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The Chern character of a transversally elliptic symbol and the equivariant index

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Abstract. Let G be a compact Lie group acting on a compact manifold M. In this article, we associate to a G-transversally elliptic symbol on M a G-invariant generalized function on G, constructed in terms of equivariant closed differential forms on the cotangent bundle T^*M.

1. Introduction

Let M be a compact smooth manifold with a smooth action of a compact Lie group G. We consider the closed subset T^*_GM of the cotangent bundle T^*M, union of the spaces $(T^*_GM)_x$, $x \in M$, where $(T^*_GM)_x \subset T^*_xM$ is the orthogonal of the tangent space at x to the orbit $G.x$. Notice that T^*_GM is not a sub-bundle in general, as the dimension of the orbit may vary. We consider $K_G(T^*_GM)$, the group of equivariant K-theory of the space T^*_GM. It is a module over $R(G)$, the representation ring of G. Let $\mathcal{E}^\pm$ be smooth G-equivariant vector bundles over M. Let P be a pseudodifferential operator on M, mapping sections of $\mathcal{E}^+$ to sections of $\mathcal{E}^-$. Let $p : T^*M \to M$ be the natural projection. The principal symbol $\sigma(P)$ of P is a bundle map $p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ defined over $T^*M\backslash 0$. The operator P is said to be G-transversally elliptic if it is G-invariant and if its principal symbol $\sigma(P)(x,\xi)$ is invertible for every $(x,\xi) \in T^*_GM$ such that $\xi \neq 0$. If P is G-invariant, its kernel $\operatorname{Ker} P$ is a G-invariant vector space. If P is moreover G-transversally elliptic, the representation of G in $\operatorname{Ker} P$ is trace class [1]. We choose G-invariant metrics on M and $\mathcal{E}^\pm$. The adjoint P^* is G-invariant and G-transversally elliptic, with principal symbol $\sigma(P^*)(x,\xi) = \sigma(P)(x,\xi)^*$. The G-equivariant index of P is the generalized function on G defined by

$$\operatorname{index}^G(P)(g) := \operatorname{Tr}(g, \operatorname{Ker} P) - \operatorname{Tr}(g, \operatorname{Ker} P^*).$$

The principal symbol $\sigma(P)$ defines an element $[\sigma(P)]$ of $K_G(T^*_GM)$. Moreover, for any $m \in \mathbb{Z}$, any element of $K_G(T^*_GM)$ can be represented by the principal symbol of an operator P of order m. As proven in [1], the G-equivariant index of P depends only on the class $[\sigma(P)] \in K_G(T^*_G M)$. Thus the G-equivariant index induces a map

$$\mathrm{index}_a^{G,M} : K_G(T^*_G M) \to C^{-\infty}(G)^G$$

which is called the *analytical index*. Clearly $\mathrm{index}_a^{G,M}$ is a homomorphism of $R(G)$-modules.

Our purpose in this article is to define the *cohomological index* $\mathrm{index}_c^{G,M}$, a homomorphism of $R(G)$-modules from $K_G(T^*_G M)$ to $C^{-\infty}(G)^G$, by means of the *bouquet integral* of a certain family of equivariant differential forms on the cotangent bundle T^*M. We make use of the concept of superconnection introduced in [16]: our family is the *bouquet* of Chern characters of a particular superconnection attached to the symbol $\sigma(P)$. In the article [11], we will show that these two maps – the analytical and the cohomological index – coincide. The bouquet integral is a generalized function on G defined locally; around $s \in G$, it is constructed by the method of descent [14] from the integral over $T^*M(s)$ of a $G(s)$-equivariant differential form on $T^*M(s)$, where $M(s) \subset M$ is the fixed point set of s and $G(s) \subset G$ is the centralizer of s. The precise result is stated in Theorem 30.

Let us describe the formula which defines the cohomological index around the identity $e \in G$. Let $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ be a G-equivariant smooth bundle map over T^*M. Let $\nabla^\pm$ be a G-invariant connection on $\mathscr{E}^\pm$. We choose a G-invariant Hermitian metric on $\mathscr{E}^\pm$. Let σ^* be the adjoint of σ with respect to the metrics. Let ω^M be the canonical 1-form on T^*M. We consider the two following superconnections $\mathbb{A}$ and $\mathbb{A}^\omega$ on the superbundle $p^*(\mathscr{E}^+ \oplus \mathscr{E}^-)$:

$$\begin{aligned}\mathbb{A} &= i\begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} + \begin{pmatrix} p^*\nabla^+ & 0 \\ 0 & p^*\nabla^- \end{pmatrix}, \\ \mathbb{A}^\omega &= \mathbb{A} - i\omega^M I_{p^*(\mathscr{E}^+ \oplus \mathscr{E}^-)}.\end{aligned}$$

Let $\mathfrak{g}$ be the Lie algebra of G and let $f : T^*M \to \mathfrak{g}^*$ be the moment map defined by $\langle f(x,\xi), X\rangle = \langle \xi, (X_M)_x\rangle$, for $X \in \mathfrak{g}$, where X_M is the vector field on M associated to the action of G. Then the Chern characters of these superconnections are related by

$$\mathrm{ch}(\mathbb{A}^\omega)(X) = e^{i\langle f, X\rangle} e^{-id\omega^M} \mathrm{ch}(\mathbb{A})(X).$$

Assume that σ is G-transversally elliptic and satisfies the growth conditions of definition 10, for instance σ coincides far from the zero section with the principal symbol of a transversally elliptic operator of order $m \geqq 1$. Our first main result, Theorem 16, is

Main Theorem 1. *If ϕ is a test function on $\mathfrak{g}$, the differential form $\int_{\mathfrak{g}} \mathrm{ch}(\mathbb{A}^\omega)(X)\phi(X)\,dX$ is rapidly decreasing on T^*M.*

This theorem follows from the assumption of transversal ellipticity and some estimates on Fourier transforms: for instance, consider the simple situation where M is a product $M = G \times M_1$ with G acting on the left on itself and acting trivially on M_1. Let σ be the pull-back to M of an elliptic symbol on M_1. In this case, the theorem amounts to estimates on the integrals $(\xi_0, \xi_1) \mapsto \int_{\mathfrak{g}} e^{i\langle X, \xi_0\rangle} e^{-\|\xi_1\|^2} \phi(X)\, dX$, for $\xi_0 \in \mathfrak{g}^*$ and $\xi_1 \in T^*M_1$, where $\phi(X)$ is a test function on $\mathfrak{g}$.

We introduce the J-genus $J(M)(X)$ of M, an equivariant differential form on M which is invertible for small $X \in \mathfrak{g}$. Then we can define a generalized function $\theta_e(\sigma)(X)$ near 0 in $\mathfrak{g}$ by

$$\theta_e(\sigma)(X) = \int_{T^*M} (2i\pi)^{-\dim M} \mathrm{ch}\,(\mathbb{A}^{\omega})(X)\, p^*(J(M)(X)^{-1}).$$

From the above theorem it follows that this integral makes sense as a *generalized* function on a neighbourhood of $0 \in \mathfrak{g}$: if ϕ is a test function on $\mathfrak{g}$ we define

$$\int_{\mathfrak{g}} \theta_e(X)\phi(X)\, dX = \int_{T^*M} \Big(\int_{\mathfrak{g}} (2i\pi)^{-\dim M} \mathrm{ch}\,(\mathbb{A}^{\omega})(X)\, p^*(J(M)(X)^{-1})\phi(X)\, dX \Big).$$

For any $s \in G$, a generalized function $\theta_s(\sigma)(Y)$ is defined for small $Y \in \mathfrak{g}(s)$ by a similar integral over $T^*M(s)$.

We prove that all the generalized functions θ_s can indeed be glued together, (Theorem 30):

Main Theorem 2. *There exists a G-invariant generalized function* $\mathrm{index}_c^{G,M}(\sigma)$ *on G whose germ at $s \in G$ is given, for $Y \in \mathfrak{g}(s)$ small enough, by* $\mathrm{index}_c^{G,M}(\sigma)(s \exp Y) = \theta_s(Y)$.

Formally, this is a consequence of the localization formula in equivariant cohomology, but it requires a rather technical proof, as we deal with equivariant differential forms on T^*M whose integrals exist as generalized functions on $\mathfrak{g}$, although they cannot be integrated pointwise. Furthermore we show that the G-invariant generalized function thus defined depends only on the class $[\sigma] \in K_G(T_G^*M)$.

In the elliptic case, the cohomological index is a smooth function, and, as proven in [10], our definition of the cohomological index at $s \in G$ coincides with the Atiyah-Segal-Singer fixed point formula, [4], [5], [6].

Thus, by the results of [11], the cohomological index defined here gives a cohomological Atiyah-Segal-Singer type formula for the equivariant index of transversally elliptic operators.

2. K_G-theory and symbols

2.1. *Bundle maps and K_G-theory*

Let $\mathcal{V}$ be a locally compact topological space with an action of a compact group G. We consider G-equivariant bundle maps $\mathcal{E}^+ \xrightarrow{\sigma} \mathcal{E}^-$, where $\mathcal{E}^\pm$ are G-equivariant complex vector bundles with base $\mathcal{V}$. The *characteristic set* of a bundle map $\mathscr{E}^+ \xrightarrow{\sigma} \mathscr{E}^-$ is the set of points $y \in \mathscr{V}$ for which $\sigma_y : \mathscr{E}^+_y \to \mathscr{E}^-_y$ is not invertible. We consider only G-bundle maps with compact characteristic set. Two G-bundle maps σ_i, $i = 0, 1$, over $\mathscr{V}$ with compact characteristic set are said to be homotopic if there exists a G-bundle map τ with compact characteristic set over $\mathscr{V} \times [0, 1]$ (with trivial G-action on $[0, 1]$) such that $\tau|_{\mathscr{V} \times (i)} \cong \sigma_i$. We denote by $\mathscr{B}_{G,c}(\mathscr{V})$ the set of homotopy classes of G-bundle maps over $\mathscr{V}$. It is a semigroup under direct sums. The subset of classes represented by bundle maps with empty characteristic set is a sub-semigroup. The quotient is the group of "equivariant K-theory with compact supports" $K_G(\mathscr{V})$. Thus two bundle maps with compact characteristic set $\mathscr{E}^+ \xrightarrow{\sigma} \mathscr{E}^-$ and $\mathscr{F}^+ \xrightarrow{v} \mathscr{F}^-$ define the same class in $K_G(\mathscr{V})$ if and only if they are stably homotopic: there exist G-equivariant vector bundles $\mathscr{M}, \mathscr{N}$ such that the G-bundle maps $\mathscr{E}^+ \oplus \mathscr{M} \xrightarrow{\sigma \oplus I_{\mathscr{M}}} \mathscr{E}^- \oplus \mathscr{M}$ and $\mathscr{F}^+ \oplus \mathscr{N} \xrightarrow{v \oplus I_{\mathscr{N}}} \mathscr{F}^- \oplus \mathscr{N}$ are homotopic. The class of a bundle map $\mathscr{E}^+ \xrightarrow{\sigma} \mathscr{E}^-$ is denoted by $[\mathscr{E}^+, \sigma, \mathscr{E}^-]$ or simply by $[\sigma]$.

Let $\mathscr{E}^+ \xrightarrow{\sigma} \mathscr{E}^-$ be a G-equivariant bundle map with compact characteristic set. Assume that $\mathscr{E}^+ = \mathscr{E}^-$ and, for some G-invariant Hermitian metric on $\mathscr{E}^+ = \mathscr{E}^-$, the map σ_y is selfadjoint for every $y \in \mathscr{V}$, then $[\sigma] = 0$ in $K_G(\mathscr{V})$. Indeed, if σ_y is selfadjoint and invertible, then for any $t \in [0, 1]$, the map $tI_{\mathscr{E}^+_y} + (1-t)i\sigma_y$ is also invertible. Thus $[\mathscr{E}^+, \sigma, \mathscr{E}^+] = [\mathscr{E}^+, I, \mathscr{E}^+] = 0$. As a consequence, for any bundle map $\mathscr{E}^+ \xrightarrow{\sigma} \mathscr{E}^-$, let $\sigma^* : \mathscr{E}^- \to \mathscr{E}^+$ be the adjoint bundle map defined with respect to a choice of G-invariant metrics, then the opposite in $K_G(\mathscr{V})$ of $[\mathscr{E}^+, \sigma, \mathscr{E}^-]$ is $[\mathscr{E}^-, \sigma^*, \mathscr{E}^-]$.

The following construction will be important: let $\sigma : \mathscr{E}^+ \to \mathscr{E}^-$ be a G-equivariant bundle map. Consider the super vector bundle $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$. By choosing G-invariant Hermitian metrics, we associate to σ the odd selfadjoint bundle endomorphism of $\mathscr{E}^+ \oplus \mathscr{E}^-$ given by

$$v(\sigma)(y) = \begin{pmatrix} 0 & \sigma_y^* \\ \sigma_y & 0 \end{pmatrix}. \tag{1}$$

Then $v(\sigma)(y)^2 = \begin{pmatrix} \sigma_y^*\sigma_y & 0 \\ 0 & \sigma_y\sigma_y^* \end{pmatrix}$ is positive and the characteristic set of σ is the complement of the set of points $y \in \mathscr{V}$ such that $v(\sigma)(y)^2$ is strictly positive.

2.2. Transversally elliptic symbols on a G-manifold

Let M be a smooth manifold and G be a compact Lie group acting on M.

A point of the cotangent bundle T^*M will be denoted by (x, ξ), where $x \in M$ and $\xi \in T^*_x M$. We denote by p the projection $T^*M \to M$.

For any X in $\mathfrak{g}$, the Lie algebra of G, we denote by X_M the vector field on M generated by the action of G:

$$X_M(x) = \frac{d}{dt} \exp(-tX) \cdot x|_{t=0} \quad \text{for } x \in M. \tag{2}$$

We denote by $f^{M,G} : T^*M \to \mathfrak{g}^*$ the moment map for the action of G on T^*M with respect to the canonical Hamiltonian structure,

$$\langle f^{M,G}(x,\xi), X\rangle = \langle \xi, X_M(x)\rangle. \tag{3}$$

We denote by $f_x^{M,G}$ the restriction of $f^{M,G}$ to T_x^*M.

For $x \in M$, we denote by $(T_G^*M)_x$ the subspace of T_x^*M which is orthogonal to the tangent space to the orbit $G.x$. Thus

$$(T_G^*M)_x = (f_x^{M,G})^{-1}(0)$$

The dimension of $(T_G^*M)_x$ is equal to the codimension of $G.x$. It depends on x. We denote by T_G^*M the closed subset of T^*M union of the spaces $(T_G^*M)_x,\ x \in M$. We call T_G^*M the *G-transversal* space.

For brevity, a smooth bundle map over T^*M of the form $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$, where $\mathscr{E}^\pm$ are smooth vector bundles over M, will be called a *symbol on* M. Remark that we do not make any homogeneity assumption, but we want σ to be defined and smooth on the whole space T^*M, not only outside of the zero section. Thus the *principal symbol* of a pseudodifferential operator must be modified near the zero section in order to define a symbol in the above sense.

Definition 1. *A symbol $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ on M is said to be G-transversally elliptic if the following conditions are satisfied*:

(1) *$\mathscr{E}^+$ and $\mathscr{E}^-$ are G-equivariant vector bundles and σ is a G-map.*

(2) *The projection $p : T^*M \to M$ is proper on the intersection of T_G^*M with the characteristic set of σ.*

For $G = \{e\}$, a transversally elliptic symbol is said to be elliptic.

If M is paracompact, every G-vector bundle over T^*M or over T_G^*M is isomorphic to the pull back of its restriction to the zero section and every G-vector bundle on M is G-homotopic to a smooth one. Therefore, if M is compact, every element of $K_G(T^*M)$ is represented by an elliptic symbol and every element of $K_G(T_G^*M)$ is represented by a transversally elliptic symbol.

3. Equivariant differential forms

3.1. Equivariant differential forms, integration

Let M be a manifold with a smooth action of a Lie group G. We denote by $\mathscr{A}(M)$ the algebra of smooth differential forms on M. If $\mathscr{E}$ is a vector bundle over M, we denote by $\mathscr{A}(M,\mathscr{E})$ the space of $\mathscr{E}$-valued differential forms on M.

A *G-equivariant differential form* on M is a smooth G-equivariant map, defined on the Lie algebra $\mathfrak{g}$, with values in the space $\mathscr{A}(M)$ of differential forms on M. We will denote by X either an element of $\mathfrak{g}$ or the function $X \to X$. Thus a map $\alpha : \mathfrak{g} \to \mathscr{A}(M)$ can be denoted also by $\alpha(X)$. Similar notations will be used also for other functions, thus the notation $f(x)$ means either the function f itself or its value at a point x, depending on the context.

We denote the space of G-equivariant differential forms by $\mathscr{A}_G^\infty(\mathfrak{g},M)$. The symbol ∞ refers to the smoothness as functions of $X \in \mathfrak{g}$. We denote by $\mathscr{A}_G^{pol}(\mathfrak{g},M)$ the subalgebra of equivariant differential forms which depend polynomially on $X \in \mathfrak{g}$. Thus, if we denote by $S(\mathfrak{g}^*)$ the algebra of polynomial functions on $\mathfrak{g}$, $\mathscr{A}_G^{pol}(\mathfrak{g},M)$ is the algebra $(S(\mathfrak{g}^*) \oplus \mathscr{A}(M))^G$.

We will consider equivariant differential forms $\alpha(X)$ which are defined only for X in a G-invariant open subset $W \subset \mathfrak{g}$. We denote by $\mathscr{A}_G^\infty(W,M) = C^\infty(W,\mathscr{A}(M))^G$ the space of these forms. An element of $C^\infty(W,\mathscr{A}(M))$ will be refered to as a differential form on M depending on $X \in W$.

The *equivariant coboundary* $d_\mathfrak{g} : \mathscr{A}_G^\infty(W,M) \to \mathscr{A}_G^\infty(W,M)$ is defined by

$$(d_\mathfrak{g}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)),$$

for $\alpha \in \mathscr{A}_G^\infty(W,M)$ and $X \in W$, where $\iota(X_M)$ is the contraction with the vector field X_M. An equivariant differential form α is said to be (equivariantly) closed if $d_\mathfrak{g}\alpha = 0$.

