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THE CHARACTERIZATION OF RIEMANNIAN METRIC ARISING FROM PHASE TRANSITION PROBLEMS

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Abstract. We present one property of the Riemannian metric which is derived from the positive power of potential functions. Then this property is applied to the study of the Γ-convergence of energy functionals which are associated with the Euler-Lagrange p-Laplacian equation.

1. Introduction. The singular perturbation nonconvex functionals,

$$E_\varepsilon(v) = \int_\Omega \left[\frac{1}{\varepsilon} W(v(x)) + \varepsilon |\nabla v(x)|^2\right] dx \tag{1}$$

with a nonnegative double-wells potential W, were studied by Modica and Mortola [22] in 1977. In 1987, Modica [21] applied the Γ-convergence theory to solve the minimal interface problem in the Van der Waals-Cahn-Hilliard theory of phase transitions. Suppose that $W(v) = 0$ if and only if $v \in \{a, b\}$. It was shown in [21, 22, 24] that the Γ-limit of $\{E_\varepsilon\}_{\varepsilon>0}$ is of the form

$$E_0(v) = \begin{cases} \beta \mathrm{Per}_\Omega(A) & \text{if} \quad v \in T\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

where $T = \{v \in \mathrm{BV}(\Omega);\, v = \chi_A a + (1 - \chi_A) b,\ A \subset \Omega \text{ with } 0 < |A| < |\Omega|\}$ and

$$\beta = 2 \int_a^b W^{1/2}(s) ds\,. \tag{2}$$

The notation $\mathrm{BV}(\Omega)$ and $\mathrm{Per}_\Omega(A)$ stand for the space of functions of bounded variation and the perimeter of A in Ω, respectively. Let $p > 1$ and q the exponent conjugate to p, that is, $1/p + 1/q = 1$, and let K be a positive integer in this whole article. Let a_i $(i = 1, \ldots, K)$ be given points in $\boldsymbol{R}^N$ and $\mathcal{A} = \{a_i\,;\, i = 1, \ldots, K\}$. We denote by $C^1_{\mathrm{pw}}(I)$ the set of piecewise C^1 curves $f : I \subset \boldsymbol{R} \to \boldsymbol{R}^N$, and we define

$$T_{F,L} \equiv \{g \in C^1_{\mathrm{pw}}([-L, L])\,|\, g(-L) = a_i\,, \text{ and } g(L) = a_j \text{ for some } i \neq j\}\,,$$

$$T_F \equiv \bigcup_{L>0} T_{F,L}\,, \tag{3}$$

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and

$$S_F \equiv \left\{ \int_{-L}^{L} \left(\frac{W(g(s))}{q} + \frac{|g'(s)|^p}{p} \right) ds \; ; \quad g \in T_F \right\} . \tag{4}$$

Barroso and Fonseca [5], Fonseca and Tartar [16] generalize the result of Γ-convergence to the vector-valued case for which $v : \Omega \to \boldsymbol{R}^N$, $\Omega \subset \boldsymbol{R}^n$ and the equation (2) is replaced by $\beta = \inf S_F$ for $p = 2$ and $K = 2$. Baldo [4] in 1990 established the Γ-limit of $\{F_\varepsilon\}_{\varepsilon>0}$ for multiple-well potential. Suppose that $W(v) = 0$ if and only if $v \in \{a_1, \ldots, a_K\}$. The Γ-limit of $\{E_\varepsilon\}_{\varepsilon>0}$ is essentially of the form

$$E_0(v) = \begin{cases} \sum_{i,j=1}^{K} d(a_i, a_j) H^{n-1}(\partial^\star S_i \cap \partial^\star S_j \cap \Omega) & \text{if} \quad v \in T , \\ +\infty & \text{otherwise} , \end{cases}$$

where $T = \{v \in \mathrm{BV}(\Omega; \boldsymbol{R}^N);\ v = \sum_{i=1}^{K} \chi_{S_i} a_i$, with $S_i \subset \Omega$ and $|\Omega - \bigcup_{i=1}^{K} S_i| = 0$, and $S_i \cap S_j = \emptyset, i \neq j\}$, $d(a_i, a_j) = \inf S(a_i, a_j)$ and

$$S(a_i, a_j) = \left\{ \int_{-L}^{L} [W(g(s)) + |g'(s)|^2] ds \; ; \; g \in C^1_{\mathrm{pw}}([-L, L]),\ g(-L) = a_i, g(L) = a_j \right\} .$$

For more generalization of this case, see Ambrosio [1] and Sternberg [24].

The development of this type of Γ-convergence for gradient vector fields, i.e., $\{E_\varepsilon(\nabla v)\}_{\varepsilon>0}$ was studied by Jin and Kohn [18], and Conti, Fonseca and Leoni [9]. There have been many works on the related subjects, for instance, Fonseca and Mantegazza [15], Fonseca [14], Ambrosio, De Lellis and Mantegazza [2], Aviles and Giga [3], and DeSimone, Kohn, Müller and Otto [11].

In this paper, we consider a family of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ given by

$$F_\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\varepsilon} \frac{W(v(x))}{q} + \varepsilon^{p-1} \frac{|\nabla v(x)|^p}{p} \right] dx & \text{if} \quad v \in \mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N) , \\ +\infty & \text{otherwise} . \end{cases} \tag{5}$$

The Euler-Lagrange equation of these functionals is the following p-Laplacian equation,

$$\varepsilon^p \Delta_p v - \frac{W'(v)}{q} = 0 \quad \text{in} \quad \Omega , \tag{6}$$

where $\Delta_p v = \mathrm{div}(|\nabla v|^{p-2} \nabla v)$. The functionals defined by Equation (1) could be regarded as a special case of the functionals defined by Equation (5) for $p = 2$. The minimizer and Γ-convergence for scalar case with a special potential $W(v) = v^2(1 - v)^2/4$ has been studied in [8]. It follows from the important role of constants $\inf S(a_i, a_j)$ in the study of Γ-convergence of functional $\{E_\varepsilon\}_{\varepsilon>0}$ that give us a motivation to find properties of the Riemannian metric which is derived from the positive power of potential functions. In this paper, we present one property of this Riemannian metric and apply it to the study of the Γ-convergence of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ for which the potential is of double-well type.

