Archives

Ref.TH.1894-CERN

THE CHRONOS PRINCIPLE

D. Christodoulou

CERN - Geneva

A B S T R A C T

The definition of time is established as a fundamental physical principle. From this principle, in connection with the quantum and relativity principles, the laws of quantum geometrodynamics, and, in the classical limit, those of general relativity theory, are uniquely deduced.

Ref.TH.1894-CERN

27 June 1974

INTRODUCTION

The problem of the canonical quantization of general relativity theory has hitherto been attacked by the following approach : 1) General relativity was brought to a Hamiltonian form by 3+1 dimensional splitting of the four-dimensional geometry of space-time [1),2)]. 2) The initial value equations were considered as equations determining the way that space-time is to be sliced into spacelike hypersurfaces, given the three-dimensional components of the metric functions of the four-geometry at two infinitesimally different values of the 4th co-ordinate [3)]. 3) A correspondence of classical dynamical variables and quantum mechanical operators was used to determine the equations which the quantum mechanical wave function must satisfy [4)]. Since the time did not appear explicitly in these equations, some aspect of the three-dimensional goemetry of space was searched in order to parametrize the evolution [5),6)].

In the preceding work [7)], a new approach was followed, based on the idea that the intrinsic geometry of three-dimensional space is the only directly measurable entity in general relativity theory : the initial value equations were understood as giving the definition of time, once a sequence of three-dimensional geometries is given. The quantum laws were then directly deduced from Feynman's formulation of the quantum principle [8)] as applied to geometrodynamics.

In this paper we take the definition of time, as deduced from general relativity theory in the preceding work, and establish it, in its more abstract form, as one of the fundamental principles of physics. In Section 1, the quantum principle is reformulated and deduced from simpler assumptions. In Section 2 the concept of distance in the space of geometries is investigated, and the "Chronos Principle" is introduced. The relativity principle [9)] is then restated in Section 3. In Section 4 the three principles are synthesized and general relativity theory is uniquely deduced in the classical limit (up to a cosmological constant). Finally, in Section 5, it is shown that all the laws of the quantum theory are included in the equation :

$$\mathcal{H}\psi = 0$$

for the invariant probability amplitude functional $\psi[\mathcal{G}]$, where $\mathcal{H}$ is the Hermitian operator

$$\mathcal{H} = \int d^3x \left\{ -\frac{4\pi}{G^{1/2}} \frac{\delta}{\delta g_{ij}} G^{1/2} G^{-1}{}_{ijmn} \frac{\delta}{\delta g_{mn}} - \frac{R\sqrt{g}}{16\pi} \right\}$$

1. THE QUANTUM PRINCIPLE RESTATED

The fundamental object in what we are about to consider is superspace, the set of all three-dimensional Riemannian geometries that have a positive definite metric. For the present, we assume that order and distance can uniquely be defined in such a set. We shall investigate these definitions in the next section.

We assert that the (quantum mechanical) history of the universe is given by a continuous one-parameter sequence of functionals $\varphi[\mathcal{G},\sigma]$, that is to say a mapping of Riemannian geometries $\mathcal{G}$ to complex numbers φ, for each value of the parameter σ.

The functional $\varphi[\mathcal{G},\sigma]$ is interpreted physically as the "probability amplitude" for the universe to arrive at the configuration $\mathcal{G}$ with the value σ of the sequential parameter.

We further introduce the assumption that there exists a unique connecting rule in the sequence $\varphi[\mathcal{G},\sigma]$, namely a rule by means of which from any given member of the sequence one can get any other member, and do so uniquely. This means that there is a linear connection between the probability amplitude at a certain value of the parameter σ and that at a value infinitesimally differing from the first. Such a connection has in general the form

$$\varphi[\mathcal{G}_1,\sigma+\delta\sigma] = \int_{all\ \mathcal{G}} \varphi[\mathcal{G}_1(\sigma+\delta\sigma),\mathcal{G}(\sigma)]\,\varphi[\mathcal{G},\sigma]\,\mathcal{D}\mathcal{G} \qquad (1.1)$$

where the integration is taken over all superspace for the geometry $\mathcal{G}$. It can be seen that the connecting factor $\varphi[\mathcal{G}_1(\sigma+\delta\sigma),\mathcal{G}(\sigma)]$ depends on the elements of an infinitesimal path in superspace, namely a supervector [7].

