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Research Article

The Characterization and Stability of g-Riesz Frames for Super Hilbert Space

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G-frames and g-Riesz frames as generalized frames in Hilbert spaces have been studied by many authors in recent years. The super Hilbert space has a certain advantage compared with the Hilbert space in the field of studying quantum mechanics. In this paper, for super Hilbert space $H \oplus K$, the definitions of a g-Riesz frame and minimal g-complete are put forward; also a characterization of g-Riesz frames is obtained. In particular, we generalize them to general super Hilbert space $L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Finally, a conclusion of the stability of a g-Riesz frame for the super Hilbert space is given.

1. Introduction

Frame as generalized basis in Hilbert space was first introduced by Duffin and Schaeffer [1] during the studied nonharmonic Fourier series in 1952. In 1986, Daubechies et al. [2] reintroduced the concept of frame. Now the theory of frames has been widely used in many areas such as the characterization of function spaces, signal processing, filter theory, image processing and quantum mechanics. We refer to [3–10] for an introduction to frame theory in Hilbert space and its application.

Sun [11] introduced g-Riesz basis and g-frame; g-frame actually generalized the concept of frame. Since then, g-frame, g-frame sequence, Besselian g-frame, near g-Riesz bases, and so on are focused on and studied by many authors. The authors [12] introduced Besselian g-frames and near g-Riesz bases in Hilbert space and gave some characterizations of them. In [13], g-Riesz frames were studied and some corresponding results were given. In [14], the concept of g-bases in Hilbert spaces was introduced and some properties about g-bases were proved. Because super Hilbert spaces arose naturally as the state space of a quantum field in the functional Schrödinger representation of spinor quantum field theory and it provided a means to bring supersymmetric quantum field theories into a form resembling standard quantum mechanics, the super Hilbert space has certain advantages compared with the Hilbert space in quantum mechanics. With the extensive research of super Hilbert space [15–20], scholars were beginning to study g-frames for super Hilbert spaces [20, 21]. Unfortunately, although g-Riesz frames were considered as a class of important frames, we have not consulted the literature of the g-Riesz frame for super Hilbert space $H \oplus K$ so far. Because g-Riesz frames play an important role in approximate calculating coefficients of g-frames, therefore, the study of g-Riesz frames for super Hilbert space $H \oplus K$ has a double meaning of theory and application. In order to enrich the frame theory, we give the concept of g-Riesz frame for super Hilbert space $H \oplus K$ and the characterization and necessary condition for g-Riesz frame. We also expand corresponding conclusions to general super Hilbert space $L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Finally, we consider the stability of a g-Riesz frame for super Hilbert space.

Throughout this paper, H and K are two complex Hilbert spaces and $\{H_j : j \in J\}$ is a sequence of closed subspaces of H. $L(H, H_j)$ is the collection of all bounded linear operators from H into H_j, where J is a subset of integers $\mathbb{Z}$. $l^2(\{H_j\}_{j\in J})$ is defined by

$$l^2\left(\{H_j\}_{j\in J}\right) = \left\{\{a_j\}_{j\in J} : a_j \in H_j, j \in J, \sum_{j\in J}\left\|a_j\right\|^2 < +\infty\right\}, \quad (1)$$

with the inner product given by

$$\left\langle \{a_j\}_{j\in J}, \{b_j\}_{j\in J}\right\rangle = \sum_{j\in J}\left\langle a_j, b_j\right\rangle_{H_j}, \quad (2)$$

and it is clear that $l^2(\{H_j\}_{j\in J})$ is a complex Hilbert space.

The literature [16] gave the definition of super Hilbert space.

Definition 1 (see [16, p.557]). Super Hilbert space is a direct sum that $H = H_0 \oplus H_1$ of two complex Hilbert spaces $(H_0, \langle\cdot,\cdot\rangle_0), (H_1, \langle\cdot,\cdot\rangle_1)$ equipped with the super Hermitian form $\langle\langle\cdot,\cdot\rangle\rangle = \langle\cdot,\cdot\rangle_0 + \langle\cdot,\cdot\rangle_1$.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Definition 2 (see [11, Definition 1.1]). A sequence $\{\Lambda_j \in L(H, H_j) : j \in J\}$ is called a g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist two positive constants A and B such that, for all $f \in H$,

$$A\|f\|^2 \le \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \le B\|f\|^2. \quad (3)$$

The constants A and B are called the lower and upper bounds of g-frame, respectively. If the right hand inequality holds, then we say that $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. If $A = B$, we call this g-frame a tight g-frame. If a g-frame ceases to be a g-frame whenever any single element is removed from $\{\Lambda_j\}_{j\in J}$, it is called an exact g-frame.

Definition 3 (see [14, Definition 2.2]). One says that $\{\Lambda_j \in L(H, H_j) : j \in J\}$ is g-complete, if $\{f \in H : \Lambda_j f = 0, \text{ for all } j\} = \{0\}$.

Definition 4 (see [11, Definition 3.1]). A sequence $\{\Lambda_j \in L(H, H_j) : j \in J\}$ is called a g-Riesz basis for H with respect to $\{H_j\}_{j\in J}$, if the sequence $\{\Lambda_j\}_{j\in J}$ is g-complete and there exist positive constants A and B such that

$$A\sum_{j\in J_1}\left\|g_j\right\|^2 \le \left\|\sum_{j\in J_1}\Lambda_j^* g_j\right\|^2 \le B\sum_{j\in J_1}\left\|g_j\right\|^2 \quad (4)$$

for all finite subset $J_1 \subset J$ and $g_j \in H_j$, $j \in J_1$. The constants A and B are called the lower and upper bounds of g-Riesz bases, respectively.

Definition 5 (see [11, Definition 3.1]). Let $\{\Lambda_j \in L(H, H_j) : j \in J\}$. Suppose that $\{e_{jk}\}_{k\in K_j}$ is an orthonormal basis for H_j, where K_j is a subset of $\mathbb{Z}$. Denote

$$u_{jk} = \Lambda_j^* e_{jk}, \quad j \in J,\ k \in K_j, \quad (5)$$

$$\Lambda_j f = \sum_{k\in K_j}\left\langle f, u_{jk}\right\rangle e_{jk}, \quad \forall f \in H. \quad (6)$$

We call $\{u_{jk}\}_{j\in J, k\in K_j}$ the sequence induced by $\{\Lambda_j\}_{j\in J}$ with respect to $\{e_{jk}\}_{j\in J, k\in K_j}$.

