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The Chow ring of the Cayley plane

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Abstract

We give a full description of the Chow ring of the complex Cayley plane $\mathbb{OP}^2$. For this, we describe explicitly the most interesting of its Schubert varieties and compute their intersection products. Translating our results into the Borel presentation, i.e. in terms of Weyl group invariants, we are able to compute the degree of the variety of reductions Y_8 introduced by the current authors in arXiv: math.AG/0306328.

1. Introduction

In this paper we give a detailed description of the Chow ring of the complex Cayley plane $X_8 = \mathbb{OP}^2$, the fourth Severi variety (not to be confused with the real Cayley plane F_4/Spin_9, the real part of $\mathbb{OP}^2$, which admits a cell decomposition $\mathbb{R}^0 \cup \mathbb{R}^8 \cup \mathbb{R}^{16}$ and is topologically much simpler). This is a smooth complex projective variety of dimension 16, homogeneous under the action of the adjoint group of type E_6. It can be described as the closed orbit in the projectivization $\mathbb{P}^{26}$ of the minimal representation of E_6.

The Chow ring of a projective homogeneous variety G/P has been described classically in two different ways.

First, it can be described as a quotient of a ring of invariants. Namely, we have to consider the action of the Weyl group of P on the character ring, take the invariant subring, and mod out by the homogeneous ideal generated by the invariants (of positive degree) of the full Weyl group of G. This is the *Borel presentation* [Bor53].

Second, the Chow ring has a basis given by the *Schubert classes*, the classes of the closures of the B-orbits for some Borel subgroup B of E_6. These varieties are the Schubert varieties. Their intersection products can in principle be computed by using Demazure operators [BGG73]. This is the *Schubert presentation*.

We give a detailed description of the Schubert presentation of the Chow ring $A^*(\mathbb{OP}^2)$ of the Cayley plane. We describe explicitly the most interesting Schubert cycles, after having explained how to understand geometrically a Borel subgroup of E_6. Then we compute the intersection numbers. In the final section, we turn to the Borel presentation and determine the classes of some invariants of the partial Weyl group in terms of Schubert classes, from which we deduce the Chern classes of the normal bundle of $X_8 = \mathbb{OP}^2$ in $\mathbb{P}^{26}$. This allows us to compute the degree of the variety of reductions $Y_8 \subset \mathbb{P}^{272}$ introduced in [IM03], which was the initial motivation for writing this paper.

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2. The Cayley plane

Let **O** denote the normed algebra of (real) octonions (see e.g. [Bae02]), and let $\mathbb{O}$ be its complexification. The space

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & c_2 & x_1 \\ x_2 & \overline{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbb{C}, x_i \in \mathbb{O} \right\} \cong \mathbb{C}^{27}$$

of $\mathbb{O}$-Hermitian matrices of order three is the exceptional simple complex Jordan algebra, for the Jordan multiplication $A \circ B = \frac{1}{2}(AB + BA)$.

The subgroup $SL_3(\mathbb{O})$ of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting of automorphisms preserving the determinant is the adjoint group of type E_6. The Jordan algebra $\mathcal{J}_3(\mathbb{O})$ and its dual are the minimal representations of this group.

The action of E_6 on the projectivization $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6-orbit. These three orbits are the sets of matrices of rank three, two, and one respectively.

The closed orbit, i.e. the (projectivization of) the set of rank-one matrices, is the *Cayley plane*. It can be defined by the quadratic equation

$$X^2 = \operatorname{trace}(X)X, \quad X \in \mathcal{J}_3(\mathbb{O}),$$

or as the closure of the affine cell

$$\mathbb{OP}^2_1 = \left\{ \begin{pmatrix} 1 & x & y \\ \overline{x} & x\overline{x} & y\overline{x} \\ \overline{y} & x\overline{y} & y\overline{y} \end{pmatrix}, \quad x, y \in \mathbb{O} \right\} \cong \mathbb{C}^{16}.$$

It is also the closure of the two similar cells

$$\mathbb{OP}^2_2 = \left\{ \begin{pmatrix} \overline{u}u & u & vu \\ \overline{u} & 1 & v \\ \overline{uv} & \overline{v} & v\overline{v} \end{pmatrix}, \quad u, v \in \mathbb{O} \right\} \cong \mathbb{C}^{16}$$

and

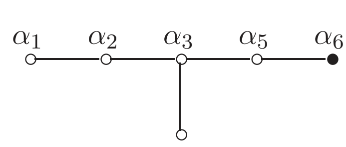
$$\mathbb{OP}^2_3 = \left\{ \begin{pmatrix} \overline{t}t & \overline{s}t & t \\ \overline{t}s & \overline{s}s & s \\ \overline{t} & \overline{s} & 1 \end{pmatrix}, \quad s, t \in \mathbb{O} \right\} \cong \mathbb{C}^{16}.$$

Unlike the ordinary projective plane, these three affine cells do not cover $\mathbb{OP}^2$. The complement of their union is

$$\mathbb{OP}^2_\infty = \left\{ \begin{pmatrix} 0 & x_3 & x_2 \\ \overline{x_3} & 0 & x_1 \\ \overline{x_2} & \overline{x_1} & 0 \end{pmatrix}, \quad \begin{matrix} q(x_1) = q(x_2) = q(x_3) = 0, \\ x_2\overline{x_3} = x_1x_3 = \overline{x_1}x_2 = 0 \end{matrix} \right\},$$

a singular codimension-3 linear section. Here, $q(x) = x\overline{x}$ denotes the non-degenerate quadratic form on $\mathbb{O}$ obtained by complexification of the norm of **O**.

Since the Cayley plane is a closed orbit of E_6, it can also be identified with the quotient of E_6 by a parabolic subgroup, namely the maximal parabolic subgroup defined by the simple root α_6 in the notation below. The semi-simple part of this maximal parabolic subgroup is isomorphic to Spin_{10}.



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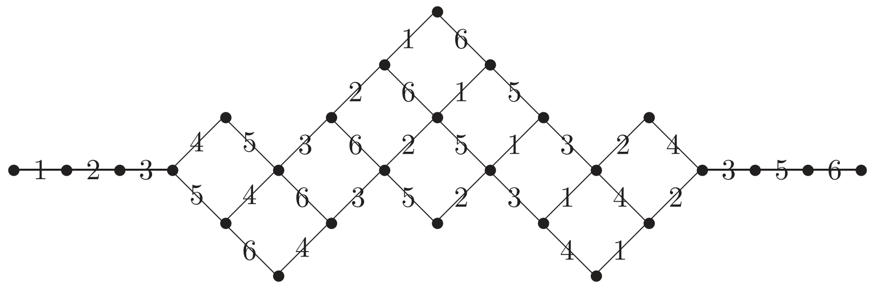

FIGURE 1. Hasse diagram of the Cayley plane $\mathbb{OP}^2$.