Define $\mathcal{A} = \{a_1, \ldots, a_K\}$ and let us introduce two hypothesis on the potential W below.

(H1) The potential $W \in \mathcal{W}^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}^N; \boldsymbol{R})$ is a nonnegative function such that $W(x) = 0$ if and only if $x = a_i$ for some $i = 1, \ldots, K$.

(H2) Let $d = \min\{\mathrm{dist}(a_i, a_j)\,;\, i \neq j,$ and $i, j = 1, \dots, K\}$. There exist $\alpha, \delta \in \boldsymbol{R}$ with $0 < \alpha \leq 1$ and $0 < \delta < d/2$ such that, if $x \in \mathcal{A}$ and $y \in \boldsymbol{R}^N \backslash \mathcal{A}$ with $|y - x| \leq \delta$, then $\alpha|y - x|^p \leq W(y) \leq \alpha^{-1}|y - x|^p$.

Furthermore, let us define two sets of functions and two sets of real numbers. They are

$$T_\infty \equiv \{g \in C^1_{\mathrm{pw}}(\boldsymbol{R})\,;\; g(-\infty) = a_i, \text{ and } \; g(\infty) = a_j \text{ for some } i \neq j\}\,, \tag{7}$$

$$T_\star \equiv \{g \in C^1_{\mathrm{pw}}([-1, 1])\,;\; g(-1) = a_i, \text{ and } \; g(1) = a_j \text{ for some } i \neq j\}\,, \tag{8}$$

and

$$S_\infty \equiv \left\{ \int_{-\infty}^{\infty} \left(\frac{W(g(s))}{q} + \frac{|g'(s)|^p}{p} \right) ds\,;\, g \in T_\infty \right\}\,, \tag{9}$$

$$S_\star \equiv \left\{ \int_{-1}^{1} \left(W(g(s))^{1/q} \cdot |g'(s)| \right) ds\,;\, g \in T_\star \right\}\,. \tag{10}$$

The following theorem states one property of our Riemannian metric. We will prove it in the next section.

THEOREM 1. *Let* S_∞, S_F *and* $S_\star$ *be the set of values defined by* (9), (4) *and* (10), *respectively. Then*

$$\inf S_\infty = \inf S_F = \inf S_\star\,.$$

We will apply Theorem 1 to the Γ-convergence theory for the case of two wells, that is $K = 2$, with $a_1 = a$ and $a_2 = b$. Let Ω be an open bounded strongly Lipschitz domain of $\boldsymbol{R}^n$. The sets of states are defined as

$$A = \{x \in \Omega\,;\; v(x) = a\}\,, \quad B = \{x \in \Omega\,;\; v(x) = b\}\,. \tag{11}$$

The role of $F_0(v)$ is played by the limit problem

$$F_0(v) = \begin{cases} (\inf S_\star)\mathrm{Per}_\Omega(A) & \text{if } \quad v \in \mathrm{BV}(\Omega; \boldsymbol{R}^N) \text{ with } v(x) \in \{a, b\}\; a.e. \\ +\infty & \text{otherwise.} \end{cases} \tag{12}$$

We will show that the functional F_0 is the $\Gamma(L^1(\Omega; \boldsymbol{R}^N))$-limit of $\{F_\varepsilon\}_{\varepsilon>0}$ (denoted by $F_0 = \Gamma(L^1(\Omega; \boldsymbol{R}^N))$-limit $\{F_\varepsilon\}_{\varepsilon>0}$). This means by definition that for each $v \in L^1(\Omega; \boldsymbol{R}^N)$,

(i) if v_ε converges to v in $L^1(\Omega; \boldsymbol{R}^N)$, then $F_0(v) \leq \liminf_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon)$,

(ii) there exists a family $\{v_\varepsilon\}_{\varepsilon>0}$ in $L^1(\Omega; \boldsymbol{R}^N)$ such that v_ε converges to v in $L^1(\Omega; \boldsymbol{R}^N)$ and $\limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon) \leq F_0(v)$.

The Γ-convergence thereom below will be proved in Section 3.

THEOREM 2. *In addition to the assumptions* (H1) *and* (H2), *assume that*

(H3) *there exist* $\alpha, C > 0$ *such that* $W(v) \geq \alpha|v|$ *whenever* $|v| > C$.

Then the functional F_0 *defined in* (12) *is the* $\Gamma(L^1(\Omega; \boldsymbol{R}^N))$*-limit of* $\{F_\varepsilon\}_{\varepsilon>0}$ *defined by* (5).

Let a, b and m be vectors in $\boldsymbol{R}^N$ with $m = \theta a + (1-\theta)b$ for some $\theta \in (0,1)$. Define the spaces V_m and V as follows.

$$V_m \equiv \left\{ u \in W^{1,p}(\Omega; \boldsymbol{R}^N)\,; \quad \int_\Omega u(x)dx = |\Omega| m \right\},$$

$$V \equiv \left\{ u \in \mathrm{BV}(\Omega; \boldsymbol{R}^N)\,; \quad u(x) \in \{a, b\} \text{ a.e. and } \int_\Omega u(x)dx = |\Omega| m \right\}.$$

We would like to give some remarks on our works for more applications and further studies.

Remark 1. For each $\varepsilon > 0$, the existence of a minimizer u_ε of F_ε on V_m can be obtained by virtue of the direct methods in the calculus of variations.

Remark 2. Using the definition of Γ-convergence, we observe that if (i) $F_0 = \Gamma(L^1(\Omega; \boldsymbol{R}^N))$-limit$\{F_\varepsilon\}_{\varepsilon>0}$, (ii) u_ε is a minimizer of F_ε on V_m for each $\varepsilon > 0$, and (iii) u_ε converges to u_0 in $L^1(\Omega; \boldsymbol{R}^N)$ as ε goes to 0^+, then

$$\lim_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) = F_0(u_0)$$

and u_0 is a minimizer of F_0 on V.