We will consider these notions in the framework of the total space of a G-equivariant Euclidean bundle $\mathscr{V}$ over M, with in mind the case where $\mathscr{V}$ is the cotangent bundle T^*M. Let $M_0 \subset M$ be a coordinate chart such that $\mathscr{V}|_{M_0}$ is trivialized in $\mathscr{V}|_{M_0} \simeq M_0 \times \mathbb{R}^n$ and let (x,ξ) be the corresponding local coordinates on the total space of $\mathscr{V}|_{M_0}$. Then a differential form on $\mathscr{V}$ is locally a map on $\mathscr{V}$ with values in the fixed vector space $\wedge\mathbb{R}^n$, with local expression $\alpha(x,\xi) = \sum_{a,\,b} \alpha_{a,\,b}(x,\xi)\, dx_a\, d\xi_b$. By a derivative of α, we mean $\partial_x^j \partial_\xi^k \alpha(x,\xi) = \sum \partial_x^j \partial_\xi^k \alpha_{a,\,b}(x,\xi)\, dx_a\, d\xi_b$.

We denote by $\mathscr{A}_{G,\,rapid}^\infty(W,\mathscr{V})$ the space of equivariant differential forms on $\mathscr{V}$ such that $\alpha(X,x,\xi)$ and its derivatives $\partial_x^j \partial_\xi^k \alpha(X,x,\xi)$ are rapidly decreasing, uniformly on compact sets of $W \times M$. Let $\alpha \in \mathscr{A}_{G,\,rapid}^\infty(W,\mathscr{V})$. Then $d\alpha(X)$ is rapidly decreasing, and so is also $\iota(X_\mathscr{V})\alpha(X)$, as the vector field $X_\mathscr{V}$ depends linearly on ξ, thus the equivariant coboundary $d_\mathfrak{g}$ maps $\mathscr{A}_{G,\,rapid}^\infty(W,\mathscr{V})$ into itself. We denote by $\mathscr{H}_{G,\,rapid}^\infty(W,\mathscr{V})$ the corresponding cohomology space $\operatorname{Ker} d_\mathfrak{g}/\operatorname{Im} d_\mathfrak{g}$.

Definition 2. *A differential form $\alpha \in C^\infty(W,\mathscr{A}(\mathscr{V}))$ is said to be rapidly decreasing in $\mathfrak{g}$-mean if, for every test function ϕ on $W \subset \mathfrak{g}$, the differential form $\int_\mathfrak{g} \alpha(X)\phi(X)\,dX$ on $\mathscr{V}$ is rapidly decreasing along the fibers, as well as all its derivatives, and such that, moreover, the map $\phi \mapsto \int_\mathfrak{g} \alpha(X)\phi(X)\,dX$ is continuous, with respect to the natural semi-norms on the space of rapidly decreasing differential forms.*

We denote by $\mathscr{A}_{G,\,mean\text{-}rapid}^\infty(W,\mathscr{V})$ the space of G-equivariant differential forms on $\mathscr{V}$ which are rapidly decreasing in $\mathfrak{g}$-mean. The equivariant coboundary $d_\mathfrak{g}$ maps $\mathscr{A}_{G,\,mean\text{-}rapid}^\infty(W,\mathscr{V})$ into itself. We denote by $\mathscr{H}_{G,\,mean\text{-}rapid}^\infty(W,\mathscr{V})$ the corresponding cohomology space $\operatorname{Ker} d_\mathfrak{g}/\operatorname{Im} d_\mathfrak{g}$.

Remark 3. Let $\alpha \in C^\infty(W,\mathscr{A}(\mathscr{V}))$ be rapidly decreasing in $\mathfrak{g}$-mean and let $q \in C^\infty(W,\mathscr{A}(\mathscr{V}))$. If $q(X)$ is the pull-back of a form on M, or if q does not depend on X and is slowly increasing along the fibers, uniformly on compact sets of M, together with its derivatives $\partial_x^j \partial_\xi^k q(x, \xi)$, then the product $\alpha(X) q(X)$ is also rapidly decreasing in $\mathfrak{g}$-mean.

Remark 4. We will often identify a differential form α on M and its pullback on $\mathscr{V}$, thus denoted also by α instead of $p^*\alpha$.

Assume that M is compact. When the total space of $\mathscr{V}$ is oriented, the integral $\int_{\mathscr{V}} \alpha$ over $\mathscr{V}$ of a form $\mathscr{A}^\infty_{G,\, mean\text{-}rapid}(W, \mathscr{V})$ is defined as a generalized function on W by

$$\int_{\mathfrak{g}} \left(\int_{\mathscr{V}} \alpha \right)(X)\, \phi(X)\, dX = \int_{\mathscr{V}} \left(\int_{\mathfrak{g}} \alpha(X)\, \phi(X)\, dX \right)$$

for any test density $\phi(X)\, dX$ on W. (We recall that the integral of a differential form on a manifold means the integral of its term of maximum exterior degree). The generalized function $\int_{\mathscr{V}} \alpha$ is G-invariant. For $\alpha \in \mathscr{A}^\infty_{G,\, mean\text{-}rapid}(W, \mathscr{V})$, we have $\int_{\mathscr{V}} d_{\mathfrak{g}} \alpha = 0$. Therefore the integral $\alpha \to \int_{\mathscr{V}} \alpha$ is a map

$$\mathscr{H}^\infty_{G,\, mean\text{-}rapid}(W, \mathscr{V}) \to C^{-\infty}(W)^G.$$

3.2. *Chern–Weil equivariant differential forms*

Let M be a manifold and G a Lie group acting on M. Let $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ be a G-equivariant vector bundle over M and let $\mathbb{A}$ be a G-invariant superconnection on $\mathscr{E}$. We recall some definitions (see [7], chapter 7). We denote by $\mathscr{L}^{\mathscr{E}}(X)$ the infinitesimal action of $X \in \mathfrak{g}$ on the space of differential forms $\mathscr{A}(M, \mathscr{E})$. The moment of $\mathbb{A}$ is the map $\mu^{\mathbb{A}} : \mathfrak{g} \to \mathscr{A}(M, \operatorname{End} \mathscr{E})$ given by

$$\mu^{\mathbb{A}}(X) = \mathscr{L}^{\mathscr{E}}(X) - (\iota(X_M)\mathbb{A} + \mathbb{A}\iota(X_M)). \tag{4}$$

This equality means that the endomorphism of $\mathscr{A}(M, \mathscr{E})$ in the right hand side is the multiplication by the form $\mu^{\mathbb{A}}(X)$.

We denote by $\mathsf{F}^{\mathbb{A}} \in \mathscr{A}(M, \operatorname{End}(\mathscr{E}))$ the curvature of the superconnection $\mathbb{A}$

$$\mathsf{F}^{\mathbb{A}} = \mathbb{A}^2.$$

The *equivariant curvature* of $\mathbb{A}$ is defined by

$$\mathsf{F}^{\mathbb{A}}(X) = \mathsf{F}^{\mathbb{A}} + \mu^{\mathbb{A}}(X) \quad \text{for } X \in \mathfrak{g}.$$

If $\Phi(z)$ is a power series in one indeterminate z with infinite radius of convergence, the Chern–Weil equivariant differential form associated to $\Phi(z)$ is defined by $\operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}}(X)))$, for $X \in \mathfrak{g}$, where Str denotes the *supertrace* map: $\mathscr{A}(M, \operatorname{End}(\mathscr{E})) \to \mathscr{A}(M)$. The equivariant differential form $\operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}}(X)))$ is equivariantly closed and its equivariant cohomology class is independent of the choice of the superconnection on $\mathscr{E}$, as a consequence of the following transgression formula. Let $\mathbb{A}_t$, $0 \leqq t \leqq 1$, be a smooth path of G-invariant superconnections on $\mathscr{E}$. Let

$$\alpha_t(X) = \operatorname{Str}\left(\frac{d\mathbb{A}_t}{dt}\, \Phi'(\mathsf{F}^{\mathbb{A}_t}(X)) \right). \tag{5}$$

Then
$$\frac{d}{dt}\operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}_t}(X))) = (d_{\mathfrak{g}}\alpha_t)(X),$$
whence
$$\operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}_1}(X))) - \operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}_0}(X))) = \left(d_{\mathfrak{g}}\int_0^1 \alpha_t\right)(X).$$

In particular the *equivariant Chern character* form of the superbundle $\mathscr{E}$ with superconnection $\mathbb{A}$ is defined by
$$\operatorname{ch}(\mathscr{E}, \mathbb{A})(X) = \operatorname{Str}(\exp(\mathsf{F}^{\mathbb{A}}(X))).$$
We warn the reader about the omission of the factor $2i\pi$ in this definition of the Chern character, in agreement with [16] and [7].

The transgression formula for the Chern character is
$$\operatorname{ch}(\mathscr{E}, \mathbb{A}_1)(X) - \operatorname{ch}(\mathscr{E}, \mathbb{A}_0)(X) = d_{\mathfrak{g}}\int_0^1 \operatorname{Str}\left(\frac{d\mathbb{A}_t}{dt}\exp(\mathsf{F}^{\mathbb{A}_t}(X))\right). \tag{6}$$
We will sometimes denote the Chern character form by $\operatorname{ch}(\mathscr{E})$ or $\operatorname{ch}(\mathbb{A})$.

We will often use superconnections of the form $\mathbb{A} = v + \nabla$, where $v = \left(\begin{smallmatrix} 0 & v^- \\ v^+ & 0 \end{smallmatrix}\right)$ is an odd endomorphism of the superbundle $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$, and $\nabla = \nabla^+ \oplus \nabla^-$ is a connection on $\mathscr{E}$ such that $\nabla^{\pm}$ is a connection on $\mathscr{E}^{\pm}$. It will be useful to observe that the associated Chern-Weil forms are 0 in the case where $\mathscr{E}^+ = \mathscr{E}^-$, $\nabla^+ = \nabla^-$ and $v^+ = v^-$; indeed, in this case the curvature $\mathsf{F}^{\mathbb{A}}(X)$ belongs to the subalgebra of elements of the form $\left(\begin{smallmatrix} \alpha & \beta \\ \beta & \alpha \end{smallmatrix}\right)$, consequently $\operatorname{Str}(\Phi(\mathsf{F}^{\mathbb{A}}(X))) = 0$ for any Φ; moreover if $v_t^+ = v^-$; is a smooth path, then the transgression form given by (5) is also 0.

Similarly one defines the *equivariant Euler form* of a G-equivariant Euclidean oriented vector bundle $\mathscr{E}$ with G-invariant metric and G-invariant metric connection ∇. Let o be an orientation on $\mathscr{E}$. The choice of o determines a square root of the determinant on $\mathfrak{so}(\mathscr{E})$:

Definition 5. *If $A \in \mathfrak{so}(\mathscr{E}_x)$ is an endomorphism with matrix in a positive basis made of diagonal blocks $\left(\begin{smallmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{smallmatrix}\right)$, then $\det_o^{1/2}(A) = \prod \theta_j$.*

The curvature $\mathbb{F}^{\nabla}$ and the moment $\mu^{\nabla}(X)$ are both elements of $\mathscr{A}(M, \mathfrak{so}(\mathscr{E}))$. The equivariant Euler form associated to the orientation o is the element of $\mathscr{A}_G^{pol}(\mathfrak{g}, M)$ given by
$$\operatorname{Eul}_0(\mathscr{E}, \nabla)(X) = (-2\pi)^{-\operatorname{rank}\mathscr{E}/2}\det{}_o^{1/2}(\mathsf{F}^{\nabla}(X)).$$
The equivariant cohomology class of the equivariant Euler form depends on the choice of the orientation, but not on the metric and connection.

Finally, we recall the construction of the equivariant Thom form of a G-equivariant oriented vector bundle, with respect to a G-invariant Euclidean metric and a G-invariant metric connection ∇. This form is the equivariant extension of the Thom form defined in [15]. The metric and the orientation determine a trivialization $\wedge^r \mathscr{E} \cong M \times \mathbb{R}$, with $r = \operatorname{rank} \mathscr{E}$. We denote the projection by $p_{\mathscr{E}} : \mathscr{E} \to M$. Let T be the map

$$\mathscr{A}(\mathscr{E}, \Lambda p_{\mathscr{E}}^*(\mathscr{E})) \to \mathscr{A}(\mathscr{E})$$

obtained by taking the term of maximal degree r in $\Lambda p_{\mathscr{E}}^*(\mathscr{E})$. Let e_i be a local positive orthonormal frame of $\mathscr{E}$ with corresponding coordinates $\xi = \sum_i \xi_i e_i$ for $(x, \xi) \in \mathscr{E}$. Let $\omega_{\mathscr{E}}$ be the connection form of ∇ with respect to this frame and let $F(X)$, $X \in \mathfrak{g}$, be the equivariant curvature of ∇.

Definition 6. *The* equivariant Thom form $u_0(\mathscr{E}, \nabla)$, *associated to a choice of an orientation o, of a metric and of a connection* ∇, *is the G-equivariantly closed differential form on the total space of* $\mathscr{E}$ *given locally for* $X \in \mathfrak{g}$ *by*

$$\begin{aligned} u_0(\mathscr{E}, \nabla)(X) &= (-1)^{r(r-1)/2} \pi^{-r/2} \\ &\times T\Big(\exp\Big(-\|\xi\|^2 + \sum_i (d\xi_i + \langle \omega_{\mathscr{E}} \xi, e_i \rangle) \otimes e_i + \tfrac{1}{2} \sum_{i<j} \langle F(X) e_i, e_j \rangle \otimes e_i \wedge e_j \Big)\Big). \end{aligned}$$

Let $\delta_t : \mathscr{E} \to \mathscr{E}$ *be the homothety* $\delta_t(x, \xi) = \delta(x, t\xi)$ *in the fibers of* $\mathscr{E}$. *The* rescaled Thom form *is defined for* $t \geqq 0$ *by*

$$u_0(\mathscr{E}, \nabla)(t)(X) = \delta_t^*(u_0(\mathscr{E}, \nabla)(X)).$$

For $t > 0$, the integral on the fibers of $u_0(\mathscr{E}, \nabla)(t)(X)$ is equal to 1, for any $x \in \mathfrak{g}$. For $t = 0$, the Thom form is equal to the pullback $p_{\mathscr{E}}^*(\mathrm{Eul}_0(\mathscr{E}, \nabla)(X))$ of the Euler form associated to the same data. There is a transgression formula for the rescaled Thom form:

$$\frac{d}{dt} u_{\mathscr{E}}(t) = d_{\mathfrak{g}} \alpha_t \tag{7}$$

where

$$\begin{aligned} \alpha_t =& (-1)^{r(r-1)/2} \pi^{-r/2} T\Big((\textstyle\sum \xi_i \otimes e_i) \\ & \times \exp\Big(-t\|\xi\|^2 + t \sum_i (d\xi_i + \langle \omega_{\mathscr{E}} \xi, e_i \rangle) \otimes e_i + \tfrac{1}{2} \sum_{i<j} \langle F(X) e_i, e_j \rangle \otimes e_i \wedge e_j \Big)\Big). \end{aligned}$$

4. The Chern character of a *G*-transversally elliptic symbol

4.1. Volterra expansion formula

We will make constant use of Volterra's expansion formula for the exponential (see [7], Chapter 2, Sect. 4). Consider the simplex

$$\Delta_k = \left\{ (s_o, \ldots, s_k) \in \mathbb{R}^{k+1} \;\middle|\; \sum_{i=0}^{k} s_i = 1,\ s_i \geqq 0 \right\}$$

with the measure $ds_i \dots ds_k$. Let A and Z be two elements of a Banach algebra. Then

$$e^{A+Z} - e^A = \sum_{k=1}^{\infty} I_k, \tag{8}$$

where

$$I_k = \int_{\Delta_k} e^{s_0 A} Z e^{s_1 A} Z \dots e^{s_{k-1} A} Z e^{s_k A}\, ds_1 \dots ds_k. \tag{9}$$

This infinite sum is convergent: the volume of the simplex Δ_k is $\frac{1}{k!}$, thus

$$\|I_k\| \leqq e^{\|A\|} \|Z\|^k / k!. \tag{10}$$

Volterra's formula can be viewed as a Taylor's formula for the exponential map. It contains Duhamel's formula for the derivative of the exponential map: assume that $A(z)$ depends smoothly on $z \in \mathsf{R}$, then

$$\frac{d}{dz} e^{A(z)} = \int_0^1 e^{sA(z)} \frac{dA}{dz}(z) e^{(1-s)A(z)}\, ds. \tag{11}$$

We will also use the expansion of the derivative: assume that Z depends on a parameter $z \in \mathsf{R}$, then by differentiating (8) we get $\frac{d}{dz} e^{A+Z(z)} = \sum_{k=1}^{\infty} \frac{d}{dz} I_k(z)$ with

$$\begin{aligned} &\frac{d}{dz} I_k(z) \qquad\qquad (12)\\ &\quad = \sum_{j=1}^{k} \int_{\Delta_k} e^{s_0 A} Z(z) e^{s_1 A} Z(z) \dots e^{s_{j-1} A} \frac{dZ}{dz} e^{s_j A} \dots e^{s_{k-1} A} Z(z) e^{s_k A}\, ds_1 \dots ds_k. \end{aligned}$$

We will use Volterra's expansion in the case of an algebra $\operatorname{End} E \otimes L$, where E is a finite dimensional Hermitian vector space and $L^\bullet = \sum_0^N L^k$ is a normed finite dimensional $\mathbb{Z}_+$-graded algebra, such that $L^0 = \mathbb{C}$ (for example an exterior algebra). We identify $\operatorname{End} E$ with the component of degree 0 of $\operatorname{End} E \otimes L^\bullet$. We consider L acting on itself by left multiplication. The norm on $\operatorname{End} E \otimes L$ is the operator norm on $\operatorname{End}(E \otimes L)$. Assume that A is a Hermitian element of $\operatorname{End} E$ and let $\lambda \in \mathbb{R}$ be the largest eigenvalue of A. Then $\|e^{sA}\| = e^{s\lambda}$ for every $s \geqq 0$, whence

$$\|I_k\| \leqq e^{\lambda} \|Z\|^k / k!, \tag{13}$$

$$\|e^{A+Z}\| \leqq e^{\lambda + \|Z\|}. \tag{14}$$

We obtain a refinement of (13) by decomposing $Z = Z_0 + Z_1$, where $Z_0 \in \operatorname{End} E$ is the degree zero component of $Z \in \operatorname{End} E \otimes L^\bullet$. The integral I_k expands in the sum of 2^k terms of the form

$$\int_{\Delta_k} e^{s_0 A} Z_{\varepsilon_1} e^{s_1 A} Z_{\varepsilon_2} \dots e^{s_{k-1} A} Z_{\varepsilon_k} e^{s_k A}\, ds_1 \dots ds_k.$$

Let N be the highest degree allowed in $L^\bullet$. Then any term which contains more than N factors Z_1 is zero, therefore there are at most N factors Z_1 and k factors Z_0. If we set $z_0 = \sup(\|Z_0\|, 1)$, and $z_1 = \sup(\|Z_1\|, 1)$, we have

$$\|I_k\| \leqq e^{\lambda} z_1^N z_0^k 2^k / k!. \tag{15}$$

4.2. Transversally good symbols

When the manifold M and the group G are compact, every element of $K_G(T^*_G M)$ is represented by a G-transversally elliptic symbol $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$. In order to define the equivariant Chern character of $[\sigma]$ in terms of differential forms, it is useful to impose on σ a stronger assumption. By choosing G-invariant Hermitian metrics on $\mathscr{E}^\pm$ we associate to σ the Hermitian endomorphism $v(\sigma)$ of $p^*\mathscr{E} = p^*(\mathscr{E}^+ \oplus \mathscr{E}^-)$ defined as in (1) by $v(\sigma)(x,\xi) = \begin{pmatrix} 0 & \sigma^*(x,\xi) \\ \sigma(x,\xi) & 0 \end{pmatrix}$. Then $v(\sigma)^2\ (x,\xi) = \begin{pmatrix} \sigma^*(x,\xi)\sigma(x,\xi) & 0 \\ 0 & \sigma(x,\xi)\sigma^*(x,\xi) \end{pmatrix}$ is a positive endomorphism of $\mathscr{E}_x$ for every $(x,\xi) \in T^*M$ and positive definite for $(x,\xi) \in T^*_G M$ with ξ large enough.