The E_6-module $\mathcal{J}_3(\mathbb{O})$ is *minuscule*, meaning that its weights with respect to any maximal torus of E_6 are all conjugate under the Weyl group action. We can easily list these weights as follows. Once we have fixed a set of simple roots of the Lie algebra, we can define the height of any weight ω as the sum of its coefficients when we express ω in the basis of simple roots. Alternatively, this is just the scalar product (ρ, ω), if ρ denotes, as usual, the sum of the fundamental weights, and the scalar product is dual to the Killing form. The highest weight ω_6 of $\mathcal{J}_3(\mathbb{O})$ is the unique weight with maximal height. We can obtain the other weights using the following process: If we have some weight ω of $\mathcal{J}_3(\mathbb{O})$, we express it in the basis of fundamental weights. For each fundamental weight ω_i on which the coefficient of ω is positive, we apply the corresponding simple reflection s_i. The result is a weight of $\mathcal{J}_3(\mathbb{O})$ of height smaller than that of ω, and we obtain all the weights in this way. Figure 1 is the result of this process, where the highest weight ω_6 corresponds to the rightmost vertex. We do not write down the weights explicitly, but we keep track of the action of the simple reflections: if we apply s_i to go from a weight to another one, we draw an edge between them, labeled with an i.

3. The Hasse diagram of Schubert cycles

Schubert cycles in $\mathbb{OP}^2$ are indexed by a subset W^0 of the Weyl group W of E_6, the elements of which are minimal-length representatives of the W_0-cosets in W. Here W_0 denotes the Weyl group of the maximal parabolic $P_6 \subset E_6$: it is the subgroup of W generated by the simple reflections $s_1, \dots, s_5$, thus isomorphic to the Weyl group of Spin_{10}.

But W_0 is also the stabilizer in W of the weight ω_6. Therefore, the weights of $\mathcal{J}_3(\mathbb{O})$ are in natural correspondence with the elements of W^0, and we can obtain very explicitly, from Figure 1, the elements of W^0. Indeed, choose any vertex of the diagram, and any chain of minimal length joining this vertex to the rightmost one. Let $i_1, \dots, i_k$ be the consecutive labels on the edges of this chain; then $s_{i_1} \cdots s_{i_k}$ is a minimal decomposition of the corresponding elements of W^0, and every such decomposition is obtained in this way.

For any $w \in W^0$, denote by σ_w the corresponding Schubert cycle of $\mathbb{OP}^2$. This cycle σ_w belongs to $A^{l(w)}(\mathbb{OP}^2)$, where $l(w)$ denotes the length of w. We have just seen that this length is equal to the distance of the point corresponding to w in Figure 1, to the rightmost vertex. In particular, the dimension of $A^k(\mathbb{OP}^2)$ is equal to 1 for $0 \leqslant k \leqslant 3$, to 2 for $4 \leqslant k \leqslant 7$, and to 3 for $k = 8$ (and by duality, this dimension is of course unaltered when k is changed into $16 - k$).

The degree of each Schubert class can be deduced from the Pieri formula, which is particularly simple in the minuscule case. Indeed, we have [Hil82, ch. V, Corollary 3.3, p. 176], if H denotes the

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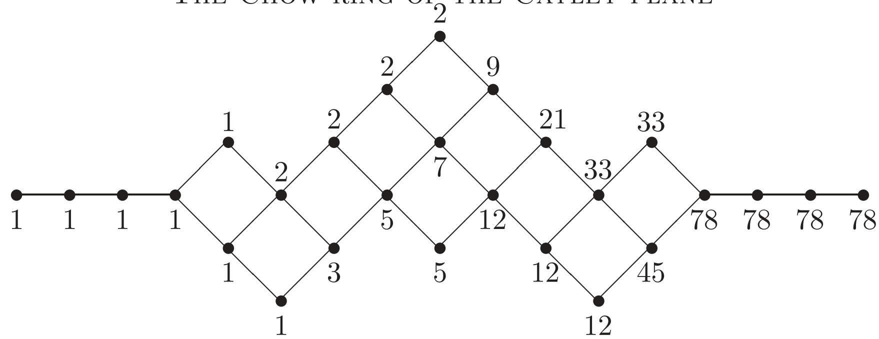


Figure 2. Degrees of the Schubert cycles in the Cayley plane $\mathbb{OP}^2$.

hyperplane class,

$$\sigma_w.H^k = \sum_{l(v)=l(w)+k} \kappa(w,v)\sigma_v,$$

where $\kappa(w,v)$ denotes the number of paths from w to v in Figure 1; that is, the number of chains $w = u_0 \to u_1 \to \cdots \to u_k = v$ in W^0 such that $l(u_i) = l(w) + i$ and $u_{i+1}u_i^{-1}$ is a simple reflection. In particular, the degree of σ_w is just $\kappa(w, w^0)$, where w^0 denotes the longest element of W^0, which corresponds to the leftmost vertex of the diagram. We include these degrees in Figure 2, the Hasse diagram of $\mathbb{OP}^2$. Note that they can very quickly be computed inductively, beginning from the left: the degree of each cycle is the sum of the degrees of the cycles connected to it in one dimension less.

We can already read several interesting pieces of information from Figure 2.

(1) The degree of $\mathbb{OP}^2 \subset \mathbb{P}^{26}$ is 78. This is precisely the dimension of E_6. Is there a natural explanation of this coincidence?

(2) One of the three Schubert varieties of dimension 8 is a quadric. This must be an $\mathbb{O}$-line in $\mathbb{OP}^2$, i.e. a copy of $\mathbb{OP}^1 \simeq \mathbb{Q}^8$. Indeed, E_6 acts transitively on the family of these lines, which is actually parametrized by $\mathbb{OP}^2$ itself. In particular, a Borel subgroup has a fixed point in this family, which must be a Schubert variety.

(3) The Cayley plane contains two families of Schubert cycles which are maximal linear subspaces: a family of projective spaces $\mathbb{P}^4$, which are maximal linear subspaces in some $\mathbb{O}$-line, and a family of projective spaces $\mathbb{P}^5$ which are not contained in any $\mathbb{O}$-line. We thus recover the results of [LM03], from which we also know that these two families of linear spaces in $\mathbb{OP}^2$ are homogeneous. Explicitly, we can describe both types in the following way.