Remark 3. Another consequence is that Γ-convergence is stable under continuous perturbations: If $\{F_\varepsilon\}_{\varepsilon>0}$ Γ-converges to F_0 in $L^1(\Omega; \boldsymbol{R}^N)$ and the functional G is continuous with respect to the $L^1(\Omega; \boldsymbol{R}^N)$-topology, then $\{F_\varepsilon + G\}_{\varepsilon>0}$ Γ-converges to $F_0 + G$ in $L^1(\Omega; \boldsymbol{R}^N)$.

Remark 4. Theorem 1 is useful for characterizing the behavior of the minimizers of Γ-limit F_0 of $\{F_\varepsilon\}_{\varepsilon>0}$ with multiple potential wells. We anticipate futher developments in this direction in the future.

2. Riemannian metric. We divide the proof of Theorem 1 into Lemmas 3, 4 and 5 in this section.

LEMMA 3. *Let S_∞ and S_F be the sets defined in Equations* (9) *and* (4), *respectively. Then* $\inf S_\infty = \inf S_F$.

PROOF. We first prove that $\inf S_\infty \le \inf S_F$ by showing

$$S_F \subset S_\infty\,. \tag{13}$$

For any $\alpha \in S_F$, Equation (4) implies that there exists a function $g_L \in T_F$ such that

$$\alpha = \int_{-L}^{L} \left(\frac{W(g_L(s))}{q} + \frac{|g_L'(s)|^p}{p} \right) ds\,.$$

The function $g_L \in T_F$ can be extended to $g_\infty \in T_\infty$ by defining

$$g_\infty(s) \equiv \begin{cases} g_L(-L) & \text{if} \quad s < -L\,, \\ g_L(s) & \text{if} \quad -L \le s \le L\,, \\ g_L(L) & \text{if} \quad s > L\,. \end{cases} \tag{14}$$

The function g_∞ belongs to T_∞ by Equations (3) and (7). Since $W(g_\infty(s)) = 0$ and $|g'_\infty(s)| = 0$ for $s \in \boldsymbol{R}\backslash[-L, L]$, we evaluate

$$\alpha = \int_{-L}^{L} \left(\frac{W(g_L(s))}{q} + \frac{|g'_L(s)|^p}{p} \right) ds = \int_{-\infty}^{\infty} \left(\frac{W(g_\infty(s))}{q} + \frac{|g'_\infty(s)|^p}{p} \right) ds\,.$$

It follows that α is in S_∞, and we get (13).

Next we show $\inf S_F \le \inf S_\infty$. Now, we assume that α is in S_∞. By Equation (9), there is a piecewise C^1 curve $g : \boldsymbol{R} \to \boldsymbol{R}^N$ with $g(-\infty) = a = a_i$ and $g(\infty) = b = a_j$ for some $i \neq j$ such that

$$\alpha = \int_{-\infty}^{\infty} \left(\frac{W(g(s))}{q} + \frac{|g'(s)|^p}{p} \right) ds < +\infty\,. \tag{15}$$

Condition (H2) implies that there exist $0 < \delta < d/2$ and $L > 0$ such that

$$|g(s) - a| < \delta \quad \text{for all } s \in (-\infty, -L) \quad \text{and} \quad |g(s) - b| < \delta \quad \text{for all } s \in (L, \infty)\,. \tag{16}$$

Without loss of generality, we may assume $g(s) \in \boldsymbol{R}^N\backslash\mathcal{A}$ for $s \in \boldsymbol{R}\backslash[-L, L]$. By (H2) again, we see that

$$\alpha|g(s) - a|^p \le W(g(s)) \le \frac{1}{\alpha}|g(s) - a|^p \quad \text{for } s \in (-\infty, -L)\,. \tag{17}$$

Using Equations (15) and (17) gives us

$$\int_{-\infty}^{-L} |g(s) - a|^p ds \le \frac{1}{\alpha} \int_{-\infty}^{-L} W(g(s)) ds < +\infty\,.$$

It follows that

$$\int_{-\infty}^{L} |g(s) - a|^p ds < +\infty \tag{18}$$

by $\int_{-L}^{L} |g(s) - a|^p ds < +\infty$. The identity $|(g(s) - a)'| = |g'(s)|$ and Equation (17) imply

$$\int_{-\infty}^{L} |(g(s) - a)'|^p ds < +\infty\,. \tag{19}$$

From Equations (18) and (19), the function $g - a$ is in $\mathcal{W}^{1,p}((-\infty, L)\,; \boldsymbol{R}^N)$. Similarly, the function $g - b$ is in $\mathcal{W}^{1,p}((-L, \infty)\,; \boldsymbol{R}^N)$. It is easy to observe that for all $\tilde{L} > 0$, $g - a$ is in $\mathcal{W}^{1,p}((-\infty, \tilde{L})\,; \boldsymbol{R}^N)$ and $g - b$ is in $\mathcal{W}^{1,p}((-\tilde{L}, \infty)\,; \boldsymbol{R}^N)$.

By the theorem on partition of unity [23], there exists a function $\gamma : \boldsymbol{R} \to \boldsymbol{R}$ in C_c^∞ with $0 \le \gamma \le 1$, $\gamma(s) = 1$ on $[-1, 1]$ and $\gamma(s) = 0$ on $\boldsymbol{R}\backslash(-2, 2)$. Let us define a sequence of functions $\{g_k\}_{k=1}^\infty$ by

$$g_k(s) \equiv \begin{cases} \gamma(s/k)g(s) + (1 - \gamma(s/k))a & \text{if} \quad s < 0\,, \\ g(s) & \text{if} \quad s = 0\,, \\ \gamma(s/k)g(s) + (1 - \gamma(s/k))b & \text{if} \quad s > 0\,. \end{cases} \tag{20}$$

Note that for $k > L$, we have

$$g_k(s) = \begin{cases} a & \text{if} \quad s < -2k\,, \\ g(s) & \text{if} \quad -k \le s \le k\,, \\ b & \text{if} \quad s > 2k\,, \end{cases} \tag{21}$$

and hence $g_k \in T_\infty$. We claim that

$$\lim_{k\to\infty} \int_{-2k}^{2k} \left(\frac{W(g_k(s))}{q} + \frac{|g_k'(s)|^p}{p} \right) ds = \int_{-\infty}^{\infty} \left(\frac{W(g(s))}{q} + \frac{|g'(s)|^p}{p} \right) ds\,. \tag{22}$$

