1}^{q} \left(\sum_{\substack{\text{over partitions} \\ \mathscr{P}_i \text{ of } P_i : \mathscr{P}_i = \{P_{i,1}, \ldots, P_{i,t_i}\}}} \int_{T^{t_i}} f^{|P_{i,1}|}(x_1) \cdot \cdots \cdot f^{|P_{i,t_i}|}(x_{t_i}) (-1)^{t_i - 1} \right.\right. \\
&\quad\quad\quad \left.\left. \times \sum_{\substack{\text{over cyclic} \\ \text{permutations } \sigma \in S_{t_i}}} \prod_{j=1}^{t_i} Q_n(x_j, x_{\sigma(j)})\, dx_1 \cdots dx_{t_i} \right) \right).
\end{aligned}
\tag{2.5}
$$

We recall that the moments are expressed in terms of cumulants as

$$
m_N = \sum_{\substack{\text{over partitions} \\ \mathscr{P} = \{P_1, \ldots, P_k\}}} C_{|P_1|} \cdot \cdots \cdot C_{|P_k|}.
$$

Comparing the last formula with (2.5) we arrive at

$$
\begin{aligned}
C_{l,n}(f) = {} & \sum_{\substack{\text{partitions} \\ \mathscr{P}=\{R_1,\dots,R_m\} \text{ of } \{1,\dots,l\}}} \int_{T^m} f^{|R_1|}(x_1)\cdot \cdots \cdot f^{|R_m|}(x_m) \\
& \times (-1)^{m-1} \sum_{\substack{\text{cyclic permutations} \\ \sigma\in S_n}} \prod_{j=1}^{m} Q_n(x_j, x_{\sigma(j)})\, dx_1\cdots dx_m \qquad (2.6)\\
= {} & \sum_{m=1}^{l} \sum_{\substack{\text{over ordered collections} \\ (l_1,\dots,l_m):\ \sum_1^m l_i = l,\ l_i\ge 1}} (-1)^{m-1} \frac{l!}{l_1!\cdots l_m!}\frac{1}{m!}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& \int_{T^m} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m)\cdot m!\frac{1}{m}\cdot \prod_{j=1}^{m} Q_n(x_j, x_{j+1})\, dx_1\cdots dx_m \\
& = \sum_{m=1}^{l} \sum_{\substack{(l_1,\dots,l_m):\ l_1+\cdots+l_m = l, \\ l_i\ge 1,\ i=1,\dots,m}} \qquad (2.7)\\
& \quad \times \frac{(-1)^{m-1}}{m} \frac{l!}{l_1!\cdot \cdots \cdot l_m!} \int_{T^m} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m) \\
& \quad \times \prod_{j=1}^{m} Q_n(x_j, x_{j+1})\, dx_1\cdots dx_m.
\end{aligned}
$$

Since $Q_n(x, y) = \sum_{j=0}^{n-1} \exp(-ij(x-y))$ we can rewrite (2.6) as

$$
\begin{aligned}
C_{l,n}(f) = {} & \sum_{m=1}^{l} \sum_{\substack{(l_1,\dots,l_m): \\ l_1+\cdots+l_m = l,\ l_i\ge 1}} \frac{(-1)^{m-1}}{m} \frac{l!}{l_1!\cdots l_m!} \\
& \times \sum_{s_1=0}^{n-1} \cdots \sum_{s_m=0}^{n-1} \widehat{f^{l_1}}(-s_m+s_1)\cdot \widehat{f^{l_2}}(-s_1+s_2)\cdot \cdots \cdot \widehat{f^{l_m}}(-s_{m-1}+s_m).
\end{aligned}
$$

Writing down the Fourier coefficients of the powers of f as the convolutions of the Fourier coefficients of f,

$$
\begin{aligned}
\widehat{f^{l_1}}(-s_m+s_1) &= \sum_{\substack{(k_1,\dots,k_{l_1}): \\ k_1+\cdots+k_{l_1} = s_1 - s_m}} \hat{f}(k_1)\cdot \cdots \cdot \hat{f}(k_{l_1}), \\
\widehat{f^{l_2}}(-s_1+s_2) &= \sum_{\substack{(k_{l_1+1},\dots,k_{l_2}): \\ k_{l_1+1}+\cdots+k_{l_2} = s_2 - s_1}} \hat{f}(k_{l_1+1})\cdot \cdots \cdot \hat{f}(k_{l_2}), \dots,
\end{aligned}
$$