Let $z \in \mathbb{O}$ be a non-zero octonion such that $q(z) = 0$. Denote by $R(z)$ and $L(z)$ the spaces of elements of $\mathbb{O}$ defined as the images of the right and left multiplication by z, respectively. Similarly, if $l \subset \mathbb{O}$ is an isotropic line, denote by $R(l)$ and $L(l)$ the spaces $R(z)$ and $L(z)$, if z is a generator of l. When l varies, $R(l)$ and $L(l)$ describe the two families of maximal isotropic subspaces of $\mathbb{O}$ (this is a geometric version of triality, see e.g. [Cha02]). Consider the sets

$$\left\{ \begin{pmatrix} 1 & x & y \\ \overline{x} & 0 & 0 \\ \overline{y} & 0 & 0 \end{pmatrix}, \ y \in l, x \in L(l) \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ x \in R(l) \right\}.$$

Their closures in $\mathbb{P}\mathcal{J}_3(\mathbb{O})$ are maximal linear subspaces of $\mathbb{OP}^2$ of respective dimensions 5 and 4.

4. What is a Borel subgroup of E_6?

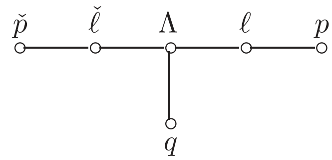
The Schubert varieties in $\mathbb{OP}^2$, by definition, are the closures of the B-orbits, where B denotes a Borel subgroup of E_6. To identify the Schubert varieties geometrically, we need to understand these Borel subgroups better.

The Cayley plane $\mathbb{OP}^2 = E_6/P_6 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$ is one of the E_6-grassmannians, if we mean by this a quotient of E_6 by a maximal parabolic subgroup. It is isomorphic to the dual plane $\check{\mathbb{O}}\mathbb{P}^2 = E_6/P_1 \subset \check{\mathbb{P}}\mathcal{J}_3(\mathbb{O})$, the closed orbit of the projectivized dual representation. By [LM03], we can identify E_6/P_5 and E_6/P_3 to the varieties $G(\mathbb{P}^1, \mathbb{OP}^2)$ and $G(\mathbb{P}^2, \mathbb{OP}^2)$ of projective lines and planes contained in $\mathbb{OP}^2$. Similarly, E_6/P_2 and E_6/P_3 can be interpreted as the varieties of projective lines and planes contained in $\check{\mathbb{O}}\mathbb{P}^2$.

The remaining E_6-grassmannian E_6/P_4 is the adjoint variety E_6^{ad}, the closed orbit in the projectivization $\mathbb{P}\mathfrak{e}_6$ of the adjoint representation. By [LM03] again, E_6/P_3 can be identified to the variety $G(\mathbb{P}^1, E_6^{\mathrm{ad}})$ of projective lines contained in $E_6^{\mathrm{ad}} \subset \mathbb{P}\mathfrak{e}_6$.

Now, a Borel subgroup B in E_6 is the intersection of the maximal parabolic subgroups that it contains, and there is one such group for each simple root. Each of these maximal parabolic subgroups can be seen as a point on an E_6-grassmannian, and the fact that these parabolic subgroups have a Borel subgroup in common means that these points are *incident* in the sense of Tits geometries [Tit95].

Concretely, a point of E_6/P_3 defines a projective plane Π in $\mathbb{OP}^2$, a dual plane $\check{\Pi}$ in $\check{\mathbb{O}}\mathbb{P}^2$, and a line Λ in E_6^{ad}. Choose a point p and a line ℓ in $\mathbb{OP}^2$ such that $p \in \ell \subset \Pi$, choose a point $\check{p}$ and a line $\check{\ell}$ in $\check{\mathbb{O}}\mathbb{P}^2$ such that $\check{p} \in \check{\ell} \subset \check{\Pi}$, and finally a point $q \in \Lambda$.



We call these data a complete E_6-flag. By [Tit95], there is a bijective correspondence between the set of Borel subgroups of E_6 and the set of complete E_6-flags: this is a direct generalization of the usual fact that a Borel subgroup of SL_n is the stabilizer of a unique flag of vector subspaces of $\mathbb{C}^n$.

We will not need this, but to complete the picture let us mention that the correspondence between Π, $\check{\Pi}$ and Λ can be described as follows:

$$\Pi = \bigcap_{z \in \check{\Pi}} (T_z \check{\mathbb{O}}\mathbb{P}^2)^{\perp} = \bigcap_{y \in \Lambda} y\mathcal{J}_3(\mathbb{O}).$$

This description of Borel subgroups will be useful to construct Schubert varieties in $\mathbb{OP}^2$. Indeed, any subvariety of the Cayley plane that can be defined in terms of a complete (or incomplete) E_6-flag must be a finite union of Schubert varieties.

Let us apply this principle in small codimension. The data $\check{p}$, $\check{\ell}$ and $\check{\Pi}$ from our E_6-flag are respectively a point, a line and a plane in $\check{\mathbb{O}}\mathbb{P}^2$. They define special linear sections of $\mathbb{OP}^2$, of respective codimensions 1, 2 and 3. We read from the Hasse diagram that these sections are irreducible Schubert varieties.

Something more interesting happens in codimension-4, since we can read from the Hasse diagram that a well-chosen codimension-4 linear section of $\mathbb{OP}^2$ should split into the union of two Schubert varieties, of degrees 33 and 45. The most degenerate codimension-4 sections must correspond to very special projective spaces $\mathbb{P}^3$ in $\check{\mathbb{O}}\mathbb{P}^2$. We know from [LM03] that $\check{\mathbb{O}}\mathbb{P}^2$ contains a whole family

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of projective spaces $\mathbb{P}^3$, in fact a homogeneous family parametrized by $E_6/P_{2,4}$. In terms of our E_6-flag, that means that a unique member of this family is defined by the pair (q, ℓ).

We can describe explicitly a $\mathbb{P}^3$ in $\mathbb{OP}^2$ in the following way. Choose a non-zero vector $z \in \mathbb{O}$, of zero norm. Then the closure of the set

$$\left\{ \begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x \in L(z) \right\},$$

is a three-dimensional projective space $\mathbb{P}^3_z$ in $\mathbb{OP}^2$. Let us take the orthogonal of this space with respect to the quadratic form $Q(X) = \mathrm{trace}(X^2)$, and cut it with $\mathbb{OP}^2$. We obtain two codimension-4 subvarieties Z_1 and Z_2, respectively the closures of the following affine cells Z_1^0 and Z_2^0:

$$Z_1^0 = \left\{ \begin{pmatrix} 1 & x & y \\ \overline{x} & 0 & \overline{x}y \\ \overline{y} & \overline{y}x & \overline{y}y \end{pmatrix}, \quad x \in L(z),\ y \in \mathbb{O} \right\}, \tag{1}$$

$$Z_2^0 = \left\{ \begin{pmatrix} 0 & u & uv \\ \overline{u} & 1 & v \\ \overline{vu} & \overline{v} & \overline{v}v \end{pmatrix}, \quad u \in L(z),\ v \in \mathbb{O} \right\}. \tag{2}$$

The sum of the degrees of these two varieties is equal to 78. The corresponding cycles are linear combinations of Schubert cycles with non-negative coefficients. But in codimension-4 we have only two such cycles, σ'_4 and σ''_4, of respective degrees 33 and 45. The only possibility is that the cycles $[Z_1]$ and $[Z_2]$ coincide, up to the order, with σ'_4 and σ''_4.