We divide the proof into the two parts

$$\lim_{k\to\infty} \int_{-2k}^{2k} W(g_k(s))ds = \int_{-\infty}^{\infty} W(g(s))ds \quad \text{and} \quad \lim_{k\to\infty} \int_{-2k}^{2k} |g_k'(s)|^p ds = \int_{-\infty}^{\infty} |g'(s)|^p ds\,.$$

It is equivalent to show that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} W(g_k(s))ds = \int_{-\infty}^{\infty} W(g(s))ds \quad \text{and} \quad \lim_{k\to\infty} \int_{-\infty}^{\infty} |g_k'(s)|^p ds = \int_{-\infty}^{\infty} |g'(s)|^p ds\,,$$

since $W(g_k(s)) = 0$ and $g_k'(s) = 0$ for all $|s| \ge 2k$ and $k \in \mathcal{N}$ by Equation (21).

Let $L > 0$ be given. By Equation (20), we have $g_k(s) - a = \gamma(s/k)(g(s) - a)$ for $s < 0$. By the inequality $0 \le \gamma \le 1$ and Condition (H2), we have

$$W(g_k(s)) \le \frac{1}{\alpha}|g_k(s) - a|^p \le \frac{1}{\alpha}|g(s) - a|^p \quad \text{on} \quad (-\infty, -L)\,, \tag{23}$$

for each $k \in \mathcal{N}$. By the continuity of the potential W and the convergence $g_k \to g$ on $\boldsymbol{R}$ as $k \to \infty$, we have

$$W \circ g_k \to W \circ g \quad \text{on } \boldsymbol{R} \text{ as } \quad k \to \infty\,. \tag{24}$$

Applying the Lebesgue Dominated Convergence Theorem and by Equations (24) and (23), we get

$$\lim_{k\to\infty} \int_{-\infty}^{-L} W(g_k(s))ds = \int_{-\infty}^{-L} W(g(s))ds\,. \tag{25}$$

A similar argument gives us

$$\lim_{k\to\infty} \int_{L}^{\infty} W(g_k(s))ds = \int_{L}^{\infty} W(g(s))ds\,. \tag{26}$$

If $k > L$, by Equation (21), we get $g_k(s) = g(s)$ for $-L \le s \le L$. Hence we have

$$\int_{-L}^{L} W(g_k(s))ds = \int_{-L}^{L} W(g(s))ds\,. \tag{27}$$

By combining Equations (25), (26) and (27), we get the equality

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} W(g_k(s))ds = \int_{-\infty}^{\infty} W(g(s))ds\,.$$

Since $\gamma \in C_c^\infty(\boldsymbol{R})$, there exists a constant M such that $|\gamma'(s)| \le M$ for all $s \in \boldsymbol{R}$. A simple calculation shows that, if $s < 0$, then $g_k'(s) = (1/k)\gamma'(s/k)(g(s)-a)^T + \gamma(s/k)g'(s)$, and $|g_k'(s)| \le M|g(s)-a| + |g'(s)|$ on $(-\infty, 0)$. This also implies $|g_k'(s)|^p \le 2^p \max\{M^p|g(s)-$

$a|^p, |g'(s)|^p\}$ on $(-\infty, 0)$. It is easy to show g'_k converges to g' on $\boldsymbol{R}$ as k go to ∞. By the fact that $g - a$ is in $\mathcal{W}^{1,p}(\boldsymbol{R}; \boldsymbol{R}^N)$ and by Lebesgue's Dominated Convergence Theorem, we conclude that

$$\lim_{k\to\infty} \int_{-\infty}^{0} |g'_k(s)|^p ds = \int_{-\infty}^{0} |g'(s)|^p ds\,.$$

Similary, $\lim_{k\to\infty} \int_0^\infty |g'_k(s)|^p ds = \int_0^\infty |g'(s)|^p ds$. Hence

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} |g'_k(s)|^p ds = \int_{-\infty}^{\infty} |g'(s)|^p ds\,.$$

Thus Equation (22) holds and the proof of Lemma 3 is completed. □

LEMMA 4. *Let S_F and $S_\star$ be the sets defined in Equations* (4) *and* (10), *respectively. Then* $\inf S_\star \le \inf S_F$.

PROOF. Let $\alpha \in S_F$. By Equation (4), there exists a function $g_L \in T_F$ such that

$$\alpha = \int_{-L}^{L} \left(\frac{W(g_L(s))}{q} + \frac{|g'_L(s)|^p}{p}\right) ds\,.$$

Using the technique of rescaling set, we define $g_{L,1}(s) = g_L(sL)$ for $s \in [-1, 1]$. Applying the change of variables and Young's inequality, we have

$$\int_{-1}^{1} (W(g_{L,1}(s)))^{1/q} \cdot |g'_{L,1}(s)| ds = \int_{-L}^{L} W(g_L(s))^{1/q} \cdot |g'_L(s)| ds\,,$$

and

$$\int_{-L}^{L} W(g_L(s))^{1/q} \cdot |g'_L(s)| ds \le \int_{-L}^{L} \left(\frac{W(g_L(s))}{q} + \frac{|g'_L(s)|^p}{p}\right) ds\,.$$

Define $\tilde{\alpha} \equiv \int_{-1}^{1} (W(g_{L,1}(s)))^{1/q} \cdot |g'_{L,1}(s)| ds$. We have shown that for each $\alpha \in S_F$, there exists a number $\tilde{\alpha} \in S_\star$ such that $\tilde{\alpha} \le \alpha$. Therefore, $\inf S_\star \le \inf S_F$. □

LEMMA 5. *Let S_∞ and $S_\star$ be the sets defined in Equations* (9) *and* (10), *respectively. Then* $\inf S_\infty \le \inf S_\star$.

