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ABSTRACT

Let X be a curve over a number field K with genus $g \geq 2$, \mathfrak{p} a prime of \mathcal{O}_K over an unramified rational prime $p > 2r$, J the Jacobian of X , $r = \text{rank } J(K)$, and \mathcal{X} a regular proper model of X at \mathfrak{p} . Suppose $r < g$. We prove that $\#X(K) \leq \#\mathcal{X}(\mathbb{F}_{\mathfrak{p}}) + 2r$, extending the refined version of the Chabauty–Coleman bound to the case of bad reduction. The new technical insight is to isolate variants of the classical rank of a divisor on a curve which are better suited for singular curves and which satisfy Clifford’s theorem.

1. Introduction

Let K be a number field and X/K be a curve (i.e. a smooth geometrically integral one-dimensional variety) of genus $g \geq 2$ and let p denote a prime which is unramified in K . Faltings’ [Fal86], Vojta’s [Voj91], and Bombieri’s [Bom90] proofs of the Mordell conjecture tell us that $X(K)$ is finite, but they are ineffective, providing no assistance in determining $X(K)$ explicitly for a specific curve. Chabauty [Cha41], building on an idea of Skolem [Sko34], gave a partial proof of the Mordell conjecture under the hypothesis that the rank r of the Jacobian of X is strictly less than the genus g . Coleman later realized that Chabauty’s proof could be modified to get an explicit upper bound for $\#X(K)$.

THEOREM 1.1 [Col85]. *Suppose $p > 2g$ and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime of good reduction which lies above p . Suppose $r < g$. Then*

$$\#X(K) \leq \#X(\mathbb{F}_{\mathfrak{p}}) + 2g - 2.$$

In [LT02], the authors ask if one can refine Coleman’s bound when the rank is small (i.e. $r \leq g - 2$). Stoll proved that by choosing, for each residue class, the ‘best’ differential one can indeed refine the bound.

THEOREM 1.2 [Sto06, Corollary 6.7]. *With the hypothesis of Theorem 1.1,*

$$\#X(K) \leq \#X(\mathbb{F}_{\mathfrak{p}}) + 2r.$$

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In another direction, Lorenzini and Tucker derive Coleman’s bound when \mathfrak{p} is a prime of bad reduction; a later, alternative proof using intersection theory on \mathcal{X} is given in [MP10, Theorem A.5]. Let \mathcal{X} be a minimal regular proper model of X at \mathfrak{p} and denote by $\mathcal{X}_{\mathfrak{p}}^{\text{sm}}$ the smooth locus of $\mathcal{X}_{\mathfrak{p}}$.

THEOREM 1.3 [LT02, Proposition 1.10]. *Suppose $p > 2g$ and let \mathfrak{p} be a prime above p . Let \mathcal{X} be a proper regular model of X over $\mathcal{O}_{K_{\mathfrak{p}}}$. Suppose $r < g$. Then*

$$\#X(K) \leq \#\mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}}) + 2g - 2.$$

1.1 Main result

Lorenzini and Tucker ask [LT02, comment after 1.11] if one can generalize Theorem 1.2 to the case when \mathfrak{p} is a prime of bad reduction; Michael Stoll remarks [Sto06, Remark 6.5] that the analogue of Theorem 1.2 is true at least for a hyperelliptic curve. The main result of this paper is such a generalization.

THEOREM 1.4. *Let X be a curve over a number field K . Suppose $p > 2r + 2$ is a prime which is unramified in K and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime above p . Let \mathcal{X} be a proper regular model of X over $\mathcal{O}_{K_{\mathfrak{p}}}$. Suppose moreover that $r < g$. Then*

$$\#X(K) \leq \#\mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}}) + 2r.$$

Remark 1.5. The condition that $p > 2r + 2$ ensures that the bounds one gets from the use of Newton Polygons in Chabauty’s method are as sharp as possible; using Proposition 2.8, one can write out weaker, but still explicit (in terms of g and p), bounds when $p \leq 2r + 2$ or when p ramifies in K (see any of [Col85, Sto06], or [LT02]).

Remark 1.6. When $\mathcal{X}_{\mathfrak{p}}$ is singular or is not hyperelliptic, one can often do better than $2r$; see § 5.1.

Remark 1.7. The charm of these theorems is that they occasionally allow one to compute $X(K)$; see [Gra94] for the first such example and Example 5.1 for another which uses the refined bound of Theorem 1.4 at a prime of bad reduction.

Example 5.1 is of course quite special, in that most of the time the bound of Theorem 1.4 is not sharp. Chabauty’s method gives a way to bound the number of rational points in each residue class (see Proposition 2.8); however, this bound is never less than 1, and yet some residue classes may not have rational points in them. In practice, one applies Chabauty’s method to each residue class. If a given residue class seems to have no rational points, a variety of methods usually allow one to prove that this is correct (the most common being Scharaschkin’s method, often called the Mordell–Weil sieve); if a given residue class does have a point, then one applies Chabauty’s method.

There are a wealth of interesting examples where a more careful analysis of Chabauty’s method allows one to determine $X(K)$. Worth noting are the works [Bru99, PSS07], where the integral coprime solutions of the generalized Fermat equations $x^2 + y^8 = z^3$ and $x^2 + y^3 = z^7$ (both of which have large, non-trivial solutions) are completely determined by reducing to curves and using Chabauty methods.

For low genus hyperelliptic curves Chabauty’s method has been made completely explicit and even implemented in [BCP97], so that for any particular low genus hyperelliptic curve X over \mathbb{Q} , $X(\mathbb{Q})$ can often be determined; see the survey [Poo02] for a general discussion of computational issues, [MP10, § 7] for a discussion of effectivity of Chabauty’s method, and the online documentation [BCP97] for many details. See [Sto11] for the most up to date survey on such issues; [Sto11, 4.2] gives a quick summary of the interplay between the Mordell–Weil sieve

and Chabauty’s method and how to handle residue classes which are empty, and [BS08, BS10] give a much more detailed explanation.

Remark 1.8. The $p = 2r + 2$ case of Theorem 1.4 (i.e. $p = 2$ and $r = 0$) is [LT02, Proposition 1.10].

1.2 New ideas; geometric rank functions

Recall the classical notion of the rank of a divisor D on a smooth projective geometrically integral curve X over an infinite field.

DEFINITION 1.9. We say that the rank $r_X(D)$ of a divisor $D \in \text{Div } X$ is -1 if D is not equivalent to an effective divisor, and we say that a divisor D which is equivalent to an effective divisor has rank $r_X(D) = r$ if for any effective divisor E of degree r on X , $D - E$ is equivalent to an effective divisor and moreover r is the largest integer with this property (i.e. there is an effective divisor E on X of degree $r + 1$ for which $D - E$ is not equivalent to an effective divisor). Alternatively, let

$$r(D) = \dim_K H^0(X, \mathcal{O}_X(D)) - 1 = \dim |D| \quad \text{where } |D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$$

($|D|$ is the linear system of effective divisors equivalent to D).

Since X is smooth, projective, geometrically integral, and defined over an infinite field, the functions r_X and r are equal. Moreover, $r_X(D)$ satisfies Clifford’s theorem [Har77, ch. III, Theorem 5.4]: if D is a special divisor (i.e. D and $K_X - D$ are each equivalent to an effective divisor, where K_X is the canonical divisor of X), then $r_X(D) \leq \frac{1}{2} \text{deg } D$.

Now let X be the curve from the setup of Theorem 1.2. Stoll’s proof of Theorem 1.2 uses Clifford’s theorem and Riemann–Roch in an essential way; he constructs a special divisor $D_{\text{chab},p}$ on the (smooth) special fiber \mathcal{X}_p of X (see § 2.5) such that:

- (i) $\#X(K) - \#X(\mathbb{F}_p) \leq \text{deg } D_{\text{chab},p}$; and
- (ii) $r(K_X - D_{\text{chab},p}) \geq g - r - 1$,

and deduces that $\text{deg } D_{\text{chab},p} \leq 2r$ from Riemann–Roch and Clifford’s theorem.

When the special fiber \mathcal{X}_p has multiple components, it is no longer true that the ranks $r_X(D)$ and $r(D)$ agree for divisors D on \mathcal{X}_p , since a section $s \in H^0(\mathcal{X}_p, \mathcal{O}_{\mathcal{X}_p}(D))$ may vanish along some component of \mathcal{X}_p ; in general $r_X(D) < \dim_K H^0(\mathcal{X}_p, \mathcal{O}_{\mathcal{X}_p}(D))$. When \mathcal{X} is regular, $r(D)$ still satisfies Riemann–Roch [Liu06, Theorem 7.3.26], and in an earlier version of this paper we gave a direct proof of Clifford’s theorem for $r(D)$. However, the divisor $D_{\text{chab},p}$ to which we want to apply Clifford’s theorem is no longer special in the usual sense; moreover, when \mathcal{X}_p has multiple components, it does not follow from Chabauty’s method that $r(K_{\mathcal{X}_p} - D_{\text{chab},p}) \geq g - r - 1$ (compare with § 3.5).

Modified rank functions. In § 3.3, motivated by the canonical decomposition of the special fiber of the Néron model of the Jacobian of X as successive extensions of Abelian, toric, unipotent, and discrete algebraic groups (see § 3.1), we define rank functions $r_{\text{num}}, r_{\text{ab}}, r_{\text{tor}}, r_X$ which satisfy $r_{\text{num}}(D) \geq r_{\text{ab}}(D) \geq r_{\text{tor}}(D) \geq r_X(D)$. (Our variant of r_X agrees with the one defined above when X has good reduction; moreover $r_X = r$ (Proposition 3.11).)

To take into account that sections can vanish along components of \mathcal{X}_p , these functions instead take as input divisors on \mathcal{X} which reduce to smooth points of \mathcal{X}_p . Moreover, the modified rank functions should be viewed as successively finer approximations to the effectivity of a divisor. For example, if a horizontal divisor \mathcal{D} on \mathcal{X} is equivalent to an effective horizontal divisor \mathcal{E} , then the degree $\text{deg}(\mathcal{E}|_{X_v})$ of the restriction of \mathcal{E} to each component X_v of \mathcal{X}_p is non-negative.