To decide which is which, let us cut Z_1 with $H_1 = \{c_1 = 0\}$.

Lemma 4.1. *The hyperplane section $Y_1 = Z_1 \cap H_1$ has two components $Y_{1,1}$ and $Y_{1,2}$. One of these two components, say $Y_{1,1}$, is the closure of*

$$Y_{1,1}^0 = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ \overline{t} & \overline{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0,\ \overline{s}t = 0 \right\}.$$

It is a cone over the spinor variety $\mathbb{S}_{10} \subset \mathbb{P}^{15}$.

The proof of this lemma follows towards the end of this section.

Recall that the spinor variety $\mathbb{S}_{10}$ is one of the two families of maximal isotropic subspaces of a smooth eight-dimensional quadric. Its appearance is not surprising, since we have seen on the weighted Dynkin diagram of $\mathbb{OP}^2 = E_6/P_6$ that the semi-simple part of P_6 is a copy of Spin_{10}. At a given point of $p \in \mathbb{OP}^2$, the stabilizer P_6 and its subgroup Spin_{10} act on the tangent space, which is isomorphic as a Spin_{10}-module to a half-spin representation, say Δ_+. From [LM03], we know that the family of lines through p that are contained in $\mathbb{OP}^2$ is isomorphic to the spinor variety $\mathbb{S}_{10}$, since it is the closed Spin_{10}-orbit in $\mathbb{P}\Delta_+$.

In particular, to each point p of $\mathbb{OP}^2$ we can associate a subvariety, the union of lines through that point, which is a cone $\mathcal{C}(\mathbb{S}_{10})$ over the spinor variety. This is precisely what $Y_{1,1}$ is. Note that we get a Schubert variety in the Cayley plane. Moreover, since we can choose a Borel subgroup of Spin_{10} inside a Borel subgroup of E_6 contained in P_6, we obtain a whole series of Schubert varieties that are isomorphic to cones over the Schubert subvarieties of $\mathbb{S}_{10}$. These Schubert varieties can be described in terms of incidence relations with an isotropic reference flag which in principle can be deduced from our reference E_6-flag. The Hasse diagram of Schubert varieties in $\mathbb{S}_{10}$ is shown in Figure 3.

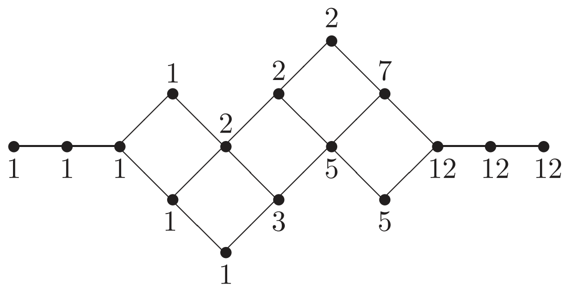

Figure 3. Degrees of the Schubert cycles in $\mathbb{S}_{10}$.

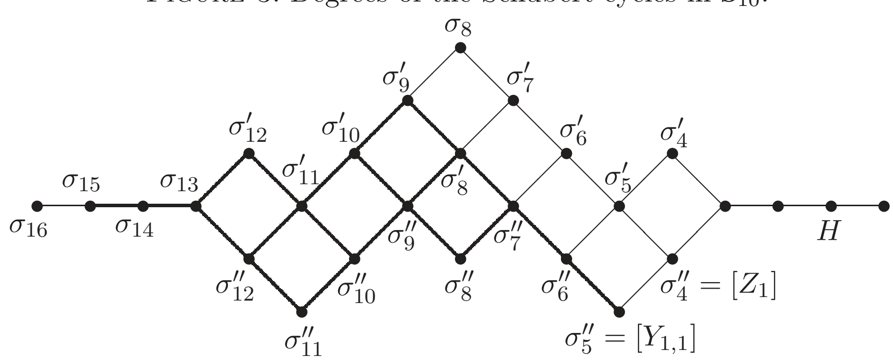

Figure 4. The Schubert cycles in $\mathbb{OP}^2$.

The identification of the cone of lines in $\mathbb{OP}^2$ through some given point, with the spinor variety $\mathbb{S}_{10}$, is not so obvious. Consider the map

$$\nu_2 : \mathbb{O} \oplus \mathbb{O} \to \mathcal{J}_2(\mathbb{O}), \qquad \nu_2(x, y) = \begin{pmatrix} x\overline{x} & x\overline{y} \\ y\overline{x} & y\overline{y} \end{pmatrix}.$$

We want to identify $\mathbb{P}\nu_2^{-1}(0)$ with $\mathbb{S}_{10}$. The following result is due to P. E. Chaput.

Proposition 4.2. *Let $(x, y) \in \nu_2^{-1}(0)$. The image of the tangent map to ν_2 at (x, y) is a five-dimensional subspace of $\mathcal{J}_2(\mathbb{O})$, which is isotropic with respect to the determinantal quadratic form on $\mathcal{J}_2(\mathbb{O})$. Moreover, this induces an isomorphism between $\mathbb{P}\nu_2^{-1}(0)$ and the spinor variety $\mathbb{S}_{10}$.*

In fact, in this way we can obtain the two families of maximal isotropic subspaces in $\mathcal{J}_2(\mathbb{O})$, just by switching the two diagonal coefficients in the definition of ν_2. The spin group Spin_{10} can also be described very nicely.

But let us come back to the Schubert varieties in $\mathbb{S}_{10}$. Taking cones over them, we get Schubert subvarieties that define a subdiagram of the Hasse diagram of $\mathbb{OP}^2$. We draw this subdiagram in thick lines in Figure 4. We also indicate in Figure 4 the indexing of Schubert classes that we use in the sequel, rather than the indexing by the Weyl group.

In principle, we are able to describe any of these Schubert varieties geometrically in terms of our reference E_6-flag.