PROOF. Let $\alpha \in S_\star$. Then there is a function $g_\star \in T_\star$ such that

$$\alpha = \int_{-1}^{1} (W(g_\star(s)))^{1/q} \cdot |g'_\star(s)| ds\,. \tag{28}$$

We change the variable to the arclength $\tau(s)$, $\tau(s) \equiv \int_{-1}^{s} |g'_\star(t)| dt$ for $-1 \le s \le 1$ and denote $L = \tau(1)$. Since $\tau'(s) = |g'_\star(s)| > 0$ on $(-1, 1)$, the function τ is monotone increasing, and its inverse function $\tau^{-1} : [0, L] \to [-1, 1]$ exists. Furthermore, $(\tau^{-1})'(s) = (|g'_\star(\tau^{-1}(s))|)^{-1} > 0$ on $(0, L)$. We define $g_L(s) = g_\star(\tau^{-1}(s))$ for $0 \le s \le L$. Then

$$g'_L(s) = \frac{g'_\star(\tau^{-1}(s))}{|g'_\star(\tau^{-1}(s))|}, \quad \text{and} \quad |g'_L(s)| = 1 \quad \text{for} \quad s \in [0, L]\,. \tag{29}$$

Using the variable of arclength to rewrite Equation (28) as

$$\alpha = \int_0^L (W(g_\star(\tau^{-1}(s)))^{1/q} \cdot |g_\star'(\tau^{-1}(s))|\,|(\tau^{-1})'(s)|ds\,.$$

Since $|g_\star'(\tau^{-1}(s))||\tau^{-1}(s)| = |g_L'(s)|$, it further implies

$$\alpha = \int_0^L (W(g_L(s)))^{1/q} \cdot |g_L'(s)|ds\,. \tag{30}$$

For the function g_L, if there is a function $g_\infty \in T_\infty$ such that

$$\int_{-\infty}^{\infty} \left(\frac{W(g_\infty(s))}{q} + \frac{|g_\infty'(s)|^p}{p}\right)ds = \int_0^L (W(g_L(s)))^{1/q} \cdot |g_L'(s)|ds = \alpha\,,$$

then $\alpha \in S_\infty$. Moreover, $S_\star \subset S_\infty$ and this will show that Lemma 5 holds. The connection between the two functions g_L and g_∞ is contained in the two equalities

$$\int_{-\infty}^{\infty} \left(\frac{W(g_\infty(s))}{q} + \frac{|g_\infty'(s)|^p}{p}\right)ds = \int_{-\infty}^{\infty} (W(g_\infty(s))^{1/q}) \cdot |g_\infty'(s)|ds \tag{31}$$

and

$$\int_{-\infty}^{\infty} (W(g_\infty(s))^{1/q}) \cdot |g_\infty'(s)|ds = \int_0^L (W(g_L(s)))^{1/q} \cdot |g_L'(s)|ds\,. \tag{32}$$

The condition for Equation (31) means that the equality holds in Young's inequality, that is,

$$|g_\infty'(t)| = W(g_\infty(t))^{1/p} \quad \text{for all} \quad t \in \boldsymbol{R}\,. \tag{33}$$

The condition for Equation (32) is that there is a monotone increasing function $h \in C^1_{\mathrm{pw}}(\boldsymbol{R})$ such that

$$g_\infty(t) = g_L(h(t)) \quad \text{for all} \quad t \in \boldsymbol{R}\,, \tag{34}$$

$$\lim_{t\to-\infty} h(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} h(t) = L\,. \tag{35}$$

Since h is increasing, Equation (33) is reduced to

$$h'(t) = |h'(t)| = W(g_L(h(t)))^{1/p} \quad \text{for all} \quad t \in \boldsymbol{R} \tag{36}$$

by Equations (29) and (34). The above discussion reduces the problem to the existence of an increasing function h which satisfies the ordinary differential equation $y' = W(g_L(y))^{1/p}$ with initial value $y(0) = L/2$. Since W is in $\mathcal{W}^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}^N; \boldsymbol{R})$ and W is locally Lipschitz on $\boldsymbol{R}^N$, it implies that $W(g_L(y))^{1/p}$ is in $W^{1,\infty}_{\mathrm{loc}}((0, L))$. Thanks to Picard's Existence and Uniqueness Theorem [17], we obtain the existence of a unique continuously differentiable function $y : I \to \boldsymbol{R}$ with $0 \in I$ satisfying the differential equation and the initial condition, and also by the Continuation Theorem [17], we extend the interval I into the maximum interval of existence, say (t_0, t_1) with $I \subset (t_0, t_1)$, and

$$\lim_{t\to t_0^+} y(t) = 0\,, \quad \lim_{t\to t_1^-} y(t) = L\,. \tag{37}$$

Define

$$h(t) \equiv \begin{cases} 0 & \text{if} \quad t \le t_0\,, \\ y(t) & \text{if} \quad t_0 < t < t_1\,, \\ L & \text{if} \quad t \ge t_1\,. \end{cases} \tag{38}$$

The function $h \in C^1_{\mathrm{pw}}(\boldsymbol{R})$ is as desired and it also satisfies Equations (36) and (35). By Equations (34), (37) and (38), it follows that g_∞ is in $C^1_{\mathrm{pw}}(\boldsymbol{R})$,

$$g_\infty(-\infty) \equiv \lim_{t\to-\infty} g_\infty(t) = g_L(0) = g_\star(-1) \in A_i\,,$$

and

$$g_\infty(\infty) \equiv \lim_{t\to\infty} g_\infty(t) = g_L(L) = g_\star(1) \in A_j\,.$$

Hence g_∞ is in T_∞ and satisfies Equations (31) and (32). As a consequence, $S_\star$ is included in S_∞ and Lemma 5 is proved. □

3. Γ-convergence theory. We first investigate the struture of a limit function.

LEMMA 6. *If the functionals F_ε are defined by Equation* (5) *with* $\liminf_{\varepsilon\to0^+} F_\varepsilon(v_\varepsilon) < \infty$ *and functions v_ε converge to v in $L^1(\Omega;\boldsymbol{R}^N)$, then the values of the function $v(x)$ belong to $\{a,b\}$ a.e. Furthermore, the function $v(x)$ is equal to $\chi_A(x)a + \chi_B(x)b$, where the set A and B are defined by* (11).