So a *necessary* condition for a horizontal divisor \mathcal{D} on \mathcal{X} to be equivalent to an effective horizontal divisor \mathcal{E} is that $\deg(\mathcal{E}|_{X_v}) \geq 0$ for every component X_v of $\mathcal{X}_{\mathfrak{p}}$; to define $r_{\text{num}}(\mathcal{D})$ we thus modify Definition 1.9 to say that $r_{\text{num}}(\mathcal{D})$ is at least r if for every horizontal divisor \mathcal{E} of degree r , $\mathcal{D} - \mathcal{E}$ has non-negative degree on *every* component X_v of $\mathcal{X}_{\mathfrak{p}}$, and that $r_{\text{num}}(\mathcal{D}) = -1$ if $\deg(\mathcal{D}|_{X_v}) < 0$ for *some* component X_v of $\mathcal{X}_{\mathfrak{p}}$. (Actually, we also allow *twisting*; i.e. modification by divisors $\sum a_v X_v$ supported on the special fiber; see § 3.2.)

Our main result about these rank functions (Proposition 3.20) is that r_{ab} (and thus $r_{\text{tor}}(D)$ and $r_X(D)$) satisfies a variant of Clifford’s theorem.

PROPOSITION 1.10. *Let $D_{\mathfrak{p}}$ be an effective divisor on $\mathcal{X}_{\mathfrak{p}}$ and let $K_{\mathcal{X}_{\mathfrak{p}}}$ be a divisor in the class of the canonical bundle. Suppose that $D_{\mathfrak{p}}$ and $K_{\mathcal{X}_{\mathfrak{p}}}$ are supported on smooth points. Then,*

$$r_{\text{ab}}(K_{\mathcal{X}_{\mathfrak{p}}} - D_{\mathfrak{p}}) \leq g - \frac{\deg D_{\mathfrak{p}}}{2} - 1.$$

With a bit more work, Theorem 1.4 follows (see § 4).

Remark 1.11 (Comparison with Baker–Norine). Baker [Bak08] defines a notion of linear equivalence, and thus of rank, for a divisor on a graph. Moreover, one can specialize a divisor D on a semistable curve to a divisor $\underline{\deg}(D)$ on its dual graph; see §§ 3.2 and 3.4.

Our rank function r_{num} agrees with theirs in the sense that $r_{\text{num}}(D) = r(\underline{\deg}(D))$. When $\mathcal{X}_{\mathfrak{p}}$ is semistable and totally degenerate, $r_{\text{num}}(D) = r_{\text{ab}}(D)$, but in general $r_{\text{num}}(D)$ is larger (see Example 5.3). In [BN07], Baker and Norine prove an analogue of Riemann–Roch (and thus Clifford’s theorem) for $r_{\text{num}}(D)$, but this is with respect to the canonical divisor K_{Γ} of the dual graph. This differs from the specialization of the divisor $K_{\mathcal{X}_{\mathfrak{p}}}$ and in particular does not take into account the genera of the components of $\mathcal{X}_{\mathfrak{p}}$. Their theorem thus suffices for $\mathcal{X}_{\mathfrak{p}}$ semistable and totally degenerate, but not for the general case.

Remark 1.12. While this paper was in preparation, we became aware of the work of Amini–Baker on metrized complexes of curves [AB12]. They introduce a rank function analogous to r_{ab} in a very slightly different context and prove a version of the Riemann–Roch theorem that implies both the Riemann–Roch theorem for r_{ab} and our Clifford bounds on r_{ab} . Their work also makes connections to the theory of limit linear series.

1.3 Organization of the paper

This paper is structured as follows. In § 2 we review the method of Chabauty and Coleman and present the main argument used to bound $\#X(K)$. In § 3 we introduce the geometric rank functions $r_{\text{num}}, r_{\text{ab}}, r_{\text{tor}}, r_X$ and prove Clifford’s theorem for r_{ab} (Proposition 3.20). In § 4 we finish the proof of Theorem 1.4. Finally, in § 5 we give an example where the refined bound of Theorem 1.4 can be used to determine $X(K)$ and examples where the inequalities $r_{\text{num}}(D) \geq r_{\text{ab}}(D) \geq r_{\text{tor}}(D) \geq r_X(D)$ are strict.

2. The method of Chabauty and Coleman

In this section we recall the method of Chabauty and Coleman. See [MP10] for many references and a more detailed account.

Some notation. Let K be a number field, p a prime integer and \mathfrak{p} a prime of K above p ; let $v = v_{\mathfrak{p}}$ be the corresponding valuation, normalized so that the value group $v(K)$ is \mathbb{Z} . Denote by $\mathbb{F}_{\mathfrak{p}}$ the residue field. For a scheme Y over K let $Y_{\mathfrak{p}}$ be the extension of scalars $Y \times_K K_{\mathfrak{p}}$, where $K_{\mathfrak{p}}$ is

the completion of K at \mathfrak{p} . For a scheme Y over a field denote by Y^{sm} its smooth locus. Let X be a smooth projective geometrically integral curve of genus $g \geq 2$ over K with Jacobian J and let $r = \text{rank } J(K)$. Suppose that there exists a rational point $P \in X(K)$ (otherwise the conclusion of Theorem 1.4 is trivially true) and let $\iota: X \rightarrow J$ be the embedding given by $Q \mapsto [Q - P]$.

2.1 Models and residue classes

Let \mathcal{X} be a proper regular model of $X_{\mathfrak{p}}$ over $\mathcal{O}_{K_{\mathfrak{p}}}$ and denote its special fiber by $\mathcal{X}_{\mathfrak{p}}$. We have the following commutative diagram.

$$\begin{array}{ccccc}
 \mathcal{X}_{\mathfrak{p}} & \longrightarrow & \mathcal{X} & \xleftarrow{i} & X_{\mathfrak{p}} \\
 \downarrow \pi_{\mathfrak{p}} & & \downarrow \pi & & \downarrow \pi_{K_{\mathfrak{p}}} \\
 \text{Spec } \mathbb{F}_{\mathfrak{p}} & \longrightarrow & \text{Spec } \mathcal{O}_{K_{\mathfrak{p}}} & \longleftarrow & \text{Spec } K_{\mathfrak{p}}
 \end{array} \tag{2.1}$$

Since \mathcal{X} is proper, the valuative criterion gives a reduction map

$$\rho: X_{\mathfrak{p}}(K_{\mathfrak{p}}) = \mathcal{X}(\mathcal{O}_{K_{\mathfrak{p}}}) \rightarrow \mathcal{X}(\mathbb{F}_{\mathfrak{p}}). \tag{2.2}$$

Alternatively, ρ is given by smearing any $K_{\mathfrak{p}}$ -point of $X_{\mathfrak{p}}$ to an $\mathcal{O}_{K_{\mathfrak{p}}}$ -point of \mathcal{X} and then intersecting with the special fiber $\mathcal{X}_{\mathfrak{p}}$; i.e. $\rho(P) = \overline{\{P\}} \cap \mathcal{X}_{\mathfrak{p}}$, where $\overline{\{P\}}$ is the closure of the point P (note that even though P is a closed point of $X_{\mathfrak{p}}$ it is not closed on \mathcal{X}). Since \mathcal{X} is regular, the image is contained in $\mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}})$. Conversely, by Hensel’s lemma any point of $\mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}})$ lifts to a point in $X_{\mathfrak{p}}(K_{\mathfrak{p}})$.

DEFINITION 2.3. For $\tilde{Q} \in \mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}})$ we define the *residue class* (or *tube*) $D_{\tilde{Q}}$ to be the preimage $\rho^{-1}(\tilde{Q})$ of \tilde{Q} under the reduction map (2.2).

DEFINITION 2.4. Let $\tilde{Q} \in \mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}})$ and let $\omega \in H^0(X_{\mathfrak{p}}, \Omega_{X_{\mathfrak{p}}/K_{\mathfrak{p}}}^1)$. Choose $t \in K_{\mathfrak{p}}^{\times}$ such that $t\omega$ (considered as a rational section of the relative dualizing sheaf of \mathcal{X}) has neither a pole nor zero along the entire component X_v of $\mathcal{X}_{\mathfrak{p}}$ containing \tilde{Q} (in other words, $t\omega$ is allowed to have poles or zeroes at some individual points of X_v , but is not allowed to be identically zero or infinite when restricted to X_v). This is possible because the relative dualizing sheaf is invertible at the generic points of components of the special fiber containing smooth points. Let $\tilde{\omega}$ be the reduction of $t\omega$ to X_v and define

$$n(\omega, \tilde{Q}) = \text{ord}_{\tilde{Q}} \tilde{\omega}.$$

2.2 p -adic integration

For a more leisurely introduction to integration on a p -adic curve see [MP10, §§ 4 and 5]. For $\omega \in H^0(X_{\mathfrak{p}}, \Omega_{X_{\mathfrak{p}}/K_{\mathfrak{p}}}^1)$ let $\eta_{\omega}: X(K_{\mathfrak{p}}) \rightarrow K_{\mathfrak{p}}$ be the function $Q \mapsto \int_P^Q \omega$. The following proposition summarizes relevant results of [LT02, § 1].

PROPOSITION 2.5. Let $\tilde{Q} \in \mathcal{X}_{\mathfrak{p}}^{\text{sm}}(\mathbb{F}_{\mathfrak{p}})$ and $Q \in D_{\tilde{Q}}$. Let $u \in \mathcal{O}_{\mathcal{X}, Q}$ such that the restriction to $\mathcal{O}_{\mathcal{X}_{\mathfrak{p}}, \tilde{Q}}$ is a uniformizer. Then the following are true.