Proof of Lemma 4.1. First note that Y_1 does not meet the two affine cells $\mathbb{OP}^2_1$ and $\mathbb{OP}^2_2$ (see § 2). Moreover, it is easy to check that $Y_1 \cap \mathbb{OP}^2_\infty$ has dimension at most 10, hence strictly smaller dimension than Y_1. Therefore, Y_1 is the closure of its intersection with $\mathbb{OP}^2_3$, namely

$$Y_1 \cap \mathbb{OP}^2_3 = \left\{ \begin{pmatrix} 0 & \overline{s}t & t \\ \overline{t}s & 0 & s \\ \overline{t} & \overline{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0,\ \overline{s}t \in L(z) \right\}.$$

For a given non-zero s, the product $\overline{s}t$ must belong to $L(z) \cap L(\overline{s})$, the intersection of two maximal isotropic spaces of the same family. In particular, this intersection has even dimension.

Generically, the intersection $L(z) \cap L(\overline{s}) = 0$, and we obtain

$$Y^0_{1,1} = \left\{ \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ \overline{t} & \overline{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0,\ \overline{s}t = 0 \right\} \subset Y_1.$$

We have seven parameters for s, and for each $s \neq 0$, t must belong to $L(s)$, which gives four parameters. In particular, $Y^0_{1,1}$ is irreducible of dimension 11, and its closure is an irreducible component of Y_1.

The intersection $L(z) \cap L(\overline{s})$ has dimension 2 exactly when the line joining z to $\overline{s}$ is isotropic, which means that $\overline{s}$ belongs to the intersection of the quadric $q = 0$ with its tangent hyperplane at z. This gives six parameters for s, and for each s, five parameters for t, which must be contained in the intersection of the quadric with a six-dimensional linear space. Therefore, the closure of

$$Y^0_{1,2} = \left\{ \begin{pmatrix} 0 & \overline{s}t & t \\ \overline{t}s & 0 & s \\ \overline{t} & \overline{s} & 1 \end{pmatrix}, \quad q(s) = q(t) = 0,\ \dim L(z) \cap L(\overline{s}) = 2 \right\}$$

is another component of Y_1.

The remaining possibility is that s be a multiple of z, but the corresponding subset of $\mathbb{OP}^2_3$ has dimension smaller than 11. Hence $Y_1 = Y_{1,1} \cup Y_{1,2}$. □

We conclude that Z_1 has degree 45, while Z_2 has degree 33. Indeed, if Z_1 had degree 33, we would read from the Hasse diagram that its proper hyperplane sections are always irreducible, and we have just verified that this is not the case.

Note that Z_1 and Z_2 look very similar at first sight. Nevertheless, a computation similar to the one we have just done shows that, if we cut Z_2 by the hyperplane $H_2 = \{c_2 = 0\}$, we get an irreducible variety, the difference with Z_1 coming from the fact that we now have to deal with maximal isotropic subspaces which are not on the same family. The difference between Z_1 and Z_2 is therefore just a question of spin.

5. Intersection numbers

We now determine the multiplicative structure of the Chow ring $A^*(\mathbb{OP}^2)$. (For general facts on Chow rings, we refer to [Ful98].) *A priori*, we have several interesting pieces of information on that ring structure. We have already seen in § 2 that the Pieri formula determines combinatorially the product with the hyperplane class. Another important property is that Poincaré duality has a very simple form in terms of Schubert cycles: the basis $(\sigma_w)_{w \in W^0}$ is, up to order, self-dual. More precisely its dual basis is $(\sigma_{w^*})_{w \in W^0}$, where the involution $w \mapsto w^*$ is very simple to define on the Hasse diagram: it is just the symmetry with respect to the vertical line passing through the cycles of middle dimension. Finally, we know from Poincaré duality and general transversality arguments that any *effective* cycle must be a linear combination of Schubert cycles with non-negative coefficients.

This is the information we have on any rational homogeneous space. For what concerns the Cayley plane, we begin with an obvious observation.

Proposition 5.1. *The Chow ring $A^*(\mathbb{OP}^2)$ is generated by the hyperplane class H, the class σ'_4, and the class σ_8 of an $\mathbb{O}$-line.*

More precisely, one can directly read from the Hasse diagram and from the Pieri formula that, as a vector space, the Chow ring is generated by classes of type H^i, $\sigma'_4 H^j$ and $\sigma_8 H^k$. For example, we have the relations

$$H^4 = \sigma'_4 + \sigma''_4, \tag{3}$$

$$\sigma'_4 H^4 = \sigma_8 + 3\sigma'_8 + 2\sigma''_8, \tag{4}$$

$$\sigma''_4 H^4 = \sigma_8 + 4\sigma'_8 + 3\sigma''_8, \tag{5}$$

$$\sigma_8 H^4 = \sigma'_{12} + \sigma''_{12}, \tag{6}$$

$$\sigma'_8 H^4 = 3\sigma'_{12} + 4\sigma''_{12}, \tag{7}$$

$$\sigma''_8 H^4 = 2\sigma'_{12} + 3\sigma''_{12}. \tag{8}$$

As a consequence, the multiplicative structure of the Chow ring will be completely determined once we have computed the intersection products $(\sigma_8)^2, \sigma'_4\sigma_8$ and $(\sigma'_4)^2$. (Note that the Hasse diagram and the Pieri formula can be used to derive relations in dimension 9 and 13, but these relations are not sufficient to determine the whole ring structure.)

Proposition 5.2. *We have the following relations in the Chow ring:*

$$\sigma_8^2 = 1, \tag{9}$$

$$\sigma'_4\sigma_8 = \sigma'_{12}, \tag{10}$$

$$\sigma''_4\sigma_8 = \sigma''_{12}. \tag{11}$$

Proof. Recall that σ_8 is the class of an $\mathbb{O}$-line in $\mathbb{OP}^2$, and that we know that the geometry of these lines is similar to the usual line geometry in $\mathbb{P}^2$: namely, two generic lines meet transversely in one point. This implies immediately that $\sigma_8^2 = 1$.