Theorem 6 (see [21, Proposition 2.8]). *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be sequences in $L(H, H_j)$ and $L(K, H_j)$, respectively, and let $\{e_{jk}\}_{k\in K_j}$ be an orthonormal basis for H_j, where K_j is a subset of $\mathbb{Z}$, and let $\psi_{jk} = \Lambda_j^* e_{jk}$, $\varphi_{jk} = \Gamma_j^* e_{jk}$, and $\Theta_j(f, g) = \Lambda_j f + \Gamma_j g$. Then $\{(\psi_{jk}, \varphi_{jk})\}_{j\in J, k\in K_j}$ is a frame (resp., Bessel sequence, Riesz basis) for super Hilbert space $H \oplus K$ if and only if $\{\Theta_j \in L(H \oplus K, H_j) : j \in J)\}$ is a g-frame (resp., g-Bessel sequence, g-Riesz basis) for $H \oplus K$ with respect to H_j.*

Proposition 7 (see [21, Proposition 2.9]). *Let $\{\Theta_j \in L(H \oplus K, H_j) : j \in J\}$, $\{(\Lambda_j \in L(H, H_j) : j \in J\}$, and $\{\Gamma_j \in L(K, H_j) : j \in J\}$ be g-frames, where $\Theta_j(f, g) = \Lambda_j f + \Gamma_j g$. Then g-frame operator for $\{\Theta_j \in L(H \oplus K, H_j) : j \in J\}$ is defined by*

$$S_\Theta(f, g) = \left(\sum_{j\in J}\left(\Lambda_j^*\Lambda_j f + \Lambda_j^*\Gamma_j g\right), \sum_{j\in J}\left(\Gamma_j^*\Lambda_j f + \Gamma_j^*\Gamma_j g\right)\right). \quad (7)$$

The literature [21] introduced the concept of disjoint g-frames. A pair of g-frames $\{\Lambda_j \in L(H, H_j) : j \in J\}$ and $\{\Gamma_j \in L(K, H_j) : j \in J\}$ is called disjoint if $\Theta_j(f, g)$ is a g-frame for $H \oplus K$, where $\Theta_j(f, g) = \Lambda_j f + \Gamma_j g$.

With the definition of g-complete of g-frame for Hilbert space, we give the definition of g-complete of g-frame for super Hilbert space as follows.

Definition 8. $\{\Theta_j \in L(H \oplus K, H_j)\}_{j=1}^{\infty}$ is called g-complete with respect to $\{H_j\}_{j=1}^{\infty}$ under the condition of that $\{(f, g) : \Theta_j(f, g) = 0, \text{for all } j\} = \{(0, 0)\}$.

3. Characterization of g-Riesz Frame for Super Hilbert Space

In this section, we first give the concept and the characterization of g-Riesz frame for super Hilbert space $H \oplus K$, and then we generalize them to super Hilbert space $L_1 \oplus L_2 \oplus \cdots \oplus L_n$.

3.1. Characterization of g-Riesz Frame for Super Hilbert Space $H \oplus K$. Before giving the characterization of g-Riesz frames for super Hilbert space $H \oplus K$, we give the definition of g-Riesz frames and some related lemmas.

Suppose that I is a subset of J, and denote

$$(H\oplus K)_I=\overline{\left\{\sum_{j\in I_1}\Theta_j^*g_j=\left(\sum_{j\in I_1}\Lambda_j^*g_j,\sum_{j\in I_1}\Gamma_j^*g_j\right):\ \text{for any finite } I_1\subset I,\ g_j\in H_j,\ j\in I_1\right\}}. \tag{8}$$

Definition 9. Suppose that $\{\Theta_j\in L(H\oplus K,H_j):j\in J\}$ is a g-frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$. One says that $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame if every subfamily $\{\Theta_j\}_{j\in I}$ of $\{\Theta_j\}_{j\in J}$ is a g-frame for $(H\oplus K)_I$ with respect to $\{H_j\}_{j\in I}$ with uniform g-frame lower bounds.

For the above $(H\oplus K)_I$, we have the following.

Lemma 10. *Suppose that for every $j\in J$, $\Lambda_j\in L(H,H_j)$, $\Gamma_j\in L(K,H_j)$, $\Theta_j\in L(H\oplus K,H_j)$, and $\{e_{jk}\}_{k\in K_j}$ is an orthonormal basis for H_j. $\psi_{jk}=\Lambda_j^*e_{jk}$ and $\varphi_{jm}=\Gamma_j^*e_{jm}$ are defined as in (5). Then $(H\oplus K)_I=\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$.*

Proof. Denote

$$\widehat{(H\oplus K)}_I=\left\{\sum_{j\in I_1}\Theta_j^*g_j=\left(\sum_{j\in I_1}\Lambda_j^*g_j,\sum_{j\in I_1}\Gamma_j^*g_j\right):\ \text{for any finite } I_1\subset I,\ g_j\in H_j,\ j\in I_1\right\}. \tag{9}$$

Since $\psi_{jk}=\Lambda_j^*e_{jk}$, $\varphi_{jm}=\Gamma_j^*e_{jm}$, $j\in I$, $k,m\in K_j$, we have $(\psi_{jk},\varphi_{jm})\in\widehat{(H\oplus K)}_I$, $j\in I$, $k,m\in K_j$. This implies that $\text{span}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}\subset\widehat{(H\oplus K)}_I$. Therefore, $\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}\subset(H\oplus K)_I$.

On the other hand, suppose that $(f,g)\in(H\oplus K)_I$, and then there exists a finite subset $I_1\subset I$ and $g_j\in H_j$, $j\in I_1$, such that $(f,g)=\sum_{j\in I_1}\Theta_j^*g_j=(\sum_{j\in I_1}\Lambda_j^*g_j,\sum_{j\in I_1}\Gamma_j^*g_j)$. For every $j\in I_1$, let $g_j=\sum_{k\in K_j}c_{jk}e_{jk}$, where $\{c_{jk}\}_{k\in K_j}\in l^2$. Then, we have

$$\begin{aligned}\Theta_j^*g_j&=\left(\Lambda_j^*\sum_{k\in K_j}c_{jk}e_{jk},\Gamma_j^*\sum_{k\in K_j}c_{jk}e_{jk}\right)\\&=\left(\sum_{k\in K_j}c_{jk}\psi_{jk},\sum_{k\in K_j}c_{jk}\varphi_{jk}\right)\\&\in\overline{\text{span}}\left\{\left(\psi_{jk},\varphi_{jm}\right)\right\}_{j\in I,k,m\in K_j}.\end{aligned} \tag{10}$$