- (1) The function u defines a bijection

$$u: D_{\tilde{Q}} \xrightarrow{\sim} \mathfrak{p}\mathcal{O}_{K_{\mathfrak{p}}}.$$

(2) There exists $I_{\omega,Q}(t) \in K_p[[t]]$ which enjoys the following properties:

- (i) for $Q' \in D_{\tilde{Q}}$, $\eta_{\omega}(Q') = I_{\omega,Q}(u(Q')) + \eta_{\omega}(Q)$;
- (ii) $w(t) := I_{\omega,Q}(t)' \in \mathcal{O}_{K_p}[[t]]$;
- (iii) if we write $w(t) = \sum_{i=0}^{\infty} a_i t^i$, then

$$\min\{i : v(a_i) = 0\} = n(\omega, \tilde{Q}).$$

The starting point of Chabauty’s method is the following proposition.

PROPOSITION 2.6 [Sto06, § 6]. Denote by V_{chab} the vector space of all $\omega \in H^0(X_p, \Omega_{X_p/K_p}^1)$ such that $\eta_{\omega}(Q) = 0$ for all $Q \in X(K)$. Then $\dim V_{\text{chab}} \geq g - r$.

2.3 Newton polygons

We will now use Newton polygons to bound the number of zeroes of $I_{\omega,Q}(u(Q'))$ as Q' ranges over $D_{\tilde{Q}}$. Following [Sto06, § 6], we let $e = v(p)$ be the absolute ramification index of K_p and make the following definitions, where v_p is the valuation of \mathbb{Q}_p (and in particular, normalized so that $v_p(p) = 1$).

DEFINITION 2.7. We set

$$\nu(Q) = \#\{Q' \in D_{\tilde{Q}} \text{ such that } I_{\omega,Q}(u(Q')) = 0 \text{ for all } \omega \in V_{\text{chab}}\}$$

and

$$\begin{aligned} \delta(v, n) &= \max\{d \geq 0 \mid n + d + 1 - v(n + d + 1) \leq n + 1 - v(n + 1)\} \\ &= \max\{d \geq 0 \mid ev_p(n + 1) + d \leq ev_p(n + d + 1)\}. \end{aligned}$$

The key proposition is [Sto06, Proposition 6.3] where a Newton polygon argument gives the following bound.

PROPOSITION 2.8. We have the bound

$$\nu(Q) \leq 1 + n(\omega, \tilde{Q}) + \delta(v, n(\omega, \tilde{Q})).$$

Furthermore, suppose $e < p - 1$. Then $\delta(v, n) \leq e \lfloor n / (p - e - 1) \rfloor$. In particular, if $p > n + e + 1$, then $\delta(v, n) = 0$.

2.4 Bounding $\#X(K)$

We bound $\#X(K)$ as follows. For each $\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)$,

$$\#(X(K) \cap D_{\tilde{Q}}) \leq \nu(Q).$$

For non-zero $\omega \in V_{\text{chab}}$ (see Definition 2.6) summing the bound of Proposition 2.8 over the residue classes of each smooth point gives

$$\#X(K) \leq \#\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p) + \sum_{\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)} (n(\omega, \tilde{Q}) + \delta(v, n(\omega, \tilde{Q}))).$$

To use this we need to bound

$$\sum_{\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)} n(\omega, \tilde{Q}).$$

As in the good reduction case of [Col85], Riemann–Roch gives, for a fixed ω , the preliminary bound

$$\sum_{\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)} n(\omega, \tilde{Q}) \leq \deg \operatorname{div} \omega = 2g - 2.$$

If $p > 2g + e - 1$, then in particular $p > n(\omega, \tilde{Q}) + e + 1$ for every \tilde{Q} and Proposition 2.8 reveals that $\delta(v, n(\omega, \tilde{Q})) = 0$, recovering the bound of Theorem 1.3,

$$\#X(K) \leq \#\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p) + 2g - 2.$$

2.5 The Chabauty divisor

The idea of the proof of Theorem 1.2 (due to [Sto06]) is to use a different differential $\omega_{\tilde{Q}}$ for each residue class to get a better bound. Stoll does this for the good reduction case [Sto06, Theorem 6.4] and what prevents his method from working in full generality is that the reduction map

$$\mathbb{P}(H^0(X_p, \Omega^1)) \rightarrow \mathbb{P}(H^0(\mathcal{X}, \Omega^1)) \tag{2.9}$$

(and thus the classical rank function $r(D)$ of Definition 1.9) is well behaved only when \mathcal{X} is smooth.

The key feature of Stoll’s proof is the following: Chabauty’s method produces a divisor D with the property that a bound on $\deg D$ gives a bound on $\#X(K)$. In this subsection we extend Stoll’s construction of this divisor to the case of bad reduction.

For a map $f : Y \rightarrow Z$ denote by ω_f the relative dualizing sheaf of f . Recall the setup of (2.1). The appropriate generalization of the reduction map (2.9) is the following. Since the structure map $\pi : \mathcal{X} \rightarrow \operatorname{Spec} \mathcal{O}_{K_p}$ is flat [Liu06, p. 347], base change for relative dualizing sheaves [Liu06, Theorem 6.4.9] gives an isomorphism $i^* \omega_\pi \simeq \omega_{\pi_{K_p}} \simeq \Omega^1_{X_p/K_p}$. One gets an inclusion of global sections

$$0 \longrightarrow H^0(\mathcal{X}, \omega_\pi) \xrightarrow{\phi} H^0(\mathcal{X}, \omega_\pi) \otimes_{\mathcal{O}_{K_p}} K_p \longleftarrow H^0(X_p, \Omega^1_{X_p/K_p}).$$

A subspace $V \subset H^0(X_p, \Omega^1_{X_p/K_p})$ thus pulls back to a submodule $V_{\mathcal{O}_{K_p}} := \phi^{-1}(V) \subset H^0(\mathcal{X}, \omega_\pi)$. Now let $V = V_{\text{chab}}$ (see Proposition 2.6), and for any $\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)$ let

$$n_{\tilde{Q}} := \min\{n(\omega, \tilde{Q}) \mid \omega \in V\}.$$

Then Proposition 2.8 becomes

$$\nu(\tilde{Q}) \leq 1 + n_{\tilde{Q}} + \delta(v, n_{\tilde{Q}})$$

and thus

$$\#X(K) \leq \#\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p) + \sum_{\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)} (n_{\tilde{Q}} + \delta(v, n_{\tilde{Q}})).$$

DEFINITION 2.10. We define the *Chabauty Divisor* to be the divisor

$$D_{\text{chab},p} = \sum_{\tilde{Q} \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)} n_{\tilde{Q}} \tilde{Q}. \tag{2.11}$$

Since $\sum n_{\tilde{Q}} = \deg D_{\text{chab},p}$, the goal is to bound $\deg D_{\text{chab},p}$.

In what follows, we introduce new ideas to show that $\deg D_{\text{chab},p} \leq 2r$.

3. Geometric rank functions

In this section we define the rank functions $r_{\text{num}}(D), r_{\text{ab}}(D), r_{\text{tor}}(D), r_X(D)$ and prove Clifford’s theorem for r_{ab} .

Let \mathcal{O}_{K_p} be a DVR with fraction field K_p and perfect, infinite residue field \mathbb{F}_p and let \mathcal{X} be a regular *semistable* curve over \mathcal{O}_{K_p} with generic fiber X_p and special fiber \mathcal{X}_p .

Remark 3.1. Note that in this section we make the additional assumptions that *the residue field is infinite* and that \mathcal{X} is *semistable*. With a view toward our intended application (bounding the degree of the Chabauty divisor of Definition 2.10) we note that the degree of $D_{\text{chab},p}$ can only *increase* after field extensions. Moreover, after a finite extension of K_p , \mathcal{X}_{K_p} is dominated by a regular semistable model \mathcal{X}' of its generic fiber, and again the degree of the Chabauty divisor of \mathcal{X}' can only grow. See § 4 for the precise argument.

3.1 Motivation; Néron models

Let $\mathcal{D}_1, \mathcal{D}_2$ be horizontal divisors on \mathcal{X} (in other words, no component of \mathcal{D}_i is contained in \mathcal{X}_p). Let D_1, D_2 be their generic fibers. Suppose $\deg(D_1) = \deg(D_2)$ and that D_1, D_2 are supported on K_p -rational points. Since \mathcal{X} is regular, the reductions $\rho(\mathcal{D}_1), \rho(\mathcal{D}_2)$ are supported on smooth points of \mathcal{X}_p .

We are interested in the following question: *are D_1, D_2 linearly equivalent?* We approach this by considering $D_1 - D_2$ and finding obstructions for $D_1 - D_2 = 0$ in the Jacobian J of X_p . Let \mathcal{J} be a Néron model for the Jacobian; $D_1 - D_2$ induces a section $d: \text{Spec } \mathcal{O}_{K_p} \rightarrow \mathcal{J}$ by the Néron mapping property. So our question becomes: *is d supported on the closure of the identity in \mathcal{J} ?*

We can refine this question by studying the structure of the Néron model of the Jacobian [BLR90, § 9.5], [Bak08, Appendix A]. The Néron model is constructed from the relative Picard scheme $\text{Pic}_{\mathcal{X}/\mathcal{O}_{K_p}}$. Let P be the open subfunctor of $\text{Pic}_{\mathcal{X}/\mathcal{O}_{K_p}}$ consisting of divisors of total degree 0. Let U be the closure of the unit section $\text{Spec } \mathcal{O}_{K_p} \rightarrow P$. Then by [BLR90, Theorem 9.5/4], $\mathcal{J} = P/U$. The \mathcal{O}_{K_p} -points of U can be described in terms of the components of \mathcal{X}_p . In fact, they are line bundles of the form $\mathcal{O}_{\mathcal{X}}(V)$ where V is a divisor of \mathcal{X} supported on the special fiber. We will call such divisors *twists* (see § 3.2). These line bundles restrict to trivial bundles on the generic fiber.