PROOF. Since v_ε converges to v in $L^1(\Omega;\boldsymbol{R}^N)$, there is a subsequence $\{v_{\varepsilon_j}\}$ which converges to v a.e. It follows that $W\circ v_{\varepsilon_j}$ converges to $W\circ v$ a.e. By Fatou's lemma, we have

$$0 \le \int_\Omega W(v(x))dx \le \liminf_{j\to\infty}\int_\Omega W(v_{\varepsilon_j}(x))dx \le \liminf_{j\to\infty} \varepsilon_j q F_{\varepsilon_j}(v_{\varepsilon_j}) = 0\,.$$

The last equality holds due to the assumption. The integral of the nonnegative potential function W over Ω vanishes, and hence $W(v(x)) = 0$. That implies $v(x)$ is in $\{a,b\}$ a.e., and this lemma is proved. □

It follows from (H3) that we can choose $r_1 > |a-b|/2 = r_0$, and $C = \max\{W^{1/q}(v);\ |v-(a+b)/2| \le r_1\}$, such that

$$\int_{r_0}^{r_1}\left[\inf_{|v-(a+b)/2|=r} W^{1/q}(v)\right]dr > \frac{1}{2}\inf S_\star\,.$$

Define $W^\star(v) \equiv \min\{W^{1/q}(v), C\}$, $T(\xi) = \{\gamma \in C^1_{\mathrm{pw}}([-1,1])\,;\, \gamma(-1) = a,\ \gamma(1) = \xi\}$, and

$$\Phi(\xi) = \inf\left\{\int_{-1}^{1} W^\star(\gamma(s))|\gamma'(s)|ds\,;\, \gamma \in T(\xi)\right\}\,. \tag{39}$$

The technique of truncation has been used to guarantee the Lipschitz continuity of the function Φ on $\boldsymbol{R}^N$. A similar argument in Fonseca and Tartar [16] establishes the properties of the function Φ in the following lemma.

LEMMA 7. *Let Φ be the function defined by* (39).

(i) *The function $\Phi : \boldsymbol{R}^N \to \boldsymbol{R}$ is Lipschitz continuous on $\boldsymbol{R}^N$.*

(ii) *If $v \in \mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N)$, then $\Phi \circ v \in \mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N)$ and*

$$|\nabla(\Phi \circ v)(x)| \leq W^{1/q}(v(x)) \cdot |\nabla v(x)| \quad \textit{for a.e. } x \in \Omega .$$

(iii) $\inf S_\star = \Phi(b)$.

Let us recall the following lemma.

LEMMA 8 (cf. [13, 20], Lower semicontinuous of variation measures). *Suppose that $\Omega \subset \boldsymbol{R}^n$ is open. Suppose f_k is in $BV(\Omega)$ for each k in N and f_k converges to f in $L^1(\Omega)$. Then $\int_\Omega |\nabla f(x)| dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla f_k(x)| dx$.*

Now, we will show the theorem below.

THEOREM 9. *Let functionals F_ε and F_0 be defined by Equations* (5) *and* (12), *respectively. If v_ε converges to v in $L^1(\Omega; \boldsymbol{R}^N)$, then $F_0(v) \leq \liminf_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon)$.*

PROOF. We first claim that the limit function v is a BV-function on Ω. If v_ε converges to v in $L^1(\Omega; \boldsymbol{R}^N)$, then $\liminf_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon)$ is finite and $\Phi \circ v_\varepsilon$ converges to $\Phi \circ v$ in $L^1(\Omega)$. It implies that v_ε is in $\mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N)$. By the fact that if u is in $\mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N)$ then $\Phi \circ u$ is in $\mathcal{W}^{1,p}(\Omega; \boldsymbol{R}^N)$ and also in $\mathrm{BV}(\Omega)$, we have $\Phi \circ v_\varepsilon \in \mathrm{BV}(\Omega)$. Applying Lemma 8 to $\Phi \circ v_\varepsilon$ and $\Phi \circ v$, we have

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx \leq \liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla(\Phi \circ v_\varepsilon)(x)| dx .$$

Using condition (ii) in Lemma 7 and Young's inequality to (5), we get

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx \leq \liminf_{\varepsilon\to 0^+} \int_\Omega W^{1/q}(v_\varepsilon(x)) |\nabla v_\varepsilon(x)| dx$$

and

$$\int_\Omega W^{1/q}(v_\varepsilon(x)) |\nabla v_\varepsilon(x)| dx \leq F_\varepsilon(v_\varepsilon) .$$

It follows that

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx \leq \liminf_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon) , \tag{40}$$

and $\Phi \circ v$ is in $\mathrm{BV}(\Omega)$. By (39), $\Phi(a)$ is zero and $\Phi(b)$ is equal to $\inf S_\star$. Hence we obtain $(\Phi \circ v)(x) = \Phi(v(x)) = \Phi(\chi_A a + \chi_B b) = \Phi(b)\chi_B$. Thus, χ_B is in $\mathrm{BV}(\Omega)$ and $\chi_A = \chi_{\Omega\setminus B}$ is in $\mathrm{BV}(\Omega)$. Furthermore, the function v is in $\mathrm{BV}(\Omega)$ since $v = \chi_A a + \chi_B b$. We have proved the claim. The next step is to show the identity

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx = F_0(v) . \tag{41}$$

Since $(\Phi \circ v)(x) = \Phi(b)\chi_B$, we have

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx = \int_\Omega \Phi(b) |\nabla \chi_B| dx = \Phi(b) \mathrm{Per}_\Omega(B) = \Phi(b) \cdot \mathrm{Per}_\Omega(A) .$$

From the equality $\Phi(b) = \inf S_\star$ again, it follows that

$$\int_\Omega |\nabla(\Phi \circ v)(x)| dx = (\inf S_\star)\mathrm{Per}_\Omega(A)\,. \tag{42}$$

Combining Lemma 7 with Equation (12), we get Equation (41). Theorem 9 follows from Equations (41), (42) and (40). □

THEOREM 10. *Let* $v \in \mathcal{W}^{1,p}(\Omega;\boldsymbol{R}^N)$. *Then there exists a family* $\{v_\varepsilon\}_{\varepsilon>0}$ *in* $L^1(\Omega;\boldsymbol{R}^N)$ *which converges to* v *in* $L^1(\Omega;\boldsymbol{R}^N)$ *and* $\limsup_{\varepsilon\to 0^+} F_\varepsilon(v_\varepsilon)$ *is less than or equal to* $F_0(v)$.