Let $\mathscr{V}$ be a Euclidean vector bundle on a manifold M, with projection $p : \mathscr{V} \to M$. Let $\mathfrak{g}$ be a Euclidean vector space with dual $\mathfrak{g}^*$ and let f be a smooth map from $\mathscr{V}$ to $\mathfrak{g}^*$, such that f is linear in each fiber $\mathfrak{g}_x$. Let $\mathscr{E}$ be a Hermitian vector bundle on M.

Definition 7. *Let h be a Hermitian endomorphism of $p^*\mathscr{E}$. We say that h is f-good, or good with respect to f, if there exist $r > 0$, $c > 0$ and $a > 0$ such that*

$$h(x,\xi) \geqq c\|\xi\|^2 I_{\mathscr{E}_x},$$

for every $(x,\xi) \in \mathscr{V}$ such that $\|f(x,\xi)\| < a\|\xi\|$ and $\|\xi\| \geqq r$.

If $f = 0$, we simply say that h is good. The condition is then $h(x,\xi) \geqq c\|\xi\|^2 I_{\mathscr{E}_x}$ for all ξ such that $\|\xi\| \geqq r$.

The inequality in this definition means that the Hermitian endomorphism $h(x,\xi) - c\|\xi\|^2 I_{\mathscr{E}_x}$ of $\mathscr{E}_x$ is positive, that is, all the eigenvalues of $h(x,\xi)$ are greater than $c\|\xi\|^2$.

Remark 8. For $a > 0$, let

$$\Gamma_a = \{(x,\xi) \in \mathscr{V}; \|f(x,\xi)\| < a\|\xi\|\}.$$

If M is compact, the sets Γ_a form a basis of conic neighbourhoods of $f^{-1}(0)\backslash 0$ in $\mathscr{V}$.

We now consider the case where $\mathscr{V} = T^*M$. We choose a Riemannian metric on M. Let $\mathscr{E}^+$ and $\mathscr{E}^-$ be two vector bundles over M.

Definition 9. *A symbol $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ on a manifold M is said to be* **good** *if*

(1) *σ is smooth,*

(2) *$\sigma(x,\xi)$ and all its derivatives $\partial_x^k \partial_\xi^\ell \sigma(x,\xi)$ are slowly increasing along the fibers,*

(3) *For some (equivalently, for any) choice of Hermitian metric on $\mathscr{E}^\pm$, the endomorphism $v(\sigma)^2$ is good.*

For the rest of the chapter, M is a manifold with an action of a compact Lie group G. All metrics are assumed to be G-invariant. Let $\mathscr{E}^+$ and $\mathscr{E}^-$ be two G-equivariant vector bundles over M.

Definition 10. *A G-transversally elliptic symbol $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ is said to be G*-transversally good *if*

(1) *σ is smooth,*

(2) *$\sigma(x,\xi)$ and all its derivatives $\partial_x^k \partial_\xi^\ell \sigma(x,\xi)$ are slowly increasing along the fibers,*

(3) *For some (equivalently, for any) choice of G-invariant metric on $\mathcal{E}^\pm$, the endomorphism $v(\sigma)^2$ is good with respect to the canonical moment map $f^{M,G} : T^*M \to \mathfrak{g}^*$.*

Remark 11. Assume that σ is a symbol on M which coincides for large ξ with the principal symbol of a pseudodifferential operator P of order $m \geqq 1$. Then $v(\sigma)^2(x,\xi)$ is homogeneous of degree $2m \geqq 2$ with respect to ξ, for ξ large. Clearly, if P is elliptic then σ is a good symbol, and if P is transversally elliptic then σ is a transversally good symbol, as $(f^{M,G})^{-1}(0) = T_G^*M$.

The following lemma is easy [10]:

Lemma 12. *Assume that M is compact.*

(i) *Every element of $K_G(T_G^*M)$ is represented by a transversally good symbol. More precisely let $\sigma : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be a transversally elliptic symbol. Assume that $\sigma(x,\xi)$ is invertible for $\xi \in (T_G^*M)_x$, $\|\xi\| \geqq 1$. Let $\vartheta \in C^\infty(\mathbb{R})$ such that $\vartheta(t) = 0$ for $t \leqq \frac{1}{2}$ and $\vartheta(t) = 1$ for $t \geqq 1$. Let*

$$\sigma_1(x,\xi) = \|\xi\|\vartheta(\|\xi\|)\sigma(x,\xi/\|\xi\|).$$

Then σ_1 is a transversally good symbol with the same class as σ in $K_G(T_G^*M)$.

(ii) *Let σ_0, $\sigma_1 : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ be two transversally good symbols. Assume that there exists a homotopy σ_t between σ_0 and σ_1 such that σ_t is G-transversally elliptic for every t. Then there exists a homotopy $\tilde{\sigma}_t : p^*\mathcal{E}^+ \to p^*\mathcal{E}^-$ such that each $\tilde{\sigma}_t$ is G-transversally good and such that furthermore $\frac{d}{dt}\tilde{\sigma}_t$ and all its derivatives $\partial_x^j \partial_\xi^k \frac{d}{dt}\tilde{\sigma}_t$ have at most polynomial growth along the fibers.*

4.3. More conventions about the cotangent bundle of a G-manifold

Let M be a manifold. We denote by ω^M or simply by ω the Liouville 1-form on T^*M. The moment map $f^{M,G}$ will sometimes also be denoted simply by f.

We make the convention that the canonical symplectic 2-form is $-d\omega^M$.

If a Lie group G acts on M, the map $f^{M,G}$ satisfies, for $X \in \mathfrak{g}$,

$$d\langle f^{M,G}, X\rangle = -\iota(X_{T^*M})\, d\omega^M,$$

thus it is a moment map for the canonical symplectic structure. The canonical equivariant two-form on T^*M is then given by

$$-(d_{\mathfrak{g}}\omega^M)(X) = -d\omega^M + \langle f^{M,G}, X\rangle. \qquad (16)$$

As in [6], the total space of T^*M is oriented by the volume element $(-d\omega^M)^{\dim M}$.

4.4. The Chern character of a transversally good symbol

Let M be a compact manifold and G be a compact Lie group acting on M. Let $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ be a G-transversally good symbol. On $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ we choose a G-equivariant connection $\nabla = \nabla^+ \oplus \nabla^-$ and a G-invariant Hermitian metric h.

We associate to these data two superconnections on the superbundle $p^*\mathscr{E}$:

$$\mathbb{A}(\sigma, \nabla, h) = iv(\sigma) + p^*\nabla \tag{17}$$

$$\mathbb{A}^{\omega}(\sigma, \nabla, h) = iv(\sigma) + p^*\nabla - i\omega^M. \tag{18}$$

When some of the data (σ, ∇, h) are understood, we may omit them in the notation $\mathbb{A}$ and $\mathbb{A}^{\omega}$. We denote by $\mathbb{F}^{\mathbb{A}}(X)$, $X \in \mathfrak{g}$ the equivariant curvature of the superconnection $\mathbb{A}(\sigma, \nabla, h)$ and by $\mathbb{F}^{\mathbb{A}^{\omega}}(X)$ that of $\mathbb{A}^{\omega}(\sigma, \nabla, h)$. Let $\mathbb{F}(X)$ be the equivariant curvature of the connection ∇. Then

$$\mathbb{F}^{\mathbb{A}}(X) = -\, v(\sigma)^2 + p^*(\mathbb{F}(X)) + i\, p^*(\nabla).v(\sigma), \tag{19}$$

$$\mathbb{F}^{\mathbb{A}^{\omega}}(X) = -\, i d_{\mathfrak{g}}\omega^M + \mathbb{F}^{\mathbb{A}}(X) \tag{20}$$

We consider the equivariant Chern characters and $\mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma, \nabla, h))(X) = \mathrm{Str}\,(\exp \mathbb{F}^{\mathbb{A}^{\omega}}(X))$ $\mathrm{ch}\,(\mathbb{A}(\sigma, \nabla, h))(X) = \mathrm{Str}\,(\exp \mathbb{F}^{\mathbb{A}}(X))$. They are related by

$$\mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma, \nabla, h)) = e^{-id_{\mathfrak{g}}\omega^M}\mathrm{ch}\,(\mathbb{A}(\sigma, \nabla, h)). \tag{21}$$

As we are going to see, (Theorem 16), the use of the 1-form $-i\omega^M$ in the superconnection $\mathbb{A}^{\omega}$ is to make the Chern character $\mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma, \nabla, h))$ rapidly decreasing in $\mathfrak{g}$-mean, if σ is transversally good.

If σ is good, not only transversally good, then, as we will see in Theorem 18, the equivariant Chern character $\mathrm{ch}\,(\mathbb{A}(\sigma, \nabla, h))(X)$ is rapidly decreasing for each $X \in \mathfrak{g}$, therefore it follows from (21) that $\mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma, \nabla, h))$ is also in $\mathscr{A}^{\infty}_{G, rapid}(\mathfrak{g}, T^*M)$, and that the two Chern characters have the same class in $\mathscr{H}^{\infty}_{G, rapid}(\mathfrak{g}, T^*M)$.

In the following lemmas 13, 36 and 15, we collect the estimates which we will need. Let $\mathfrak{g}$ be a Euclidean vector space with dual $\mathfrak{g}^*$. Let F be a Euclidean vector space. On the space $C^{\infty}(\mathfrak{g}, F)$ we consider the family of semi-norms, for m an integer and K a compact subset of $\mathfrak{g}$,

$$N_{m,K}(\psi) = \sup_{X \in K,\, |\alpha| \leqq m} \|\partial^{\alpha}\psi(X)\|.$$

where ∂^{α} is a partial derivative. Let V be a Euclidean vector space. The elements of V are denoted by ξ. Let M_0 be an open subset of M and let $f : M_0 \times V \to \mathfrak{g}^*$ be a smooth map, linear with respect to the second variable. The following lemma is the crucial technical point of this article.

Lemma 13. *Let $q(X,x,\xi)$ be a smooth F-valued function on $\mathfrak{g} \times M_0 \times V$ with the following properties*:

(1) *there exists an integer P and, for every integer $m \geqq 0$ and every compact subset $K \subset \mathfrak{g}$, there exists a constant $C_{m,K}$ such that*

$$N_{m,K}(q(.,x,\xi)) \leqq C_{m,K}(1 + \|\xi\|^2)^P \text{ for all } (x,\xi) \in M_0 \times V, \qquad (22)$$

(2) *there exists $a > 0$, $c > 0$ and, for every integer $m \geqq 0$ and every compact subset $K \subset \mathfrak{g}$, there exists a constant $C'_{m,K}$ such that*

$$N_{m,K}(q(.,x,\xi)) \leqq C'_{m,K} e^{-c\|\xi\|^2} \quad \text{if } \|f(x,\xi)\| < a\|\xi\|. \qquad (23)$$

Let

$$\alpha(X,x,\xi) = e^{i\langle f(x,\xi),X\rangle} q(X,x,\xi). \qquad (24)$$

Then the function $\alpha(X,x,\xi)$ has the following property of decrease in $\mathfrak{g}$-mean on $M_0 \times V$: for every integer $d > 0$ and for every compact subset $K \subset \mathfrak{g}$ there exists a constant $C''_{d,K}$ such that, for every $\phi \in C_c^\infty(\mathfrak{g})$ with support in K,

$$\left\| \int_{\mathfrak{g}} \alpha(X,x,\xi)\phi(X)\,dX \right\| \leqq C''_{d,K}(1 + \|\xi\|^2)^{-d} N_{2(d+P),K}(\phi) \qquad (25)$$

for all $(x,\xi) \in M_0 \times V$.

Proof. Let $\phi \in C_c^\infty(\mathfrak{g})$. For $(x,\xi) \in M_0 \times V$, let

$$v(x,\xi) = \int_{\mathfrak{g}} e^{i\langle f(x,\xi),X\rangle} q(X,x,\xi)\phi(X)\,dX.$$

Then $v(x,\xi)$ is the value at the point $-f(x,\xi) \in \mathfrak{g}^*$ of the Fourier transform of the function $X \mapsto q(X,x,\xi)\phi(X)$ on $\mathfrak{g}$. Therefore, for every compact subset $K \subset \mathfrak{g}$ and for every integer $m \geqq 0$, there exists a constant $C_1 = C_1(m,K) > 0$ such that, if ϕ is supported in K,

$$\begin{aligned}\|v(x,\xi)\| &\leqq C_1(1 + \|f(x,\xi)\|^2)^{-m} N_{2m,K}(q(.,x,\xi)\phi)\\ &\leqq 2^m C_1(1 + \|f(x,\xi)\|^2)^{-m} N_{2m,K}(q(.,x,\xi)) N_{2m,K}(\phi).\end{aligned}$$

Let $m = d + P$ and let a, c be as in condition (2). We may assume that $a < 1$. If $\|f(x,\xi)\| > a\|\xi\|$ we have the majorations

$$(1 + \|f(x,\xi)\|^2)^{-(d+P)} \leqq (1 + a^2\|\xi\|^2)^{-(d+P)} \leqq a^{-2(d+P)}(1 + \|\xi\|^2)^{-(d+P)},$$

$$N_{2(d+P),K}(q(.,x,\xi)) \leqq C_{2(d+P),K}(1 + \|\xi\|^2)^P,$$

thus the inequality (25). If $\|f(x,\xi)\| < a\|\xi\|$ we have the majorations

$$(1 + \|f(x,\xi)\|^2)^{-m} \leqq 1 \text{ and } N_{2(d+P),K}(q(.,x,\xi)) \leqq C'_{2(d+P),K} e^{-c\|\xi\|^2}$$

thus a fortiori (25). $\square$

We will apply this lemma when $\mathfrak{g}$ is the Lie algebra of G and $f = f^{M,G}$: $T^*M \to \mathfrak{g}^*$ is the moment map.

Let $\mathscr{E}$ be a vector bundle on M. Let $M_0 \subset M$ be a relatively compact open coordinate chart, let $V = \mathbb{R}^{\dim M}$ so that $T^*M|_{M_0}$ is identified with $M_0 \times V$. We assume that $\mathscr{E}$ is trivialized above M_0 so that $\mathscr{E}|_{M_0} \simeq M_0 \times E$. Let $F = \Lambda(V \oplus V) \otimes \operatorname{End} E$. Recall $p : T^*M \to M$. Then above M_0 an element of $C^\infty(\mathfrak{g}, \mathscr{A}(T^*M, \operatorname{End}(p^*\mathscr{E})))$ is identified with a smooth F-valued function on $\mathfrak{g} \times M_0 \times V$. Thus it makes sense to consider those forms $q \in C^\infty(\mathfrak{g}, \mathscr{A}(T^*M, \operatorname{End}(p^*\mathscr{E})))$ which satisfy the conditions of Lemma 13 locally, that is, above any small enough open subset $M_0 \subset M$, with $\mathfrak{g}$ the Lie algebra of G and f the moment map $f^{M,G}$.

Definition 14. *We denote by* $Q(\mathscr{E}) \subset C^\infty(\mathfrak{g}, \mathscr{A}(T^*M, \operatorname{End}(p^*\mathscr{E})))$ *the space of forms which satisfy the conditions of Lemma* 13 *locally, with* $\mathfrak{g}$ *the Lie algebra of* G *and* f *the moment map* $f^{M,G}$.

Lemma 15. *Let* $\mathscr{E}$ *be a Hermitian vector bundle on* M. *Let* h *be a positive Hermitian endomorphism of* $p^*\mathscr{E}$, *where* $p : T^*M \to M$. *Assume that* h *is good with respect to* $f^{M,G}$. *Let* $Z \in C^\infty(\mathfrak{g}, \mathscr{A}(T^*M, \operatorname{End}(p^*\mathscr{E})))$ *be a differential form depending on* $X \in \mathfrak{g}$. *Denote by* $Z_0(X)$ *the component of exterior degree 0 of* Z *and let* $Z_1(X) = Z(X) - Z_0(X)$. *Assume that the zero degree term* $Z_0(X)$ *is the pull back of a form on* M, *that the term* Z_1 *does not depend on* X *and has at most polynomial growth along the fibers. Then the form*

$$q(X) = e^{-h+Z(X)} \tag{26}$$

belongs to the subspace $\mathscr{Q}(\mathscr{E})$.

Proof. We fix $(x, \xi) \in T^*M$ and we bound $e^{-h(x,\xi)+Z(X)}$ by means of Volterra expansion

$$e^{-h(x,\xi)+Z(X,x,\xi)} = e^{-h(x,\xi)} + \sum_{k=1}^{\infty} I_k(X, x, \xi)$$

applied to $A = -h(x, \xi)$ and $Z = Z(X, x, \xi) = Z_0(X, x) + Z_1(x, \xi)$.