The identity component of the Néron model can be related to the Picard scheme. By [BLR90, Theorem 9.5.4b], $\mathcal{J}^0 = \text{Pic}_{\mathcal{X}/\mathcal{O}_{K_p}}^0$, where the latter scheme denotes the subscheme of the Picard scheme consisting of line bundles whose restriction to every component of the special fiber has degree 0. Let $\pi: \widetilde{\mathcal{X}}_p \rightarrow \mathcal{X}_p$ be the normalization. Then there is an exact sequence [Liu06, Theorem 7.5.19]

$$0 \rightarrow T \rightarrow \text{Pic}_{\mathcal{X}_p}^0 \rightarrow \text{Pic}_{\widetilde{\mathcal{X}}_p}^0 \rightarrow 0$$

where T is a toric group (i.e. geometrically isomorphic to a power of \mathbb{G}_m). Note that $\text{Pic}_{\widetilde{\mathcal{X}}_p}^0$ is an Abelian variety. We can use this structure of \mathcal{J} to give a hierarchy of conditions required for $D_1 - D_2$ to be the trivial divisor in J :

- (1) $\rho(\mathcal{D}_1 - \mathcal{D}_2)$ must be in the identity component of \mathcal{J}_p ;
- (2) the image of $\rho(\mathcal{D}_1 - \mathcal{D}_2)$ in $\text{Pic}_{\widetilde{\mathcal{X}}_p}^0$ must be the identity;
- (3) $\mathcal{D}_1 - \mathcal{D}_2$ must be the identity in \mathcal{J}_p ;
- (4) $\mathcal{D}_1 - \mathcal{D}_2$ must extend as the identity in \mathcal{J} .



FIGURE 1. A curve and its dual graph.

We call the first three steps in the hierarchy *numeric*, *Abelian*, *toric*. We can rephrase the steps in the hierarchy as finding a better and better approximation to a section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1 - \mathcal{D}_2)$.

- (1) There is a twist $\mathcal{O}_{\mathcal{X}}(V)$ such that $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1 - \mathcal{D}_2 + V)$ has degree 0 on every component of $\mathcal{X}_{\mathfrak{p}}$.
- (2) There is a twist $\mathcal{O}_{\mathcal{X}}(V)$ together with a section s_v of $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1 - \mathcal{D}_2 + V)|_{X_v}$ for each component X_v of $\mathcal{X}_{\mathfrak{p}}$.
- (3) The sections can be chosen above to agree across nodes.
- (4) The section s formed from the sections s_v , considered as a section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D}_1 - \mathcal{D}_2 + V)|_{\mathcal{X}_{\mathfrak{p}}}$ extends to \mathcal{X} .

Note that steps (1)–(3) are only concerned with $\rho(\mathcal{D}_1), \rho(\mathcal{D}_2)$. Step (1) has been studied in the context of linear systems on graphs by Baker–Norine. Baker–Norine’s theory works best when all components of $\mathcal{X}_{\mathfrak{p}}$ are rational. In this case step (2) imposes no constraints. See § 3.4 for a short review.

In this paper, we will advocate for including step (2) when not all components of the special fiber are rational.

3.2 Dual graphs

Let $\mathcal{X}/\mathcal{O}_{K_{\mathfrak{p}}}$ be a regular, semistable curve. We define the *dual graph* Γ of \mathcal{X} to be the graph whose vertices are the irreducible components X_v of $\mathcal{X}_{\mathfrak{p}}$ and with an edge between the vertices corresponding to nodes of $\mathcal{X}_{\mathfrak{p}}$ (see Figure 1).

Remark 3.2. Our usage differs from that of Baker [Bak08] for the following reasons: we do not require the models to be strictly semistable; vertices in the dual graph correspond to irreducible components of $\mathcal{X}_{\mathfrak{p}}$, not to components of the normalization. Consequently, components X_v of $\mathcal{X}_{\mathfrak{p}}$ may have nodes.

Because there are multiple edges in our graph, we will treat edges as a multiset in $\text{Sym}^2 V(\Gamma)$, so there may be many edges e satisfying $e = vw$ for a pair of vertices v, w .

Let $\mathcal{M}(\Gamma)$ be the Abelian group of integer-valued functions on the vertices of Γ . Let $\text{Div}(\Gamma)$ be the free Abelian group generated by the vertices of Γ . For $D = \sum_v a_v v \in \text{Div}(\Gamma)$, write $D(v)$ for the coefficient a_v . We say that D is effective and write $D \geq 0$ if $D(v) \geq 0$ for all $v \in V(\Gamma)$. The *Laplacian* on Γ is defined to be the map

$$\Delta: \mathcal{M}(\Gamma) \rightarrow \text{Div}(\Gamma)$$

$$\varphi \mapsto \sum_v \left(\sum_{e=vw} (\varphi(w) - \varphi(v)) \right) (v).$$

Let $\mathcal{M}(\Gamma)_{\sim}$ be the quotient of $\mathcal{M}(\Gamma)$ by constant functions. Note that Δ is defined on $\mathcal{M}(\Gamma)_{\sim}$.

We can view $\varphi \in \mathcal{M}(\Gamma)$ as a *twist* (i.e. a vertical divisor on \mathcal{X}) by setting $V_\varphi = \sum_v \varphi(v)X_v$. Since the special fiber of \mathcal{X} is linearly equivalent to 0, elements of $\mathcal{M}(\Gamma)$ that differ by a constant function induce linearly equivalent twists. For a divisor \mathcal{D} on \mathcal{X} , we define the *twist* of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ by φ to be $\mathcal{O}_{\mathcal{X}}(\mathcal{D})(\varphi) = \mathcal{O}_{\mathcal{X}}(\mathcal{D} + V_\varphi)$.

Let $\text{Div}(X_p(K_p))$ be the divisors of X_p supported on K_p -points (i.e. each individual point is defined over K_p). There is a *reduction* map

$$\rho: X_p(K_p) \rightarrow \mathcal{X}_p(\mathbb{F}_p)$$

given by $P \mapsto \overline{\{P\}} \cap \mathcal{X}_p$. By regularity of \mathcal{X} , this map is surjective onto $\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)$ and therefore induces a surjective map

$$\rho: \text{Div}(X_p(K_p)) \rightarrow \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)).$$

There is a *multidegree* map

$$\begin{aligned} \underline{\text{deg}}: \text{Div}(X_p(K_p)) &\rightarrow \text{Div}(\Gamma) \\ D &\mapsto (v \mapsto \text{deg}(\mathcal{O}_{\mathcal{X}}(D)|_{X_v})) \end{aligned}$$

that factors through ρ . Note that our usage differs from that of [Bak08] in that what we call the multidegree map is there called the specialization map. The canonical bundle $K_{\mathcal{X}_p}$ of the special fiber has the following multidegree:

$$\underline{\text{deg}}(K_{\mathcal{X}_p}) = \sum_v (2g(X_v) + \text{deg}(v) - 2)(v).$$

Since \mathcal{X} is proper, $X_p(K_p) = \mathcal{X}(\mathcal{O}_{K_p})$ and thus $\text{Div} X_p(K_p)$ is naturally identified with horizontal divisors in $\text{Div} \mathcal{X}(\mathcal{O}_{K_p})$. In particular, we will freely pass between divisors supported on $X_p(K_p)$ and their closures. For example, for $\varphi \in \mathcal{M}(\Gamma)$ and $D \in \text{Div} X_p(K_p)$, the twist $D(\varphi)$ refers to the divisor $\mathcal{D} + V_\varphi$, where \mathcal{D} is the closure of D . Moreover, we will apply ρ and $\underline{\text{deg}}$ to horizontal divisors.

3.3 Rank functions

We can use the steps in the hierarchy of §3.1 to define geometric rank functions. Let D be a divisor in $\text{Div}(X_p(K_p))$ and E be an effective divisor in $\text{Div}(X_p(K_p))$. Let \mathcal{D}, \mathcal{E} denote the closures of D, E on \mathcal{X} . We consider the following conditions.

- (1) There is a twist φ such that $\mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{E})(\varphi)$ has non-negative degree on every component of \mathcal{X}_p .
- (2) There is a regular section s_v of $\mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{E})(\varphi)|_{X_v}$ for each $v \in V(\Gamma)$.
- (3) The sections s_v can be chosen to agree across nodes.
- (4) The section s formed from the sections s_v extends to \mathcal{X} as a section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{E})(\varphi)$.

DEFINITION 3.3. We define *rank functions* r_1, r_2, r_3, r_4 , which we denote by $r_{\text{num}}, r_{\text{ab}}, r_{\text{tor}}, r_X$, as follows. Let $D \in \text{Div}(X_p(K_p))$. We say $r_i(D) = -1$ if and only if D fails to satisfy steps (1) to i , and for $r \geq 0$, we say $r_i(D) = r$ if and only if for any effective divisor E of degree r on $X_p(K_p)$, $D - E$ satisfies steps (1) to i and there is an effective divisor E on $X_p(K_p)$ of degree $r + 1$ for which $D - E$ fails to satisfy one of steps (1) to i .

DEFINITION 3.4. We say two divisors $D, E \in \text{Div}(X_p(K_p))$ of the same degree are (num, ab, tor)-*linearly equivalent* if the difference of their closures on \mathcal{X} , $\mathcal{D} - \mathcal{E}$, satisfies steps (1) to (1), (2), or (3) respectively.

These rank functions do not all depend on the geometry of the generic fiber X_p . Instead, some of them capture the combinatorics and the geometry of the special fiber. First, we note that the restriction of a twist to the special fiber depends only on the special fiber.