Therefore,

$$(f,g)=\left(\sum_{j\in I_1}\Lambda_j^*g_j,\sum_{j\in I_1}\Gamma_j^*g_j\right)\in\overline{\text{span}}\left\{\left(\psi_{jk},\varphi_{jm}\right)\right\}_{j\in I,k,m\in K_j}. \tag{11}$$

This implies that $(H\oplus K)_I\subset\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$. The proof of Lemma 10 is completed. □

Lemma 11. *Let W be a closed subspace of $H\oplus K$ and $\{\Theta_j\in L(H\oplus K,H_j)\}_{j\in J}$. Suppose that $(\psi_{jk},\varphi_{jm})\in W$ for every $j\in I$, $k,m\in K_j$, where I is a subset of J and $\{\psi_{jk}\}_{j\in J,k\in K_j}$, $\{\varphi_{jm}\}_{j\in J,m\in K_j}$ are defined as in (5). If $\{\Theta_j\}_{j\in I}$ is g-complete in W, then $\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$ is complete in W.*

Proof. For any $j\in I$, $k,m\in K_j$, we have $(\psi_{jk},\varphi_{jm})\in W$, where W is a closed subspace of $H\oplus K$. Furthermore, we obtain

$$\overline{\text{span}}\left\{\left(\psi_{jk},\varphi_{jm}\right)\right\}_{j\in I,k,m\in K_j}\subset W. \tag{12}$$

It is enough to prove that if $(f,g)\in W$ and $(f,g)\perp\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$, then $\langle(f,g),(\psi_{jk},\varphi_{jm})\rangle=0$ for $j\in I$, $k,m\in K_j$. By equality (6), we obtain

$$\begin{aligned}\Theta_j(f,g)&=\Lambda_jf+\Gamma_jg\\&=\sum_{k\in K_j}\left\langle f,\psi_{jk}\right\rangle e_{jk}+\sum_{k\in K_j}\left\langle g,\varphi_{jk}\right\rangle e_{jk}=0,\\&\qquad\forall j\in I.\end{aligned} \tag{13}$$

Since $\{\Theta_j\}_{j\in I}$ is g-complete in W, we have $(f,g)=(0,0)$. Hence, $\{(\psi_{jk},\varphi_{jm})\}_{j\in J,k,m\in K_j}$ is complete in W. □

Inspired by the concept of minimal g-complete of g-frame for Hilbert space, we give the definition of minimal g-complete of g-frame for super Hilbert space.

Definition 12. Let W be a closed subspace of $H\oplus K$ and I be a subset of J. If $\{\Theta_j\}_{j\in I}$ is g-complete in W, but $\{\Theta_j\}_{j\in I\setminus\{j_0\}}$ is not g-complete in W for any $j_0\in I$, then one says that $\{\Theta_j\}_{j\in I}$ is minimal g-complete in W.

Lemma 13. *Suppose that $\{\Theta_j\}\in L(H\oplus K,H_j)$ for $j\in J$ and I is any finite nonempty subset of J. Then there exists* *a finite nonempty subset $I_0 \subset I$ such that $\{\Theta_j\}_{j\in I_0}$ is minimal g-complete in $(H\oplus K)_{I_0}$ and $(H\oplus K)_{I_0} = (H\oplus K)_I$, where $(H\oplus K)_I$ is defined as in (8).*

Proof. We prove Lemma 13 in two cases.

Case 1. $\{\Theta_j\}_{j\in I}$ is g-complete in $(H\oplus K)_I$ for any $j_0 \in I$, but $\{\Theta_j\}_{j\in I\setminus\{j_0\}}$ is not g-complete in $(H\oplus K)_I$. Let $I_0 = I$. Then the conclusion is right.

Case 2. Suppose that there exists $j_0 \in I$ such that $\{\Theta_j\}_{j\in I\setminus\{j_0\}}$ is g-complete in $(H\oplus K)_I$. Let $I_1 = I\setminus\{j_0\}$. If there still exists $j_1 \in I_1$ such that $\{(\psi_{jk},\varphi_{jm})\}_{j\in I_1\setminus\{j_1\},k,m\in K_j}$ is complete in $\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$, then we remove j_2 from I_1 in the same way. Repeat the operation above. Because I is nonempty and finite, this process must stop after finite steps. Assume that we remove $\{j_0, j_1,\dots,j_n\}$ from I, where $n\in N$ and $I_0 = I\setminus\{j_0,j_1,\dots,j_n\}$. Then, we obtain that $I_0\subset I$ satisfies the following two statements:

(1) $\{\Theta_j\}_{j\in I_0}$ is g-complete in $(H\oplus K)_I$.

(2) For any $i_0\in I_0$, $\{\Theta_j\}_{j\in I_0\setminus\{i_0\}}$ is not g-complete in $(H\oplus K)_I$.

By Lemmas 10 and 11, the statement (1) implies that

(3) $\{(\psi_{jk},\varphi_{jm})\}_{j\in I_0,k,m\in K_j}$ is complete in $\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$.

Obviously, I_0 is nonempty. The proof is by contradiction.

Suppose $I_0 = \emptyset$, by (1), we obtain that $\{\Theta_j\}_{j\in I_0}$ is g-complete in $(H\oplus K)_I$. It is obvious that this is impossible. So I_0 is nonempty. Now we prove $(H\oplus K)_{I_0} = (H\oplus K)_I$. By Definition 12 and Lemma 10, $\{\Theta_j\}_{j\in I}$ is minimal g-complete in $\overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$. By (3), we get

$$\overline{\text{span}}\left\{\left(\psi_{jk},\varphi_{jm}\right)\right\}_{j\in I_0,k,m\in K_j} = \overline{\text{span}}\left\{\left(\psi_{jk},\varphi_{jm}\right)\right\}_{j\in I,k,m\in K_j}. \tag{14}$$

By Lemma 10, we get $(H\oplus K)_{I_0} = (H\oplus K)_I$. The proof of Lemma 13 is completed. □

Based on this, we can obtain a characterization of g-Riesz frame for super Hilbert space $H\oplus K$.

Theorem 14. *Let $\{\Theta_j \in L(H\oplus K, H_j) : j\in J\}$ be a g-frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$. Then the following two statements are equivalent.*

(1) $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$.