When X remains in K the term $Z_0(X, x)$ is bounded in norm by a constant z_0, which we choose $\geqq 1$, and the term $Z_1(x, \xi)$ is bounded in norm by some power $c_1(1 + \|\xi\|^2)^{N_1}$. By using the positivity of $h(x, \xi)$ and applying the inequality (15), we get $\|I_k\| \leqq 2^k z_0^k c_1^N (1 + \|\xi\|^2)^{N_1 N}/k!$, where $N = 2 \dim M$, whence, for every (x, ξ),

$$\|e^{-h(x,\xi)+Z(X,x,\xi)}\| \leqq e^{2z_0} c_1 (1 + \|\xi\|^2)^{N_1 N}.$$

Consider the neighbourhood Γ_a of $T^*_G M \backslash 0$ and the number $r > 0$ from the definition of a good bundle endomorphism (Definition 7). For $(x, \xi) \in \Gamma_a$, $\|\xi\| \geqq r$, the smallest eigenvalue of $h(x, \xi)$ is greater than $c\|\xi\|^2$, thus from inequality (15) we obtain in this case

$$\|e^{-h(x,\xi)+Z(X,x,\xi)}\| \leqq e^{-c\|\xi\|^2} e^{2z_0} c_1^N (1 + \|\xi\|^2)^{N_1 N},$$

hence, there exists a constant $C > 0$ such that

$$\|e^{-h(x,\xi)+Z(X,x,\xi)}\| \leqq Ce^{-\frac{c}{2}\|\xi\|^2} \quad \text{for all } X \in \mathfrak{g},\ (x,\xi) \in \Gamma_a.$$

In order to estimate the derivatives with respect to $X \in \mathfrak{g}$ we use formula (12) and we bound $\partial_X I_k$ similarly to (15), by decomposing $Z = Z_0 + Z_1$. As there can be at most $N = 2\dim M$ terms Z_1, we get, for every (x,ξ),

$$\|\partial_X e^{-h(x,\xi)+Z(X,x,\xi)}\| \leqq e^{2z_0} c_1^N (1 + \|\xi\|^2)^{N_1 N} \|\partial_X Z_0(X,x)\|,$$

and, for $(x,\xi) \in \Gamma_a$, $\|\xi\| \geqq r$,

$$\|\partial_X e^{-h(x,\xi)+Z(X,x,\xi)}\| \leqq e^{-c\|\xi\|^2} e^{2k_0} c_1^N (1 + \|\xi\|^2)^{N_1 N} \|\partial_X Z_0(X,x)\|.$$

The higher derivatives with respect to X are bounded by similar computations. Thus we obtain the conditions of Lemma 13, with $P = 2N_1 N$. □

Theorem 16. *Let $\sigma : p^*\mathscr{E}+ \to p^*\mathscr{E}^-$ be a G-transversally good symbol. Assume that $\mathscr{E}^+ \oplus \mathscr{E}^-$ is endowed with a G-invariant metric and a G-invariant connection $\nabla = \nabla^+ \oplus \nabla^-$. Then the Chern character $ch(\mathbb{A}^\omega(\sigma,\nabla))$ is a G-equivariant differential form on T^*M which is rapidly decreasing in $\mathfrak{g}$ mean. Moreover its cohomology class in $\mathscr{H}^\infty_{G,\, mean\text{-}rapid}(\mathfrak{g}, T^*M)$ depends only on the class $[\sigma]$ in $K_G(T^*_G M)$.*

Proof. In order to prove the first claim, it is enough to show that the $\mathrm{End}(p^*\mathscr{E})$-valued differential form on T^*M

$$X \to \exp \mathbb{F}^{\mathbb{A}^\omega}(X)$$

is rapidly decreasing in $\mathfrak{g}$-mean.

Let $\mathbb{F}(X) = \mathbb{F} + \mu(X)$ be the equivariant curvature of the connection ∇ on the bundle $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$, with $\mathbb{F}$ the curvature of ∇ and $\mu(X)$ its moment. By (19) and (16), we have

$$\mathbb{F}^{\mathbb{A}^\omega}(X) = -v(\sigma)^2 + p^*\mathbb{F} - id\omega^M + i(p^*\nabla) \cdot v(\sigma) + i\langle f^{M,G}, X\rangle + p^*\mu(X).$$

We write

$$\mathbb{F}^{\mathbb{A}^\omega}(X) = i\langle f^{M,G}, X\rangle - v(\sigma)^2 + Z(X)$$

where $Z(X)$ is defined by

$$Z(X) = -id\omega^M + ip^*\nabla \cdot v(\sigma) + p^*\mathbb{F}(X). \tag{27}$$

Thus

$$\exp \mathbb{F}^{\mathbb{A}^\omega}(X) = e^{i\langle f^{M,G}, X\rangle} q(X)$$

with $q(X) = e^{-v(\sigma)^2 + Z(X)}$.

Let Z_0 be the zero exterior degree term of Z and let $Z_1 = Z - Z_0$. Then $Z_0(X) = p^*\mu(X)$, while $Z_1 = -id\omega^M + ip^*\nabla \cdot v(\sigma) + p^*\mathbb{F}$ does not depend on X. Moreover, the polynomial growth hypothesis on $\sigma(x,\xi)$ (Definition 10) implies that $Z_1(x,\xi)$ has at most polynomial growth with respect to ξ. Thus we may apply Lemma 15 to $q = e^{-h+Z}$, with $h = v(\sigma)^2$. The form $\alpha(X)$ in Lemma 13 is then $\alpha(X) = \exp(\mathbb{F}^{\mathbb{A}^\omega}(X))$, thus by Lemma 13, for any test function $\phi \in C_c^\infty(\mathfrak{g})$, the differential form $\int_{\mathfrak{g}} \exp(\mathbb{F}^{\mathbb{A}^\omega}(X))\phi(X)\,dX$ on T^*M is rapidly decreasing along the fibers.

We must also show that all derivatives $\int_{\mathfrak{g}} \partial_x^j \partial_\xi^k \exp(\mathbb{F}^{\mathbb{A}^\omega}(X,x,\xi))\phi(X)\,dX$ are rapidly decreasing with respect to ξ (with estimates depending on ϕ as in Lemma 13). In the derivative $\partial_x^j \partial_\xi^k (e^{i\langle f^{M,G}(x,\xi),X\rangle} q(X,x,\xi))$, the derivatives of $\langle f^{M,G}(x,\xi),X\rangle$ cause no problem since $f^{M,G}(x,\xi)$ is linear with respect to ξ. As for the derivatives of $q = e^{-v(\sigma)^2+Z(X)}$, we compute them using Duhamel formula (11). Then we proceed with computations similar to the above ones. This proves the first claim.

In order to prove the second claim, we consider two transversally good symbols σ_0 and σ_1 with the same class in $K_G(T_G^*M)$. We must show that

$$\operatorname{ch}(\mathbb{A}^\omega(\sigma_1,\nabla_1,h_1)) - \operatorname{ch}(\mathbb{A}^\omega(\sigma_0,\nabla_0,h_0)) = d_{\mathfrak{g}}\beta, \tag{28}$$

with an equivariant differential form β rapidly decreasing in $\mathfrak{g}$-mean.

The equality of the classes in $K_G(T_G^*M)$ means that there exist G-equivariant bundles $\mathscr{F}_0$ and $\mathscr{F}_1$ on M, G-equivariant isomorphisms

$$\mathscr{E}_0^+ \oplus \mathscr{F}_0 \simeq \mathscr{E}_1^+ \oplus \mathscr{F}_1,$$
$$\mathscr{E}_0^- \oplus \mathscr{F}_0 \simeq \mathscr{E}_1^- \oplus \mathscr{F}_1$$

and a homotopy between the bundle maps

$$\sigma_0 \oplus I_{p^*\mathscr{F}_0} : p^*(\mathscr{E}_0^+ \oplus \mathscr{F}_0) \to p^*(\mathscr{E}_0^- \oplus \mathscr{F}_0),$$
$$\sigma_1 \oplus I_{p^*\mathscr{F}_1} : p^*(\mathscr{E}_1^+ \oplus \mathscr{F}_1) \to p^*(\mathscr{E}_1^- \oplus \mathscr{F}_1)$$

made of G-bundle maps which are invertible outside of a fixed compact subset of T_G^*M. In order to deal only with transversally good symbols, we replace $\sigma_0 \oplus I_{p^*\mathscr{F}_0}$ by the homotopic $\sigma_0 \oplus \|\xi\|\vartheta(\|\xi\|)I_{p^*\mathscr{F}_0}$ and we do the same for σ_1. (The function ϑ is that of Lemma 12). After a change of notations, we must prove (28) in the three following cases:

(1) $\sigma_0 = \sigma_1$, but the connections and the metrics on $\mathscr{E}_0^\pm$ and $\mathscr{E}_1^\pm$ are not necessarily the same.
(2) $\sigma_1 = \sigma_0 \oplus \|\xi\|\vartheta(\|\xi\|)I_{p^*\mathscr{F}}$.
(3) σ_0 and σ_1 are G-transversally good symbols which are linked by a homotopy made of G-transversally elliptic symbols.

The second case follows immediately from the additivity of the Chern character and the following remark:

Remark 17. Assume that $\mathscr{E}^+ = \mathscr{E}^-$ with the same Hermitian metric and connection and that $\sigma(x,\xi)$ is selfadjoint for all $(x,\xi) \in T^*M$. Then $\operatorname{ch}(\mathbb{A}^\omega(\sigma,\nabla,h)) = \operatorname{ch}(\mathbb{A}(\sigma,\nabla,h)) = 0$, as observed in Sect. 3.2.

We deal with cases 1 and 3 at the same time, by making use of the transgression formula for the equivariant Chern character. Applying Lemma 12, we may assume that σ_0 and σ_1 are linked by a homotopy σ_t which satisfies the conditions in part (2) of this lemma. Let h_t be the segment, in the space of Hermitian metrics on $\mathscr{E}$, which joints the given metrics and let $v_t = \begin{pmatrix} 0 & \sigma_t^* \\ \sigma_t & 0 \end{pmatrix}$, where σ_t^* is the adjoint of σ_t with respect to h_t. Let $\mathbb{A}_t^\omega$ be the superconnection $A_t^\omega = iv_t + p^*\nabla_t - i\omega^M$, with $\nabla_t = \nabla_0 + t(\nabla_1 - \nabla_0)$. Let $F^{\mathbb{A}_t^\omega}(X)$ be the equivariant curvature of $\mathbb{A}_t^\omega$. The transgression formula reads here

$$\mathrm{ch}\,(\mathbb{A}^\omega(\sigma_1, \nabla_1)) - \mathrm{ch}\,(\mathbb{A}^\omega(\sigma_0, \nabla_0)) = d_{\mathfrak{g}}\beta,$$

with

$$\beta(X) = \int_0^1 \mathrm{Str}\left(\left(i\frac{dv_t}{dt} + p^*(\nabla_1 - \nabla_0)\right) \exp F^{\mathbb{A}_t^\omega}(X)\right) dt.$$

Thanks to the assumption that $\frac{dv_t}{dt}$ and its derivatives with respect to (x, ξ) have at most polynomial growth with respect to ξ, we show that β is rapidly decreasing in $\mathfrak{g}$-mean, in a manner similar to the proof of the first claim. □

Remark 18. Let $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ be a G-invariant symbol which is good, not only transversally good. Then, by computations similar to those in the above proof, we show that the Chern character $\mathrm{ch}\,(\mathbb{A}(\sigma, \nabla))$ is a rapidly decreasing G-equivariant differential form on T^*M and that, moreover, its cohomology class in $\mathscr{H}^\infty_{G,\,rapid}(\mathfrak{g}, T^*M)$ depends only on the class $[\sigma]$ in $K_G(T^*M)$.

From (21) it follows that, in this case, $\mathrm{ch}\,(\mathbb{A}^\omega(\sigma, \nabla))$ is also rapidly decreasing and has the same cohomology class as $\mathrm{ch}\,(\mathbb{A}(\sigma, \nabla))$ in $\mathscr{H}^\infty_{G,\,rapid}$ $(\mathfrak{g}, T^*M)$.

If σ is a G-invariant elliptic symbol, its equivariant Chern character can also be defined as a compactly supported equivariant form on T^*M by the excision procedure of embedding T^*M as an open subset of a compact manifold. By using the transgression formula (6) and estimates derived from the Volterra expansion, we can prove that $\mathrm{ch}\,(\mathbb{A}(\sigma))$ has the same cohomology class in $\mathscr{H}^\infty_{G,\,rapid}(\mathfrak{g}, T^*M)$ as the Chern character defined through excision. A detailed proof is given in [10]. The case $G = \{e\}$ follows from a result of Quillen [16].

4.5. Fixed points submanifold

Let G be a compact Lie group acting on a manifold M. In this section M is not necessarily compact. Let $s \in G$, and let $M(s)$ be the fixed points set $\{x \in M \mid sx = x\}$. Let $G(s)$ be the centralizer of s in G. As G is compact, $M(s)$ is a submanifold of M and $T(M(s)) = (TM)(s)$. (The manifold $M(s)$ may have several connected components of various dimensions). Moreover, the action of s on T^*M determines an identification

$$T^*M(s) \simeq (T^*M)(s) \tag{29}$$

with the set of fixed points of s in T^*M. Indeed for $x \in M(s)$, the action s_x^{TM} on the fiber T_xM is semi-simple, so that the decomposition in eigenspaces defines a canonical supplementary subspace of $(T_xM)(s)$ invariant under the action of s:

$$T^*M_x = (T_xM)(s) \oplus (I - s_x^{TM}) \,.\, T_xM \tag{30}$$

thus a canonical identification of $T_x^*(M(s))$ with $(T_x^*M)(s)$.

The centralizer $G(s)$ acts on the fixed points set $M(s)$. If σ is a G-invariant symbol on M then the restriction of σ to $T^*M(s) \subset T^*M$ is a $G(s)$-invariant symbol on $M(s)$.

Lemma 19. (i) *The Liouville 1-form $\omega^{M(s)}$ on $T^*M(s)$ is the restriction of the Liouville 1-form on T^*M.*

(ii) *Let π_s be the projection $\mathfrak{g}^* \to \mathfrak{g}(s)^*$. Then the moment map $f^{M(s),\,G(s)}$ is $\pi_s f^{M,\,G}|_{T^*M(s)}$.*

(iii) *The $G(s)$-transversal space $T^*_{\mathfrak{g}(s)}M(s)$ coincides with the intersection $T^*_GM \cap T^*M(s)$.*

(iv) *If σ is G-transversally elliptic, then $\sigma|_{T^*M(s)}$ is $G(s)$-transversally elliptic. If σ is G-transversally good, then $\sigma|_{T^*M(s)}$ is $G(s)$-transversally good.*

Proof. The first two statements follow immediately from the identification (30). Let (x, ξ) be an element of $(T^*M)(s)$. Then $\langle \xi, Y_M(x) - sY_M(x)\rangle = 0$ for any $Y \in \mathfrak{g}$. As $\mathfrak{g} = \mathfrak{g}(s) \oplus (I - \operatorname{ad} s)\mathfrak{g}$, we see that $\xi \in T_x^*M(s)$ is transverse to the G-orbit of x if and only if it is transverse to the $G(s)$-orbit, whence (iii). The last statement follows from the third. $\square$

Similarly, let S be an element of the Lie algebra $\mathfrak{g}$. We denote by $G(S)$ the stabiliser of S in G and by $\mathfrak{g}(S)$ the Lie algebra of $G(S)$. We denote by $M(S) \subset M$ the set of $x \in M$ such that $S_M(x) = 0$. Then $M(S)$ is a submanifold of M and the infinitesimal action of S determines a canonical identification of $T(M(S))$ with the zero set $(TM)(S)$ and of $T^*M(S)$ with the zero set $(T^*M)(S)$. One can observe that $M(S) = M(\exp tS)$ for generic t.

Furthermore, the action of S determines also an orientation o_S of the conormal bundle $\mathcal{N} = \mathcal{N}(M/M(S))$ of $M(S)$ in M, as follows. For $x \in M(S)$, the one parameter group $(\exp tS)^{TM}$ preserves the fiber T_xM. Denote by S^{T_xM} the infinitesimal action of S on T_xM thus defined. Then S^{T_xM} is semi-simple. The decomposition in eigenspaces defines a canonical supplementary subspace of the 0-eigenspace $\operatorname{Ker} S^{T_xM} = (T_xM)(S)$

$$T_xM = \operatorname{Ker} S^{T_xM} \oplus S^{T_xM}.T_xM, \tag{31}$$

thus

$$(T_xM)^* = T_x^*M(S) \oplus \mathcal{N}_x.$$

Moreover, for $x \in M(S)$, the action $S^{\mathcal{N}_x}$ of S on $\mathcal{N}_x$ has eigenvalues $\pm i\theta_j$ with $\theta_j \neq 0$. The orientation is defined by setting positive a basis of $\mathcal{N}_x$ such that $S^{\mathcal{N}_x}$ is made of diagonal blocks $\left(\begin{smallmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{smallmatrix}\right)$ with $\theta_j > 0$. In other words, if we recall the definition of the square root of the determinant, Definition 5, the orientation o_S is defined by the condition

$$\det{}_{o_S}^{\frac{1}{2}}(S^{\mathcal{N}_x}) > 0.$$

4.6. The bouquet of Chern characters

Let G be a compact Lie group acting on a manifold $\mathcal{V}$. We recall the definition of a *bouquet* of equivariant differential forms on $\mathcal{V}$ [14]. Let $s \in G$ and $S \in \mathfrak{g}(s)$. If S is sufficiently small, then

$$G(se^S) = G(s) \cap G(S), \quad \mathfrak{g}(se^S) = \mathfrak{g}(s) \cap \mathfrak{g}(S) \quad \text{and } \mathcal{V}(se^S) = \mathcal{V}(s) \cap \mathcal{V}(S).$$

Definition 20. *A bouquet of equivariant forms is a family $(\alpha_s)_{s\in G}$ where each $\alpha_s \in \mathcal{A}^\infty_{G(s)}(\mathfrak{g}(s), \mathcal{V}(s))$ is a closed $G(s)$-equivariant form, which satisfies the following conditions:*

(1) *Invariance. For all $g \in G$ and $s \in G$,*

$$\alpha_{gsg^{-1}} = g \cdot \alpha_s.$$

(2) *Compatibility: for all $s \in G$, for all $S \in \mathfrak{g}(s)$ and for all $Y \in \mathfrak{g}(s) \cap \mathfrak{g}(S)$,*

$$\alpha_{se^S}(Y)|_{\mathcal{V}(s)\cap\mathcal{V}(S)} = \alpha_s(S+Y)|_{\mathcal{V}(s)\cap\mathcal{V}(S)}.$$

Remark 21. This definition of a bouquet is more restrictive than the definition in [14], where the forms $\alpha_s(Y)$ are defined only for small $Y \in \mathfrak{g}(s)$.