LEMMA 3.5. Let $\varphi \in \mathcal{M}(\Gamma)$, considered as the twist $\sum \varphi(v)X_v$. Let X_{v_0} be a component of the special fiber. For an edge e of Γ at v_0 , let p_e be the corresponding node on X_{v_0} . Then,

$$\mathcal{O}_{\mathcal{X}}(\varphi)|_{X_{v_0}} = \sum_{e=v_0w} (\varphi(w) - \varphi(v_0))(p_e),$$

where the sum is over the edges at v_0 . Moreover,

$$\underline{\deg}(\mathcal{O}_{\mathcal{X}}(\varphi)) = \Delta(\varphi).$$

Proof. Because the special fiber of \mathcal{X} is linearly equivalent to 0 on \mathcal{X} , φ is linearly equivalent to $\sum(\varphi(v) - \varphi(v_0))X_v$. The lemma follows from the fact that

$$\mathcal{O}_{\mathcal{X}}(X_w)|_{X_{v_0}} = \sum_{e=v_0w} p_e,$$

and the claim about the multidegree of $\mathcal{O}_{\mathcal{X}}(\varphi)$ now follows from definitions. □

Remark 3.6. Because steps (1)–(3) only depend on the reduction of E and the reduction map is surjective, to determine the rank, one need only look at classes of effective divisors in $\text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$. Furthermore, r_{num} is only sensitive to the dual graph Γ and the multidegree $\underline{\deg}(\mathcal{D})$ of \mathcal{D} and is equal to the rank function of Baker–Norine [BN07] on the dual graph of \mathcal{X}_p . Because the rank functions $r_{\text{ab}}, r_{\text{tor}}$ only depend on $\rho(\mathcal{D})$, they factor through the reduction map. We may therefore view $r_{\text{ab}}, r_{\text{tor}}$ as defined on $\text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$.

Remark 3.7. One can also study the ranks associated to a regular proper minimal model. In this case, the Jacobian will have unipotent factors coming from non-reduced components. The combinatorics of bad reduction in this case have been studied by Lorenzini [Lor90, Lor12] and are quite involved.

3.4 Rank of a divisor on a graph

We review some facts from the theory of linear systems on graphs [Bak08, BN07], and in particular the identification of r_{num} with the rank of a divisor on a graph.

DEFINITION 3.8. Two divisors D_1, D_2 on a graph are said to be linearly equivalent if $D_1 - D_2 = \Delta(\varphi)$ for some $\varphi \in \mathcal{M}(\Gamma)$. For a non-negative integer r , a divisor D on Γ is said to have rank at least r if for any effective divisor E on Γ of degree r , the divisor $D - E$ is linearly equivalent to an effective divisor. We define the rank $r(D)$ of D to be -1 if D is not linearly equivalent to an effective divisor, and otherwise we define $r(D)$ to be the largest integer r such that D has rank at least r .

Remark 3.9. Let $\mathcal{X}/\mathcal{O}_{K_p}$ be a regular, semistable curve and let $D \in \text{Div } X_p(K_p)$ be a divisor. Then it follows from Lemma 3.5 that

$$r_{\text{num}}(D) = r(\underline{\deg}(D)).$$

The canonical divisor of Γ is the divisor

$$K_{\Gamma} = \sum_v (\deg(v) - 2)(v).$$

The degree of this divisor is given by

$$\deg K_\Gamma = 2g(\Gamma) - 2$$

where $g(\Gamma)$ is the first Betti number of the graph. We have the following comparison between the multidegree of the canonical bundle of \mathcal{X}_p and the canonical divisor on Γ :

$$\underline{\deg}(K_{\mathcal{X}_p}) = K_\Gamma + \sum_v 2g(X_v).$$

This formula follows by the arguments of [Bak08, Lemma 4.15]. If \mathcal{X}_p is totally degenerate (i.e. $g(X_v) = 0$ for every component X_v of \mathcal{X}_p), then we do have $\underline{\deg}(K_{\mathcal{X}_p}) = K_\Gamma$.

We say that a divisor D on Γ is *special* if $r(K_\Gamma - D) \geq 0$. We will make extensive use of the following theorem.

THEOREM 3.10 (Clifford’s theorem for graphs [BN07, Corollary 3.5]). *Let Γ be a graph and let $D \in \text{Div}(\Gamma)$ be an effective special divisor. Then*

$$r(D) \leq \frac{1}{2} \deg D.$$

When \mathcal{X}_p is totally degenerate, this theorem is sufficient for the proof of Theorem 1.4. In general though, the dual graph Γ of \mathcal{X}_p may have *smaller* genus than \mathcal{X}_p , and the bound on the degree $\deg D_{\text{chab},p}$ of the Chabauty divisor will be $2r + 2(g(\mathcal{X}_p) - g(\Gamma)) > 2r$, which is not sufficient for the application to the Chabauty–Coleman bound.

3.5 Upper bounds

We now observe that these rank functions provide upper bounds for the dimensions of linear subspaces on X_p . For r_{num} , this is the specialization lemma of Baker [Bak08].

PROPOSITION 3.11 (Compare [Bak08, Lemma 2.4]). *Suppose that $X_p(K_p)$ is infinite. Let D be a divisor in $\text{Div}(X_p(K_p))$. Then $r_X(D) = h^0(X_p, \mathcal{O}_{X_p}(D)) - 1$.*

Proof. Let $r = h^0(X_p, \mathcal{O}_{X_p}(D)) - 1$. Let $E \in \text{Div}(X_p(K_p))$ be an effective divisor of degree r . Because the condition that a section of $\mathcal{O}_{X_p}(D)$ vanishes at E imposes at most r conditions, one can find a section s on X_p vanishing at E . Let $\mathcal{D} = \overline{D}$ and extend the section s as a rational section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$. Decompose the divisor of s as the sum of a horizontal divisor H and a vertical divisor $V = -\sum_v \varphi(v)X_v$ for some $\varphi \in \mathcal{M}(\Gamma)$. If s is considered as a section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})(\varphi)$, then it is a regular section and does not vanish along any component of the special fiber. Using that convention, we set $s_v = s|_{X_v}$ for each component X_v of the special fiber. Then the sections s_v satisfy conditions (1)–(4). We conclude that $r_X(D) \geq r$.

Now we show $r_X(D) \leq r$. Because $X_p(K_p)$ is infinite, there is a divisor $E \in \text{Div}(X_p(K_p))$ of degree $r + 1$ such that there is no non-zero section of $\mathcal{O}_{X_p}(D)$ vanishing on E . If $r_X(D) \geq r + 1$, then there would be a twist φ such that $\mathcal{O}_{\mathcal{X}}(\mathcal{D} - \mathcal{E})(\varphi)$ has a section s that is non-zero on every component of \mathcal{X}_p . Such a section s restricts to the generic fiber as a non-zero section of $\mathcal{O}_{X_p}(D - E)$. This is impossible. \square

Remark 3.12. If the residue field \mathbb{F}_p is algebraically closed, then $\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)$ has infinitely many points. By the surjectivity of the reduction map, $X_p(K_p)$ is infinite and the above proposition applies.

Since, by Definition 3.3, $r_1(D) \geq r_2(D) \geq r_3(D) \geq r_4(D)$, we have the following corollary.

COROLLARY 3.13. *Suppose that \mathbb{F}_p is algebraically closed. Let D be a divisor in $\text{Div}(X_p(K_p))$. Then $r_{\text{num}}(\rho(D)), r_{\text{ab}}(\rho(D)), r_{\text{tor}}(\rho(D))$ provide upper bounds for $h^0(X_p, \mathcal{O}_{X_p}(D)) - 1$.*

Remark 3.14. There are thus inequalities $r_{\text{num}}(D) \geq r_{\text{ab}}(D) \geq r_{\text{tor}}(D) \geq r_X(D)$; see § 5 for examples where the inequalities are strict.

Remark 3.15. The rank functions r_{ab} and r_{tor} are sensitive to unramified field extensions. When one extends the residue field, one allows more choices for the effective divisor E , so the rank functions do drop with field extensions. This is not a problem since they do stabilize at a finite extension. The proof of that requires results from the next subsection. To avoid this issue, we can pass to the maximal unramified field extensions. This does not affect the specialization lemma or the rank r_X . In fact, extensions of the residue field give better bounds for r_X .

3.6 Twist general position

Recall that the residue field of \mathcal{O}_{K_p} is infinite. All divisors D_p on \mathcal{X}_p will be supported at smooth points.

LEMMA 3.16. *Let D be a divisor on Γ . Then the set of twists that make D effective,*

$$S_D = \{\varphi \in \mathcal{M}(\Gamma) \sim \mid \Delta(\varphi) + D \geq 0\}$$

is finite.

Proof. Given $\varphi \in \mathcal{M}(\Gamma)$, we may translate to suppose $\max(\varphi(v)) = 0$. We show that φ is bounded below by a constant in terms of $\text{deg}(D_p)$. Suppose $\max(\varphi)$ is achieved at some vertex v_0 . Then we have

$$\sum_{e=vw} (\varphi(w) - \varphi(v)) + D(v) \geq 0.$$

for all v . For w_0 adjacent to v , we have

$$\varphi(w_0) \geq \varphi(v) - D(v) + \sum_{\substack{e=vw \\ w \neq w_0}} (\varphi(v) - \varphi(w)) \geq \text{deg}(v)\varphi(v) - D(v).$$

Taking $v = v_0$ and continuing outward, we see that φ is bounded below. □

DEFINITION 3.17. Let D_p be a divisor on $\mathcal{X}_p^{\text{sm}}$. An effective divisor E_p is said to be in *twist general position* with respect to D_p if for all components X_v and all twists $\varphi \in \mathcal{M}(\Gamma)$ with $\text{deg}(\mathcal{O}_{\mathcal{X}_p}(D_p)(\varphi)) \geq 0$,

$$h^0(\mathcal{O}_{\mathcal{X}_p}(D_p - E_p)(\varphi)|_{X_v}) = \max(0, h^0(\mathcal{O}_{\mathcal{X}_p}(D_p)(\varphi)|_{X_v}) - \text{deg}(\mathcal{O}_{\mathcal{X}_p}(E_p)|_{X_v})).$$

In other words E_p imposes $\text{deg}(E_p|_{X_v})$ conditions on sections of $\mathcal{O}_{\mathcal{X}_p}(D_p)|_{X_v}$. There exist twist general position divisors of any multidegree. Note also that to make sense of the expression $\text{deg}(\mathcal{O}_{\mathcal{X}_p}(D_p)(\varphi))$, it suffices to pick a lift \mathcal{D} of D_p and consider $\text{deg}(\mathcal{O}_{\mathcal{X}}(\mathcal{D})(\varphi))$.