Before the proof of Theorem 10, let us recall two lemmas.

LEMMA 11 ([4, 21]). *Let* $\Omega \subset \boldsymbol{R}^n$ *be open, and* A *be a polygonal domain in* $\boldsymbol{R}^n$ *with the compact boundary* ∂A *and* $H^{n-1}(\partial A \cap \partial\Omega) = 0$. *Then the following two statements hold*:

(i) *There exists a constant* $\eta > 0$ *such that the function* $h(x)$ *defined by*

$$h(x) \equiv \begin{cases} -d(x,\partial A) & \text{if } x \in A\,, \\ \ \ d(x,\partial A) & \text{if } x \notin A \end{cases}$$

is Lipshitz continuous on $D_\eta \equiv \{x \in \boldsymbol{R}^n;\ |h(x)| < \eta\}$ *and* $|\nabla h(x)| = 1$ *a.e. on* D_η.

(ii) *If* $S_t \equiv \{x \in \boldsymbol{R}^n;\ h(x) = t\}$, *then* $\lim_{t\to 0} H^{n-1}(S_t \cap \Omega) = H^{n-1}(\partial A \cap \Omega)$.

LEMMA 12 (cf. [13, 20], Coarea formula). *Suppose that* $f : \boldsymbol{R} \to \boldsymbol{R}$ *is a measurable function and that* $h : \boldsymbol{R}^n \to \boldsymbol{R}$ *is a Lipschitz function. Then*

$$\int_\Omega f(h(x))|\nabla h(x)| dx = \int_{-\infty}^{\infty} f(t) H^{n-1}(\{x \in \Omega\,;\, h(x) = t\}) dt$$

for each measurable subset Ω *of* $\boldsymbol{R}^n$. *In particular, we have*

$$\int_\Omega |\nabla h(x)| dx = \int_{-\infty}^{\infty} H^{n-1}(\{x \in \Omega\,;\, h(x) = t\}) dt$$

for each measurable subset Ω *of* $\boldsymbol{R}^n$.

PROOF OF THEOREM 10. The strategy for the proof is to observe the form of the limit function v, $v = \chi_A a + (1-\chi_A)b$, then to construct a sequence of functions $v_\varepsilon^g \in L^1(\Omega;\boldsymbol{R}^N)$ which converges to v in $L^1(\Omega;\boldsymbol{R}^N)$ starting from a given function g in T_F. Moreover, this sequence of functions satisfies

$$\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon^g) = \mathrm{Per}_\Omega(A) \int_{-L}^{L} \left(\frac{W(g(s))}{q} + \frac{|g'(s)|^p}{p} \right) ds\,. \tag{43}$$

Finally, we apply the above result to construct a sequence which we want.

By the assumption, $v \in \mathcal{W}^{1,p}(\Omega;\boldsymbol{R}^N)$ and Lemma 6, v must be of the form

$$v = \chi_A a + (1-\chi_A) b \qquad \text{a.e. in } \Omega\,. \tag{44}$$

Without loss of generality, we assume that $\partial A \cap \Omega \in C^2$ and $H^{n-1}(\partial A \cap \partial\Omega) = 0$. We define the function $h : \Omega \to \boldsymbol{R}$ by

$$h(x) \equiv \begin{cases} -d(x, \partial A) & \text{if} \quad x \in A\,, \\ d(x, \partial A) & \text{if} \quad x \notin A\,. \end{cases} \tag{45}$$

Now, let $g \in T_F$, where T_F was defined by (3). That is, there is $L > 0$ such that $g(-L) = a$ and $g(L) = b$. For this function g, we define a sequence of functions $v_\varepsilon^g : \Omega \to \boldsymbol{R}^N$ by

$$v_\varepsilon^g(x) = \begin{cases} a & \text{if} \quad h(x) \le -\varepsilon L\,, \\ g_\varepsilon(h(x)) & \text{if} \quad -\varepsilon L < h(x) < \varepsilon L\,, \\ b & \text{if} \quad \varepsilon L \le h(x)\,, \end{cases} \tag{46}$$

where $g_\varepsilon(s) = g(s/\varepsilon)$ for all s in $[-L, L]$. For the simplification of notation, we define $f : \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(t) = \begin{cases} |g_\varepsilon(t) - a| & \text{if} \quad -\varepsilon L < t \le 0\,, \\ |g_\varepsilon(t) - b| & \text{if} \quad 0 < t < \varepsilon L\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $|v_\varepsilon^g - v|$ is equal to $f \circ h$ on Ω. By Lemma 11, we have

$$\int_\Omega |v_\varepsilon^g - v|(x)dx = \int_\Omega f \circ h(x)dx = \int_\Omega f(h(x))|\nabla h|(x)dx\,,$$

and by Lemma 12, we have

$$\begin{aligned} \int_\Omega |v_\varepsilon^g - v|(x)dx &= \int_{-\infty}^{\infty} f(t) \cdot H^{n-1}(\{x \in \Omega\,;\ h(x) = t\})dt \\ &= \int_{-\varepsilon L}^{0} |g_\varepsilon(t) - a| \cdot H^{n-1}(\{x \in \Omega\,;\ h(x) = t\})dt \\ &\quad + \int_0^{\varepsilon L} |g_\varepsilon(t) - b| \cdot H^{n-1}(\{x \in \Omega\,; h(x) = t\})dt\,. \end{aligned}$$

Using the change of variables for the last term in the last equality, we get the equality

$$\begin{aligned} &\int_0^{\varepsilon L} |g_\varepsilon(t) - b| \cdot H^{n-1}(\{x \in \Omega\,; h(x) = t\})dt \\ &\qquad = \varepsilon \int_0^L |g(t) - b|\, H^{n-1}(\{x \in \Omega\,; h(x) = \varepsilon t\})dt\,. \end{aligned}$$