Let $\mathcal{E}$ be a G-equivariant superbundle on $\mathcal{V}$ with a G-invariant superconnection $\mathbb{A}$ of equivariant curvature $\mathbb{F}^{\mathbb{A}}(X)$. For $s \in G$ the action $s^{\mathcal{E}}$ of s on the bundle $\mathcal{E}$ preserves the fibers of $\mathcal{E}|_{\mathcal{V}(s)}$. The $G(s)$-equivariant differential form on $\mathcal{V}(s)$ defined by

$$\mathrm{ch}_s(\mathbb{A})(X) = \mathrm{Str}\,(s^{\mathcal{E}} e^{\mathbb{F}^{\mathbb{A}}(X)}|_{\mathcal{V}(s)}) \quad \text{for } X \in \mathfrak{g}(s) \tag{32}$$

is closed. Furthermore the family $(\mathrm{ch}_s(\mathbb{A}))_{s\in G}$ is a bouquet of equivariant differential forms. We call it the bouquet of Chern characters of the superconnection $\mathbb{A}$.

We introduce the notations of section 4.4 and we consider the bouquet of Chern characters of the superconnection $\mathbb{A}^\omega(\sigma)$ on T^*M.

Lemma 22. *Consider the bundles $\mathcal{E}_s^\pm = \mathcal{E}^\pm|_{M(s)}$ with the restricted connections and denote by σ_s the restriction of σ to $T^*M(s)$. Consider the superconnection $\mathbb{A}^\omega(\sigma_s)$ on $\mathcal{E}_s$. Its $G(s)$-equivariant curvature $\mathbb{F}^{\mathbb{A}^\omega(\sigma_s)}$ coincides with the restriction to $T^*M(s)$ and $\mathfrak{g}(s)$ of the G-equivariant curvature $\mathbb{F}^{\mathbb{A}^\omega(\sigma)}$.*

Proof. This is a consequence of Lemma 19. $\square$

Theorem 23. *Let s be an element of G. The $G(s)$-equivariant differential form* $\mathrm{ch}_s(\mathbb{A}^\omega(\sigma))$ *is rapidly decreasing in* $\mathfrak{g}(s)$*-mean. Its class in* $\mathscr{H}^\infty_{G(s),\,mean\text{-}rapid}(\mathfrak{g}(s), T^*M(s))$ *depends only on the class* $[\sigma]$ *in* $K_G(T^*_G M)$.

Proof. The proof is similar to the proof of Theorem 16, using the fact that σ_s is $G(s)$-transversally good, and Lemma 22. □

5. The cohomological index

5.1. Descent of generalized functions on G and $\mathfrak{g}$

Let G be a compact Lie group, s an element of G with centralizer $G(s) \subset G$ and let U be a $G(s)$-invariant open neighbourhood of s in $G(s)$. If U is small enough, the set $W = G.U$ is a G-invariant open neighbourhood of the orbit $G.s$. The method of descent sets up a one to one correspondance between the G-invariant generalized functions on W and the $G(s)$-invariant generalized functions on U, as follows. Let Θ be a G-invariant generalized function on W. Then Θ restricts to the submanifold $W \cap G(s)$ which contains U as an open set. In the other direction, the correspondance comes from the integration formula associated to the diffeomorphism $(g,u) \to gug^{-1}$ from $G \times_{G(s)} U$ onto W, as follows. Let the left invariant measures on G, $G(s)$ and $G/G(s)$ be choosen in a compatible way. If ϕ is a test function on W we have

$$\int_G \phi(g)\,dg = \int_{G/G(s)} \Big(\int_U \phi(gug^{-1}) \mathrm{det}_{\mathfrak{g}/\mathfrak{g}(s)}(1-u)\,du \Big) d\dot{g}$$

If η is a $G(s)$-invariant generalized function on U, we define a G-invariant generalized function Θ on W by setting for any test function ϕ on W

$$\int_W \Theta(g)\phi(g)\,dg = \int_{G/G(s)} \left(\int_U \eta(u)\phi(gug^{-1}) \mathrm{det}_{\mathfrak{g}/\mathfrak{g}(s)}(1-u)\,du \right) d\dot{g}. \qquad (33)$$

The restriction of Θ to U is equal to η and every G-invariant generalized function on W is obtained by this way.

Let $U_s(0)$ be an open neighbourhood of 0 in $\mathfrak{g}(s)$ such that the map $Y \mapsto se^Y$ is a diffeomorphism of $U_s(0)$ on the neighbourhood U of s in $G(s)$. If $\Theta \in C^{-\infty}(W)^G$, we denote by Θ_s the generalized function on $U_s(0)$ defined by $\Theta_s(Y) = \Theta|_U(se^Y)$.

There is a similar correspondance for invariant functions on the Lie algebra. Let S be an element of the Lie algebra $\mathfrak{g}$ and let U be a $G(S)$-invariant open neighbourhood of S in $\mathfrak{g}(S)$. If U is small enough, the map $(g,Y) \mapsto g.Y$ is a diffeomorphism of $G \times_{G(S)} U$ on the open subset $W = G.U \subset \mathfrak{g}$. Every G-invariant generalized function Θ on W restricts to U. Let the Lebesgue measures on $\mathfrak{g}$ and $\mathfrak{g}(s)$ and the left invariant measure $d\dot{g}$ on $G/G(S)$ be chosen in a compatible way. For $\eta \in C^{-\infty}(U)^{G(S)}$ we define $\Theta \in C^{-\infty}(W)^G$ by

$$\int_W \Theta(X)\phi(X)\,dX = \int_{G/G(S)} \int_U \eta(Y)\phi(g.Y)\mathrm{det}_{\mathfrak{g}/\mathfrak{g}(s)}(Y)\,dY\,d\dot{g}. \qquad (34)$$

The restriction of Θ to U is equal to η and every G-invariant generalized function on W is obtained by this way.

If $\Theta \in C^{-\infty}(W)^G$, we denote by Θ_S the $G(S)$-generalized function defined on the neighbourhood $-S + U$ of 0 in $\mathfrak{g}(S)$ by $\Theta_S(Y) = \Theta|_U(S+Y)$.

We recall the conditions which ensure that a family of invariant generalized functions $\theta_s \in C^{-\infty}(U_s(0))^{G(s)}$, $s \in G$, is the family Θ_s associated in this manner to a global generalized function $\Theta \in C^{-\infty}(G)^G$ ([14], Definition 46 and Theorem 47). Consider a family $U_s(0), s \in G$, where $U_s(0)$ is an open neighbourhood of $0 \in \mathfrak{g}(s)$, such that the map $(g, Y) \mapsto gse^Y g^{-1}$ is a diffeomorphism of $G \times_{G(s)} U_s(0)$ on a neighbourhood of $G.s$ in G and such that $U_{gsg^{-1}}(0) = g.U_s(0)$, for any g and $s \in G$. Such a family will be called a *bunch of neighbourhoods*. We have then $G(se^s) = G(s) \cap G(S)$ for all $S \in U_s(0)$. If θ_s is a $G(s)$-invariant generalized function on the neighbourhood $U_s(0) \subset \mathfrak{g}(s)$ and if $S \in U_s(0)$, we can define the generalized function $(\theta_s)_S(Y) = \theta_s(S+Y)$ on a neighbourhood of 0 in $\mathfrak{g}(s) \cap \mathfrak{g}(S)$.

Theorem 24. *Let G be a compact Lie group. Consider a family $(U_s(0), \theta_s)_{s\in G}$, where $(U_s(0))_{s\in G}$ is a bunch of neighbourhoods, and, for every $s \in G$, $\theta_s \in C^{-\infty}(U_s(0))^{G(s)}$, such that the following conditions are verified.*

Invariance: for any g and $s \in G$, $\theta_{gsg^{-1}}(g.Y) = \theta_s(Y)$ on $U_s(0)$.

Compatibility: for every $S \in U_s(0)$, there exists a neighbourhood $U' \subset U_{s \exp S}(0)$ such that $S + U' \subset U_s(0)$ and such that we have the equality of generalized functions on U'

$$\theta_{s \exp S}(Y) = \theta_s(S+Y).$$

Then there exists a unique generalized function $\Theta \in C^{-\infty}(G)^G$ such that, for all $s \in G$, the equality $\theta_s(Y) = \Theta(se^Y)$ holds in $U_s(0)$.

5.2. Bouquet-integral of the bouquet of Chern characters of a symbol

Let σ be a G-transversally good symbol, representing the element $[\sigma] \in K_G(T^*_G M)$. Our purpose in this section is to construct a G-invariant generalized function on G by applying to the bouquet of Chern characters $\mathrm{ch}_s(\mathbb{A}^\omega(\sigma))$ the *bouquet-integral* construction introduced in [14]. The construction is the following. Let $\mathcal{V}$ be a G-manifold and $(\alpha_s)_{s\in G}$ a family where α_s is a $G(s)$-equivariantly closed differential form on $\mathcal{V}(s)$ defined near 0 in $\mathfrak{g}(s)$. For each $s \in G$, we multiply α_s by an adequate equivariant differential form and we integrate on $\mathcal{V}(s)$ (Definition 29 below). We thus obtain a germ of generalized function near 0 as $\mathfrak{g}(s)$. If the family α_s is a bouquet, ([14] introduces the notion of twisted bouquet, to avoid the difficulty of orienting the submanifolds $\mathcal{V}(s)$), and if $\mathcal{V}$ is compact, the compatibility condition in Theorem 24 is verified as a consequence of the localisation formula (Theorem 25). Here, we are dealing with the manifold $\mathscr{V} = T^*M$, we will thus need to prove a localisation formula for certain equivariant differential forms which are rapidly decreasing in mean. On the other hand, we do not need twisted bouquets, as $T^*M(s)$ is naturally oriented.

First, we recall the localisation formula for G-equivariant differential forms on a compact manifold $\mathscr{V}$.

Let $S \in \mathfrak{g}$ and consider the zero set $\mathscr{V}(S)$. The normal bundle $\mathscr{N} = \mathscr{N}(\mathscr{V}, \mathscr{V}(S))$ is orientable as explained in section 4.5. We choose a $G(S)$-invariant metric connection on $\mathscr{N}$ and we denote by $R_{\mathscr{N}}(Y)$ its $G(S)$-equivariant curvature. We choose an orientation o on $\mathscr{N}$ and we denote the $\mathrm{Eul}_{\mathscr{N},o}(Y)$ the $G(S)$-equivariant Euler form of the bundle $\mathscr{N}$ on $\mathscr{V}(S)$, (with respect to the choosen orientation and connection), thus

$$\mathrm{Eul}_{\mathscr{N},o}(Y) = (-2\pi)^{-\mathrm{rank}(\mathscr{N})/2} \det\nolimits_o^{\frac{1}{2}} R_{\mathscr{N}}(Y)$$

where $\mathrm{rank}(\mathscr{N})$ is the rank of the normal bundle $\mathscr{N} \to \mathscr{V}(S)$ and the square root of the determinant is determined by the orientation o as in Definition 5.

For $Y = S$ the zero exterior degree term of $\mathrm{Eul}_{\mathscr{N},o}(Y)$ is non zero; it remains non zero for Y in a neighbourhood of S in $\mathfrak{g}(S)$, thus the inverse $\mathrm{Eul}_{\mathscr{N},o}(Y)^{-1}$ is defined for Y near S, as $\mathscr{V}$ is compact.

The localization formula is the following (cf. [7], Theorem 7.13 and [12]):

Theorem 25. *Let $\mathscr{V}$ be a compact oriented manifold and G a compact Lie group acting on $\mathscr{V}$. Let S be an element of the Lie algebra $\mathfrak{g}$. Let $\alpha \in \mathscr{A}^{\infty}_{G(S)}(U, \mathscr{V})$ be an equivariantly closed $G(S)$-equivariant differential form on $\mathscr{V}$, defined on a neighbourhood U of S in $\mathfrak{g}(S)$. Then for $Y \in \mathfrak{g}(S)$ sufficiently close to S, we have*

$$\int_{\mathscr{V}} \alpha(Y) = \int_{\mathscr{V}(S)} \alpha(Y)|_{\mathscr{V}(S)} \,\mathrm{Eul}_{\mathscr{N},o}(Y)^{-1}.$$

In this formula, the orientations on $\mathscr{V}$, $\mathscr{V}(S)$ and the orientation o on $\mathscr{N}$ must be chosen in a compatible way.

If α is a G-equivariantly closed differential form on $\mathscr{V}$, defined on $W \subset \mathfrak{g}$, its integral on $\mathscr{V}$ is a smooth function on W, defined pointwise by $(\int_{\mathscr{V}} \alpha)(X) = \int_{\mathscr{V}} \alpha(X)$. It is G-invariant and depends only on the $d_{\mathfrak{g}}$ cohomology class of α. The localisation formula computes the restriction of $\int_{\mathscr{V}} \alpha$ to a neighbourhood of S in $\mathfrak{g}(S)$. When $\mathscr{V}$ is no longer compact, we may sometimes still define $\int_{\mathscr{V}} \alpha$ as a generalized function on W as in Section 3.1, although it may no longer be defined pointwise. Since it is G-invariant it has a restriction to a neighbourhood of S in $\mathfrak{g}(S)$, as explained in Section 5.1. When the inverse of the Euler class makes sense, one expects that the localisation formula should still hold as an equality of generalized functions. As we will see, it holds for the forms on $T^*M(s)$ involved in the bouquet integral of the Chern character of a transversally good symbol.

The definition of the bouquet-integral requires the following two particular Chern-Weil equivariant differential forms.

Definition 26. *Let $\mathcal{V}$ be a G-equivariant real bundle $\mathcal{V}$ to M, with a G-equivariant connection ∇. Let $R(X)$, $X \in \mathfrak{g}$, be the equivariant curvature of ∇. The J-genus of the bundle $\mathcal{V}$, with respect to the connection ∇, is the G-equivariantly closed form on M defined by*

$$J(\mathcal{V}, M)(X) = \det \frac{e^{R(X)/2} - e^{-R(X)/2}}{R(X)}.$$

The J-genus $J(TM, M)$ of the tangent bundle TM is denoted simply by $J(M)$.

Definition 27. *Let $s \in G$. Denote by $\mathcal{N}$ the normal bundle to $M(s)$ in M and by $R_{\mathcal{N}}(Y)$, $Y \in \mathfrak{g}(s)$, the equivariant curvature of $\mathcal{N}$ with respect to a $G(s)$-equivariant connection. We define a $G(s)$-equivariantly closed form on $M(s)$ by*

$$D_s(\mathcal{N})(Y) = \det(1 - s^{\mathcal{N}} \exp R_{\mathcal{N}}(Y)) \quad \textit{for } Y \in \mathfrak{g}(s).$$

As for all Chern-Weil forms, the cohomology classes do not depend on the choice of connection. The forms $J(M)(X)$ and $D_s(\mathcal{N}(Y)$ have a non vanishing zero exterior degree term for $X = 0$ and $Y = 0$; as M is compact, they have inverses for X and Y sufficiently close to 0.

In the application, we will use these Chern-Weil differential forms in the case where M is replaced with the total space of T^*M. Then they are given by pull-backs of Chern-Weil forms on M itself, in the following way:

Let $\mathcal{V}$ be a G-equivariant real vector bundle over M, with projection $p : \mathcal{V} \to M$. Let us choose a G-invariant connection for the fibration $\mathcal{V} \to M$. Then we can write $T\mathcal{V} = p^*TM \oplus p^*\mathcal{V}$ as a sum of two G-equivariant vector bundles, where p^*TM is identified to the horizontal tangent bundle to $\mathcal{V}$ and $p^*\mathcal{V}$ is identified to the vertical tangent bundle. From the multiplicativity of the J-genus and the obvious relationship between pullbacks and Chern-Weil forms, we see that the G-equivariant J-genus of the tangent bundle $T\mathcal{V}$ to the total space of $\mathcal{V}$ is equal to the product $p^*(J(\mathcal{V}, M))p^*(J(M))$.

In the particular case of $\mathcal{V} = T^*M$ we obtain

$$J(T^*M) = J(M)^2. \tag{35}$$

Similarly, let $s \in G$ and let $\mathcal{N}$ be the normal bundle to $M(s)$ in M. Let p_s be the projection $T^*M(s) \to M(s)$. A choice of G-invariant metric and connection on TM allows us to write the normal bundle $\mathcal{N}(T^*M, T^*M(s))$ to $T^*M(s)$ in T^*M as

$$\mathcal{N}(T^*M, T^*M(s)) = p_s^*\mathcal{N} \oplus p_s^*\mathcal{N}.$$

Thus

$$D_s(\mathcal{N}(T^*M, T^*M(s)) = D_s(\mathcal{N})^2. \tag{36}$$

Finally, let $s \in \mathfrak{g}$. We compute the $G(S)$-equivariant Euler form of the normal bundle $\mathcal{N}(T^*M, T^*M(s))$ to $T^*M(S)$ in T^*M. Let $\mathcal{N}$ denote now the normal bundle to $M(S)$ in M and let p_S be the projection $T^*M(S) \to M(S)$. A choice of G-invariant metric and connection on TM allows us to write

$$\mathcal{N}(T^*M, T^*M(s)) = p_S^*\mathcal{N} \oplus p_S^*\mathcal{N}.$$

Lemma 28. *Let the total space of T^*M and $T^*M(S)$ be oriented by the symplectic orientation* (*Sect.* 4.3) *and the normal bundle be oriented by the quotient orientation. Then*

$$\mathrm{Eul}_{\mathcal{N}(T^*M,T^*M(s))}(Y) = (-2\pi)^{-\mathrm{rank}\,\mathcal{N}} \det(R_{\mathcal{N}}(Y)).$$

Proof. This follows easily from the following computation. Let $e_1, e_2, \ldots$ be an orthonormal basis of $\mathcal{N}_x$. Let R be an endomomorphism of $\mathcal{N}_x$ with matrix made of diagonal blocks $\left(\begin{smallmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{smallmatrix}\right)$, then $\det(R) = \prod \theta_j^2$. On the other side, let $e_1^*, e_2^*, \ldots$ be the dual basis of $\mathcal{N}_x^*$, then the basis $(e_1, e_1^*, e_2, e_2^*, \ldots)$ is positive for the canonical symplectic orientation o of $\mathcal{N}_x \oplus \mathcal{N}_x^* \simeq \mathcal{N}_x \oplus \mathcal{N}_x$, thus for the endomorphism $R \oplus R$ of $\mathcal{N}_x \oplus \mathcal{N}_x$ we have $\det_o^{\frac{1}{2}}(R \oplus R) = \prod(-\theta_j^2)$. $\square$

The general definition of the bouquet-integral on a manifold $\mathcal{V}$ involves square roots of the Chern-Weil forms $J(\mathcal{V}(s))$ and $D_s(\mathcal{N}(\mathcal{V}, \mathcal{V}(s))$, [14]. In the case of the manifold $\mathcal{V} = T^*M$, as we have a canonical choice of these square roots by (35) and (36), we introduce the following forms:

Definition 29. *Let σ be a G-transversally good symbol on M. Let $\mathbb{A}(\sigma)$ and $\mathbb{A}^\omega(\sigma)$ be the superconnections associated to σ as in Sect.* 4.4, (*the other data are kept implicit*). *Let $s \in G$ and $\mathcal{N}$ be the normal bundle to $M(s)$ in M. The following two $G(s)$-equivariantly closed forms on $T^*M(s)$ are defined on a sufficiently small neighbourhood $U_s(0)$ of 0 in $\mathfrak{g}(s)$:*

$$I(s,\sigma)(Y) = \mathrm{ch}_s(\mathbb{A}(\sigma))(Y)\, J(M(s))(Y)^{-1} D_s(\mathcal{N})(Y)^{-1}, \tag{37}$$

$$I^\omega(s,\sigma)(Y) = \mathrm{ch}_s(\mathbb{A}^\omega(\sigma))(Y)\, J(M(s))(Y)^{-1} D_s(\mathcal{N})(Y)^{-1}. \tag{38}$$

As $J(M(s))(Y)$ and $D_s(\mathcal{N})(Y)$ are pullbacks of forms on $M(s)$, it follows from Theorem 23 and Remark 3 that the form $I^\omega(s,\sigma)$ belongs to $\mathcal{A}^\infty_{G(s),\,mean\text{-}rapid}(U_s(0), T^*M(s))$. Therefore we define a $G(s)$-invariant generalized function on $U_s(0)$ as the integral

$$\theta_s(Y) = \int_{T^*M(s)} (2i\pi)^{-\dim M(s)} I^\omega(s,\sigma)(Y) \tag{39}$$

where the manifold $T^*M(s)$ is oriented by the volume form $(-d\omega^{M(s)})^{\dim M(s)}$. Moreover the cohomology class in $\mathcal{H}^\infty_{G,\,mean\text{-}rapid}(U_s(0), T^*M(s))$ of $I^\omega(s,\sigma)$ depends only on the class $[\sigma]$, therefore the function (39) depends only on $[\sigma] \in K_G(T^*_G M)$.