LEMMA 3.18. *Let D_p be a divisor on \mathcal{X}_p . Let $E \in \text{Div}(\Gamma)$. Then there exists a divisor $E_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ such that $\text{deg}(E_p) = E$ and E_p is in twist general position with respect to D_p .*

Proof. For any twist φ and component X_v , the set of effective divisors F_v supported on $X_v \cap \mathcal{X}_p^{\text{sm}}$ of degree $E(v)$ with

$$h^0(\mathcal{O}_{X_p}(D_p)(\varphi)|_{X_v} - F_v) = \max(0, h^0(\mathcal{O}_{X_p}(D_p)(\varphi)|_{X_v}) - E(v))$$

is non-empty and Zariski open in $\text{Sym}^{E(v)}(X_v^{\text{sm}})$. Call this set $U_{\varphi,v}$. Now let $U_v = \bigcap_{\varphi} U_{\varphi,v}$ where the intersection is over the finite set constructed in Lemma 3.16. Choose a divisor E_v in U_v for each v . The sum of divisors, $E_p = \sum E_v$ is the desired divisor. □

3.7 Clifford bounds

We prove the Clifford bound for r_{ab} . By Remark 3.14 this implies the corresponding bounds for r_{tor} . We suppose that \mathbb{F}_p is algebraically closed. We also pick once and for all a (not necessarily effective) divisor $K_{\mathcal{X}_p} \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ in the class of the canonical bundle on the special fiber.

Such a divisor exists by Bertini’s theorem; indeed, by [Liu06, 7.1, Lemma 1.31] any line bundle on a projective variety is a difference of very ample line bundles \mathcal{L} and \mathcal{L}' , and by Bertini’s theorem [Har77, Theorem 8.18] we can choose a section of each which does not vanish at the finitely many singular points of \mathcal{X}_p .

DEFINITION 3.19. Let L be a line bundle on \mathcal{X}_p . For a vertex v of the dual graph Γ , we say v is *Clifford with respect to L* if $\text{deg}(L|_{X_v}) < 2g$.

In this case, the usual Clifford bounds (if L is special) or the Riemann–Roch theorem (if L is non-special) imply

$$h^0(L|_{X_v}) < g + 1.$$

Consequently, for a generic degree g divisor D on X_v , we do not expect $h^0(L|_{X_v} - D)$ to have a non-zero section.

PROPOSITION 3.20. Let D_p be an effective divisor on \mathcal{X}_p supported at smooth points. Then,

$$r_{\text{ab}}(K_{\mathcal{X}_p} - D_p) \leq g - \frac{\text{deg } D_p}{2} - 1.$$

Remark 3.21. The idea of the proof is the following. For a vertex v corresponding to a component X_v of \mathcal{X}_p ,

$$\text{deg}(K_{\mathcal{X}_p})|_{X_v} = K_\Gamma(v) + 2g(X_v);$$

in particular, if $K_\Gamma - \underline{\text{deg}}(D_p)$ is not equivalent to an effective divisor on Γ (e.g., it may have negative degree), then for every twist φ , there is a component X_{v_φ} of \mathcal{X}_p such that

$$\text{deg}((K_{\mathcal{X}_p} - D_p)(\varphi)|_{X_{v_\varphi}}) < 2g(X_{v_\varphi}).$$

In particular, Clifford’s theorem or the Riemann–Roch theorem for X_{v_φ} implies that

$$h^0((K_{\mathcal{X}_p} - D_p)(\varphi)|_{X_{v_\varphi}}) < g(X_{v_\varphi}) + 1.$$

The upshot is that, if Q is a divisor in twist general position such that $\text{deg } Q|_{X_v} = g(X_v)$ for all v , then for every twist φ , $(K_{\mathcal{X}_p} - D_p - Q)(\varphi)|_{X_{v_\varphi}}$ has no sections, so $r_{\text{ab}}(K_{\mathcal{X}_p} - D_p) < \text{deg } Q = \sum g(X_v)$.

This does not quite give the desired bound, since it may happen that $\sum g(X_v) > g - \text{deg } D_p/2 - 1$; the subtleties of the proof are:

- (1) to find a component X_v of \mathcal{X}_p such that fewer than $g(X_v)$ points are necessary to force the rank of $(K_{\mathcal{X}_p} - D_p - Q)(\varphi)|_{X_{v_\varphi}}$ to be -1 ; and
- (2) when $K_{\mathcal{X}_p} - D_p$ has sections on every component, we can add a divisor Q' in twist general position such that $K_{\mathcal{X}_p} - D_p - Q'$ has no sections and apply (1); we can keep the degree of Q' small by applying the Riemann–Roch theorem for graphs to $K_\Gamma - \underline{\text{deg}}(D_p)$.

Proof of Proposition 3.20. We first consider the case $\text{deg}(K_\Gamma - \underline{\text{deg}}(D_p)) < 0$. Let $S_{\text{deg}(D_p)}$ be the finite set of Lemma 3.16. If $S_{\text{deg}(D_p)}$ is empty, the rank of D_p is -1 and we are done. So suppose $S_{\text{deg}(D_p)}$ is non-empty. For $\varphi \in S_{\text{deg}(D_p)}$, let $\text{Cliff}(\varphi)$ be the set of vertices that are Clifford with respect to $(K_{\mathcal{X}_p} - D_p)(\varphi)$. We call such vertices φ -Clifford. By degree considerations, $\text{Cliff}(\varphi) \neq \emptyset$

for all $\varphi \in S_{\underline{\deg}(D_p)}$. Let m be the minimum of $|\text{Cliff}(\varphi)|$ taken over all $\varphi \in S_{\underline{\deg}(D_p)}$ and

$$S_m = \{\varphi \in S_{\underline{\deg}(D_p)} \mid |\text{Cliff}(\varphi)| = m\}.$$

Now, let

$$M(\varphi) = \max\{\deg((K_{\mathcal{X}_p} - D_p)(\varphi)|_{X_v}) \mid v \in \text{Cliff}(\varphi)\}.$$

Pick $\varphi \in S_m$ maximizing $M(\varphi)$. Let v_φ be a φ -Clifford vertex such that

$$\deg((K_{\mathcal{X}_p} - D_p)(\varphi)|_{X_{v_\varphi}}) = M(\varphi).$$

Now let Q_p be a divisor consisting of $\lfloor M(\varphi)/2 + 1 \rfloor$ points on X_{v_φ} and $g(X_w)$ points on X_w for each non- φ -Clifford w , chosen in twist general position with respect to $K_{\mathcal{X}_p} - D_p$. Note that $\deg(Q) \leq g - \deg D_p/2$.

We claim that $r_{\text{ab}}(K_{\mathcal{X}_p} - D_p - Q_p) < 0$. Suppose to the contrary that there is a twist φ' such that $(K_{\mathcal{X}_p} - D_p - Q_p)(\varphi')$ has a section on every component. By construction $K_{\mathcal{X}_p} - D_p$ has at least m φ' -Clifford vertices. If it has more than m φ' -Clifford vertices, then there is some vertex w that is φ' -Clifford but not φ -Clifford. Therefore, Q_p contains $g(X_w)$ points on X_w . Since $h^0((K_{\mathcal{X}_p} - D_p)(\varphi')|_{X_w}) < g(X_w) + 1$ by Clifford's theorem or the Riemann–Roch theorem on X_w , we can conclude that $h^0((K_{\mathcal{X}_p} - D_p - Q_p)(\varphi')|_{X_w}) = 0$.

We may suppose that $\text{Cliff}(\varphi') = m$. Now, if v_φ is φ' -Clifford, then

$$\deg((K_{\mathcal{X}_p} - D_p)(\varphi')|_{X_{v_\varphi}}) \leq M(\varphi') \leq M(\varphi)$$

by construction. But then by Clifford's theorem

$$h^0((K_{\mathcal{X}_p} - D_p)(\varphi')|_{X_{v_\varphi}}) \leq \frac{M(\varphi)}{2} + 1$$

so

$$h^0((K_{\mathcal{X}_p} - D_p - Q)(\varphi')|_{X_{v_\varphi}}) = 0.$$

Therefore, we may suppose that v_φ is not φ' -Clifford. But because $|\text{Cliff}(\varphi')| = |\text{Cliff}(\varphi)|$, there is some vertex w that is φ' -Clifford but not φ -Clifford. By the reasoning above, there cannot be a section of $(K_{\mathcal{X}_p} - D_p - Q)(\varphi')$ on that component.

We now consider the case $\deg(K_\Gamma - \underline{\deg}(D_p)) \geq 0$. By Baker–Norine's Clifford bounds (Theorem 3.10),

$$r_{\text{num}}(K_\Gamma - \rho(D_p)) \leq g(\Gamma) - \frac{\deg D_p}{2} - 1.$$

Therefore, if we set $r = \lfloor g(\Gamma) - \deg D_p/2 - 1 \rfloor$, we may pick a divisor $Q' \in \text{Div}(\Gamma)$ of degree $r + 1$ such that $K_\Gamma - \underline{\deg}(D_p) - Q'$ is not num-linearly equivalent to an effective divisor on Γ . Now, pick $Q'_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ with $\underline{\deg}(Q'_p) = Q'$. Now, for each v , let $Q_v \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ be a divisor of degree $g(v)$ in twist general position with respect to $K_{\mathcal{X}_p} - D_p - Q'_p$. Let

$$Q_p = Q'_p + \sum_v Q_v.$$

Note

$$\deg(Q_p) = r + 1 + \sum g(X_v) = \left\lfloor g(\Gamma) + \sum g(X_v) - \frac{\deg D_p}{2} - 1 \right\rfloor + 1 \leq g - \frac{\deg(D_p)}{2}.$$

We claim that $r_{\text{ab}}(K_{\mathcal{X}_p} - D_p - Q_p) < 0$. In other words, even after twisting by some φ , there is some component X_v such that $K_{\mathcal{X}_p} - D_p - Q_p|_{X_v}$ has no section. By the definition of Baker–Norine rank, for any twist φ there is a vertex v such that $(K_\Gamma - \underline{\deg}(D_p) - \underline{\deg}(Q_p))(v) < 0$.