$$\widehat{f^{l_m}}(-s_{m-1}+s_m)=\sum_{\substack{(k_{l_{m-1}+1},\dots,k_{l_m}):\\ k_{l_{m-1}+1}+\cdots+k_{l_m}=s_m-s_{m-1}}} \hat{f}(k_{l_{m-1}+1})\cdot\dots\cdot\hat{f}(k_{l_m}),$$

we obtain

$$\begin{aligned} C_{l,n}(f)=&\sum_{k_1+\cdots+k_l=0}\hat{f}(k_1)\cdot\dots\cdot\hat{f}(k_l)\sum_{m=1}^{l}\frac{(-1)^{m-1}}{m}\sum_{\substack{(l_1,\dots,l_m):\\ l_1+\cdots+l_m=l,\, l_i\ge 1}} \\ &\times\frac{l!}{l_1!\cdots l_m!}\cdot\#\Big\{u\colon 0\le u\le n-1,\ 0\le u+\sum_1^{l_1}k_i\le n-1,\dots, \\ &\qquad 0\le u+\sum_1^{l_1+\cdots+l_{m-1}}k_i\le n-1\Big\}. \end{aligned}\tag{2.8}$$

The last factor in (2.8) is equal to

$$\begin{aligned} &n-\max\Big(0,\sum_1^{l_1}k_i,\dots,\sum_1^{l_1+\cdots+l_{m-1}}k_i\Big) \\ &\quad-\max\Big(0,\sum_1^{l_1}(-k_i),\dots,\sum_1^{l_1+\cdots+l_{m-1}}(-k_i)\Big) \end{aligned}\tag{2.9}$$

if the expression in (2.9) is nonnegative, or zero otherwise.

Lemma 1 is proved. □

3. Proof of the main combinatorial lemma. First we show that $G(k_1,\dots,k_l)$ is a linear combination of terms $|k_{i_1}+\cdots+k_{i_s}|$. Then we compute the coefficient in front of every such term and show it to be equal to zero.

Assume $l>2$. Consider a partition $\mathscr{P}=\{P_1,\dots,P_m\}$ of the set $\{1,2,\dots,l\}$. Let us denote $v_1=\sum_{j\in P_1}k_j,\dots,v_m=\sum_{j\in P_m}k_j$. The expression for G can be transformed into

$$\begin{aligned} G(k_1,\dots,k_l)=&\sum_{m=1}^{l}\sum_{\mathscr{P}=\{P_1,\dots,P_m\}}\frac{(-1)^m}{m}\sum_{\tau\in S_m} \\ &\times\max\big(0,v_{\tau(1)},v_{\tau(1)}+v_{\tau(2)},\dots,v_{\tau(1)} \\ &\qquad+v_{\tau(2)}+\cdots+v_{\tau(m-1)}\big). \end{aligned}\tag{3.1}$$

In [27] Rudnick and Sarnak, following the ideas of [22] and [32] (see also [6, 1]), used the following identity for the set of real numbers $v_1,\dots,v_m$ with zero sum:

$$\begin{aligned}(3.2)\quad &\frac{1}{m}\sum_{\tau\in S_m}\max\bigl(0, v_{\tau(1)}, v_{\tau(1)}+v_{\tau(2)},\dots, v_{\tau(1)}+v_{\tau(2)}+\cdots v_{\tau(m-1)}\bigr)\\ &\quad=\frac{1}{4}\sum_{\substack{F\subset\{1,\dots,m\},\\ F,\,F^C\neq\varnothing}}(|F|-1)!(m-|F|-1)!\biggl|\sum_{l\in F}v_l\biggr|.\end{aligned}$$

The last formula gives us

$$\begin{aligned}(3.3)\quad G(k_1,\dots,k_l)&=\frac{1}{4}\sum_{m=1}^{l}\sum_{\mathscr{P}=\{P_1,\dots,P_m\}}\sum_{\substack{F\subset\{1,\dots,m\},\\ F,\,F^C\neq\varnothing}}(-1)^{|F|-1}\\ &\quad\times(|F|-1)!\biggl|\sum_{i\in\bigcup_{j\in F}P_j}k_i\biggr|\cdot(-1)^{(m-|F|-1)}\cdot(m-|F|-1)!.\end{aligned}$$

Let us denote by A the subset $\bigcup_{j\in F}P_j$ of $\{1,2,\dots,l\}$. Then $\{P_j\}_{j\in F}$ defines a partition of A, and $\{P_j\}_{j\in F^C}$ a partition of $A^C=\{1,2,\dots,l\}\setminus A$.