Similarly, if σ is a G-invariant good symbol, the form $I(s,\sigma)$ is rapidly decreasing and we define pointwise a smooth $G(s)$-invariant function on $U_s(0)$, depending only on $[\sigma] \in K_G(T^*M)$, as the integral

$$\int_{T^*M(s)} (2i\pi)^{-\dim M(s)} I(s,\sigma)(Y). \tag{40}$$

At last, we are ready to state the main theorem of this article.

Theorem 30. *Let M be a compact manifold, let G be a compact Lie group acting on M. Let σ be a G-transversally good symbol on M. Then there exists a G-invariant generalized function on the group G, which we denote by* $\mathrm{index}_c^{G,M}(\sigma)$, *whose germ at $s \in G$ is given by*

$$\mathrm{index}_c(\sigma)(se^Y) = \int_{T^*M(s)} (2i\pi)^{-\dim M(s)} \mathrm{ch}_s\,(\mathbb{A}^{\omega}(\sigma))(Y)\, J(M(s))(Y)^{-1} D_s(\mathcal{N})(Y)^{-1}$$

for $Y \in \mathfrak{g}(s)$ near 0.

The generalized function $\mathrm{index}_c^{G,M}(\sigma)$ depends only on the class $[\sigma] \in K_G(T^*_G M)$.

Proof. We will verify the conditions of Theorem 24 for the family of generalized functions

$$\theta_s(Y) = \Big(\int_{T^*M(s)} (2i\pi)^{-\dim M(s)} I^{\omega}(s,\sigma)\Big)(Y).$$

We have already explained that $\theta_s(Y)$ depends only on $[\sigma]$.

The invariance condition of Theorem 24 is clear. There remains to prove the compatibility condition $\theta_{(se^S)}(Y) = \theta_s(S+Y)$, for $Y \in \mathfrak{g}(se^S) = \mathfrak{g}(s) \cap \mathfrak{g}(S)$, Y small enough.

Recall that $M(se^S) = M(s) \cap M(S)$ if $S \in \mathfrak{g}(s)$ is sufficiently small. Formally, the compatibility of the family of functions θ_s is a consequence of the localisation formula and of the following lemma.

Lemma 31. *Let $U_s(0)$ be a bunch of neighbourhoods such that $J(M(s))(Y)$ and $D_s(\mathcal{N})(Y)$ are invertible for $Y \in U_s(0)$ and such that $M(se^S) = M(s) \cap M(S)$ for $S \in U_s(0)$. Let $s \in G$. Then, for all $S \in U_s(0)$ and $Y \in U_{se^S}(0)$ such that $S + Y \in U_s(0)$, we have*

$$(2i\pi)^{-\dim M(s)} I^{\omega}(s,\sigma)(S+Y)|_{(T^*M(s))(S)} \mathrm{Eul}^{-1}_{\mathcal{N}(T^*M(s),\,T^*M(s)(S))}(S+Y)$$
$$= (2i\pi)^{-\dim (M(s))(S)} I^{\omega}(se^S,\sigma)(Y).$$

Proof. Let us consider the case $s = e$. For $Y \in \mathfrak{g}(S)$ we have

$$\mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma))(S+Y)|_{M(S)} = \mathrm{ch}_{e^S}\,(\mathbb{A}^{\omega}(\sigma))(Y).$$

Thus we need to verify

$$(2i\pi)^{\dim M(s)} J(M)(S+Y) \mathrm{Eul}_{\mathcal{N}(T^*M,\, T^*M(S))}(S+Y)$$
$$= (2i\pi)^{\dim M(S)} J(M(S))(Y) \det(1 - e^S \exp R_{\mathcal{N}(M,\, M(S))}(Y)).$$

From the decomposition $TM|_{M(S)} = TM(S) \oplus \mathcal{N}$ and the fact that $\mathcal{N}$ has even rank, we see that

$$J(M)(S+Y) = J(M(S))(Y) \frac{\det(1 - e^S \exp R_{\mathcal{N}(M,\, M(S))}(Y))}{\det R_{\mathcal{N}(M,\, M(S))}(S+Y)}.$$

Thus the equality follows from Lemma 28. $\square$

There remains to prove the localisation formula in our situation:

Lemma 32. *Let $U_s(0)$ be a bunch of neighbourhoods satisfying the conditions of the previous lemma. Let $s \in G$ and $S \in U_s(0)$. Then there exists a neighbourhood U of S in $\mathfrak{g}(s) \cap \mathfrak{g}(S)$ such that the following equality of generalized functions on U holds*:

$$\theta_s|_U(Y) = \int_{T^*M(s)(S)} (2i\pi)^{-\dim M(s)} I^{\omega}(s,\sigma)(Y) \mathrm{Eul}^{-1}_{\mathcal{N}(T^*M(s),\, T^*M(s)(S))}(Y).$$

Proof. Let us consider the case $s = e$. We observe that the $G(S)$-equivariant Euler form $\mathrm{Eul}_{\mathcal{N}(T^*M,\, T^*M(S))}(Y)$ is indeed the pull back of a form on $M(S)$, therefore it is invertible when $Y \in \mathfrak{g}(S)$ is close enough to S and the above integral is well defined. Formally Lemma 32 is the localization formula of Theorem 25 applied to the equivariant form $I^{\omega}(e,\sigma)(X)$. But, as here the integrals do not make sense for each Y individually, we first replace $I^{\omega}(e,\sigma)(X)$ by a form which is rapidly decreasing in $\mathfrak{g}(S)$-mean and not only in $\mathfrak{g}$-mean. Once this is done, it is easy to adapt the proof of Theorem 25 given in [13], Proposition 25.

We write $Z = G(S)$ and $\mathfrak{z} = \mathfrak{g}(S)$. We fix a G-invariant Euclidean metric on $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{q}$ where $\mathfrak{q} = [S, \mathfrak{g}]$ is the orthogonal of $\mathfrak{z}$ in $\mathfrak{g}$. We fix an orientation on $\mathfrak{q}$.

Let $u_{\mathfrak{q}}(Y)$ be the Z-equivariant Thom form of the Euclidean space $\mathfrak{q}$ (see [7]). Let $e_1, \ldots, e_{2d}$ be a positive orthonormal basis of $\mathfrak{q}$. Let $T : \mathscr{A}(\mathfrak{q}) \otimes \Lambda\mathfrak{q} \to \mathscr{A}(\mathfrak{q})$ denote the coefficient of $e_1 \wedge e_2 \wedge \cdots \wedge e_{2d}$. For $Y \in \mathfrak{z}$ and $f_1 = \sum f^i e_i \in \mathfrak{q}$,

$$u_{\mathfrak{q}}(Y)(f_1) =$$
$$(-\pi)^{-d} e^{-\|f_1\|^2} T\Big(\exp\Big(\sum_i df^i \otimes e_i + \tfrac{1}{2}\sum_{i<j} \langle \mathrm{ad}_{\mathfrak{q}} Y e_i, e_j\rangle e_i \wedge e_j\Big)\Big).$$

We denote by $f_{\mathfrak{q}} : T^*M \to \mathfrak{q}$ the composition of the moment map $f^{M,G} : T^*M \to \mathfrak{g}^*$ and the projection $p_{\mathfrak{q}} : \mathfrak{g}^* \to \mathfrak{q}^* \simeq \mathfrak{q}$. Let

$$u(Y) = f^*_{\mathfrak{q}}(u_{\mathfrak{q}}(Y)) = (f^{M,G})^*(p^*_{\mathfrak{q}}(u_{\mathfrak{q}}(Y))).$$

We choose an open neighbourhood $U \subset \mathfrak{z}$ of S such that the open subset $W = G.U \subset U_e(0)$ is a tubular neighbourhood of the orbit $G.S$, diffeomorphic to $G \times_Z U$, and such that $\mathrm{Eul}^{-1}_{\mathcal{N}(T^*M,\, T^*M(S))}(Y)$ is defined for $Y \in U$.

Let $\alpha(X) = I^{\omega}(e, \sigma)(X)$. By means of the form u we give a first expression for the restriction of $\left(\int_{T^*M} \alpha\right)(X)$ to the open subset $U \subset \mathfrak{z}$.

Lemma 33. (i) *The Z-equivariant form $u(Y)\alpha(Y)$ on T^*M is rapidly decreasing in $\mathfrak{z}$-mean.*

(ii) *We have the equality of generalized functions on $U \subset \mathfrak{z}$*

$$\det{}_{\mathfrak{q}}^{1/2}(Y)\Big(\int_{T^*M} \alpha\Big)_U(Y) = (-2\pi)^d \int_{T^*M} u(Y)\alpha(Y)$$

Proof. We write $f^{M,G}(x,\xi) = f_0(x,\xi) + f_1(x,\xi)$ with $f_0(x,\xi) \in \mathfrak{z}^*$ and $f_1(x,\xi) \in \mathfrak{q}^*$. Then

$$u(Y)\alpha(Y) = \sum P_I(Y) e^{-\|f_1\|^2} \alpha(Y)(d f_1)_I$$

where $P_I(Y)$ depend polynomially on Y. Thus, as $f_1(x,\xi)$ is linear in ξ, it is sufficient to prove that $e^{-\|f_1\|^2}\alpha(Y)$ is rapidly decreasing in $\mathfrak{z}$-mean. We trivialise T^*M and $\mathcal{E}$ in a neighbourhood of $x \in M$ and we employ the notations of Lemma 13. We write

$$\alpha(X) = e^{i\langle f(x,\xi), X\rangle} q(X, x, \xi)$$

with $q(X,x,\xi) = \mathrm{Str}\,(e^{-h(x,\xi)+Z(Y,x,\xi)})$. Here, $Z(X,x,\xi) = Z_0(X,x) + Z_1(x,\xi)$, the term $Z_0(X,x) = p^*\mu(X)$ is linear in X, the term $Z_1(x,\xi)$ does not depend on X and $h = v(\sigma)^2$. Let $h_1(x,\xi) = \|f_1(x,\xi)\|^2 + h(x,\xi)$. Then for $Y \in \mathfrak{z}$

$$e^{-\|f_1\|^2}\alpha(Y) = e^{i\langle f_0(x,\xi), Y\rangle} q_0(X,x,\xi)$$

with $q_0(Y,x,\xi) = \mathrm{Str}\,(e^{-h_1(x,\xi)+Z(Y,x,\xi)})$. Let us observe that h_1 is good with respect to the map $f_0 : T^*M \to \mathfrak{z}^*$. Indeed, let a and c be such that $h(x,\xi) \geqq c\|\xi\|^2$ if $\|f(x,\xi)\| \leqq a\|\xi\|$. Let $\|f_0(x,\xi)\| \leqq \frac{a}{2}\|\xi\|$. If $\|f_1(x,\xi)\| \geqq \frac{a}{2}\|\xi\|$ then $h_1(x,\xi) \geqq \frac{a^2}{4}\|\xi\|^2$. If $\|f_1(x,\xi)\| \leqq \frac{a}{2}\|\xi\|$ then $\|f^{M,G}(x,\xi)\| \leqq a\|\xi\|$, hence $h_1(x,\xi) \geqq v(\sigma)^2(x,\xi) \geqq c\|\xi\|^2$.

Thus we can apply Lemma 15 to the situation where f is f_0 and conclude that $e^{-\|f_1\|^2}\alpha(Y)$ is rapidly decreasing in $\mathfrak{z}$-mean.

Let us prove (ii). Formally, this equality is a consequence of the equality in Z-equivariant cohomology of the forms $(-2\pi)^d u(Y)$ and $\det{}_{\mathfrak{q}}^{\frac{1}{2}}(Y)$ on T^*M. Our method will be to use a homotopy of G-equivariant differential forms on the manifold $G \times_Z T^*M$. Let us consider on the principal bundle $G \to G/Z$ the connection defined by the decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{q}$. We denote its curvature by Ω and its moment by $\mu(X)$, for $X \in \mathfrak{g}$. We view Ω and $\mu(X)$ as $\mathfrak{z}$-valued differential forms on G. We denote by $\chi_0(X)$, $X \in \mathfrak{g}$, the Euler form of the tangent bundle $T(G/Z)$. Then χ_0 corresponds by the Chern-Weil homomorphism $S(\mathfrak{z}^*)^Z \to \mathscr{A}_G^{pol}(\mathfrak{g}, G/Z)$ to the polynomial function $(-2\pi)^{-d}\det_{\mathfrak{q}}^{\frac{1}{2}}(Y)$ on $\mathfrak{z}$:

$$\chi_0(X) = (-2\pi)^{-d}\det{}_{\mathfrak{q}}^{\frac{1}{2}}(\Omega + \mu(X)). \tag{41}$$

The subalgebra $\mathfrak{z}$ has same rank as $\mathfrak{g}$. Hence the map $G \times_Z \mathfrak{z} \to \mathfrak{g}$ given by $[g, Y] \mapsto g.Y$ is surjective. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{z}$. Let $N(G, \mathfrak{h})$ be the normalizer of $\mathfrak{h}$ in G. Let w be the number of elements in $N(G, \mathfrak{h})/Z \cap N(G, \mathfrak{h})$. Then w is the number of elements in the generic fiber of the map $G \times_Z \mathfrak{z} \to \mathfrak{g}$. Hence, if Θ is a G-invariant generalized function on $\mathfrak{g}$, we have the following integral formula for the restriction of Θ to $W \cap \mathfrak{z}$ (compare with (34)). Let the Lebesgue measures on $\mathfrak{g}$ and $\mathfrak{z}$ and the left invariant measure $d\dot{g}$ on G/Z be chosen in a compatible way. Then for any test function on $\mathfrak{g}$ with support in W

$$\int_{\mathfrak{g}} \Theta(X)\phi(X)\,dX = w^{-1} \int_{G/Z}\int_{\mathfrak{z}} \Theta|_{W\cap\mathfrak{z}}(Y)\phi(g.Y)\det{}_{\mathfrak{q}}(Y)\,dY\,d\dot{g}. \tag{42}$$

For a regular $X \in \mathfrak{g}$, the number of zeroes of the vector field $X_{G/Z}$ is equal to w. By applying Theorem 25 to $\chi_0(X)$, for X regular, we obtain

$$\int_{G/Z} \chi_0(Z) = w.$$

By continuity this formula holds also for any $X \in \mathfrak{g}$.

Let us consider the manifold $(G/Z) \times T^*M$ with the diagonal action of G. We have

$$\int_{(G/Z)\times T^*M} \chi_0(X)\alpha(X) = w \int_{T^*M} \alpha(X). \tag{43}$$

Let us consider the manifold $(G/Z) \times \mathfrak{g}^*$ with the diagonal action of G. We consider it as a vector bundle over G/Z. We identify $\mathfrak{g}^*$ with the Z-invariant submanifold $\{e\} \times \mathfrak{g}^*$. In the next lemma we use the following notation: for $\mathscr{V}$ a Euclidean vector bundle, we denote by $\mathscr{A}_{slow}(\mathscr{V})$ the space of differential forms on the total space $\mathscr{V}$ which are slowly increasing on the fibers as well as their derivatives and we get

$$\mathscr{A}_{G,slow}^{pol}(\mathfrak{g}, \mathscr{V}) = (S(\mathfrak{g}^*) \otimes \mathscr{A}_{slow}(\mathscr{V}))^G.$$

Lemma 34. *There exists a G-equivariantly closed differential form $\chi_t(X)$, $X \in \mathfrak{g}$, on $(G/Z) \times \mathfrak{g}^*$ depending smoothly on $t \geqq 0$ such that*

(i) *for $t = 0$, the form χ_0 is the pullback to $(G/Z) \times \mathfrak{g}^*$ of the form χ_0 defined in* (41).