Consequently,

$$\deg((K_{\mathcal{X}_p} - D_p - Q'_p)(\varphi)|_{X_v}) < 2g$$

and the usual Clifford or Riemann–Roch bounds apply to show that

$$h^0((K_{\mathcal{X}_p} - D_p - Q'_p)(\varphi)|_{X_v}) < g + 1.$$

By our choice of Q_p and the definition of twist general position,

$$h^0((K_{\mathcal{X}_p} - D_p - Q_p)(\varphi)|_{X_v}) = 0. \quad \square$$

Remark 3.22. It follows from the proof that one can in fact choose the divisor E_p to avoid any finite set of points on \mathcal{X}_p . In other words, there exists an effective divisor $E_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ such that:

- (i) $\deg E_p \leq \frac{1}{2} \deg(K_{X_p} - D_p)$;
- (ii) for any twist φ , $\mathcal{O}_{\mathcal{X}}(\varphi)|_{X_v} \otimes \mathcal{O}_{\mathcal{X}_p}(K_{\mathcal{X}_p} - D_p - E_p)|_{X_v}$ has no non-zero sections for some component X_v of \mathcal{X}_p ; and
- (iii) $\text{Supp}(E_p) \cap \text{Supp}(K_{\mathcal{X}_p} - D_p) = \emptyset$.

Remark 3.23. One can use our arguments to prove the analogue of Clifford’s theorem in its usual form: let $D_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$; if $r_{\text{ab}}(K_{\mathcal{X}_p} - D_p) \geq 0$, then $r_{\text{ab}}(D_p) \leq \deg(D_p)/2$. The hypothesis produces a twist φ such that $(K_{\mathcal{X}_p} - D_p)(\varphi)$ has a section s_v on every component X_v . One applies the above argument to $(K_{\mathcal{X}_p})|_{X_v} - (s_v)$ on each component. Unfortunately, the zero-locus of s_v is not necessarily supported on smooth points of \mathcal{X}_p so the above proposition does not directly apply.

We easily recover the following expected corollary.

COROLLARY 3.24. $r_{\text{ab}}(K_{\mathcal{X}_p}) = g - 1$.

Proof. By the specialization lemma, $r_{\text{ab}}(K_{\mathcal{X}_p}) \geq g - 1$. The opposite inequality is given by Proposition 3.20. □

Remark 3.25. It is a result of Amini–Baker [AB12] that r_{ab} satisfies the appropriate version of the Riemann–Roch theorem. They note that the Clifford bounds follow from this Riemann–Roch theorem. We include an independent proof for the sake of completeness. Additionally, our proof contains an algorithm that may give bounds sharper than Theorem 1.4 in specific examples.

Remark 3.26. We do not know if r_{tor} satisfies the Riemann–Roch theorem. In practice, one may obtain better bounds on r_X by considering r_{tor} . See § 5.1 for examples.

4. Bounding the degree of the Chabauty divisor

Here we show that the degree of the Chabauty divisor $D_{\text{chab},p}$ of § 2.5 is at most $2r$, completing the proof of Theorem 1.4. The rough idea of the proof is the following two steps: (i) passing to a semistable model so that our results about geometric rank functions of § 3 apply; and (ii) using the subspace V_{chab} (see Proposition 2.6) to show that $g - r - 1 \leq r_{\text{ab}}(K_{\mathcal{X}_p} - D_{\text{chab},p})$, which, combined with the Clifford bound $r_{\text{ab}}(K_{\mathcal{X}_p} - D_{\text{chab},p}) \leq \frac{1}{2} \deg(K_{\mathcal{X}_p} - D_{\text{chab},p})$ and a rearrangement, directly proves the theorem. (Actually, step (ii) is a bit more subtle than this, and we must make use of the ‘avoidance’ variant of Clifford’s theorem of Remark 3.22.)

In the following we continue with the setup of § 2 to prove the theorem in the semistable case.

PROPOSITION 4.1. *Let \mathcal{X} be a regular and semistable curve over \mathcal{O}_{K_p} . Let $D_{\text{chab},p}$ be the Chabauty divisor (see (2.11)). Then $\deg D_{\text{chab},p} \leq 2r$.*

Proof. By using the avoidance form of the Clifford bounds as in Remark 3.22, we obtain an effective divisor $E_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ such that:

- (i) $\deg E_p \leq \frac{1}{2} \deg(K_{\mathcal{X}_p} - D_{\text{chab},p})$;
- (ii) for any twist φ , $\mathcal{O}_{\mathcal{X}}(\varphi)|_{X_v} \otimes \mathcal{O}_{\mathcal{X}_p}(K_{\mathcal{X}_p} - D_{\text{chab},p} - E_p)|_{X_v}$ has no non-zero sections for some component X_v of \mathcal{X}_p ; and
- (iii) $\text{Supp}(E_p) \cap \text{Supp}(K_{\mathcal{X}_p} - D_{\text{chab},p}) = \emptyset$.

Moreover, we claim that $\deg(E_p) \geq \dim V_{\text{chab}} - 1$. Indeed, suppose not, and let $E \in \text{Div}(X_p(K_p))$ be an effective divisor such that $\rho(E) = E_p$. Then there would exist an element ω of V_{chab} vanishing on E (with the correct multiplicities). We may view ω as a rational section of $K_{\mathcal{X}}$. By picking a twist φ as in the proof of Proposition 3.11, we can treat ω as a regular section of $K_{\mathcal{X}}(\varphi)$ vanishing along no component of the special fiber. For any component X_v , ω vanishes on E_p and on $D_{\text{chab},p}$ by construction. Since $\text{Supp } \rho(E_p)$ is disjoint from $\text{Supp } D_{\text{chab},p}$, ω restricts to a non-zero section of $\mathcal{O}_{\mathcal{X}}(\varphi)|_{X_v} \otimes \mathcal{O}_{\mathcal{X}_p}(K_{\mathcal{X}_p} - D_{\text{chab},p} - E_p)|_{X_v}$ for each v . This contradicts condition (ii) in the choice of E_p .

Putting these inequalities together gives

$$\deg(K_{\mathcal{X}_p} - D_{\text{chab},p}) \geq 2 \deg(E_p) \geq 2 \dim V_{\text{chab}} - 2 \geq 2g - 2r - 2.$$

Rearranging completes the proof. □

Remark 4.2. One can obtain this bound using linear systems on weighted graphs as developed by Amini and Caporaso [AC11]. Their theory is more combinatorial than ours and does not consider the algebraic geometry of the irreducible components. Our approach, however, may give sharper bounds in examples when one has an understanding of the geometry of the components.

Remark 4.3. We describe in this remark a rank function that is constructed to give the sharpest bound in situations such as those considered above. Note that the divisor $D_{\text{chab},p}$ is supported on the special fiber. Given a line bundle \mathcal{L} on a semistable model \mathcal{X} and an effective divisor D_p supported on smooth points of the special fiber, one can define a rank function $r(\mathcal{L}, D_p)$ by saying that $r(\mathcal{L}, D_p) \geq r$ if and only if for every effective divisor $E_p \in \text{Div}(\mathcal{X}_p^{\text{sm}}(\mathbb{F}_p))$ of degree r , there is a section s of $\mathcal{L}|_X$ such that $D_p + E_p$ is contained (with multiplicity) in the reduction of the divisor (s) . Indeed, the proof of Proposition 4.1 uses a bound on $r(K_{\mathcal{X}_p}, D_{\text{chab},p})$.

Now we remove the semistability hypothesis of Proposition 4.1.

THEOREM 4.4. *Let \mathcal{X} be a proper regular curve over \mathcal{O}_{K_p} with Chabauty divisor $D_{\text{chab},p}$. Then $\deg D_{\text{chab},p} \leq 2r$.*

Proof. By replacing K_p by its maximal unramified extension, we may suppose that the residue field of \mathcal{O}_{K_p} is algebraically closed. Let K'_p be a finite extension of K_p over which the base extension $X_p \times_{K_p} K'_p$ has regular semistable reduction. Denote the valuation on K'_p by v' . Let \mathcal{X}' be the base change $\mathcal{X} \times_{\mathcal{O}_{K_p}} \mathcal{O}_{K'_p}$ of \mathcal{X} to $\mathcal{O}_{K'_p}$. (Note that $\mathcal{O}_{K'_p}$ is a DVR with infinite residue field \mathbb{F}_p .) Then \mathcal{X}' may no longer be regular. However, there exists a dominant proper map $\mathcal{X}'' \rightarrow \mathcal{X}'$ of curves over \mathcal{O}'_{K_p} such that \mathcal{X}'' is regular and *semistable* (though probably not minimal) and which is an isomorphism on generic fibers. Moreover, we may chose the map $\mathcal{X}'' \rightarrow \mathcal{X}'$ so that no component of \mathcal{X}'' is contracted to a smooth point of \mathcal{X}'_p by the composition $\mathcal{X}'' \rightarrow \mathcal{X}'$.