By Lemma 11 again, $\lim_{\tau\to 0} H^{n-1}(\{x \in \Omega\,; h(x) = \tau\}) = H^{n-1}(\partial A \cap \Omega) = \mathrm{Per}_\Omega(A)$. Taking $\tau = \varepsilon t$ for each $\varepsilon > 0$ with $|\varepsilon L| < \delta$, we have $H^{n-1}(\{x \in \Omega\,; h(x) = \varepsilon t\}) < \mathrm{Per}_\Omega(A) + 1$ for all $-L \le t \le L$. Therefore, we have

$$\varepsilon \int_0^L |g(t) - b|\, H^{n-1}(\{x \in \Omega\,; h(x) = \varepsilon t\})dt \le \varepsilon L(\mathrm{Per}_\Omega(A) + 1)\|g - b\|_\infty\,.$$

Similary, we have

$$\varepsilon \int_{-L}^0 |g(t) - a|\, H^{n-1}(\{x \in \Omega\,; h(x) = \varepsilon t\})dt \le \varepsilon L(\mathrm{Per}_\Omega(A) + 1)\|g - a\|_\infty\,.$$

The above argument gives us the estimate

$$\int_{\Omega} |v_\varepsilon^g - v|(x)dx \leq \varepsilon L(\mathrm{Per}_{\Omega}(A) + 1)[\|g - a\|_\infty + \|g - b\|_\infty]\,,$$

provided $\varepsilon > 0$ small enough. Furthermore, we know $\lim_{\varepsilon\to 0}\int_{\Omega} |v_\varepsilon^g - v|(x)dx$ is equal to zero, and the convergence of v_ε^g to v in $L^1(\Omega; \boldsymbol{R}^N)$ is proved. Next step is to show Equation (43).

By Equation (5), we evaluate

$$\begin{aligned}
F_\varepsilon(v_\varepsilon^g) &= \int_{\Omega}\left[\frac{1}{\varepsilon}\frac{W(v_\varepsilon^g(x))}{q} + \varepsilon^{p-1}\frac{|\nabla v_\varepsilon^g|^p(x)}{p}\right]dx\\
&= \int_{\Omega}\left[\frac{1}{\varepsilon}\frac{W(g_\varepsilon(h(x)))}{q} + \varepsilon^{p-1}\frac{|\nabla v_\varepsilon^g(h(x))|^p(x)|\nabla h(x)|}{p}\right]dx\\
&= \int_{\Omega}\left[\frac{1}{\varepsilon}\frac{W(g_\varepsilon(h(x)))}{q} + \varepsilon^{p-1}\frac{|\nabla v_\varepsilon^g(h(x))|^p(x)}{p}\right]|\nabla h(x)|dx\,.
\end{aligned}$$

The last equality follows from $|\nabla h| = 1$ a.e. (cf. Lemma 11) whenever $\varepsilon > 0$ small enough. By using Lemmas 11, 12 and technique of change of variables to the last term, we rewrite $F_\varepsilon(v_\varepsilon^g)$ as

$$F_\varepsilon(v_\varepsilon^g) = \int_{-L}^{L}\left(\frac{W\circ g}{q} + \frac{|g'|^p}{p}\right)(t)\ H^{n-1}(\{x \in \Omega; h(x) = \varepsilon t\})dt\,.$$

Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon^g) &= \lim_{\varepsilon\to 0}\int_{-L}^{L}\left(\frac{W\circ g}{q} + \frac{|g'|^p}{p}\right)(t)\ H^{n-1}(\{x \in \Omega; h(x) = \varepsilon t\})dt\\
&= \int_{-L}^{L}\left(\frac{W\circ g}{q} + \frac{|g'|^p}{p}\right)(t)\ \lim_{\varepsilon\to 0} H^{n-1}(\{x \in \Omega; h(x) = \varepsilon t\})dt\\
&= \mathrm{Per}_{\Omega}(A)\int_{-L}^{L}\left(\frac{W(g(t))}{q} + \frac{|g'(t)|^p}{p}\right)dt\,.
\end{aligned}$$

Let $\tilde{\varepsilon}$ be a positive number. Let us denote $\tilde{S}_F = \mathrm{Per}_{\Omega}(A)S_F$, $\tilde{S}_\infty = \mathrm{Per}_{\Omega}(A)S_\infty$ and $\tilde{S}_\star = \mathrm{Per}_{\Omega}(A)S_\star$. By Theorem 1, we have $\inf\tilde{S}_F = \inf\tilde{S}_\infty = \inf\tilde{S}_\star = F_0(v)$. Since $F_0(v) = \inf\tilde{S}_F$, it follows that there exists a number $\eta = \eta(\tilde{\varepsilon})$ in $\tilde{S}_F$ and g_η in T_F such that $\eta < F_0(v) + \tilde{\varepsilon}$ and

$$\eta = \mathrm{Per}_{\Omega}(A)\int_{-L}^{L}\left[\frac{W(g_\eta(s))}{q} + \frac{|g_\eta'(s)|^p}{p}\right]ds \quad \text{for some} \quad L > 0\,.$$

Furthermore, for $\eta \in \tilde{S}_F$, there exists a family $\{v_\tau^{g_\eta}\}_{\tau>0}$ in $L^1(\Omega; \boldsymbol{R}^N)$ such that $v_\tau^{g_\eta}$ converges to v in $L^1(\Omega; \boldsymbol{R}^N)$ and $F_\tau(v_\tau^{g_\eta})$ converges to η as τ goes to 0^+. We get that for each $\tilde{\varepsilon} > 0$, there is a $\eta < F_0(v) + \tilde{\varepsilon}$ and $v_{\tau_\eta}^{g_\eta}$ in $L^1(\Omega; \boldsymbol{R}^N)$ such that $\|v_{\tau_\eta}^{g_\eta} - v\|_{L^1(\Omega;\boldsymbol{R}^N)} < \tilde{\varepsilon}$ and $F_{\tau_\eta}(v_{\tau_\eta}^{g_\eta}) < F_0(v) + \tilde{\varepsilon}$. Let $v_{\tilde{\varepsilon}} = v_{\tau_\eta}^{g_\eta}$. The sequence $\{v_{\tilde{\varepsilon}}\}_{\tilde{\varepsilon}>0}$ has the desired property and Theorem 10 is proved. □

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