To compute $\sigma'_4\sigma_8$ and $\sigma''_4\sigma_8$, we cut the Schubert varieties Z_1 and Z_2 introduced in § 4, whose class we know to be σ'_4 and σ''_4, with the $\mathbb{O}$-line L defined in $\mathbb{OP}^2$ by the conditions $x_1 = x_2 = r_3 = 0$. We get transverse intersections

$$Z_1 \cap L = \left\{ \begin{pmatrix} r & y & 0 \\ \overline{y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \in L(z) \right\},$$

$$Z_2 \cap L = \left\{ \begin{pmatrix} 0 & y & 0 \\ \overline{y} & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \in L(z) \right\}.$$

These are two four-dimensional projective spaces $\mathbb{P}^4_1$ and $\mathbb{P}^4_2$ inside $\mathbb{OP}^2$, which look very similar. But there is actually a big difference: $\mathbb{P}^4_1$ is extendable, but $\mathbb{P}^4_2$ is not! Indeed, a $\mathbb{P}^5$ in $\mathbb{OP}^2$ containing $\mathbb{P}^4_1$ or $\mathbb{P}^4_2$ must be of the form, respectively,

$$\left\{ \begin{pmatrix} r & y & s \\ \overline{y} & 0 & 0 \\ \overline{s} & 0 & 0 \end{pmatrix}, \ y \in L(z) \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & y & 0 \\ \overline{y} & r & s \\ 0 & \overline{s} & 0 \end{pmatrix}, \ y \in L(z) \right\},$$

where s describes some line in $\mathbb{O}$. In the second case, the equation $sy = 0$ must be verified identically, and we can take s on the line $\mathbb{C}\overline{z}$: thus $\mathbb{P}^4_2$ is extendable. But in the first case, we need the identity $y\overline{s} = 0$ for all $y \in L(z)$, which would imply that $L(z) \subset R(s)$: this is impossible, and $\mathbb{P}^4_1$ is not extendable. The proposition follows (see the third observation at the end of § 3). ∎

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We now have enough information to complete the multiplication table. First, we know by Poincaré duality that

$$(\sigma_8)^2 = (\sigma_8')^2 = (\sigma_8'')^2 = 1, \tag{12}$$
$$\sigma_8\sigma_8' = \sigma_8'\sigma_8'' = \sigma_8\sigma_8'' = 0, \tag{13}$$
$$\sigma_4'\sigma_{12}' = \sigma_4''\sigma_{12}'' = 1, \tag{14}$$
$$\sigma_4'\sigma_{12}'' = \sigma_4''\sigma_{12}' = 0. \tag{15}$$

Suppose that we have

$$(\sigma_4')^2 = \mu_0\sigma_8 + \mu_1\sigma_8' + \mu_2\sigma_8'',$$
$$(\sigma_4'')^2 = \nu_0\sigma_8 + \nu_1\sigma_8' + \nu_2\sigma_8'',$$
$$\sigma_4'\sigma_4'' = \gamma_0\sigma_8 + \gamma_1\sigma_8' + \gamma_2\sigma_8'',$$

for some coefficients to be determined. Cutting with σ_8, we get $\mu_0 = \nu_0 = 1$. Equations (3), (4) and (5) give the relations

$$\mu_0 + \gamma_0 = 1, \quad \mu_1 + \gamma_1 = 3, \quad \mu_2 + \gamma_2 = 2, \quad \nu_0 + \gamma_0 = 1, \quad \nu_1 + \gamma_1 = 4, \quad \nu_2 + \gamma_2 = 3.$$

In particular, $\gamma_0 = 0$. Now, we compute $(\sigma_4')^2(\sigma_4'')^2$ in two ways to obtain the relation

$$\gamma_1^2 + \gamma_2^2 = \mu_0\nu_0 + \mu_1\nu_1 + \mu_2\nu_2.$$

Eliminating the μ_i and ν_i, we get that $7\gamma_1 + 5\gamma_2 = 19$. But γ_1 and γ_2 are non-negative integers, so the only possibility is that $\gamma_1 = 2$, $\gamma_2 = 1$. Thus

$$(\sigma_4')^2 = \sigma_8 + \sigma_8' + \sigma_8'', \tag{16}$$
$$(\sigma_4'')^2 = \sigma_8 + 2\sigma_8' + 2\sigma_8'', \tag{17}$$
$$\sigma_4'\sigma_4'' = 2\sigma_8' + \sigma_8'' \tag{18}$$

and this easily implies that

$$\sigma_4'\sigma_8' = \sigma_{12}' + 2\sigma_{12}'', \tag{19}$$
$$\sigma_4'\sigma_8'' = \sigma_{12}' + \sigma_{12}'', \tag{20}$$
$$\sigma_4''\sigma_8' = 2\sigma_{12}' + 2\sigma_{12}'', \tag{21}$$
$$\sigma_4''\sigma_8'' = \sigma_{12}' + 2\sigma_{12}''. \tag{22}$$

6. The Borel presentation

We now turn to the Borel presentation of the Chow ring of $\mathbb{OP}^2$. This is the ring isomorphism

$$A^*(\mathbb{OP}^2)_{\mathbb{Q}} \simeq \mathbb{Q}[\mathcal{P}]^{W_0}/\mathbb{Q}[\mathcal{P}]^W_+,$$

where $\mathbb{Q}[\mathcal{P}]^{W_0}$ denotes the ring of W_0-invariant polynomials on the weight lattice, and $\mathbb{Q}[\mathcal{P}]^W_+$ is the ideal of $\mathbb{Q}[\mathcal{P}]^{W_0}$ generated by W-invariants without constant term (see [Bor53, Proposition 27.3], or [BGG73, Theorem 5.5]).

The ring $\mathbb{Q}[\mathcal{P}]^{W_0}$ is easily determined: it is generated by ω_6, and the subring of W_0-invariants in the weight lattice of Spin_{10}. It is therefore the polynomial ring in the elementary symmetric functions $e_{2i} = c_i(\varepsilon_1, \dots, \varepsilon_5)$, $1 \leqslant i \leqslant 4$, and in $e_5 = \varepsilon_1 \cdots \varepsilon_5$.

The invariants of W, the full Weyl group of E_6, are more difficult to determine, although we know their fundamental degrees. But since we know how to compute the intersection products of any two Schubert cycles, we just need to express the W_0-invariants in terms of the Schubert classes. This can be achieved, following [BGG73], by applying suitable difference operators to these invariants.

Since we give a prominent role to the subsystem of E_6 of type D_5, it is natural to choose for the first five simple roots the usual simple roots of D_5, that is, in a Euclidian six-dimensional space with orthonormal basis $\varepsilon_1, \dots, \varepsilon_6$,

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = \varepsilon_4 - \varepsilon_5,$$
$$\alpha_5 = \varepsilon_4 + \varepsilon_5, \quad \alpha_6 = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6.$$

The fundamental weights are given by the dual basis:

$$\begin{aligned}
\omega_1 &= \varepsilon_1 + \frac{1}{\sqrt{3}}\varepsilon_6,\\
\omega_2 &= \varepsilon_1 + \varepsilon_2 + \frac{2}{\sqrt{3}}\varepsilon_6,\\
\omega_3 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \frac{3}{\sqrt{3}}\varepsilon_6,\\
\omega_4 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6,\\
\omega_5 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{5}{\sqrt{3}}\varepsilon_6,\\
\omega_6 &= -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) + \frac{\sqrt{3}}{2}\varepsilon_6.
\end{aligned}$$

The action of the fundamental reflections on the weight lattice is especially simple in the basis $\varepsilon_1, \dots, \varepsilon_5, \alpha_6$. Indeed, s_1, s_2, s_3 and s_4 are just the transpositions (12), (23), (34) and (45). The reflection s_5 affects $\varepsilon_4, \varepsilon_5$ and α_6, which are changed into $-\varepsilon_5, -\varepsilon_4$ and $\alpha_6 + \varepsilon_4 + \varepsilon_5$. Finally, s_6 changes each ε_i into $\varepsilon_i + \alpha_6/2$, and of course α_6 into $-\alpha_6$.