Define $V'_{\text{chab}} = K'_p \otimes_{K_p} V_{\text{chab}}$ and define $D'_{\text{chab},p}$ as in Definition 2.10; i.e.,

$$D'_{\text{chab},p} = \sum_{\tilde{Q}' \in \mathcal{X}''^{\text{sm}}(\mathbb{F}'_p)} (\min\{n(\omega', \tilde{Q}') \mid \omega' \in V'_{\text{chab}}\}) \tilde{Q}'.$$

We claim that $\deg D_{\text{chab},p} \leq \deg D'_{\text{chab},p}$, in which case the theorem would follow from Proposition 4.1. In fact, the pullback of $D_{\text{chab},p}$ to \mathcal{X}'' is a subdivisor of $D'_{\text{chab},p}$. Indeed, let \tilde{Q} be a point of the support of $D_{\text{chab},p}$. Let X_w be the component of \mathcal{X}_p containing \tilde{Q} . Because X_w contains smooth points of \mathcal{X}_p , the special fiber \mathcal{X}_p must be reduced along X_w . Since the desingularization $\mathcal{X}'' \rightarrow \mathcal{X}'$ is chosen not to affect smooth points of \mathcal{X}'_p , the inverse image of \tilde{Q} under $\mathcal{X}'' \rightarrow \mathcal{X}$ is a point \tilde{Q}' of $\mathcal{X}''^{\text{sm}}(\mathbb{F}'_p)$. Now, let $\omega' \in V'_{\text{chab}}$. Then ω' is a finite sum $\sum a_i \otimes \omega_i$, for $\omega_i \in V_{\text{chab}}$ and $a_i \in K'_p$. Recall that every differential on X_p has an extension to \mathcal{X} as a rational section of the relative dualizing sheaf. By associating elements of K_p across the tensor product, we may suppose that the extensions of the sections ω_i have neither poles nor zeroes along X_w . Moreover, we can choose an expression $\sum a_i \otimes \omega_i$ for ω' such that $\min_i v'(a_i)$ is maximized. Now, express each a_i as $(\pi')^{k_i} u'_i$ where π' is a uniformizer of K'_p and u'_i has valuation equal to 0. Let k be the minimum of the integers k_i .

Let X'_w be the component of \mathcal{X}'' containing \tilde{Q}' . Note that for any differential ω on X_p , its extension to \mathcal{X} vanishes along X_p if and only if its extension to \mathcal{X}'' vanishes along X'_w . We claim that $\sum_{i|k_i=k} u'_i \otimes \omega_i$ does not vanish along X'_w . If it did, for each i with $k_i = k$, take $u_i \in K_p$ that has the same residue as u'_i and rewrite the sum of terms with minimum $v'(a_i)$ as

$$(\pi')^k \left(\sum_{i|k_i=k} (u'_i - u_i) \otimes \omega_i + \sum_{i|k_i=k} u_i \otimes \omega_i \right).$$

Since $\sum u_i \otimes \omega_i = 1 \otimes \sum u_i \omega_i$ vanishes along X_w , it can be written as $(\pi')^l \otimes \tilde{\omega}$ for some $l > 0$ and $\tilde{\omega}$ in V_{chab} where $\tilde{\omega}$ does not vanish along X_w . Moreover, each $u'_i - u_i$ has positive valuation. Consequently, ω can be written as a linear combination with higher $\min(v'(a_i))$ contradicting our choice of expression for ω .

Now, since $\sum_{i|k_i=k} u'_i \otimes \omega_i$ does not vanish along X'_w , by the non-Archimedean property of vanishing orders,

$$n(\omega', \tilde{Q}') = v_{\tilde{Q}'} \left(\sum_{i|k_i=k} u_i \omega_i \right) \geq \min_{i|k_i=k} v_{\tilde{Q}'}(\omega_i) = \min_{i|k_i=k} v_{\tilde{Q}}(\omega_i) \geq n_{\tilde{Q}}.$$

To justify the second equality, it suffices to note that the map of complete local rings

$$\widehat{\mathcal{O}}_{\mathcal{X}, \tilde{Q}} \rightarrow \widehat{\mathcal{O}}_{\mathcal{X}'', \tilde{Q}'}$$

is isomorphic to the base extension map

$$K_p[[t]] \rightarrow K'_p[[t]]$$

(see for instance the proof of Proposition 10.4.6 [Liu06]). □

5. Examples

Example 5.1. Here we give an example of a hyperelliptic curve with bad reduction where the refined bound of Theorem 1.4 is sharp. Let X be the smooth genus 3 hyperelliptic curve over \mathbb{Q} with affine piece

$$-2 \cdot 11 \cdot 19 \cdot 173 \cdot y^2 = (x - 50)(x - 9)(x - 3)(x + 13)(x^3 + 2x^2 + 3x + 4).$$

This curve has bad cuspidal reduction at the prime 5 and its regular proper minimal model \mathcal{X} over \mathbb{Z}_5 is given by the same equation as the above Weierstrass model. A descent calculation using Magma’s `TwoSelmerGroup` function shows that its Jacobian has rank 1. A point count reveals that

$$X(\mathbb{Q}) \supset \{\infty, (50, 0), (9, 0), (3, 0), (-13, 0), (25, 20\,247\,920), (25, -20\,247\,920)\}$$

(and in particular $7 \leq \#X(\mathbb{Q})$) and $\#\mathcal{X}_5^{\text{sm}}(\mathbb{F}_5) = 5$. Theorem 1.4 reads

$$7 \leq \#X(\mathbb{Q}) \leq \#\mathcal{X}_5^{\text{sm}}(\mathbb{F}_5) + 2 = 7,$$

which determines $X(\mathbb{Q})$.

Let J be the Jacobian of X . Then J is absolutely simple. Indeed, J has good reduction at 13 and for $i \in \{1, \dots, 30\}$ a computation reveals that the characteristic polynomial of Frobenius for $J_{\mathbb{F}_{13^i}}$ is irreducible. By an argument analogous to [PS97, Proposition 14.4] (see also [Sto08, Lemma 3]) we conclude that $J_{\mathbb{F}_{13}}$ (and hence J) is absolutely simple.

One can check that 5 is the only prime at which our refinement of the Chabauty–Coleman bound is sharp. Thus, one can use neither a map to a curve of smaller genus nor the Chabauty–Coleman bound at a prime of good reduction to determine $X(\mathbb{Q})$.

5.1 Remarks on obtaining better bounds

Remark 5.2. Let X be a smooth projective geometrically integral curve over an algebraically closed field. For an integer $0 \leq i \leq g - 1$, let

$$r(i) = \max\{\deg D \mid D \geq 0 \text{ and } r(K_X - D) \geq g - i - 1\};$$

Clifford’s theorem implies that $r(i) \leq 2i$. In fact $r(i) = 2i$ if and only if X is hyperelliptic, with the maximum witnessed when D is a sum of pairs of points which are conjugate under the hyperelliptic involution; see [Har77, ch. III, Theorem 5.4]. The true value of $r(i)$ depends on the geometry of X ; see [Sto06, § 3].

The situation is even more delicate (but usually favorable) when \mathcal{X}_p has multiple components; next, we give examples where the inequalities in $r_{\text{num}}(D) \geq r_{\text{ab}}(D) \geq r_{\text{tor}}(D) \geq r_X(D)$ are strict.

Example 5.3 ($r_{\text{num}}(D) > r_{\text{ab}}(D)$). In general, on a semistable curve \mathcal{X} which is not totally degenerate (i.e. $g(X_v) > 0$ for some component X_v of \mathcal{X}_p) one expects to find divisors D such that $r_{\text{num}}(D) > r_{\text{ab}}(D)$. For example, let \mathcal{X} be a genus 1 curve with good reduction, so that Γ is a single vertex. Pick $P \in X_p(K_p)$. Then $r_{\text{ab}}(P) = 0$ since there is no non-constant section of $\mathcal{O}_{\mathcal{X}_p}(P)$ while $r_{\text{num}}(P) = 1$.

Example 5.4 ($r_{\text{ab}}(D) > r_{\text{tor}}(D)$). Let \mathcal{X} be a regular curve which is a regular proper minimal model of its generic fiber and whose special fiber has the following form: one component X_v is a hyperelliptic curve; the other component X_w is a rational curve; the two components meet in two nodes which are neither fixed points of the hyperelliptic involution on X_v nor hyperelliptically conjugate on X_v . Note that the special fiber is not hyperelliptic since the hyperelliptic involution does not extend from X_v (which would follow from the universal property of the regular proper minimal model). Let $P, Q \in \mathcal{X}_p^{\text{sm}}(\mathbb{F}_p)$ be hyperelliptically conjugate points on X_v . Since there is a section of $\mathcal{O}_{X_v}(P + Q)$, $r_{\text{ab}}(P + Q) = 0$. We claim that $r_{\text{tor}}(P + Q) = -1$. Notice that no non-trivial twist of $P + Q$ has non-negative degree on both components. Therefore, we only have to consider sections of $\mathcal{O}_{\mathcal{X}_p}(P + Q)$. A section of $\mathcal{O}_{\mathcal{X}_p}(P + Q)|_{X_v}$ gives a hyperelliptic projection which takes different values on the nodes. A section of $\mathcal{O}_{\mathcal{X}_p}(P + Q)|_{X_w} = \mathcal{O}_{X_w}$ is just a constant function. Therefore, no choice of sections match across both nodes.

Example 5.5 ($r_{\text{tor}}(D) > r_X(D)$). Let \mathcal{X} be a non-hyperelliptic curve with good reduction whose special fiber is hyperelliptic. Let $P \in \mathcal{X}(K_{\mathfrak{p}})$ be a point whose reduction $\rho(P)$ to the special fiber is fixed by the hyperelliptic involution. Then $r_{\text{tor}}(2P) = 1$ since for any $Q \in \mathcal{X}_{\mathfrak{p}}(\mathbb{F}_{\mathfrak{p}})$, $\mathcal{O}_{\mathcal{X}_{\mathfrak{p}}}(\rho(2P) - Q)$ has a non-vanishing section. However, any section of $\mathcal{O}_{\mathcal{X}}(2P)$ must restrict to the special fiber as a constant section. If such a section vanishes on Q , then it must vanish identically on \mathcal{X} .

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