We change now the order of summation in (3.3): first we sum over all nonempty subsets A of $\{1,2,\dots,l\}$ and then over all partitions of A and A^C:

$$\begin{aligned}(3.4)\quad G(k_1,\dots,k_l)&=\frac{1}{4}\sum_{\substack{A\subset\{1,\dots,l\},\\ A,\,A^C\neq\varnothing}}\Biggl(\sum_{\substack{\text{over partitions}\\ \mathscr{U}=\{U_1,\dots,U_r\}\text{ of }A}}(-1)^{|\mathscr{U}|-1}(|\mathscr{U}|-1)!\Biggr)\\ &\quad\times\Biggl(\sum_{\substack{\text{over partitions}\\ \mathscr{U}'\text{ of }A^C}}(-1)^{|\mathscr{U}'|-1}\cdot(|\mathscr{U}'|-1)!\Biggr)\biggl|\sum_{i\in A}k_i\biggr|.\end{aligned}$$

Finally we note that

$$\begin{aligned}&\sum_{\mathscr{U}=\{U_1,\dots,U_r\}}(-1)^{|\mathscr{U}|-1}\cdot(|\mathscr{U}|-1)!\\ &\quad=\sum_{r=1}^{|A|}\sum_{\substack{(t_1,\dots,t_r):\\ \sum_{i=1}^r t_i=|A|,\,t_i\geq 1}}(-1)^{r-1}(r-1)!\frac{|A|!}{t_1!\cdot\cdots\cdot t_r!}\frac{1}{r!},\end{aligned}$$

the expression we already considered in (1.14). Indeed, there are exactly $(|A|!/t_1!\cdot\cdots\cdot t_r!)(1/r!)$ different partitions of A such that $\{t_1,\dots,t_r\}=\{|U_1|,\dots,|U_r|\}$. By (1.14) if $|A|\geq 2$ this sum is zero. If $|A|=1$, then $|A^C|=l-|A|\geq 2$ and the second factor in (3.4) equals zero by the same argument. □

Now we turn to a combinatorial lemma first formulated in [33]. Let us denote by $\alpha=(\alpha_1,\dots,\alpha_l)$, $\beta=(\beta_1,\dots,\beta_l)$ vectors with entries $\alpha_j\in\{0,1\}$. We consider a lexicographic order on the set of such vectors: $\alpha<\beta$ iff $\alpha_j\leq\beta_j$, $j=1,\dots,l$ and at least for one j_0 $\alpha_{j_0}<\beta_{j_0}$. Following [33] we call such nonzero vectors branches and a set T of ordered branches $T=\{\alpha^{(1)},\dots,\alpha^{(m)}\}$,

$\alpha^{(1)} < \alpha^{(2)} < \cdots < \alpha^{(m)}$, $|T| = m \leq l$, a tree. We denote by $T(l)$ the set of all trees formed by l-dimensional vectors (branches). A combinatorial sum in question is

$$U(k_1, \ldots, k_l) = \sum_{T \in T(l)} (-1)^{|T|-1} \cdot \max(0, \alpha \cdot k | \alpha \in T). \tag{3.5}$$

Here we used the notation $\alpha \cdot k = \sum_{j=1}^{l} \alpha_j \cdot k_j$. We call $\max(0, \alpha \cdot k | \alpha \in T)$ the maximum of the tree T. For a warm-up we prove the following.

PROPOSITION 1.

$$U(k_1, \ldots, k_l) + U(-k_1, \ldots, -k_l) = G(k_1, \ldots, k_l, k_{l+1}) + G(-k_1, \ldots, -k_l, -k_{l+1}),$$

where $k_{l+1} = -k_1 - k_2 - \cdots - k_l$.

REMARK 7. Once the proposition is proved we see of course that $U(k_1, \ldots, k_l) + U(-k_1, \ldots, -k_l)$ is zero for $l \geq 2$.

PROOF. In the above notations,

$$G(k_1, \ldots, k_l, k_{l+1}) = \sum_{T \in T(l+1)}{}' \frac{(-1)^{|T|-1}}{|T|} \cdot \max(0, \alpha \cdot k' | \alpha \in T),$$