It is then reasonably simple to compute the corresponding divided differences with Maple. We obtain the following proposition.

Proposition 6.1. *The fundamental W_0-invariants are given, in the Chow ring of the Cayley plane, in terms of Schubert cycles by:*

$$e_2 = -\frac{3}{4}H^2, \tag{23}$$
$$e_4 = -\frac{27}{8}\sigma_4' + \frac{21}{8}\sigma_4'', \tag{24}$$
$$e_5 = \frac{3}{16}\sigma_5' - \frac{21}{32}\sigma_5'', \tag{25}$$
$$e_6 = -\frac{27}{16}\sigma_6' + \frac{87}{32}\sigma_6'', \tag{26}$$
$$e_8 = \frac{21}{128}\sigma_8 + \frac{291}{256}\sigma_8' - \frac{519}{256}\sigma_8''. \tag{27}$$

This allows one to compute any product in the Borel presentation of the Chow ring of $\mathbb{OP}^2$.

7. Chern classes of the normal bundle

Let $\mathcal{N}$ denote the normal bundle to the Cayley plane $\mathbb{OP}^2 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$. We want to compute its Chern classes (see e.g. [Ful98]).

First note that the restriction of $\mathcal{J}_3(\mathbb{O})$ to the Levi part $L \simeq \mathrm{Spin}_{10} \times \mathbb{C}^*$ of the parabolic subgroup P_6 of E_6 is

$$\mathcal{J}_3(\mathbb{O})_{|L} \simeq W_{\omega_6} \oplus W_{\omega_5 - \omega_6} \oplus W_{\omega_1 - \omega_6}.$$

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Indeed, there is certainly the line generated by the highest weight vector, which gives a stable line on which L acts through the character ω_6. After ω_6, there is in $\mathcal{J}_3(\mathbb{O})$ a unique highest weight, $\omega_5-\omega_6$, which generates a 16-dimensional half-spin module. Finally, the lowest weight of $\mathcal{J}_3(\mathbb{O})$ is $-\omega_1$, whose highest W_0-conjugate is $\omega_1-\omega_6$ and generates a copy of the natural 10-dimensional representation of Spin_{10}. Since these three modules give $1+16+10=27$ dimensions, we have the full decomposition.

Geometrically, this decomposition of $\mathcal{J}_3(\mathbb{O})$ must be interpreted as follows. We have chosen a point p of $\mathbb{OP}^2$, corresponding to the line $\hat{p}=W_{\omega_6}$. The tangent space to $\mathbb{OP}^2$ at that point is given by the factor $W_{\omega_5-\omega_6}$. (More precisely, only the affine tangent space $\hat{T}$ is a well-defined P_6-submodule of $\mathcal{J}_3(\mathbb{O})$, and it coincides with $W_{\omega_6}\oplus W_{\omega_5-\omega_6}$.) The remaining term $W_{\omega_1-\omega_6}$ corresponds to the normal bundle. To be precise, if $\mathcal{N}_p$ denotes the normal space to $\mathbb{OP}^2$ at p, there is a canonical identification

$$\mathcal{N}_p \simeq \mathcal{H}om(\hat{p},\mathcal{J}_3(\mathbb{O})/\hat{T}) = \mathcal{H}om(W_{\omega_6},W_{\omega_1-\omega_6}).$$

In other words, the normal bundle $\mathcal{N}$ to $\mathbb{OP}^2$ is the homogeneous bundle $\mathcal{E}_{\omega_1-2\omega_6}$ defined by the irreducible P_6-module $W_{\omega_1-2\omega_6}$.

Since $\omega_1=\varepsilon_1+\frac{1}{2}\omega_6$, the weights of the normal bundle are the $\pm\varepsilon_i-\frac{3}{2}\omega_6$, and its Chern class is

$$\begin{aligned} c(\mathcal{N}) &= \prod_{i=1}^{5}(1+\varepsilon_i-\tfrac{3}{2}\omega_6)(1-\varepsilon_i-\tfrac{3}{2}\omega_6) \\ &= \prod_{i=1}^{5}((1+\tfrac{3}{2}H)^2-\varepsilon_i^2) \\ &= \sum_{i=0}^{5}(-1)^i(1+\tfrac{3}{2}H)^{10-2i}e_{2i}, \end{aligned}$$

where $e_{10}=e_5^2$. We know how to express this in terms of Schubert classes, and the result is as follows.

PROPOSITION 7.1. *In terms of Schubert cycles, the Chern classes of the normal bundle to $\mathbb{OP}^2\subset\mathbb{P}\mathcal{J}_3(\mathbb{O})$ are:*

$$\begin{aligned} c_1(\mathcal{N}) &= 15H, \\ c_2(\mathcal{N}) &= 102H^2, \\ c_3(\mathcal{N}) &= 414H^3, \\ c_4(\mathcal{N}) &= 1107\sigma_4'+1113\sigma_4'', \\ c_5(\mathcal{N}) &= 2025\sigma_4'H+2079\sigma_4''H, \\ c_6(\mathcal{N}) &= 5292\sigma_6'+8034\sigma_6'', \\ c_7(\mathcal{N}) &= 4698\sigma_6'H+7218\sigma_6''H, \\ c_8(\mathcal{N}) &= 2751\sigma_8+9786\sigma_8'+7032\sigma_8'', \\ c_9(\mathcal{N}) &= 963\sigma_8H+3438\sigma_8'H+2466\sigma_8''H, \\ c_{10}(\mathcal{N}) &= 153\sigma_8H^2+549\sigma_8'H^2+387\sigma_8''H^2. \end{aligned}$$

Note that, as expected, we get integer coefficients, while the fundamental W^0-invariants are only rational combinations of the Schubert cycles. This is a strong indication that our computations are correct.