Keeping in mind the aforementioned physical interpretation of $\varphi[\mathcal{G},\sigma]$, it is seen that the connecting factor $\varphi[\mathcal{G}_1(\sigma+\delta\sigma),\mathcal{G}(\sigma)]$ can most easily be interpreted as the probability amplitude for the universe to follow, during the parameter interval $d\sigma$, this infinitesimal path $\mathcal{G}=\mathcal{G}(\sigma)$ in superspace, with $\mathcal{G}(\sigma)=\mathcal{G}$ and $\mathcal{G}(\sigma+\delta\sigma)=\mathcal{G}_1$.

Applying the connection rule (1.1) n times, we obtain the following expression for $\varphi[\mathcal{G}_n,\sigma+n\delta\sigma]$ in terms of $\varphi[\mathcal{G},\sigma]$:

$$\varphi[\mathcal{G}_n, \sigma+n\delta\sigma] = \int_{\mathcal{G}_{n-1}} \cdots \int_{\mathcal{G}} \varphi[\mathcal{G}_n(\sigma+n\delta\sigma), \mathcal{G}_{n-1}(\sigma+(n-1)\delta\sigma)]$$
$$\times \cdots \times \varphi[\mathcal{G}_1(\sigma+\delta\sigma), \mathcal{G}(\sigma)]\ \varphi[\mathcal{G},\sigma]\ \mathcal{D}\mathcal{G}_{n-1} \cdots \mathcal{D}\mathcal{G} \tag{1.2}$$

If we let n become arbitrarily large while keeping $\delta\tau$ infinitesimal, given the probability amplitude distribution at a certain value of the parameter σ, Eq. (1.2) would give that at a finite parameter interval away.

The above equation can most naturally be interpreted as follows : we define the probability amplitude of the path $\mathcal{G} = \mathcal{G}(\sigma)$, for the finite interval between σ and $\sigma+n\delta\sigma$ to be the product of the probability amplitudes of the infinitesimal supervectors that constitute the given finite path :

$$\varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+n\delta\sigma} = \varphi[\mathcal{G}_n(\sigma+n\delta\sigma), \mathcal{G}_{n-1}(\sigma+(n-1)\delta\sigma)]$$
$$\times \cdots \times \varphi[\mathcal{G}_1(\sigma+\delta\sigma), \mathcal{G}(\sigma)] \tag{1.3}$$

Then Eq. (1.2) can be expressed in the form (substituting σ' for $\sigma+n\delta\sigma$ and $\mathcal{G}'$ for $\mathcal{G}_n$) :

$$\varphi[\mathcal{G}',\sigma'] = \int_{\mathcal{G}} \sum_{\text{all paths}} \varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma'}\ \varphi[\mathcal{G},\sigma]\ \mathcal{D}\mathcal{G} \tag{1.4}$$

where the summation is over all paths that begin from $\mathcal{G}(\sigma)$ and end at $\mathcal{G}'(\sigma')$, namely all histories of the universe with these fixed end points.

Now let us return to Eq. (1.1), and let us specify the general form of the connecting factor $\varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+\delta\sigma}$. First it is evident from our physical interpretation of this connecting factor that it can differ from unity only infinitesimally. We then postulate that this difference must be proportional to the length $d\mathcal{L}$ of the infinitesimal supervector to which the connecting factor refers. Thus, we express $\varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+\delta\sigma}$ in the form :

$$\varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+\delta\sigma} = 1 + i A\, d\mathcal{L} \tag{1.5}$$

where A is an, as yet, arbitrary, complex in general, functional of the geometry $\mathcal{G}$.

If we than take into account Eq. (1.3), we can determine the general form of the probability amplitude of a finite path. From Eq. (1.3), for a given path $\mathcal{G} = \mathcal{G}(\sigma)$ in superspace, it follows that

$$\varphi[\mathcal{G}(\sigma)]_{\sigma_0}^{\sigma+\delta\sigma} = \varphi[\mathcal{G}(\sigma)]_{\sigma_0}^{\sigma} \; \varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+\delta\sigma}$$

The factor $\varphi[\mathcal{G}(\sigma)]_{\sigma_0}^{\sigma}$ for a fixed σ_0 and a fixed path, can be considered as a function simply of σ, which we call $\varphi(\sigma)$. Then $\varphi[\mathcal{G}(\sigma)]_{\sigma_0}^{\sigma+\delta\sigma} = \varphi(\sigma+\delta\sigma)$, and substituting for $\varphi[\mathcal{G}(\sigma)]_{\sigma}^{\sigma+\delta\sigma}$ the expression given in Eq. (1.5) we obtain the difference equation

$$\varphi(\sigma+\delta\sigma) - \varphi(\sigma) = i A \varphi(\sigma)\, d\mathcal{L}$$

which after integrating gives

$$\varphi(\sigma) = \varphi[\mathcal{G}(\sigma)]_{\sigma_0}^{\sigma} = \exp\left\{ i \int_{\sigma_0}^{\sigma} A\, d\mathcal{L} \right\} \tag{1.6}$$

In other words, the probability amplitude that the history of the universe is a certain path in superspace is the exponential of an, as yet, arbitrary integral over that path.