where $k' = (k_1, \ldots, k_l, k_{l+1})$, and the sum $\sum'$ is over all trees $T \in T(l+1)$ such that the largest branch of T, $\alpha^{(|T|)}$ is less than $D = (1, 1, \ldots, 1)$. Similarly, we can write $U(k_1, \ldots, k_l) = \sum''_{T \in T(l+1)} (-1)^{|T|-1} \cdot \max(0, \alpha \cdot k' | \alpha \in T)$, where the sum $\sum''$ is over the trees $T \in T(l+1)$ such that the $(l+1)$th coordinate of $\alpha^{(|T|)}$ is zero. We define a "rotation" on the set of all trees such that $\alpha^{(|T|)} \neq D$: $W((\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(|T|)})) = (\alpha^{(2)} - \alpha^{(1)}, \alpha^{(3)} - \alpha^{(1)}, \ldots, \alpha^{(|T|)} - \alpha^{(1)}, D - \alpha^{(1)})$. It is easy to see that W rotates the spacings $(\alpha^{(1)}, \alpha^{(2)} - \alpha^{(1)}, \ldots, \alpha^{(|T|)} - \alpha^{(|T|-1)}, D - \alpha^{(|T|)})$ of T. Since $\sum_{j=1}^{l+1} k_j = 0$, we observe that

$$\begin{aligned} &\max(0, \alpha \cdot k' | \alpha \in T) + \max(0, \alpha \cdot (-k') | \alpha \in T) \\ &\quad = \max\bigl(0, \alpha \cdot k' | \alpha \in W(T)\bigr) + \max\bigl(0, \alpha \cdot (-k') | \alpha \in W(T)\bigr). \end{aligned} \tag{3.6}$$

The last equality implies

$$\begin{aligned} &U(k_1, \ldots, k_l) + U(-k_1, \ldots, -k_l) \\ &\quad = \sum_{T \in T(l+1)}{}'' (-1)^{|T|-1} \cdot (\max(0, \alpha \cdot k' | \alpha \in T) + \max(0, \alpha \cdot (-k') | \alpha \in T)) \\ &\quad = \sum_{T \in T(l+1)}{}'' (-1)^{|T|-1} \frac{1}{|T|} \sum_{p=0}^{|T|-1} \bigl(\max(0, \alpha \cdot k' | \alpha \in W^p(T)) \\ &\qquad + \max(0, \alpha \cdot (-k') | \alpha \in W^p(T))\bigr) \end{aligned}$$

$$= \sum_{\substack{T\in T(l+1)\\ \alpha^{(|T|)}\neq D}} \frac{(-1)^{|T|-1}}{|T|}\cdot\big(\max(0,\alpha\cdot k'|\alpha\in T)+(\alpha\cdot(-k')|\alpha\in T)\big)$$

$$= G(k_1,\dots,k_{l+1})+G(-k_1,\dots,-k_{l+1}).$$

Here we used that for any T' with $\alpha^{(|T'|)}\neq D$ there exists a unique T with $\alpha_{l+1}^{(|T|)}=0$ and $0\le p<|T|$ such that $T'=W^p(T)$. □

PROPOSITION 2.

$$U(k_1,\dots,k_l)=0 \quad \text{if } l\ge 2. \tag{3.7}$$

We proceed by induction.

It is easy to check the case $l=2$. Let us assume that the proposition is true for some $l\ge 2$. Consider $U(k_0,k_1,\dots,k_l)$. Since U is a symmetric function we may assume $k_0\le k_1\le\cdots k_l$. The continuity of U implies that it is enough to check (3.7) for nondegenerate vectors $(k_0,k_1,\dots,k_l)$. Therefore we may assume that the coordinates $k_1,\dots,k_l$ are linearly independent over the integers. Fix such $k_1,\dots,k_l$ and consider U as a piecewise linear function of $y=k_0$, $U(y,k)=U(y,k_1,k_2,\dots,k_l)$. Our first claim is that $U(y,k)$ is zero for all negative y. To show this we write

$$U(y,k)=\sum_{T\in T(l+1)}(-1)^{|T|-1}\max(0,\alpha\cdot(y,k)|\alpha\in T)$$

$$=\sum_{\substack{T\in T(l+1):\\ \alpha_1^{(|T|)}=0}}+\sum_{\substack{T\in T(l+1):\\ \alpha_1^{(|T|)}=1,\,\alpha_1^{(|T|)-1}=0}}+\sum_{\substack{T\in T(l+1):\\ \alpha_1^{(|T|)}=1,\,\alpha_1^{(|T|-1)}=1}}.$$