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8. The final computation

We shall compute the degree of the variety of reductions Y_8 introduced in [IM03]: we refer to that paper for the definitions, notations, and the proof of the facts we use in this section. This variety Y_8 is a smooth projective variety of dimension 24, embedded in $\mathbb{P}^{272}$. A $\mathbb{P}^1$-bundle Z_8 over Y_8 can be identified with the blow-up of the projected Cayley plane $\overline{X_8}$ in $\mathbb{P}\mathcal{J}_3(\mathbb{O})_0$, the projective space of trace-zero Hermitian matrices of order three, with coefficients in the Cayley octonions.

Let H denote the pull-back to Z_8 of the hyperplane class of $\mathbb{P}\mathcal{J}_3(\mathbb{O})_0$, and E the exceptional divisor of the blow-up. We want to compute

$$\deg Y_8 = H(3H-E)^{24}.$$

We use the fact that the Chow ring of the exceptional divisor $E \subset Z_8$, since it is the projectivization of the normal, is the quotient of the ring $A^*(\mathbb{OP}^2)[e]$ by the relation given by the Chern classes of the normal bundle $\overline{\mathcal{N}}$ of $\overline{X_8}$, namely

$$e^9 + \sum_{i=1}^{9}(-1)^i c_i(\overline{\mathcal{N}})e^{9-i} = 0.$$

The normal bundle $\overline{\mathcal{N}}$ of $\overline{X_8}$ is related to the normal bundle $\mathcal{N}$ of $X_8 = \mathbb{OP}^2$ by an exact sequence $0 \to \mathcal{O}(1) \to \mathcal{N} \to \overline{\mathcal{N}} \to 0$, from which we can compute the Chern classes of $\overline{\mathcal{N}}$:

$$\begin{aligned}
c_1(\overline{\mathcal{N}}) &= 14H,\\
c_2(\overline{\mathcal{N}}) &= 88H^2,\\
c_3(\overline{\mathcal{N}}) &= 326H^3,\\
c_4(\overline{\mathcal{N}}) &= 781\sigma_4' + 787\sigma_4'',\\
c_5(\overline{\mathcal{N}}) &= 1244\sigma_4'H + 1292\sigma_4''H = 2536\sigma_5' + 1292\sigma_5'',\\
c_6(\overline{\mathcal{N}}) &= 2756\sigma_6' + 4206\sigma_6'',\\
c_7(\overline{\mathcal{N}}) &= 1942\sigma_6'H + 3012\sigma_6''H = 1942\sigma_7' + 4954\sigma_7'',\\
c_8(\overline{\mathcal{N}}) &= 809\sigma_8 + 2890\sigma_8' + 2078\sigma_8'',\\
c_9(\overline{\mathcal{N}}) &= 154\sigma_8H + 548\sigma_8'H + 388\sigma_8''H = 702\sigma_9' + 936\sigma_9'',\\
c_{10}(\overline{\mathcal{N}}) &= -\sigma_8H^2 + \sigma_8'H^2 - \sigma_8''H^2 = 0.
\end{aligned}$$

The fact that we get $c_{10}(\overline{\mathcal{N}}) = 0$, which must hold since $\overline{\mathcal{N}}$ has rank nine, is again a strong indication that we have made no mistake.

To complete our computation, we must compute the intersection products $H^{25-i}E^i$ in the Chow ring of Z_8. For $i > 0$, this can be computed on the exceptional divisor; since the restriction of the class E to the exceptional divisor is just the relative hyperplane section, that is, the class e, we have $H^{25-i}E^i = H^{25-i}e^{i-1}$, the later product being computed in $A^*(E)$. We still denoted by H the pull-back of the hyperplane section from $\mathbb{OP}^2$.

Lemma 8.1. *Let $\sigma \in A^{16-k}(\mathbb{OP}^2)$. Then $\sigma e^{8+k} = \sigma s_k(\overline{\mathcal{N}})$, where $s_k(\overline{\mathcal{N}})$ denotes the kth Segre class of the normal bundle $\overline{\mathcal{N}}$. The former product is computed in $A^*(E)$, and the latter in $A^*(\mathbb{OP}^2)$.*

Proof. The proof is by induction, using the relation $e^9 + \sum_{i=1}^{9}(-1)^i c_i(\overline{\mathcal{N}})e^{9-i} = 0$, and the fact that the Segre classes are related to the Chern classes by the formally similar relation $s_k(\overline{\mathcal{N}}) + \sum_{i=1}^{9}(-1)^i c_i(\overline{\mathcal{N}})s_{k-i}(\overline{\mathcal{N}}) = 0$. ∎

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We use the later relation to determine the Segre classes inductively. We obtain

$$\begin{aligned}
s_1(\overline{\mathcal{N}}) &= 14H,\\
s_2(\overline{\mathcal{N}}) &= 108H^2,\\
s_3(\overline{\mathcal{N}}) &= 606H^3,\\
s_4(\overline{\mathcal{N}}) &= 2763\sigma_4' + 2757\sigma_4'',\\
s_5(\overline{\mathcal{N}}) &= 21\,624\sigma_5' + 10\,752\sigma_5'',\\
s_6(\overline{\mathcal{N}}) &= 75\,492\sigma_6' + 112\,602\sigma_6'',\\
s_7(\overline{\mathcal{N}}) &= 240\,534\sigma_7' + 596\,598\sigma_7'',\\
s_8(\overline{\mathcal{N}}) &= 711\,489\sigma_8 + 2462\,397\sigma_8' + 1750\,947\sigma_8'',\\
s_9(\overline{\mathcal{N}}) &= 8768\,196\sigma_9' + 11\,600\,304\sigma_9'',\\
s_{10}(\overline{\mathcal{N}}) &= 53\,127\,900\sigma_{10}' + 30\,193\,704\sigma_{10}'',\\
s_{11}(\overline{\mathcal{N}}) &= 206\,857\,602\sigma_{11}' + 74\,823\,228\sigma_{11}'',\\
s_{12}(\overline{\mathcal{N}}) &= 491\,985\,531\sigma_{12}' + 669\,523\,221\sigma_{12}'',\\
s_{13}(\overline{\mathcal{N}}) &= 2657\,712\,312\sigma_{13},\\
s_{14}(\overline{\mathcal{N}}) &= 5875\,513\,812\sigma_{14},\\
s_{15}(\overline{\mathcal{N}}) &= 12\,591\,161\,406\sigma_{15}.
\end{aligned}$$

This immediately gives the degree of Y_8:

$$\deg Y_8 = 3^{24} + \sum_{k=9}^{24} (-1)^k \binom{24}{k} 3^{24-k} H^{25-k} s_{k-9}(\overline{\mathcal{N}}).$$

THEOREM 8.2. *The degree of the variety of reductions Y_8 is*

$$\deg Y_8 = 1047\,361\,761.$$

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