Since $\varphi[\mathcal{G},\sigma]$ is interpreted as probability amplitude, $|\varphi[\mathcal{G},\sigma]|^2$ must be interpreted as probability density for the universe to arrive at the configuration $\mathcal{G}$ having the value σ of the sequential parameter. Then, since at any value of the parameter σ the universe must assume some configuration, it must be true that $|\varphi[\mathcal{G},\sigma]|^2$ when integrated over all superspace must give unity, independently of the parameter σ. Therefore, the following condition must be satisfied, if the physical interpretation of $\varphi[\mathcal{G},\sigma]$ is to hold true

$$\int_{all\,\mathcal{G}'} |\varphi[\mathcal{G}',\sigma']|^2 \mathcal{D}\mathcal{G}' = \int_{all\,\mathcal{G}} |\varphi[\mathcal{G},\sigma]|^2 \mathcal{D}\mathcal{G} \tag{1.7}$$

It can be seen from Eqs. (1.4) and (1.6) that a necessary condition for the above equation to be satisfied is $\mathrm{Im}\, A = 0$.

We are therefore led to the following statement

$$\varphi[\mathcal{G}',\sigma'] = \int_{\mathcal{G}} \Big(\sum_{\text{all paths}} \exp\Big\{ i \int A \, d\mathcal{L} \Big\} \Big) \varphi[\mathcal{G},\sigma] \, \mathcal{D}\mathcal{G} \tag{1.8}$$

(where A is an, as yet, arbitrary real functional of the geometry), which constitutes the quantum principle.

Finally, we must impose the requirement that from the functionals $\varphi[\mathcal{G},\sigma]$ that are consistent with the above equation we must accept as physically meaningful only those which are independent of the parameter σ. In other words we do not consider as physically meaningful to ask what is the probability amplitude for the universe to arrive at the configuration $\mathcal{G}$ with the value σ of the parameter, but viewing the parameter as non-measurable, we ask what is the probability amplitude of arrival at the configuration $\mathcal{G}$ independently of the value of the parameter. We thus define the physical probability amplitude $\psi[\mathcal{G}]$ as follows :

$$\psi[\mathcal{G}] = \int_{\text{all } \sigma} \varphi[\mathcal{G},\sigma] \, d\sigma \tag{1.9}$$

The above requirement ensures the consistency of the quantum principle with the chronos principle to be stated in the next section.

2. INTRODUCTION OF THE CHRONOS PRINCIPLE

In the previous section we spoke of the necessity of the existence of a definition of distance in superspace. Here, we shall elaborate on this definition.

First it is necessary to point out that an acceptable definition of the concept of distance of two elements of superspace must be consistent with the requirement that it be a measure of the difference of the two three-dimensional Riemannian geometries which these elements represent.

In order that the concept of distance of two geometries is definable, it is necessary that there exists a one-to-one correspondence of the points of one of these geometries to the points of the other. This correspondence must take the form of a continuous mapping whose inverse is also continuous. As a consequence, the distance of two geometries is definable only if the geometries belong to the same topology. It follows that superspace is not a connected set, but rather it is composed of disjoint connected sets, each encompassing geometries of a distinct topology.

In the preceding paper [7] we introduced the concept of a supervector, associated with a (differentiable) line in superspace. Here, we define the concept of orthogonality of two supervectors $d\mathcal{G}$ and $d\mathcal{G}'$, both referring to the same element of superspace (the geometry $\mathcal{G}$), by a relation of the form :

$$K\,[\mathcal{G}, d\mathcal{G}, d\mathcal{G}'] = 0 \tag{2.1}$$

K being a functional of the geometry and the two supervectors. Further we postulate that K is linear in both supervectors. This assumption, motivated from the theory of spaces of finite dimension, and satisfied by Riemannian geometry, contains the physical requirement that the number of dimensions (number of numbers needed to specify a point, number of components of a vector) be equal to the number of mutually orthogonal vectors defined at a certain point.