(2) There exists $A>0$ such that $\{\Theta_j\}_{j\in I}$ is minimal g-complete in $(H\oplus K)_I$ for any nonempty subset I of J. And

$$A\left\|(f,g)\right\|^2 \le \sum_{j\in I}\left\|\Theta_j(f,g)\right\|^2,\quad \forall (f,g)\in (H\oplus K)_I, \tag{15}$$

where $(H\oplus K)_I$ is defined as in (8).

Proof. (1) ⇒ (2). Since $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$, there exists $A>0$ such that $\{\Theta_j\}_{j\in I}$ is a g-frame for $(H\oplus K)_I$ with respect to $\{H_j\}_{j\in I}$ for any nonempty I of J. Then, we obtain

$$A\left\|(f,g)\right\|^2 \le \sum_{j\in I}\left\|\Theta_j(f,g)\right\|^2,\quad \forall (f,g)\in (H\oplus K)_I. \tag{16}$$

(2) ⇒ (1). Suppose that $\{\Theta_j\}_{j\in J}$ is a g-frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$ with upper bound B. Then

$$\sum_{j\in J}\left\|\Theta_j(f,g)\right\|^2 \le B\left\|(f,g)\right\|^2,\quad \forall (f,g)\in H\oplus K. \tag{17}$$

There are two cases to prove that $\{\Theta_j\}_{j\in I}$ is a g-frame for $(H\oplus K)_I$ with respect to $\{H_j\}_{j\in I}$, where I is any nonempty subset of J.

Case 1. When I is any finite nonempty subset of J, by Lemma 13, there exists a finite nonempty subset $I_0\subset I$ such that $\{\Theta_j\}_{j\in I_0}$ is minimal g-complete in $(H\oplus K)_{I_0}$ and $(H\oplus K)_{I_0} = (H\oplus K)_I$. By (15), we have

$$A\left\|(f,g)\right\|^2 \le \sum_{j\in I_0}\left\|\Theta_j(f,g)\right\|^2, \qquad \forall (f,g)\in (H\oplus K)_{I_0} = (H\oplus K)_I. \tag{18}$$

Again by (15), for $\forall (f,g)\in (H\oplus K)_I$, we have

$$\begin{aligned} A\left\|(f,g)\right\|^2 &\le \sum_{j\in I_0}\left\|\Theta_j(f,g)\right\|^2 \le \sum_{j\in I}\left\|\Theta_j(f,g)\right\|^2 \\ &\le \sum_{j\in J}\left\|\Theta_j(f,g)\right\|^2 \le B\left\|(f,g)\right\|^2. \end{aligned} \tag{19}$$

Case 2. Let I be any infinite subset of J and $(f,g)\in(H\oplus K)_I$. Then, for any $\varepsilon>0$, there exists a finite subset $I_1\subset I$ and $g_j\in H_j$, $j\in I_1$, such that $\|(f,g)-\sum_{j\in I_1}\Theta_j^* g_j\|\le\varepsilon$. Denote

$$(H\oplus K)_{I_1} = \overline{\left\{\left(\sum_{j\in I_1}\Lambda_j^* g_j, \sum_{j\in I_1}\Gamma_j^* g_j\right) : \forall g_j\in H_j,\ j\in I_1\right\}}. \tag{20}$$

Clearly, $(H\oplus K)_I = (H\oplus K)_{I_1}\oplus(H\oplus K)_{I_1}^{\perp}$. Suppose that there exists $(f_1,g_1)\in(H\oplus K)_{I_1}$ and $(f_2,g_2)\in(H\oplus K)_{I_1}^{\perp}$ such that $(f,g)-(\sum_{j\in I_1}\Lambda_j^* g_j,\sum_{j\in I_1}\Gamma_j^* g_j) = (f_1,g_1)+(f_2,g_2)$. By $\|(f,g)-\sum_{j\in I_1}\Theta_j^* g_j\|\le\varepsilon$, we get $\|(f_1,g_1)\|^2+\|(f_2,g_2)\|^2\le\varepsilon^2$. Let $(f_0,g_0) = (\sum_{j\in I_1}\Lambda_j^* g_j,\sum_{j\in I_1}\Gamma_j^* g_j)+(f_1,g_1)$, we get $(f,g) = (f_0,g_0)+(f_2,g_2)$, where $(f_0,g_0)\in(H\oplus K)_{I_1}$. By Case 1, we have

$$\begin{aligned} A\left\|(f,g)\right\|^2 &= A\left\|(f_0,g_0)\right\|^2 + A\left\|(f_2,g_2)\right\|^2 \\ &\le \sum_{j\in I_1}\left\|\Theta_j(f_0,g_0)\right\|^2 + A\varepsilon^2. \end{aligned} \tag{21}$$

Now we prove $\sum_{j\in I_1}\|\Theta_j(f,g)\|^2 = \sum_{j\in I_1}\|\Theta_j(f_0,g_0)\|^2$. We only need to prove that $\Theta_j(f,g) = \Theta_j(f_0,g_0)$ for every $j\in I_1$. From (6), we obtain

$$\Theta_j(f,g) = \sum_{k\in K_j}\left\langle (f,g),(\psi_{jk},\varphi_{jk})\right\rangle e_{jk} = \sum_{k\in K_j}\left\langle (f_0,g_0)+(f_2,g_2),(\psi_{jk},\varphi_{jk})\right\rangle e_{jk}, \quad \forall j\in I_1. \tag{22}$$

By Lemma 10, we have $(\psi_{jk},\varphi_{jk})\in(H\oplus K)_{I_1}$ for any $j\in I_1$, $k\in K_j$. From (22), we get

$$\Theta_j(f,g) = \sum_{k\in K_j}\left\langle (f_0,g_0),(\psi_{jk},\varphi_{jk})\right\rangle e_{jk}, \quad \forall j\in I_1. \tag{23}$$

Again by (6), we have

$$\Theta_j(f,g) = \sum_{k\in K_j}\left\langle (f_0,g_0),(\psi_{jk},\varphi_{jk})\right\rangle e_{jk} = \Theta_j(f_0,g_0), \quad \forall j\in I_1. \tag{24}$$

Using (21), we get

$$\begin{aligned} A\|(f,g)\|^2 &\le \sum_{j\in I_1}\|\Theta_j(f_0,g_0)\|^2 + A\varepsilon^2 \\ &= \sum_{j\in I_1}\|\Theta_j(f,g)\|^2 + A\varepsilon^2 \\ &\le \sum_{j\in I}\|\Theta_j(f,g)\|^2 + A\varepsilon^2 \\ &\le \sum_{j\in J}\|\Theta_j(f,g)\|^2 + A\varepsilon^2 \\ &\le B\|(f,g)\|^2 + A\varepsilon^2. \end{aligned} \tag{25}$$