(ii) *the form χ_t belongs to the space $\mathscr{A}_{G,slow}^{pol}(\mathfrak{g}, (G/Z) \times \mathfrak{g}^*)$; moreover $\chi_1 - \chi_0 = d_{\mathfrak{z}}\,\alpha$, with $\alpha \in \mathscr{A}_{G,slow}^{pol}(\mathfrak{g}, (G/Z) \times \mathfrak{g}^*)$.*

(iii) *for $t = 1$ and $Y \in \mathfrak{z}$, the restriction of $\chi_1(Y)$ to $\mathfrak{g}^*$ is equal to $p_{\mathfrak{q}}^* u_{\mathfrak{q}}(Y)$.*

Proof. Consider the diffeomorphism $m : (G/Z) \times \mathfrak{g}^* \to G \times_Z \mathfrak{g}^*$ defined by $m(g, f) = [g, g^{-1} f]$. It induces an isomorphism of the complexes of equivariant differential forms

$$m^* : (\mathscr{A}^{pol}_{G, slow}(\mathfrak{g}, G \times_Z \mathfrak{g}^*), d_{\mathfrak{g}}) \to (\mathscr{A}^{pol}_{G, slow}(\mathfrak{g}, (G/Z) \times \mathfrak{g}^*), d_{\mathfrak{g}}).$$

Let us recall the Chern-Weil homomorphism of complexes ([7])

$$C : (\mathscr{A}^{pol}_{Z}(\mathfrak{z}\ , \mathfrak{g}^*), d_{\mathfrak{z}}\ ) \to (\mathscr{A}^{pol}_{G}(\mathfrak{g}, G \times_Z \mathfrak{g}^*), d_{\mathfrak{g}}).$$

determined by the choice of a connection on G/Z. Let $\kappa(Y)$, $Y \in \mathfrak{z}$, be a Z-equivariant form on $\mathfrak{g}^*$, we consider $\kappa(\Omega + \mu(X))$ as a G-equivariant form on $G \times \mathfrak{g}^*$. The connection on G/Z determines a horizontal projection h from the space of Z-invariant forms on $G \times \mathfrak{g}^*$ to the space of Z-basic forms on $G \times \mathfrak{g}^*$, identified with the space of forms on $G \times_Z \mathfrak{g}^*$. The map C is defined by $C(\kappa)(X) = h(\kappa(\Omega + \mu(X)))$. One easily verifies that C sends $\mathscr{A}^{pol}_{G, slow}(\mathfrak{z}\ , \mathfrak{g}^*)$ to $\mathscr{A}^{pol}_{G, slow}(\mathfrak{g}, G \times_Z \mathfrak{g}^*)$.

Let $u_{q, t}(Y)$ be the rescaled Z-equivariant Thom form of $\mathfrak{q}$,

$$u_{\mathfrak{q}, t}(Y)(f_1) = (-\pi)^{-d} e^{-\|t f_1\|^2} T\Big(\exp\Big(t \sum_i d f^i \otimes e_i + \tfrac{1}{2} \sum_{i, j} \langle \mathrm{ad}_{\mathfrak{q}} Y e_i,\ e_j \rangle e_i \wedge e_j\Big)\Big).$$

We have

$$\frac{d}{dt} u_{\mathfrak{q}, t} = d_{\mathfrak{z}}\, \alpha_t,$$

where

$$\alpha_t = (-\pi)^{-d} e^{-\|t f_1\|^2} \times T\Big(\Big(\sum f^i \otimes e_i\Big) \exp\Big(t \sum_i df^i \otimes e_i + \tfrac{1}{2} \sum_{i<j} \langle \mathrm{ad}_{\mathfrak{q}} Y e_i,\ e_j \rangle e_i \wedge e_j\Big)\Big).$$

Thus we define $\chi_t = m^* C(p_{\mathfrak{q}}^*(u_{\mathfrak{q}, t}))$ and χ_t has the required properties. $\square$

We also denote by $f^{M, G}$ the map $(G/Z) \times T^*M \to (G/Z) \times \mathfrak{g}^*$ which extends the moment map $f^{M, G} : T^*M \to \mathfrak{g}^*$. We take $t = 1$. Let $\chi(X) = (f^{M, G})^* \chi_1(X)$. By Lemma 34 and Equation (43), we have

$$\int_{T^*M} \alpha(X) = w^{-1} \int_{(G/Z) \times T^*M} \chi_0(X) \alpha(X) = w^{-1} \int_{(G/Z) \times T^*M} \chi(X) \alpha(X), \tag{44}$$

thus we need to compute the restriction to $U \subset \mathfrak{z}$ of $\int_{(G/Z) \times T^*M} \chi(X) \alpha(X)$.

We denote by $(.,.)$ the G-invariant Riemannian metric on G/Z determined by the chosen Z-invariant Euclidean metric on $\mathfrak{q}$. We define a G-invariant one-form θ on G/Z by $\langle \theta(X), \xi \rangle = (X_{(G/Z)}, \xi)$ for $X \in \mathfrak{g}$. If we consider $d_{\mathfrak{g}} \theta$ as an equivariant form on $(G/Z) \times T^*M$ we have, by Remark 3, for every $t \geqq 0$,

$$\int_{(G/Z) \times T^*M} \chi(X) \alpha(X) = \int_{(G/Z) \times T^*M} \chi(X) e^{t(d_{\mathfrak{g}} \theta)(X)} \alpha(X).$$

Analyzing the behavior of $e^{td_{\mathfrak{g}}\theta(X)}$ when $t \to \infty$ is the last step in the proof of Lemma 33.

Denote by $T_{G/Z} : \mathscr{A}(G/Z \times T^*M) \to \mathscr{A}(G/Z \times T^*M)$ the projection on the component of exterior degree $2d$ with respect to G/Z. Let ϕ be a test function on $\mathfrak{g}$. As α is the pullback of a form on T^*M, we have

$$\int_{(G/Z)\times T^*M} \int_{\mathfrak{g}} \chi(X) e^{td_{\mathfrak{g}}\theta(X)} \alpha(X)\phi(X)\, dX$$
$$= \int_{(G/Z)\times T^*M} \int_{\mathfrak{g}} T_{G/Z}(\chi(X) e^{td_{\mathfrak{g}}\theta(X)}) \alpha(X)\phi(X)\, dX.$$

We write $T_{G/Z}(\chi(X) e^{td_{\mathfrak{g}}\theta(X)})_e = \chi_{\{1,2,\dots,2d\}}(t,X)(d\dot{g})_e$, with $\chi_{\{1,2,\dots,2d\}} \in \mathscr{A}^{pol}_{G,slow}(\mathfrak{g}, T^*M)$, where we denote by $d\dot{g}$ the G-invariance volume form on G/Z. By G-invariance we have

$$\int_{(G/Z)\times T^*M} \int_{\mathfrak{g}} \chi(X) e^{td_{\mathfrak{g}}\theta(X)}) \alpha(X)\phi(X)\, dX$$
$$= \int_{G/Z} \left(\int_{T^*M} \int_{\mathfrak{g}} \chi_{\{1,2,\dots,2d\}}(t,X)\alpha(X)\phi(g.X)\, dX \right) d\dot{g}.$$

Then we have for every t,

$$\int_{(G/Z)\times T^*M} \int_{\mathfrak{g}} \chi(X)\alpha(X)\phi(X)\, dX$$
$$= \int_{G/Z} \left(\int_{T^*M} \int_{\mathfrak{g}} \chi_{1,2,\dots,2d}(t,X)\alpha(X)\phi(g.X)\, dX \right) d\dot{g}. \tag{45}$$

Let us describe $e^{t(d_{\mathfrak{g}}\theta)(X)})$, (see [14]). Let $e_1, \dots, e_{2d}$ be our orthonormal basis of $\mathfrak{q}$ and let us denote by $dq_1, \dots, dq_{2d}$ the dual basis of $\mathfrak{q}^*$. If $X \in \mathfrak{g}$, we write $X = Y + Q$, with $Y \in \mathfrak{z}$ and $Q \in \mathfrak{q}$. Let $Q = \sum_i Q_i e_i$. Then $\theta_e(X) = \sum_i Q_i\, dq_i$, and $(d\theta)_e(X) = \sum_{i<j} \theta_{i,j}(X) dq_i \wedge dq_j$, where the $\theta_{i,j}$ are linear forms on $\mathfrak{g}$. Furthermore for $Y \in \mathfrak{z}$ we have

$$(d\theta)_e(Y) = -2 \sum_{i<j} \langle \mathrm{ad}_{\mathfrak{q}}(Y) e_i, e_j \rangle\, dq_i \wedge dq_j.$$

As $(\iota(X_{G/Z})\theta(X))_e = (X_{G/Z}, X_{G/Z})_e = \|Q\|^2$, we obtain

$$(e^{t(d_{\mathfrak{g}}\theta)(X)})_e = e^{-t\|Q\|^2} \sum_I P_I(Y,Q) t^{|I|/2} (dq)_I,$$

where $P_I(Y,Q)$ are polynomials in Y, Q. Furthermore, for $Y \in \mathfrak{z}$ and $Q = 0$, we have

$$P_{1,\dots,2d}(Y,0) = (-2)^d \det\nolimits_{\mathfrak{q}}^{\frac{1}{2}}(Y).$$

From the definition of the Chern homomorphism C and of the map m we have

$$\chi_1(X)_e = \sum_{I,J} P_{I,J}(Y,f) e^{-\|f_1\|^2} (df_1)_I \wedge (dq)_J,$$

where $P_{I,J}(Y,f)$ are polynomials in $Y \in \mathfrak{z}$ and $f \in \mathfrak{g}^*$ and $f = f_0 + f_1$ is the decomposition according to $z^* \oplus \mathfrak{q}^*$. For $Y \in \mathfrak{z}$, the term of exterior degree zero with respect to (dq_i) in $\chi_1(Y)_e|_{\mathfrak{g}^*}$ is equal to $p_{\mathfrak{q}}^*(u_{\mathfrak{q}}(Y))$. We obtain

$$\chi_{\{1,\dots,2d\}}(t,X) = e^{-t\|Q\|^2} \sum_{I,J} P_I(Y,Q) P_{I',J}(Y,f) t^{|I|/2} e^{-\|f_1\|^2} (df_1)_J, \tag{46}$$

where I' is the complement of I in $\{1,\dots,2d\}$. Thus the form $\chi_{\{1,\dots,2d\}}(t,X)$ belongs to the space $\mathscr{A}^{pol}_{G,slow}(\mathfrak{g}, T^*M)$ and is of the form

$$\sum_{I,J,K,\,l \leqq d} e^{-\|f_1\|^2} t^l P_I(X) \Psi_J(f) e^{-t\|Q\|^2} (df_1)_K,$$

where P_I and Ψ_J are polynomials. Furthermore, for $Y \in \mathfrak{z}$, the coefficient of t^d in $\chi_{\{1,\dots,2d\}}(t,Y)$ is equal to

$$e^{-t\|Q\|^2} (-2)^d \det\nolimits_{\mathfrak{q}}^{\frac{1}{2}}(Y) u(Y). \tag{47}$$

The right hand side of Formula (45) is thus equal to

$$\begin{aligned} &\sum_{I,J,K,\,l \leqq d} \int_{G/Z} \int_{T^*M} e^{-\|f_1\|^2} t^l \Psi_J(f) (df_1)_K \\ &\quad \times \Big(\int_{\mathfrak{z} \times \mathfrak{q}} e^{-t\|Q\|^2} \alpha(Y+Q) P_I(Y+Q) \phi(g.Y)\, dY\, dQ \Big) d\dot{g}. \end{aligned} \tag{48}$$

Thus if ψ is a test function on $\mathfrak{g}$, we introduce

$$\begin{aligned} v(t) &= \int_{\mathfrak{z} \times \mathfrak{q}} e^{-t\|Q\|^2} \alpha(Y+Q) \phi(Y+Q)\, dY\, dQ \\ &= t^{-d} \int_{\mathfrak{z} \times \mathfrak{q}} e^{-t\|Q\|^2} \alpha\left(Y + t^{-\frac{1}{2}} Q\right) \psi\left(Y + t^{-\frac{1}{2}} Q\right) dY\, dQ. \end{aligned}$$

Let us show that when $t \to \infty$ the form $e^{-\|f_1\|^2} t^d v(t)$ has a limit in the space of rapidly decreasing differential forms on T^*M. For $Y + Q \in \mathfrak{z} \oplus \mathfrak{q}$, we have

$$e^{-\|f_1\|^2} \alpha(Y+Q) = e^{i\langle f_0, Y\rangle} q_0(Y,Q,x,\xi),$$

with $q_0(Y,Q,x,\xi) = \mathrm{Str}(e^{-h_1(x,\xi) + R_1(Y,Q,x,\xi)})$, with $h_1(x,\xi) = \|f_1\|^2 + h(x,\xi)$ and $R_1(Y,Q,x,\xi) = Z_0(Y,x) + Z_0(Q,x) + i\langle f_1, Q\rangle + Z_1(x,\xi)$.

Recall that h_1 is good with respect to the map $f_0 : T^*M \to \mathfrak{z}^*$. As $Z_0(Q,x)$ is linear in Q we have a majoration $\|Z_0(Q,x)\| \leqq \lambda \|Q\|$. By a method similar to that of Lemma 15, we prove the following estimates for the function q_0, where we denote by $N^0_{m,k}$ the seminorms on $C^\infty(\mathfrak{z})$.

(i) There exists an integer P and $c > 0$ and, for every integer $m \geqq 0$ and every compact subset $K \subset \mathfrak{z}$, there exists a constant $C_{m,K}$ such that, for all Q, x, ξ,

$$N^0_{m,K}(q_0(.,Q,x,\xi)) \leqq C_{m,K} e^{\lambda\|Q\|} (1 + \|\xi\|^2)^P.$$

(ii) There exists $a' > 0$, $c' > 0$ and, for every integer $m \geqq 0$ and every compact subset $K \subset \mathfrak{z}$, there exists a constant $C'_{m,K}$ such that

$$N^0_{m,K}(q_0(.,Q,x,\xi)) \leqq C'_{m,K} e^{\lambda\|Q\|} e^{-c'\|\xi\|^2}$$

if $\|f_0(x,\xi)\| < a'\|\xi\|$.

Then, by a proof similar to that of Lemma 13, we conclude that for every integer $d \geqq 0$ and every compact subset $K \subset \mathfrak{z}$, there exists a constant $C''_{d,K}$ such that, for every $Q \in \mathfrak{q}$, for every smooth function β on $\mathfrak{z}$ supported in K,

$$\left\| \int_z e^{-\|f_1\|^2} \alpha(Y+Q)\beta(Y)\,dY \right\| \leqq C''_{d,K}(1+\|\xi\|^2)^{-d} e^{\lambda\|Q\|} N^0_{2(d+P),K}(\beta).$$

With these estimates, we see that

$$e^{-\|f_1\|^2} t^d v(t) = \int_{\mathfrak{z}\times\mathfrak{q}} e^{-\|f_1\|^2} \alpha(Y+t^{-\frac{1}{2}}Q)\psi(Y+t^{-\frac{1}{2}}Q) e^{-\|Q\|^2}\,dY\,dQ$$

has a limit in the space of rapidly decreasing differential forms on T^*M, when $t \to \infty$. Furthermore, the limit is

$$e^{-\|f_1\|^2} \int_{\mathfrak{z}} \alpha(Y)\psi(Y) e^{-\|Q\|^2}\,dY\,dQ = e^{-\|f_1\|^2} \pi^d \int_{\mathfrak{z}} \alpha(Y)\psi(Y)\,dY.$$

As $\chi_{\{1,\dots,2d\}}(t,X)$ is a polynomial in t of degree $\leqq d$, by Formula (48) it follows that $\int_{\mathfrak{g}} \chi_{\{1,\dots,2d\}}(t,X)\alpha(X)\phi(X)\,dX$ has a limit when $t \to \infty$ and we see that the coefficient of t^d is the only one to contribute to the limit. Thus we obtain

$$\begin{aligned}\lim_{t\to\infty} \int_{\mathfrak{g}} \chi_{\{1,\dots,2d\}}(t,X)\alpha(X)\phi(X)\,dX &= \pi^d \int_{\mathfrak{z}} P_{\{1,\dots,2d\}}(Y)u(Y)\alpha(Y)\phi(Y)\,dY \\ &= (-2\pi)^d \int_{\mathfrak{z}} \det{}_{\mathfrak{q}}^{\frac{1}{2}}(Y)u(Y)\alpha(Y)\phi(Y)\,dY.\end{aligned}$$

We can do similar computations for the derivatives with respect to (x,ξ). Taking the limit in Equation (45) we obtain

$$\begin{aligned}\int_{\mathfrak{g}} \Big(\int_{T^*M} \alpha\Big)(X)\phi(X)\,dX &= w^{-1} \int_{(G/Z)\times T^*M} \int_{\mathfrak{g}} \chi(X)\alpha(X)\phi(X)\,dX \\ &= w^{-1}(-2\pi)^d \int_{G/Z} \Big(\int_{T^*M} \Big(\int_{\mathfrak{z}} \det{}_{\mathfrak{q}}^{\frac{1}{2}}(Y)u(Y)\alpha(Y)\phi(g.Y)\,dY\Big)\Big)d\dot{g}.\end{aligned}$$

By the formula (42), we have

$$\int_{\mathfrak{g}} \Big(\int_{T^*M} \alpha\Big)(X)\phi(X)\,dX = w^{-1} \int_{G/Z} \int_{\mathfrak{z}} \Big(\int_{T^*M} \alpha\Big)(Y)\det{}_{\mathfrak{q}}(Y)\phi(g.Y)\,dY\,d\dot{g},$$

if ϕ is a test function on $\mathfrak{g}$ with support in the tubular neighbourhood $G.U$ of $G.S$. Thus we have proven the equality of generalized functions stated in Lemma 33. □

The next step is to prove the localization formula for the Z-equivariant differential form $\beta(Y) = (-2\pi)^d \det_{\mathfrak{q}}^{-\frac{1}{2}}(Y)u(Y)\alpha(Y)$. As $T^*M(S)$ is mapped into $\mathfrak{z}^*$ by $f^{M,G}$, we have $(-2\pi)^d \det_{\mathfrak{q}}^{-\frac{1}{2}}(Y)u(Y)|_{T^*M(S)} = 1$. Thus the following lemma is the localisation formula for β. It completes the proof of the crucial Lemma 32, for $s = e$.