If now there is a third supervector $d\mathcal{G}''$ which is the sum [in the sense of Ref. 7), Section 3] of the other two aforementioned supervectors $d\mathcal{G}$ and $d\mathcal{G}'$, then it follows from the definition of orthogonality and the assumption of the existence of a definition of distance, the Pythagorean theorem of superspace :

$$d\mathcal{L}''^2 = d\mathcal{L}^2 + d\mathcal{L}'^2 \qquad (2.2)$$

where $d\mathcal{L}$, $d\mathcal{L}'$ and $d\mathcal{L}''$ refer to the length of the supervectors $d\mathcal{G}$, $d\mathcal{G}'$ and $d\mathcal{G}''$, respectively.

Let us be given two geometries $\mathcal{G}$ and $\mathcal{G}+d\mathcal{G}$ infinitesimally differing from each other, and let us do a one-to-one intercorrespondence of their points. Then the components of the supervector $d\mathcal{G}$ representing their difference can be expressed as $:dg_{ij}({}^3x)$, namely the differences of the metric functions of the two three-geometries at the corresponding points.

Now let us consider a special supervector $d\mathcal{G}_1$, one that represents an infinitesimal change in the geometry of only a localized neighbourhood of space. In other words, being such that its components $dg^1_{ij}({}^3x)$ are different from zero only in the neighbourhood of the point ${}^3x_1$, let us say, in a (co-ordinate) volume element Δx_1 of which the point ${}^3x_1$ is the centre. We postulate that the length of such a localized supervector is independent of the geometry at all other neighbourhoods. It then follows from the definition of orthogonality that this length can only be of the form :

$$d\mathcal{L}_1^2 = G^{ijmn}({}^3x_1)\, dg^1_{ij}({}^3x_1)\, dg^1_{mn}({}^3x_1)\, \Delta^3 x_1 \qquad (2.3)$$

where G^{ijmn} is an, as yet, arbitrary geometric tensor density, built out of the metric tensor and its derivatives, having the symmetries of the product $dg_{ij}dg_{mn}$.

Consider now another such supervector $d\mathcal{G}_2$, of components $dg^2_{ij}({}^3x)$ different from zero only in a volume element Δ^3x_2 with centre at the point ${}^3x_2$, representing, as $d\mathcal{G}_1$, a localized infinitesimal geometrical change, however, referring to a neighbourhood disjointed from that of $d\mathcal{G}_1$. We demand these two supervectors to be orthogonal, since they represent independent geometrical changes. It then follows from the Pythagorean theorem of superspace that the square of the length of a supervector that has components $dg_{ij}({}^3x) = dg^1_{ij}+dg^2_{ij}$, non-vanishing in both neighbourhoods, is the sum of the squares of the lengths of the component supervectors $d\mathcal{G}_1$ and $d\mathcal{G}_2$. Analogously, it must be true that since a general supervector $d\mathcal{G}$ [with components $dg_{ij}({}^3x)$], representing a general infinitesimal geometrical change, can be decomposed into local supervectors $d\mathcal{G}_\nu$, representing localized geometrical changes in volume elements Δ^3x_ν, centred about points ${}^3x_\nu$:

$$dg_{ij}({}^3x) = \lim_{\substack{N\to\infty \\ \Delta^3x_\nu \to 0}} \sum_{\nu=1}^{N} dg_{ij}^{\nu}({}^3x) \tag{2.4}$$

the length of that general supervector must be given by

$$\begin{aligned} d\mathcal{L}^2 &= \lim_{\substack{N\to\infty \\ \Delta^3x_\nu \to 0}} \sum_{\nu=1}^{N} G^{ijmn}({}^3x_\nu)\, dg_{ij}({}^3x_\nu)\, dg_{mn}({}^3x_\nu)\, \Delta^3x_\nu \\ &= \int G^{ijmn}\, dg_{ij}\, dg_{mn}\, d^3x \end{aligned} \tag{2.5}$$

This will therefore be the form of the element of distance in superspace. It then follows that the functional K that is involved in the definition of orthogonality in superspace, Eq. (2.1), must assume the form

$$K[\mathcal{G}, d\mathcal{G}, d\mathcal{G}'] = \int G^{ijmn}\, dg_{ij}\, dg'_{mn}\, d^3x \tag{2.6}$$

(cosine of angle between two supervectors at the same point in superspace).