Letting $\varepsilon\to 0$, $A\|(f,g)\|^2 \le \sum_{j\in I}\|\Theta_j(f,g)\|^2 \le B\|(f,g)\|^2$ for any $(f,g)\in(H\oplus K)_I$. The proof of Theorem 14 is completed. □

3.2. Characterization of g-Riesz Frame for Super Hilbert Space $L_1\oplus L_2\oplus\cdots\oplus L_n$

Definition 15. Let $\{\Theta_j\in L(L_1\oplus L_2\oplus\cdots\oplus L_n,H_j): j\in J\}$ be a g-frame for $L_1\oplus L_2\oplus\cdots\oplus L_n$ with respect to $\{H_j\}_{j\in J}$. If any subsequence $\{\Theta_j\}_{j\in I, I\subset J}$ is also a g-frame for $(L_1\oplus L_2\oplus\cdots\oplus L_n)_I$ with respect to $\{H_j\}_{j\in I}$ with uniform g-frame lower bound, then one says that $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $L_1\oplus L_2\oplus\cdots\oplus L_n$ with respect to $\{H_j\}_{j\in J}$.

Like (8), we can define $(L_1\oplus L_2\oplus\cdots\oplus L_n)_I$ as follows:

$$(L_1\oplus L_2\oplus\cdots\oplus L_n)_I = \overline{\left\{\sum_{j\in I_1}\Theta_j^* g_j = \left(\sum_{j\in I_1}\Lambda_j^{(1)*}g_j, \sum_{j\in I_1}\Lambda_j^{(2)*}g_j,\ldots,\sum_{j\in I_1}\Lambda_j^{(n)*}g_j\right): I_1\subset I,\ \forall g_j\in H_j,\ j\in I_1\right\}}, \tag{26}$$

where I_1 is any finite subset of I and $\{\Lambda_j^{(p)}\in L(L_p,H_j): j\in J\}$, $p=1,2,\ldots,n$. Similarly, we give two lemmas.

Lemma 16. *For every $\{\Theta_j\in L(L_1\oplus L_2\oplus\cdots\oplus L_n,H_j): j\in J\}$, let $\{e_{jk}\}_{k\in K_j}$ be an orthonormal basis of H_j and $\psi_{jk_p}^{(p)} = \Lambda_j^{(p)*}e_{jk_p}$, where $\{\Lambda_j^{(p)}\in L(L_p,H_j): j\in J\}$, $p=1,2,\ldots,n$. Then*

$$(L_1\oplus L_2\oplus\cdots\oplus L_n)_I = \overline{\operatorname{span}}\left\{\left(\psi_{jk_1}^{(1)},\psi_{jk_2}^{(2)},\ldots,\psi_{jk_n}^{(n)}\right)\right\}_{j\in I,k_1,k_2,\ldots,k_n\in K_j}. \tag{27}$$

Lemma 17. *Let W be a closed subspace of $L_1\oplus L_2\oplus\cdots\oplus L_n$ and let $\{e_{jk}\}_{k\in K_j}$ be an orthonormal basis of H_j. Suppose that I is a subset of J and $\{\Theta_j\in L(L_1\oplus L_2\oplus\cdots\oplus L_n,H_j), j\in J\}$. For every $j\in I$, $k_1,k_2,\ldots,k_n\in K_j$, one has $(\psi_{jk_1}^{(1)},\psi_{jk_2}^{(2)},\ldots,\psi_{jk_n}^{(n)})\in W$, where $\psi_{jk_p}^{(p)} = \Lambda_j^{(p)*}e_{jk_p}$. If $\{\Theta_j\}_{j\in I}$ is g-complete in W, then $\{(\psi_{jk_1}^{(1)},\psi_{jk_2}^{(2)},\ldots,\psi_{jk_n}^{(n)})\}_{j\in I,k_1,k_2,\ldots,k_n\in K_j}$ is complete in W.*

In terms of the concept of minimal g-complete of g-frame for Hilbert space, we give the definition of minimal g-complete of g-frame for super Hilbert space $L_1\oplus L_2\oplus\cdots\oplus L_n$.

Definition 18. Let W be a closed subspace of $L_1\oplus L_2\oplus\cdots\oplus L_n$ and $I\subset J$. If $\{\Theta_j\}_{j\in I}$ is g-complete in W, but $\{\Theta_j\}_{j\in I\setminus\{j_0\}}$ is not g-complete in W for any $j_0\in I$, and then one says that $\{\Theta_j\}_{j\in I}$ is minimal g-complete in W.

Lemma 19. *Suppose that $\{\Theta_j\in L(L_1\oplus L_2\oplus\cdots\oplus L_n,H_j): j\in J\}$ and I is any finite nonempty subset of J. Then there exists a finite nonempty subset $I_0\subset I$ such that $\{\Theta_j\}_{j\in I_0}$ is minimal g-complete in $(L_1\oplus L_2\oplus\cdots\oplus L_n)_{I_0}$ and*

$$(L_1\oplus L_2\oplus\cdots\oplus L_n)_{I_0} = (L_1\oplus L_2\oplus\cdots\oplus L_n)_I, \tag{28}$$

where $(L_1\oplus L_2\oplus\cdots\oplus L_n)_I$ is defined as (26).

Proof. The proof is similar to proof of Lemma 10. □

By above lemma, we can get a characterization of g-Riesz frame for super Hilbert space $L_1 \oplus L_2 \oplus \cdots \oplus L_n$. Obviously, Theorem 14 is the special case of Theorem 20.

Theorem 20. *Let $\{\Theta_j \in L(L_1 \oplus L_2 \oplus \cdots \oplus L_n, H_j) : j \in J\}$ be a g-frame for $L_1 \oplus L_2 \oplus \cdots \oplus L_n$ with respect to $\{H_j\}_{j\in J}$. Then the following two statements are equivalent.*

(1) *$\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $L_1 \oplus L_2 \oplus \cdots \oplus L_n$ with respect to $\{H_j\}_{j\in J}$.*

(2) *For any nonempty subset I of J, there exists $A > 0$, if $\{\Theta_j\}_{j\in J}$ is minimal g-complete in $(L_1 \oplus L_2 \oplus \cdots \oplus L_n)_I$. And*

$$A \left\|(f_1, f_2, \ldots, f_n)\right\|^2 \le \sum_{j\in I} \left\|\Theta_j (f_1, f_2, \ldots, f_n)\right\|^2, \quad \forall (f_1, f_2, \ldots, f_n) \in (L_1 \oplus L_2 \oplus \cdots \oplus L_n)_I . \tag{29}$$

Proof. The proof is analogous to proof of Theorem 14. □

4. Stability of g-Riesz Frames for Super Hilbert Space $H \oplus K$

In this section, we use the characterization of g-Riesz frame for super Hilbert space $H \oplus K$ in Section 3 to study the stability of g-Riesz frame for super Hilbert space $H \oplus K$.

