The Choquet boundary of an operator system

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joint work with Matthew Kennedy

This is dedicated to the memory of William B. Arveson

Two recent surveys of Bill's work in JOT:

- K.R. Davidson, *The mathematical legacy of William Arveson*, J. Operator Theory 68 (2012), 307–334.
- M. Izumi, *E_0-semigroups: around and beyond Arveson's work*, J. Operator Theory 68 (2012), 335–363.

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Corollary (Generalized von Neumann inequality)

If $[p_{ij}]$ is a matrix of polynomials, and $\|T\| \leq 1$, then

$$\big\| \big[p_{ij}(T) \big] \big\| \leq \sup_{|z| \leq 1} \big\| \big[p_{ij}(z) \big] \big\|.$$

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Hence this can be considered as a study of representations of the disk algebra $\mathrm{A}(\mathbb{D})$.

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- Operator algebra $\mathcal{A}$: unital subalgebra of a C*-algebra $\mathrm{C}^*(\mathcal{A})$.
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- The role of completely positive and completely bounded maps.

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$$\varphi_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{(n)})$$

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Say φ is completely bounded (c.b.) if

$$\|\varphi\|_{cb} = \sup_{n\geq 1} \|\varphi_n\| < \infty.$$

Say φ is completely contractive (c.c.) if $\|\varphi\|_{cb} \leq 1$.

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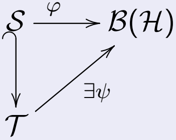
Theorem (Arveson's Extension Theorem)

If $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is c.p. and $\mathcal{S} \subset \mathcal{T}$, then there is a c.p. map $\psi : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ s.t. $\psi|_{\mathcal{S}} = \varphi$. i.e. $\mathcal{B}(\mathcal{H})$ is injective.



- A dilation of a c.c. representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a c.c. representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K} = \mathcal{K}_- \oplus \mathcal{H} \oplus \mathcal{K}_+$, and

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- Note that if $\sigma \succ \rho$, then $\tilde{\sigma} \succ \tilde{\rho}$.
  But $\psi \succ \tilde{\rho}$ may not be multiplicative on $\mathcal{A}$.

Theorem (Arveson's Dilation Theorem)

Let $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation. TFAE

1. ρ *is c.c.*
2. $\tilde{\rho}$ *is c.p.*
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- boundary representations
- the C*-envelope

Bill was able to verify this in many concrete examples. See also Subalgebras of C*-algebras II, Acta Math. **128** (1972), 271–308.

A u.c.p. map $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ or a c.c. repn. $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ has the unique extension property (u.e.p) if

1. φ has a unique u.c.p. extension to $\mathrm{C}^*(\mathcal{S})$ (or $\mathrm{C}^*(\mathcal{A})$)
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It is a boundary representation if it has u.e.p. and

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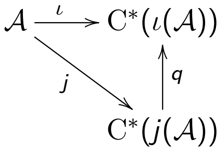
The boundary representations form the Choquet boundary of $\mathcal{S}$.

The C*-envelope of $\mathcal{A}$ is a pair $(\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}), \iota)$ where $\iota : \mathcal{A} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ is comp. isom. iso., $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) = \mathrm{C}^*(\iota(\mathcal{A}))$, with universal property: if $j : \mathcal{A} \to \mathfrak{B} = \mathrm{C}^*(j(\mathcal{A}))$ comp. isom. iso. then $\exists q : \mathfrak{B} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ $*$-homomorphism s.t. $q\,j = \iota$.



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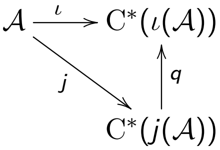

If there are sufficiently many boundary representations $\{\pi_\lambda\}$
to completely norm $\mathcal{S}$, let $\pi = \bigoplus \pi_\lambda$. Then

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{S}) = \mathrm{C}^*(\pi(\mathcal{S})).$$

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Muhly-Solel (1998) gave a homological characterization of boundary representations.

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Muhly-Solel result says: a repn. has u.e.p. $\iff$
it is an extremal extension and an extremal coextension.

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Theorem (Arveson (JAMS 2008))

If $\mathcal{S}$ is separable, then there are sufficiently many boundary representations.

Our approach

- We give a dilation theory proof of the existence of boundary representations.
- It works in complete generality.
- The argument is conceptual and natural.

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If φ is maximal at every (s, x), then φ is maximal.

Key Lemma

If φ is pure, and $(s_0, x_0) \in \mathcal{S} \times \mathcal{H}$, then there is a pure dilation $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$ s.t. $\psi \succ \varphi$ and ψ is maximal at (s_0, x_0).

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- $\{\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi\}$ is BW-compact.
  Hence $\exists \psi$ s.t. $\psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta$ with η maximal.

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If φ is pure, and $(s_0, x_0) \in \mathcal{S} \times \mathcal{H}$, then there is a pure dilation $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C})$ s.t. $\psi \succ \varphi$ and ψ is maximal at (s_0, x_0).

- If $\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$, then compression to $\text{span}\{\mathcal{H}, \psi(s_0)x_0\}$ has same norm at (s_0, x_0).
- $\{\psi : \mathcal{S} \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi\}$ is BW-compact.
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- Delicate argument to show that ψ_0 is pure.

Theorem 1

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- routine transfinite induction to obtain dilation maximal at every pair (s, x)
- if S is separable and $\dim \mathcal{H} < \infty$, then can produce the maximal dilation as limit of sequence of finite dim. maps.

Theorem 2

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- Dilate it to a boundary repn. σ of $\mathcal{M}_n(\mathcal{S})$ by Theorem 1. Then $\sigma \simeq \pi^{(n)}$, where π is irreducible repn. of $\mathrm{C}^*(\mathcal{S})$.
- If φ is u.c.p. dilation of $\pi|_{\mathcal{S}}$, then $\varphi^{(n)}$ dilates $\sigma|_{\mathcal{M}_n(\mathcal{S})}$. Hence $\varphi = \pi$. So π is the desired boundary repn. (This is easy direction of a result of Hopenwasser.)

Second method to get sufficiently many boundary repns.
A matrix state is a u.c.p. map of $\mathcal{S}$ into $\mathcal{M}_n$.

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Theorem

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- Finite dimensional compressions of a faithful repn. of $\mathrm{C}^*(\mathcal{S})$ completely norm $\mathcal{S}$. So matrix states completely norm $\mathcal{S}$.
- The collection of all matrix states $(S_n(\mathcal{S}))_{n\geq 1}$ is C*-convex: If $\gamma_j \in \mathcal{M}_{n_j,n}$, $\sum_{j=1}^k \gamma_j^* \gamma_j = I_n$ and $\psi_j \in S_{n_j}(\mathcal{S})$, then

$$\psi = \sum_{j=1}^{k} \gamma_j^* \psi_j \gamma_j \in S_n(\mathcal{S}).$$

Can define C*-convex hull.

There is a notion of C*-extreme point of a C*-convex set.

Farenick (2000) shows that the C*-extreme points of $(S_n(\mathcal{S}))_{n\geq 1}$ coincide with the pure matrix states.

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Corollary

The C*-envelope of every operator system and every unital operator algebra is obtained from a direct sum of boundary representations.

Where does this get us?

- Over four decades, we developed many techniques to get our hands on the C*-envelope of an operator algebra without using boundary representations.

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- The Choquet boundary, peak points and representing measures play a central role in the study of function algebras.
- Perhaps now, we can more diligently pursue the use of boundary representations in non-commutative dilation theory. This was central to Arveson's vision of the subject.

The end.