We denote the three subsums by U_1,U_2,U_3. The first subsum is equal to

$$\sum_{T\in T(l)}(-1)^{|T|-1}\cdot\max(0,\alpha\cdot k|\alpha\in T),$$

the second

$$\sum_{T\in T(l)}(-1)^{|T|}\cdot\max(0,\alpha\cdot k|\alpha\in T)+\max(0,y),$$

and by the induction assumptions both are zero. Now we split the third subsum in two. Consider the smallest branch $\alpha\in T$ such that the first coordinate of α is 1, denote this branch by α' and denote the preceding (may be empty) branch by α''. We write $U_3=U_{3,1}+U_{3,2}$, where in $U_{3,1}$ the summation is over $T\in T(l+1)$, such that $\alpha_1^{(|T|)}=1$, $\alpha_1^{(|T|-1)}=1$ and $\alpha'-\alpha''>(1,0,\dots,0)$, and in $U_{3,2}$ the summation is over all other trees from U_3. We establish a one-to-one correspondence between the terms of $U_{3,1}$ and $U_{3,2}$: for any tree T_1 with $\alpha'-\alpha''>(1,0,\dots,0)$ we construct $T_2=\{\alpha^{(1)},\dots,\alpha'',\alpha''+(1,0,\dots,0),\alpha',\dots,\alpha^{|T|}\}$. Clearly, $|T_2|=|T_1|+1$, therefore

$$(-1)^{|T_1|-1}\cdot\max(0,\alpha\cdot(y,k)|\alpha\in T_1)=-(-1)^{|T_2|-1}\cdot\max(0,\alpha\cdot(y,k)|\alpha\in T_2),$$

and $U_{3,1}$ and $U_{3,2}$ cancel each other.

Now we assume that y is nonnegative and $0 \le y \le k_1 < k_2 < \cdots < k_l$. As we already noted $U(y, k_1, \ldots, k_l)$ is a piecewise linear continuous function. We claim that it can change its slope only at $y = 0$. Indeed, $U(y, k_1, \ldots, k_\ell)$ can change its slope only at the points of degeneracy of $(y, k_1, \ldots, k_l)$, where $\alpha_0 \cdot y + \alpha \cdot k = \alpha'_0 \cdot y + \alpha'_0 \cdot k$ and the coordinates of (α_0, α), (α'_0, α') take values zero and one. Because k is a nondegenerate vector we must have $y + \alpha \cdot k = \alpha' \cdot k$ (or $\alpha \cdot k = y + \alpha' k$). Since the tree T contains both branches $(1, \alpha)$ and $(0, \alpha')$ only if $\alpha' \le \alpha$, the only solution for nonnegative vector (y, k) must be $y = 0$, $\alpha' = \alpha$. We will finish the proof of the proposition if we show that $U(y, k) = 0$ for sufficiently small positive y. We again write $U = U_1 + U_2 + U_3$ as before. Then $U_1 = 0$ by inductive assumption and U_3 is zero for sufficiently small y ($U_{3,1}$ and $U_{3,2}$ still cancel each other). We can write the second subsum U_2 as

$$\begin{aligned} &\sum_{T \in T(l)} (-1)^{|T|} \cdot \big(\max(0, \alpha \cdot k | \alpha \in T) + y\big) \\ &\quad = \sum_{T \in T(l)} (-1)^{|T|} \cdot \big(\max(0, \alpha \cdot k | \alpha \in T)\big) + y \sum_{T \in T(l)} (-1)^{|T|} \end{aligned} \tag{3.8}$$

(the last sum includes empty tree). The first term in (3.8) is zero by inductive assumption and the second is also zero since

$$\sum_{T \in T(l)} (-1)^{|T|} = \sum_{\substack{l_1 + \cdots + l_m = l+1, \\ l_i \ge 1}} \frac{(-1)^{m-1}}{m} \frac{(l+1)!}{l_1! \cdot \cdots \cdot l_m!} = 0.$$

Proposition 2 is proved. □

4. Orthogonal and symplectic groups. We start with the orthogonal case. The eigenvalues of matrix $M \in SO(2n)$ can be arranged in pairs:

$$\exp(i\theta_1), \exp(-i\theta_1), \ldots, \exp(i\theta_n), \exp(-i\theta_n), \qquad 0 \le \theta_1, \theta_2, \ldots, \theta_n < \pi.$$

Consider the normalized Haar measure on $SO(2n)$. The probability distribution of the eigenvalues is defined by its density (see [36]):

$$P_{SO(2n)}(\theta_1, \ldots, \theta_n) = 2\left(\frac{1}{2\pi}\right) \prod_{1 \le i < j \le n} (2\cos\theta_i - 2\cos\theta_j)^2. \tag{4.1}$$