The stability of g-frames is important in practice which is wildly studied by many authors; for example, see [22–24]. The following is a fundamental result in the study of the stability of g-frames.

Proposition 21 (see [23, Theorem 3.1]). *Suppose that $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds A and B. There exists $\lambda, \beta, \mu \ge 0$ such that $\max\{\lambda + \mu/\sqrt{A}, \beta\} < 1$. If $\{\Gamma_j \in L(H, H_j)\}_{j\in J}$ satisfies*

$$\left(\sum_{j\in J} \left\|(\Lambda_j - \Gamma_j) f\right\|^2\right)^{1/2} \le \lambda \left(\sum_{j\in J} \left\|\Lambda_j f\right\|^2\right)^{1/2} + \beta \left(\sum_{j\in J} \left\|\Gamma_j f\right\|^2\right)^{1/2} + \mu \left\|f\right\| \tag{30}$$

for $f \in H$, then $\{\Gamma_j\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$ with bounds

$$\left(\frac{(1-\lambda)\sqrt{A} - \mu}{1+\beta}\right)^2, \quad \left(\frac{(1+\lambda)\sqrt{B} + \mu}{1-\beta}\right)^2. \tag{31}$$

Example 22 illustrates a g-Riesz frame of super Hilbert space $H \oplus K$ has no result of stability like Proposition 21.

Example 22. Suppose that $\{(e_j, 0)\} \cup \{(0, e_j)\}$ is an orthonormal basis of $H \oplus H$, where $j \in J$ and $J = \mathbb{N}$. Let $H_j = \overline{\operatorname{span}}\{e_j, e_{j+1}\}$. Define the bounded linear operator as follows:

$$\Theta_j (f, f) = 2\Lambda_j f = 2 \left\langle f, e_j \right\rangle e_j, \quad \forall (f, f) \in H \oplus H. \tag{32}$$

First, we prove that $\{\Theta_j\}_{j\in J}$ is a g-frame for $H \oplus H$ with respect to $\{H_j\}_{j\in J}$. In fact, for any $(f, f) \in H \oplus H$, we have

$$\begin{aligned} \sum_{j\in J} \left\|\Theta_j (f, f)\right\|^2 &= \sum_{j\in J} \left\|2 \left\langle f, e_j \right\rangle e_j\right\|^2 \\ &= 4\sum_{j\in J} \left\|\left\langle f, e_j \right\rangle e_j\right\|^2 = 4 \left\|f\right\|^2 \\ &= 2 \left\|(f, f)\right\|^2 . \end{aligned} \tag{33}$$

Thus, $\{\Theta_j\}_{j\in J}$ is a g-frame for $H \oplus H$ with respect to $\{H_j\}_{j\in J}$.

For any $g_j \in H_j$, let $g_j = c_j e_j + c_{j+1} e_{j+1}$, and we have

$$\begin{aligned} \left\langle \Theta_j^* g_j, (f, f) \right\rangle &= \left\langle g_j, \Theta_j (f, f) \right\rangle = \left\langle g_j, 2\Lambda_j f \right\rangle \\ &= \left\langle c_j e_j + c_{j+1} e_{j+1}, 2 \left\langle f, e_j \right\rangle e_j \right\rangle \\ &= 2c_j \overline{\left\langle f, e_j \right\rangle} = 2c_j \left\langle e_j, f \right\rangle \\ &= \left\langle \left(c_j e_j, c_j e_j\right), (f, f) \right\rangle . \end{aligned} \tag{34}$$

It implies that $\Theta_j^* g_j = (c_j e_j, c_j e_j)$. Since $\{e_j, e_{j+1}\}$ is an orthonormal basis for H_j, we have

$$\begin{aligned} \psi_{j_1} &= \Lambda_j^* e_j = e_j, \\ \psi_{j_2} &= \Lambda_j^* e_{j+1} = 0. \end{aligned} \tag{35}$$

So $\{(\psi_{jk}, \psi_{jm})\}_{j\in J, k,m\in K_j} = \{(e_j, 0), (0, e_j), (0, 0), (e_j, e_j)\}_{j=1}^{\infty}$.

Next, we prove that $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $H \oplus H$ with respect to $\{H_j\}_{j\in J}$. Let I be any finite subset of J. From Lemma 10, we obtain that

$$\begin{aligned} (H \oplus H)_I &= \overline{\operatorname{span}} \left\{\left(\psi_{jk}, \psi_{jm}\right)\right\}_{j\in I, k,m\in K_j} \\ &= \overline{\operatorname{span}} \left\{\left(e_j, 0\right), \left(0, e_j\right), (0, 0), \left(e_j, e_j\right)\right\}_{j\in I} . \end{aligned} \tag{36}$$

Hence, for every $j_0 \in J$, we have $\overline{\operatorname{span}}\{(e_j, 0), (0, e_j)\}_{j\in I\setminus\{j_0\}} \ne \overline{\operatorname{span}}\{(e_j, 0), (0, e_j)\}_{j\in I}$. Therefore, $\{(\psi_{jk}, \psi_{jm})\}_{j\in I, k,m\in K_j}$ is minimal g-complete in $(H \oplus H)_I$. Now we prove that $\{\Theta_j\}_{j\in I}$ is minimal g-complete in $(H \oplus H)_I$.