Lemma 35. *Let* $\beta(Y) = (-2\pi)^d \det_{\mathfrak{q}}^{-\frac{1}{2}}(Y)u(Y)\alpha(Y)$. *On a sufficiently small neighbourhood of* S *in* $\mathfrak{z}$, *we have the equality of generalized functions*

$$\Big(\int_{T^*M} \alpha\Big)(Y) = \int_{T^*M} \beta(Y) = \int_{T^*M(S)} \alpha(Y)\mathrm{Eul}^{-1}_{T^*M/T^*M(S)}(Y).$$

Proof. We adapt the proof of Theorem 25 given in [13], Proposition 25. It relies on the Thom isomorphism in equivariant cohomology. Let N be a tubular neighbourhood of $M(S)$ in M. Then $p^{-1}(N)$ is a tubular neighbourhood of $T^*M(S)$ in T^*M, isomorphic with the normal bundle of $T^*M(S)$ in T^*M. The first step is to replace β with a Z-equivariant form β' on T^*M supported in $p^{-1}(N)$. For this purpose, we introduce the one-form τ on M which corresponds to the vector field S_M by means of the Riemannian metric, that is

$$\tau(\xi) = (S_M, \xi) \quad \text{for } \xi \in TM.$$

As τ is Z-invariant, its coboundary $d_{\mathfrak{z}}\tau$ is Z-equivariant; it is given by $(d_{\mathfrak{z}}\tau)(Y) = d\tau - (S_M, Y_M)$ for $Y \in \mathfrak{z}$. Let $a(t)$ be a smooth function on $\mathbb{R}$ such that $a(0) = 1$. Let $b(t) = (1 - a(t))/t$. We define the Z-equivariant form $a(d_{\mathfrak{z}}\tau)(Y) = a(d\tau - (S_M, Y_M))$ by means of the Taylor series of a at the point $t_0 = -(S_M(x), Y_M(x))$, for $x \in M$. We define $b(d_{\mathfrak{z}}\tau)$ in the same way. Let

$$\beta' = p^*(a(d_{\mathfrak{z}}\tau)) \wedge \beta.$$

A straightforward computation gives

$$\beta = \beta' + d_{\mathfrak{z}}\beta'',$$

where the form $\beta'' = p^*(\tau \wedge b(d_{\mathfrak{z}}\tau)) \wedge \beta$ is rapidly decreasing in $\mathfrak{z}$-mean. Therefore we have the equality of generalized functions on $\mathfrak{z}$:

$$\int_{T^*M} \beta(Y) = \int_{T^*M} \beta'(Y).$$

The condition $a(0) = 1$ implies that the forms $\beta(Y)$ and $\beta'(Y)$ have the same restriction to $T^*M(S)$. Moreover, we may choose the neighbourhood $U \subset \mathfrak{z}$ of S and $\varepsilon > 0$ such that $(S_M(x), Y_M(x)) \geqq \varepsilon$ for $Y \in U$ and for all $x \in M \backslash N$. If we choose a with support contained in $]-\varepsilon, \varepsilon[$, then the form $a(d_{\mathfrak{z}}\tau)(Y)$ is supported in N when Y remains in U. Thus $\beta'(Y)$ is supported in $p^{-1}(N)$ for $Y \in U$.

We identify N with the normal bundle $\mathcal{N}$ of $M(S)$ in M. Recall that the projection $T^*M(S) \to M(S)$ is denoted by p_S. By means of the metric and a G-invariant connection on TM, we identify $p^{-1}(N)$ with the vector bundle $p_S^*(\mathcal{N} \oplus \mathcal{N}) \to T^*M(S)$. We consider on the bundle $p_S^*(\mathcal{N} \oplus \mathcal{N})$ the quotient orientation deduced from the symplectic orientation of the total space of T^*M and that of $T^*M(S)$, as in Lemma 28. Let $v(Y)$ be the Thom form of the oriented bundle $p_S^*(\mathcal{N}\oplus\mathcal{N})$. Let $\delta(t)$ be the homothety on the fibers of the bundle $p_S^*(\mathcal{N}\oplus\mathcal{N})$. Then, by the transgression formula for the Thom form (7), we have for every $t \geqq 0$

$$\delta(t)^*(v)\mathrm{Eul}^{-1}_{\mathcal{N}(T^*M,\,T^*M(S))}\beta' - \beta' = d_{\mathfrak{z}}(\gamma_t)$$

where $\gamma_t(Y)$ is rapidly decreasing in $\mathfrak{z}$ -mean. Thus for $Y \in U$,

$$\int_{p_S^*(\mathcal{N}\oplus\mathcal{N})} \beta'(Y) = \int_{p_S^*(\mathcal{N}\oplus\mathcal{N})} \delta(t)^*(v(Y))\mathrm{Eul}^{-1}_{\mathcal{N}(T^*M,\,T^*M(S))}(Y)\beta'(Y).$$

Applying the inverse homothety $\delta(t^{-1})$ we obtain, as $\mathrm{Eul}^{-1}_{\mathcal{N}(T^*M,\,T^*M(S))}(Y)$ is a form on the basis $M(S)$,

$$\int_{p_S^*(\mathcal{N}\oplus\mathcal{N})} \beta'(Y) = \int_{p_S^*(\mathcal{N}\oplus\mathcal{N})} v(Y)\mathrm{Eul}^{-1}_{\mathcal{N}(T^*M,\,T^*M(S))}(Y)\delta(t^{-1})^*(\beta'(Y)).$$

We let t tend to $+\infty$. Then $\delta(t^{-1})^*(\beta'(Y))$ tends to the restriction $\beta'(Y)|_{T^*M(S)} = \beta(Y)|_{T^*M(S)}$. As the integral on the fiber of the Thom form is 1, the lemma will be proven if we can take the limit $t \to \infty$ under the integral. Thus, changing to $\varepsilon = t^{-1}$, we must show that the form $v(Y)\delta(\varepsilon)^*(\beta'(Y))$ is rapidly decreasing in $\mathfrak{z}$-mean uniformly with respect to $\varepsilon \geqq 0$. Let us trivialize locally T^*M with coordinates $(x_0, \xi_1', \xi_0, \xi_1)$ where (x_0, ξ_1') are coordinates on $N \simeq \mathcal{N}$ and $\xi = \xi_0 + \xi_1$ is the decomposition according to $T^*_{x_0}M(S) \oplus \mathcal{N}_{x_0}$. Then $v(Y)$ is of the form $e^{-\|\xi_1\|^2 - \|\xi_1'\|^2} \sum P_I(Y)\Psi_J(\xi_1', \xi_1)(d\xi_1')_I(d\xi_1)_J$ where the P_I are polynomials in Y and Ψ_J are polynomials in (ξ_1', ξ_1). Therefore, if ψ is a test function on $\mathfrak{z}$, we have

$$\begin{aligned}&\sum_{\mathfrak{z}} v(Y)\delta(\varepsilon)^*\beta'(Y)\psi(Y)\,dY\\ &\quad = e^{-\|\xi_1\|^2 - \|\xi_1'\|^2}\sum \Psi_J(\xi_1', \xi_1)(d\xi_1')_I(d\xi_1)_J\delta(\varepsilon)^*\Big(\int_{\mathfrak{z}} \beta'(Y)P_I(Y)\psi(Y)\,dY\Big).\end{aligned}$$

There exist constants $C_{p,K}$ such that $\| \int_{\mathfrak{z}} \beta'(Y)P_I(Y)\psi(Y)\,dY\| \leqq C_{p,K}(1 + \|\xi\|^2)^{-p}N_K^0(P_I\psi)$. Then for any $\varepsilon \geqq 0$, we have $\|\delta(\varepsilon)^* \int_{\mathfrak{z}} \beta'(Y)P_I(Y)\psi(Y)\,dY\| \leqq C_{p,K}(1 + \|\xi_0\|^2 + \varepsilon^2\|\xi_1\|^2)^{-p}N_K^0(P_I\psi)$. This proves our claim. □

We have thus proven the case $s = e$ of Theorem 30. The general case is similar, with $M(s)$ in place of M and $G(s)$ in place of G. □

5.3. *The wave front set of the cohomological index*

The purpose of this section is to give some information about the wave front set of the generalized function $\mathrm{index}_c(\sigma)$ on G. For this we will make use of the following refinement of Lemma 13. We take the notations of Lemma 13.

Lemma 36. *Let $\mathscr{C}$ be a closed cone in $M_0 \times V \backslash 0$ such that $f^{-1}(0) \cap \mathscr{C} = \emptyset$. Let $q(X, x, \xi)$ be a smooth F-valued function on $\mathfrak{g} \times M_0 \times V$ which satisfies Condition* (1) *of Lemma* 13 *and the following condition*:

(2) *For every $(x_0, \xi_0) \notin \mathscr{C}$, $\xi \neq 0$, there exists $c > 0$ and a conic neighbourhood $\Lambda(x_0, \xi_0)$ of (x_0, ξ_0) in $M_0 \times V$ and, for every integer $m \geqq 0$ and every compact subset $K \subset \mathfrak{g}$, there exists a constant $C'_{m,K}$ such that*

$$N_{m,K}(q(\,.\,,x,\xi)) \leqq C'_{m,K} e^{-c\|\xi\|^2} \quad \text{if } (x,\xi) \in \Gamma(x_0,\xi_0). \tag{49}$$

Let $\alpha(X, x, \xi)) = e^{i\langle f(x,\xi),X\rangle} q(X, x, \xi)$. Let $f_0 \notin f(\mathscr{C})$, $f_0 \neq 0$. Let $M_1 \subset M_0$ be a compact subset. Then there exists a conic neighbourhood $\Gamma(f_0)$ of f_0 in $\mathfrak{g}^*$ and for every integer d and for every compact subset $K \subset \mathfrak{g}$, there exists a constant $C''_{d,K}$ such that, for every $\phi \in C^\infty(\mathfrak{g})$ with support in K, for all $\rho \in \Gamma(f_0)$ and $(x, \xi) \in M_1 \times V$,

$$\left\| \int_{\mathfrak{g}} e^{-i\langle\rho,X\rangle} \alpha(X,x,\xi)\phi(X)\,dX \right\| \leqq C''_{d,K}(1+\|\rho\|^2)^{-d}(1+\|\xi\|^2)^{-d} N_{4(d+P),K}(\phi). \tag{50}$$

Proof. Let $\phi \in C_c^\infty(\mathfrak{g})$. For $\rho \in \mathfrak{g}^*$ and $(x, \xi) \in M_0 \times V$, let

$$v(\rho,x,\xi) = \int_{\mathfrak{g}} e^{-i\langle\rho,X\rangle} \alpha(X,x,\xi)\phi(X)\,dX.$$

Then $v(\rho, x, \xi)$ is the value at the point $\rho - f(x, \xi)$ of the Fourier transform of the function $X \mapsto q(\rho, x, \xi)\phi(X)$ on $\mathfrak{g}$. Therefore, for every compact subset $K \subset \mathfrak{g}$ and for every integer $m \geqq 0$, there exists a constant $C_1 = C_1(m, K) > 0$ such that, if ϕ is supported in K,

$$\begin{aligned} \|v(\rho,x,\xi)\| &\leqq C_1(1+\|\rho - f(x,\xi)\|^2)^{-m} N_{2m,K}(q(\,.\,,x,\xi)\phi) \\ &\leqq 2^m C_1(1+\|\rho - f(x,\xi)\|^2)^{-m} N_{2m,K}(q(\,.\,,x,\xi)) N_{2m,K}(\phi). \end{aligned} \tag{51}$$

Let $(x_0, \xi_0) \in M_0 \times V$. Assume first that $(x_0, \xi_0) \in \mathscr{C}$. Then $(x_0, \xi_0) \notin f^{-1}(0)$, therefore there exists a conic neighbourhood $\Gamma(f_0)$ of f_0 in $\mathfrak{g}^*$ and a conic neighbourhood $\Delta(x_0, \xi_0)$ of (x_0, ξ_0) in $M_0 \times V$ such that $\|f(x,\xi)\| \geqq a\|\xi\|$ for $(x, \xi) \in \Delta(x_0, \xi_0)$ for some $a > 0$ and such that

$$\Gamma(f_0) \cap f(\Delta(x_0,\xi_0)) = \{0\}.$$

Assuming that these neighbourhoods are closed, we can find a constant $a' > 0$ such that, for $\rho \in \Gamma(f_0)$ and $(x, \xi) \in \Delta(x_0, \xi_0)$,

$$\|\rho - f(x,\xi)\|^2 \geqq a'(\|\rho\|^2 + \|f(x,\xi)\|^2) \geqq a'(\|\rho\|^2 + a^2\|\xi\|^2),$$

hence, assuming $a \leqq 1$, we have

$$\|\rho - f(x,\xi)\|^2 \geqq \lambda(\|\rho\|^2 + \|\xi\|^2),$$

for $\lambda = a'a^2$ which we may assume $\leqq 1$. Then for all u, v we have $(1+\lambda(u^2+v^2))^2 \geqq \lambda^2(1+u^2)(1+v^2)$. Thus we set $m = 2(d+P)$ in (51). We have

$$(1+\|\rho - f(x,\xi)\|^2)^{-2(d+P)} \leqq \lambda^{-2(d+P)}(1+\|\rho\|^2)^{-(d+P)}(1+\|\xi\|^2)^{-(d+P)},$$
$$N_{4(d+P),K}(q(.,x,\xi)) \leqq C_{4(d+P),K}(1+\|\xi\|^2)^P,$$

hence we obtain (50) for $(x,\xi) \in \Delta(x_0,\xi_0)$ and $\rho \in \Gamma(f_0)$.

In the other case, when (x_0,ξ_0) is not in $\mathscr{C}$, we set $m = d$ in (51), and we use Peetre inequality

$$(1+\|\rho - f(x,\xi)\|^2)^{-d} \leqq 2^d(1+\|\rho\|^2)^{-d}(1+\|f(x,\xi)\|^2)^d.$$

We consider a conic neighbourhood $\Gamma(x_0,\xi_0)$ as in condition (2), such that

$$N_{2d,K}(q(.,x,\xi)) \leqq C'_{2d,K}e^{-c\|\xi\|^2}$$

for $(x,\xi) \in \Gamma(x_0,\xi_0)$. As M_1 is compact there is a constant $\lambda > 0$ such that $\|f(x,\xi)\| \leqq \lambda\|\xi\|$ for $(x,\xi) \in M_1 \times V$. Then we obtain (50) for $(x,\xi) \in \Gamma(x_0,\xi_0)$, $x \in M_1$ and any ρ. As M_1 is compact, this completes the proof. □

Let $\sigma : p^*\mathscr{E}^+ \to p^*\mathscr{E}^-$ be a symbol on M. We define the closed conic subset $\mathscr{C}(\sigma) \subset T^*M\backslash 0$ by

Definition 37. *$\mathscr{C}(\sigma)$ is the complement in $T^*M\backslash 0$ of the set of elements (x_0,ξ_0) which satisfy the following condition: there exist a conic neighbourhood $\Gamma(x_0,\xi_0)$ of (x_0,ξ_0) in $T^*M\backslash 0$, $r > 0$ and $c > 0$ such that*

$$v(\sigma)(x,\xi)^2 \geqq c\|\xi\|^2 I_{\mathscr{E}_x}$$

for every $(x,\xi) \in \Gamma(x_0,\xi_0)$ with $\|\xi\| \geqq r$.

Thus a symbol σ is G-transversally good if and only if $\mathscr{C}(\sigma)$ is disjoint from $T^*_G M$. If $\sigma(x,\xi)$ is positively homogeneous of degree $m \geqq 1$ for large ξ, then clearly $\mathscr{C}(\sigma)$ is the set of $(x,\xi) \in T^*M\backslash 0$ such that $\sigma(x,t\xi)$ is not invertible for large $t > 0$. In particular if $\sigma(x,\xi)$ coincides for large ξ with the principal symbol of a pseudodifferential operator P of order $m \geqq 1$, then $\mathscr{C}(\sigma) = \operatorname{Char} P$.

Proposition 38. *Let M be a compact manifold and G a compact Lie group acting on M. Let σ be a G-transversally good symbol on M. Then the generalized function $\mathrm{index}_c(\sigma)$ has its wave front set at $s \in G$ contained in $f^{M(s),G(s)}(\mathscr{C}(\sigma) \cap T^*M(s)) \subset \mathfrak{g}(s)^* \subset \mathfrak{g}^*$.*

Proof. We consider first $s = e$. In a neighbourhood of the origin we have $\mathrm{index}_c(\sigma)(\exp X) = \theta_e(X)$, where

$$\theta_e(X) = \int_{T^*M} (2i\pi)^{-\dim M} I^{\omega}(e,\sigma)(X).$$

Thus we study the wave front set at $0 \in \mathfrak{g}$ of the generalized function θ_e. Let $\tilde{q}(X,x,\xi)$ be the form given by

$$I^{\omega}(e,\sigma)(X) = \mathrm{ch}\,(\mathbb{A}^{\omega}(\sigma))(X)J(M,X)^{-1} = e^{i\langle f^{M,G}(x,\xi),X\rangle}\tilde{q}(X,x,\xi).$$

Thus

$$\tilde{q}(X,x,\xi) = \mathrm{Str}\,(q(X,x,\xi))J(M,X)^{-1},$$

where $q(X,x,\xi)$ has been computed in the proof of Theorem 16. By arguments similar to Lemma 15 we see that the form q satisfies locally the conditions of Lemma 36. Let $f_0 \in \mathfrak{g}^*$ such that $f_0 \notin f^{M,G}(\mathscr{C}(\sigma))$. By Lemma 36 and the compacity of M, we conclude that f_0 has a conic neighbourhood $\Gamma(f_0) \subset \mathfrak{g}^*$ such that for any $\phi \in C^{\infty}(\mathfrak{g})$ with small support, for any integer $d > 0$ there exists a constant C_d such that, for $\sigma \in \Gamma(f_0)$ and for all $(x,\xi) \in T^*M$.

$$\left\| \int_{\mathfrak{g}} e^{-i\langle\rho,X\rangle} I^{\omega}(e,\sigma)(X)(x,\xi)\phi(X)\,dX \right\| \leqq C_d(1+\|\rho\|^2)^{-d}(1+\|\xi\|^2)^{-d}.$$

Clearly it follows that f_0 does not belong to the wave front set at 0 of θ.

Let $s \in G$. We identify $\mathfrak{g}(s)^*$ with a subspace of $\mathfrak{g}^*$. From the definition of the topological index by descent, it follows that the wave front set of $\mathrm{index}_c(\sigma)$ at s coincides with the wave front set at $Y = 0$ of the generalized function θ_s on $\mathfrak{g}(s)$ defined in (39). This is computed as in the case $s = e$. □

Remark 39. In particular, let M be a $G \times H$-manifold, where H is a second compact Lie group, let σ be a G-transversally elliptic symbol on M and assume that σ is also H-equivariant. Then the $G\times H$-equivariant cohomological index of σ is a generalized function on $G\times H$ which is in fact smooth with respect to the H-variable, as its wave front set is contained in $T^*_H(G\times H) = G\times\mathfrak{g}^*\times H\times\{0\}$. This property is easily seen on the definition.

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