The k-point correlation functions are given by (see [23, 30])

$$\rho_{n,k}(\theta_1, \ldots, \theta_k) = \det\big(K^+_{2n-1}(\theta_i, \theta_j)\big)_{1 \le i, j \le n}, \tag{4.2}$$

where

$$\begin{aligned} K^+_{2n-1}(x, y) &= K_{2n-1}(x, y) + K_{2n-1}(x, -y) \\ &= \frac{1}{2\pi}\left(\frac{\sin\big((2n-1)(x-y)/2\big)}{\sin\big((x-y)/2\big)} + \frac{\sin\big((2n-1)(x+y)/2\big)}{\sin\big((x+y)/2\big)}\right). \end{aligned} \tag{4.3}$$

In [13] and [20] Diaconis and Shahshahani and Johansson studied asymptotic properties of linear statistics $\sum_{j=1}^{n} f(\theta_j)$ where for simplicity we may assume that f is a real even trigonometric polynomial, $f(\theta) = \sum_{k=1}^{m} a_k(e^{ik\theta} + e^{-ik\theta})$, $a_k = \hat{f}(k)$, $k = 1, 2, \dots, m$. As before we denote the linear statistics by $S_n(f)$. Then $S_n(f) = \mathrm{Trace}(\sum_{k=1}^{m} a_k M^k)$. It was shown that

$$
\begin{aligned}
&E_{2n} \exp\left(t \sum_{j=1}^{n} f(\theta_j)\right) \\
&\qquad = \exp\left(t \frac{1}{2} \sum_{k=1}^{m} (1 + (-1)^k) \hat{f}(k) + \frac{t^2}{2} \sum_{k=1}^{m} k \hat{f}(k)^2 + \bar{o}(1)\right),
\end{aligned}
\tag{4.4}
$$

which implies the convergence in distribution of $\sum_{j=1}^{n} f(\theta_j)$ to the normal law

$$
N\left(\frac{1}{2} \sum_{k=1}^{m} (1 + (-1)^k) \hat{f}(k), \sum_{k=1}^{m} k \cdot \hat{f}(k)^2\right).
$$

[Actually (4.4) holds under much weaker conditions; it is enough to assume $f \in C^{1+\alpha}([0, \pi])$, $\alpha > 0$.]

REMARK 8. Similarly to the unitary case (4.4) is equivalent to the large n asymptotics result for some determinants, this time Hankel determinants (see [19, 20]).

Our combinatorial approach allows proving the CLT for all $f \in C^1([0, \pi])$ as well as studying the local linear statistics $\sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta))$, $0 < \theta < \pi$. In particular we establish the following.

THEOREM 2. *Let g be a smooth function with a compact support, $L_n \to +\infty$, $L_n/n \to 0$ and $0 < \theta < \pi$. Then $E_{2n} \sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta)) = n/L_n \cdot \pi$, $\int_{-\infty}^{\infty} g(x)\,dx + \bar{o}(1)$ and the centralized random variable $\sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta)) - E_{2n} \sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta))$ converges in distribution to the normal law $N(0, (1/2\pi) \int_{-\infty}^{\infty} |\hat{g}(t)|^2 |t|\,dt)$.*

REMARK 9. To modify the results of the Theorem 2 for the case $\theta = 0, \pi$, one has to consider instead the Fourier transform of the even part of g, $1/2g(x) + 1/2g(-x)$ and replace the integration over R^1 by the integration over the semiaxis.

Theorem 2 also holds (with obvious modifications) for $SO(2n+1)$ and $\mathrm{Sp}(n)$.

Let $M \in SO(2n+1)$. Then one of the eigenvalues of M is 1 and the other $2n$ eigenvalues can be arranged in pairs as before. The density of the eigenvalues is equal to

$$
\begin{aligned}
&P_{SO(2n+1)}(\theta_1, \dots, \theta_n) \\
&\qquad = \left(\frac{2}{\pi}\right)^2 \prod_{1 \le i < j \le n} (2\cos\theta_i - 2\cos\theta_j)^2 \prod_{i=1}^{n} \sin^2\left(\frac{\theta_i}{2}\right).
\end{aligned}
\tag{4.5}
$$

The formula for the k-point correlation function is

$$\rho_{n,k}(\theta_1, \ldots, \theta_k) = \det\big(K^-_{2n}(\theta_i, \theta_j)\big)_{i,\,j=1,\ldots,k}, \tag{4.6}$$

where

$$\begin{aligned} K^-_{2n}(x, y) &= K_{2n}(x, y) - K_{2n}(x, -y) \\ &= \frac{1}{2\pi}\left(\frac{\sin\big(n(x-y)\big)}{\sin\big((x-y)/2\big)} - \frac{\sin\big(n(x+y)\big)}{\sin\big((x+y)/2\big)}\right). \end{aligned} \tag{4.7}$$