If there exists $(f, f) \in (H \oplus H)_I$ such that $\Theta_j(f, f) = 2\Lambda_j f = 0$, then we have

$$\begin{aligned} 0 &= \Theta_j (f, f) = 2\Lambda_j f = 2 \left\langle f, e_j \right\rangle e_j \\ &= 2 \left\langle (f, f), \left(e_j, 0\right) \right\rangle e_j, \\ 0 &= \Theta_j (f, f) = 2\Lambda_j f = 2\overline{\left\langle e_j, f \right\rangle} e_j \\ &= 2\overline{\left(\langle 0, f \rangle + \left\langle e_j, f \right\rangle\right)} e_j = 2\overline{\left\langle \left(0, e_j\right), (f, f) \right\rangle} e_j . \end{aligned} \tag{37}$$

Since $\{e_j, e_{j+1}\}$ is an orthonormal basis for H_j, we have $\langle (f,f), (e_j,0)\rangle = 0$ and $\overline{\langle (0,e_j),(f,f)\rangle} = 0$. From the minimal complete of $\{(\psi_{jk},\psi_{jm})\}_{j\in I}$ for $k,m \in K_j$ in $(H\oplus H)_I$ and $\overline{\operatorname{span}}\{(\psi_{jk},\psi_{jm})\}_{j\in I,k,m\in K_j} = \overline{\operatorname{span}}\{(e_j,0),(0,e_j)\}_{j\in I}$, we can obtain $(f,f) = (0,0)$. This implies that $\{\Theta_j\}_{j\in I}$ is minimal g-complete in $(H\oplus H)_I$. Then, for any $(f,f) \in (H\oplus H)_I$, we have

$$\sum_{j\in I} \left\|\Theta_j(f,f)\right\|^2 = \sum_{j\in I} \left\|2\left\langle f,e_j\right\rangle e_j\right\|^2 = 2\left\|(f,f)\right\|^2. \quad (38)$$

By Theorem 14, $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus H$ with respect to $\{H_j\}_{j\in J}$.

Let $\varepsilon \in (0,1)$. Define the bounded linear operator $\Omega_j : H\oplus H \to H_j$ as follows:

$$\Omega_j(f,f) = \Theta_j(f,f) + \frac{2\varepsilon}{j+1}\left\langle f,e_{j+1}\right\rangle e_{j+1}. \quad (39)$$

By direct calculation, for any $g_j \in H_j$, suppose that $g_j = c_je_j + c_{j+1}e_{j+1}$, and we have $\Omega_j^* g_j = (c_je_j + (\varepsilon/(j+1))c_{j+1}e_{j+1}, c_je_j + (\varepsilon/(j+1))c_{j+1}e_{j+1})$. Let $\widetilde{\psi}_{j_1} = e_j$, $\widetilde{\psi}_{j_2} = (\varepsilon/(j+1))e_{j+1}$. Then

$$\begin{aligned} \left\{\left(\widetilde{\psi}_{jk},\widetilde{\psi}_{jm}\right)\right\}_{j\in J,k,m\in K_j} = \Bigg\{&\left(e_j,\frac{\varepsilon}{j+1}e_{j+1}\right),\left(e_j,e_j\right), \\ &\left(\frac{\varepsilon}{j+1}e_{j+1},\frac{\varepsilon}{j+1}e_{j+1}\right),\left(\frac{\varepsilon}{j+1}e_{j+1},e_j\right)\Bigg\}_{j=1}^{\infty}. \end{aligned} \quad (40)$$

Now we prove that $\{\Omega_j\}_{j\in J}$ is not a g-Riesz frame for $H\oplus H$ with respect to $\{H_j\}_{j\in J}$.

The proof is by contradiction. Suppose that $\{\Omega_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus H$ with respect to $\{H_j\}_{j\in J}$. Let $I = \{j\}$. By Lemma 10, we have

$$\begin{aligned} (H\oplus H)_I &= \overline{\operatorname{span}}\left\{\left(\psi_{jk},\psi_{jm}\right)\right\}_{j\in I,k,m\in K_j} \\ &= \overline{\operatorname{span}}\Bigg\{\left(e_j,\frac{\varepsilon}{j+1}e_{j+1}\right),\left(e_j,e_j\right), \\ &\qquad\left(\frac{\varepsilon}{j+1}e_{j+1},\frac{\varepsilon}{j+1}e_{j+1}\right),\left(\frac{\varepsilon}{j+1}e_{j+1},e_j\right)\Bigg\} \\ &= \overline{\operatorname{span}}\left\{\left(e_j,e_j\right),\left(e_j,e_{j+1}\right),\left(e_{j+1},e_j\right),\left(e_{j+1},e_{j+1}\right)\right\}. \end{aligned} \quad (41)$$

Choosing $(e_{j+1},e_{j+1}) \in (H\oplus H)_I$, then

$$\left\|\Omega_j\left(e_{j+1},e_{j+1}\right)\right\|^2 = \frac{4\varepsilon^2}{(j+1)^2}. \quad (42)$$

Since $\{\Omega_j\}_{j\in I}$ is a g-frame for $(H\oplus H)_I$ with respect to $\{H_j\}_{j\in I}$, there exists $A > 0$ such that

$$\begin{aligned} A = A\left\|\left(e_{j+1},e_{j+1}\right)\right\|^2 &\le \sum_{k\in I}\left\|\Omega_k\left(e_{j+1},e_{j+1}\right)\right\|^2 \\ &= \left\|\Omega_j\left(e_{j+1},e_{j+1}\right)\right\|^2 = \frac{4\varepsilon^2}{(j+1)^2}. \end{aligned} \quad (43)$$

Letting $j \to +\infty$, it implies that $A = 0$. But this is a contradiction. We conclude that $\{\Omega_j\}_{j\in J}$ is not a g-Riesz frame for $H\oplus H$ with respect to $\{H_j\}_{j\in J}$.

On the other hand, for any $(f,f) \in H\oplus H$, we have

$$\begin{aligned} &\left(\sum_{j\in J}\left\|\left(\Theta_j-\Omega_j\right)(f,f)\right\|^2\right)^{1/2} \\ &\quad= \left(\sum_{j\in J}\left\|\frac{2\varepsilon}{j+1}\left\langle f,e_{j+1}\right\rangle e_{j+1}\right\|^2\right)^{1/2} \le 2\varepsilon\left\|(f,f)\right\|. \end{aligned} \quad (44)$$

From Example 22, we can realize that $\lambda,\beta = 0$, $\mu \ne 0$. Suppose that $\{\Theta_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$. Even if $\{\Theta_j\}_{j\in J}$ and $\{\Omega_j\}_{j\in J}$ satisfy the inequality of Proposition 21, it is uncertain to get that $\{\Omega_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$.

Theorem 23. *Let $\{\Theta_j\}_{j\in J}$ be a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$ with bounds B and A, $\lambda,\beta \in [0,1)$. If $\{\Omega_j \in L(H\oplus K, H_j)\}_{j\in J}$ satisfies*

$$\begin{aligned} &\left(\sum_{j\in J_1}\left\|\left(\Theta_j-\Omega_j\right)(f,g)\right\|^2\right)^{1/2} \\ &\quad\le \lambda\left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} \\ &\qquad+ \beta\left(\sum_{j\in J_1}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2} \end{aligned} \quad (45)$$

for any $(f,g) \in H\oplus K$, where J_1 is any finite subset of J, then $\{\Omega_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$ with bounds

$$\begin{aligned} &A\left(\frac{1-\lambda}{1+\beta}\right)^2, \\ &B\left(\frac{1+\lambda}{1-\beta}\right)^2. \end{aligned} \quad (46)$$

Proof. Since $\{\Theta_j\}_{j\in J}$ is a g-frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$ with bounds A and B, for any finite subset $J_1 \subset J$, then

$$\begin{aligned} \sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2 \le \sum_{j\in J}\left\|\Theta_j(f,g)\right\|^2 \le B\left\|(f,g)\right\|^2, \\ \forall (f,g) \in H\oplus K. \end{aligned} \quad (47)$$

By the triangle inequality, we have

$$\begin{aligned} &\left(\sum_{j\in J_1}\left\|\left(\Theta_j-\Omega_j\right)(f,g)\right\|^2\right)^{1/2} \\ &\quad\ge \left(\sum_{j\in J_1}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2} \\ &\qquad- \left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2}. \end{aligned} \quad (48)$$

From (45), (47), and (48), we have

$$\begin{aligned}\left(\sum_{j\in J_1}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2} &\le \left(\sum_{j\in J_1}\left\|\left(\Theta_j-\Omega_j\right)(f,g)\right\|^2\right)^{1/2} + \left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} \\ &\le \frac{\lambda+1}{1-\beta}\left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} \\ &\le \frac{(\lambda+1)\sqrt{B}}{1-\beta}\left\|(f,g)\right\|.\end{aligned} \tag{49}$$

Therefore, the series $\sum_{j\in J_1}\|\Omega_j(f,g)\|^2$ is convergent. By (45), we have

$$\left(\sum_{j\in J}\left\|\left(\Theta_j-\Omega_j\right)(f,g)\right\|^2\right)^{1/2} \le \lambda\left(\sum_{j\in J}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} + \beta\left(\sum_{j\in J}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2}. \tag{50}$$

By Proposition 21, we get that $\{\Omega_j\}_{j\in J}$ is a g-frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$. For any finite subset $J_1 \subset J$, by the triangle inequality, we obtain

$$\left(\sum_{j\in J_1}\left\|\left(\Theta_j-\Omega_j\right)(f,g)\right\|^2\right)^{1/2} \ge \left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} - \left(\sum_{j\in J_1}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2}. \tag{51}$$

Using (45) and (51), we have

$$\left(\sum_{j\in J_1}\left\|\Theta_j(f,g)\right\|^2\right)^{1/2} \le \frac{1+\beta}{1-\lambda}\left(\sum_{j\in J_1}\left\|\Omega_j(f,g)\right\|^2\right)^{1/2}. \tag{52}$$

For any finite subset $I\subset J$, denote

$$U_I = \overline{\left\{\sum_{j\in I}\Omega_j^* g_j : \forall g_j \in H_j,\ j\in I\right\}}. \tag{53}$$

If $\{\Omega_j\}_{j\in I}$ is minimal g-complete in U_I, we need to prove that $U_I \subset (H\oplus K)_I$; that is, $(H\oplus K)_I^{\perp} \subset U_I^{\perp}$. For any $(f,g) \in (H\oplus K)_I^{\perp}$, since $H\oplus K = U_I \oplus U_I^{\perp}$, there exist $(f_1,g_1)\in U_I$ and $(f_2,g_2)\in U_I^{\perp}$ such that $(f,g) = (f_1,g_1)+(f_2,g_2)$. From Lemma 10, we obtain that $(H\oplus K)_I = \overline{\text{span}}\{(\psi_{jk},\varphi_{jm})\}_{j\in I,k,m\in K_j}$ and $U_I = \overline{\text{span}}\{(\widetilde{\psi}_{jk},\widetilde{\varphi}_{jm})\}_{j\in I,k,m\in K_j}$, where $\widetilde{\psi}_{jk} = \Lambda_j^* e_{jk}$, $\widetilde{\varphi}_{jm} = \Gamma_j^* e_{jm}$ for $j\in I$, $k,m\in K_j$. Therefore, for any finite $j\in I$, we have

$$\begin{aligned}\Theta_j(f,g) &= \sum_{k\in K_j}\left\langle (f,g),\left(\psi_{jk},\varphi_{jk}\right)\right\rangle e_{jk} = 0, \\ \Omega_j(f,g) &= \sum_{k\in K_j}\left\langle (f,g),\left(\widetilde{\psi}_{jk},\widetilde{\varphi}_{jk}\right)\right\rangle e_{jk} \\ &= \sum_{k\in K_j}\left\langle (f_1,g_1)+(f_2,g_2),\left(\widetilde{\psi}_{jk},\widetilde{\varphi}_{jk}\right)\right\rangle e_{jk} \\ &= \sum_{k\in K_j}\left\langle (f_1,g_1),\left(\widetilde{\psi}_{jk},\widetilde{\varphi}_{jk}\right)\right\rangle e_{jk} \\ &= \Omega_j(f_1,g_1).\end{aligned} \tag{54}$$

By (45), we have $(\sum_{j\in I}\|\Omega_j(f,g)\|^2)^{1/2} \le \beta(\sum_{j\in I}\|\Omega_j(f_1,g_1)\|^2)^{1/2}$.

Then for any $j\in I$, it follows that $\Omega_j(f_1,g_1) = 0$. Since $\{\Omega_j\}_{j\in I}$ is g-complete in U_I, we have $(f_1,g_1) = 0$. Therefore $(f,g) = (f_2,g_2)\in U_I^{\perp}$, and it implies that $(H\oplus K)_I^{\perp}\subset U_I^{\perp}$.

For any $f\in U_I\subset (H\oplus K)_I$, since $\{\Theta_j\}_{j\in I}$ is a g-frame for $(H\oplus K)_I$ with respect to $\{H_j\}_{j\in I}$ with lower bound A, by (52), we have

$$A\left(\frac{1-\lambda}{1+\beta}\right)^2\left\|(f,g)\right\|^2 \le \sum_{j\in I}\left\|\Omega_j(f,g)\right\|^2, \quad \forall (f,g)\in U_I. \tag{55}$$

By Theorem 14, we get that $\{\Omega_j\}_{j\in J}$ is a g-Riesz frame for $H\oplus K$ with respect to $\{H_j\}_{j\in J}$. The proof of Theorem 23 is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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