The analogue of (4.4) reads

$$\begin{aligned} &E_{2n+1}\left(\exp\left(t\sum_{j=1}^{n} f(\theta_j)\right)\right) \\ &\quad = \exp\left(t\frac{1}{2}\sum_{k=1}^{m}\big(-1+(-1)^k\big)\hat{f}(k) + \frac{t^2}{2}\sum_{k=1}^{m} k\hat{f}(k)^2 + \bar{o}(1)\right). \end{aligned} \tag{4.8}$$

In the symplectic case $M \in \mathrm{Sp}(n)$ the $2n$ eigenvalues again can be arranged in pairs:

$$\exp(i\cdot\theta_i), \exp(-i\cdot\theta_1), \ldots, \exp(i\cdot\theta_n), \exp(-i\cdot\theta_n), \qquad 0 \le \theta_1, \theta_2, \ldots, \theta_n < \pi;$$

their density is equal to

$$\begin{aligned} &P_{\mathrm{Sp}(n)}(\theta_1, \ldots, \theta_n) \\ &\quad = \left(\frac{2}{\pi}\right)^n \prod_{1\le i<j\le n}(2\cos\theta_i - 2\cos\theta_j)^2 \prod_{i=1}^{n}\sin^2(\theta_i) \end{aligned} \tag{4.9}$$

and the formula for k-point correlation function is

$$\rho_{n,k}(\theta_1, \ldots, \theta_k) = \det\big(K^-_{2n+1}(\theta_i, \theta_j)\big)_{i,\,j=1,\ldots,k}. \tag{4.10}$$

The analogue of (4.4) reads

$$\begin{aligned} &E_{n}\left(\exp\left(t\sum_{j=1}^{n} f(\theta_j)\right)\right) \\ &\quad = \exp\left(-t\frac{1}{2}\sum_{k=1}^{m}\big(1+(-1)^k\big)\hat{f}(k) + \frac{t^2}{2}\sum_{k=1}^{m} k\hat{f}(k)^2 + \bar{o}(1)\right). \end{aligned} \tag{4.11}$$

We will prove Theorem 2 for $SO(2n)$. The proofs for $SO(2n+1)$ and $\mathrm{Sp}(n)$ are almost identical.

Proof of Theorem 2. The arguments from Section 1 imply that it is enough to prove the following.

LEMMA 3. *Let $C_{l,n}(f)$ be the lth cumulant of $\sum_{j=1}^{n} f(\theta_j)$, $l \geq 2$. Then*

$$\begin{aligned}
&\Bigg| C_{l,n}(f) - \sum_{k_1+\cdots+k_l=0} \hat{f}(k_1)\cdot \cdots \cdot \hat{f}(k_l) \\
&\qquad + \frac{1}{2}\Big(G(k_1,\ldots,k_l) + G(-k_1,\ldots,-k_l)\Big)\Bigg| \\
&\qquad \leq \mathrm{const}_l \sum_{\substack{k_1+\cdots+k_l=0\\ |k_1|+\cdots+|k_l|>n}} |k_1||\hat{f}(k_1)|\cdot \cdots \cdot |\hat{f}(k_l)| \\
&\qquad\quad + \mathrm{const}'_l \sum_{|k_1|+\cdots+|k_l|>n} |\hat{f}(k_1)|\cdot \cdots \cdot |\hat{f}(k_l)|.
\end{aligned} \tag{4.12}$$

We start with the formula (2.6) which holds for general determinantal random point fields:

$$\begin{aligned}
C_{l,n}(f) = \sum_{m=1}^{l} \sum_{\substack{l_1+\cdots+l_m=l,\\ l_i\geq 1}} (-1)^m \frac{l!}{l_1!\cdot \cdots \cdot l_m!} \frac{1}{m} \int_{[0,\pi]^m} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m) \\
\times \prod_{j=1}^{m} \big(K_{2n-1}(x_j, x_{j+1}) + K_{2n-1}(x_j, -x_{j+1})\big) dx_1 \cdots dx_m
\end{aligned}$$

(we always assume $x_{m+1} = x_1$)

$$\begin{aligned}
&= \sum_{m=1}^{l} \sum_{\substack{l_1+\cdots+l_m=l\\ l_i\geq 1}} (-1)^m \frac{1}{m} \frac{l!}{l_1!\cdot \cdots \cdot l_m!} \\
&\quad \times \sum_{\varepsilon_1=\pm 1} \sum_{\varepsilon_2=\pm 1} \cdots \sum_{\varepsilon_m=\pm 1} \int_{[0,\pi]^m} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m) \\
&\quad \times \prod_{j=1}^{m} K_{2n-1}(x_j, \varepsilon_j \cdot x_{j+1})\, dx_1 \cdot \cdots \cdot dx_m.
\end{aligned} \tag{4.13}$$

Each term in the last sum with $\prod_{i=1}^{m} \varepsilon_i = 1$ is equal to

$$\begin{aligned}
&\int_{\prod_{i=1}^{m} \varepsilon_{i-1}\cdot[0,\pi]} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m) \prod_{j=1}^{m} K_{2n-1}(x_j, x_{j+1}) \prod_{i=1}^{m} d(\varepsilon_{i-1}\cdot x_i) \\
&\quad = \frac{1}{2^m} \int_{[0,2\pi]^m} f^{l_1}(x_1)\cdot \cdots \cdot f^{l_m}(x_m) \prod_{j=1}^{m} K_{2n-1}(x_j, x_{j+1}) dx_1 \cdot \cdots \cdot dx_m
\end{aligned}$$

[we use the fact that $f(x)$ is even]. Combining these terms together we obtain the same expression as for $\frac{1}{2}\cdot C_{l,2n-1}(\sum_{j=1}^{2n-1} f(\theta_j))$ in the case of $U(2n-1)$, which gives vanishing contribution if $l > 2$. Finally we claim that the contribution from the terms with $\prod_{i=1}^{m} \varepsilon_i = -1$ can be bounded from above by

$$\text{const}_l' \sum_{|k_1|+\cdots+|k_l|>n} |\hat{f}(k_1)| \cdot \cdots \cdot |\hat{f}(k_l)|.$$

Indeed, the integral

$$\int_{[0,\pi]^m} f^{l_1}(x_1) \cdot \cdots \cdot f^{l_m}(x_m) \prod_{j=1}^{m} \left(\frac{1}{2\pi} \sum_{s_j=-n}^{n} \exp\big(is_j(x_j - \varepsilon_j \cdot x_{j+1})\big) \right) dx_1 \cdots dx_m$$

can be rewritten as

$$\frac{1}{2^m} \sum_{s_1=-n}^{n} \cdots \sum_{s_m=-n}^{n} \widehat{f^{l_1}}(s_1 - \varepsilon_m \cdot s_m) \cdot \widehat{f^{l_2}}(s_2 - \varepsilon_1 \cdot s_1) \cdot \cdots \cdot \widehat{f^{l_m}}(s_m - \varepsilon_{m-1} \cdot s_{m-1}).$$

Consider the Euclidian basis $\{e_j\}_{j=1}^m$ in $\mathbb{R}^m$ and define $f_j = e_j - \varepsilon_{j-1} e_{j-1}$, $\varepsilon_0 = \varepsilon_m$. The vectors $\{f_j\}_{j=1}^m$ form a basis in $\mathbb{R}^m$ iff $\prod_{j=1}^m \varepsilon_j = -1$. Then for any m-tuple $(t_1, \dots, t_m)$ there exists the only m-tuple $(s_1, \dots, s_m)$ such that $t_j = s_j - \varepsilon_{j-1} \cdot s_{j-1}$, $j = 1, \dots, m$. We write $\widehat{f^{l_j}}(t_j) = \sum \hat{f}(k_{l_1+\cdots+l_{j-1}+1}) \cdot \cdots \cdot \hat{f}(k_{l_1+\cdots+l_j})$, where the sum is over k_i such that $\sum_{l_1+\cdots+l_{j-1}+1}^{l_1+\cdots+l_j} k_i = t_j$. When we plug this into (4.13) we obtain a linear combination of

$$\hat{f}(k_1) \cdot \cdots \cdot \hat{f}(k_m). \tag{4.14}$$

It is easy to see that for $|k_1|+\cdots+|k_m| \le n$ the coefficient with the term (4.14) is equal to

$$\frac{1}{2m} \sum_{m=1}^{l} \sum_{\substack{l_1+\cdots+l_m=l,\\ l_i \ge 1}} (-1)^m \frac{1}{m} \frac{l!}{l_1! \cdot \cdots \cdot l_m!} = 0.$$

For $|k_1|+\cdots+|k_m| > n$ the coefficient is bounded from above by some constant. This finishes the proof of Lemma 3. □

Similar to Section 1 we obtain the proof of Theorem 2 by applying the lemma to $\sum_{j=1}^n g(L_n \cdot (\theta_j - \theta))$.

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