As has been mentioned above, the quantities dg_{ij}, and therefore also Eqs. (2.5) and (2.6), have meaning only after a certain one-to-one correspondence has been established between the points of the geometries in question. Now this intercorrespondence can a priori be done in an infinity of ways and different dg_{ij} functions will result for the same two geometries. If we parametrize the two geometries by letting them be two nearby points on a path in superspace, the parametric equations of which are $\mathcal{G} = \mathcal{G}(\sigma)$, thus letting the geometry $\mathcal{G}$ be that at the value σ of the parameter, and $\mathcal{G}+d\mathcal{G}$ be that at $\sigma+d\sigma$, then a change in the way of doing the intercorrespondence will be a transformation in the co-ordinate systems in which the geometries are described

$$x^i = f^i({}^3x^*, \sigma) \tag{2.7}$$

the transformation depending on σ, and thus being different for σ and $\sigma+d\sigma$. Thus the change of the intercorrespondence is given by the infinitesimal vectors

$$\xi^i = \frac{\partial f^i}{\partial \sigma}\, d\sigma \tag{2.8}$$

and the new metric differences dg^*_{mn} are given in terms of the original ones dg_{ij} by [7] :

$$dg^*_{mn} = \frac{\partial x^i}{\partial x^{*m}} \frac{\partial x^j}{\partial x^{*n}} (dg_{ij} + \xi_{i;j} + \xi_{j;i}) \tag{2.9}$$

Thus a different value of $d\mathcal{L}^2$ will be obtained through Eq. (2.5) by changing the way of doing the intercorrespondence of the two given geometries. On the other hand, it is necessary to demand the distance of two geometries infinitesimally different from each other to be unique. Therefore one out of all the possible ways of doing the correspondence should be preferred. In addition, we must demand the distance of two geometries which are the same (and indeed are one geometry instead of two) to be zero. It can be seen that it is necessary in order that these demands be met that we accept only the intercorrespondence which extremizes the expression on the right-hand side of Eq. (2.5), indeed that only the $d\mathcal{L}^2$ calculated through such an intercorrespondence is the true distance of the two given geometries.

Let us therefore do an arbitrary intercorrespondence of the points of the two geometries and obtain metric differences dg_{ij}. Let us then change the way of doing that intercorrespondence, the change given by the vectors ξ^i, and let us vary those vectors ξ^i in order to extremize the expression $d\mathcal{L}^2$ of Eq. (2.5). The extremization conditions are :

$$(G^{*ijmn} dg^*_{mn})_{;*j} = \left[G^{ijmn}(dg_{mn} + \xi_{m;n} + \xi_{n;m})\right]_{;j} = 0 \tag{2.10}$$

The above equations can be considered as equations determining the vectors ξ^i, namely the necessary correction to the way of doing the correspondence that must be made so that a consistent definition of distance may be obtained. However, in order that our afore-mentioned demands for such a definition are indeed met, it is necessary to show that either the above equations have a unique solution, or, in the case that two ways of doing the intercorrespondence both satisfy the above equations, that both ways give the same distance for the two geometries in question. Proving uniqueness is equivalent to proving that once a way of doing the intercorrespondence that satisfies Eq. (2.10), is found, one cannot make a further change so that the new way still satisfies the same equations. In other words, that the equations

$$\left[G^{ijmn}(\xi_{m;n} + \xi_{n;m})\right]_{;j} = 0 \tag{2.11}$$

have only the trivial solution $\xi^i = 0$. Clearly, if for one of the given geometries the equation

$$\xi_{m;n} + \xi_{n;m} = 0 \tag{2.12}$$

possesses non-trivial solutions, then there are many ways of doing the intercorrespondence that satisfy the extremization equations, one obtainable from the other by changes given by vectors ξ^i that satisfy Eq. (2.12). However, it is evident from Eq. (2.9) that all such alternative intercorrespondences give the same definition of distance for the two given geometries. [Equations (2.12) are Killing's equations, so the alternative intercorrespondences merely refer to corresponding a point on one geometry with points on the other geometry that can be reached one from the other through a translation by a Killing vector.]

We are now in a position to formulate the main idea of this work. Consider a continuous and piece-wise differentiable sequence of Riemannian geometries. Such a sequence represents a line in superspace possessing length. It is in relation to such sequences that we introduce the concept of time, defined through the concept of distance in superspace by :

$$d\tau^2 = \frac{d\mathcal{L}^2}{B} \tag{2.13}$$

The constant of proportionality B is an as yet arbitrary functional of the geometry $\mathcal{G}$ to which the infinitesimal change in the geometry $d\mathcal{G}$, whose measure is $d\mathcal{L}$, refers.

Equation (2.13) constitutes the "Chronos Principle". It contains the statement that time is not, as has been assumed by all physical theories to date, a separate physical entity, existing independently of the physical system, in which the changing of the physical system takes place. It is the measure of the changing of the physical system itself, that is time.

We introduce further the assumption that the functional B has the form of a simple integral (as $d\mathcal{L}^2$ was proved to be) :

$$B = \int b \sqrt{g}\, d^3x \tag{2.14}$$

where b is an, as yet, arbitrary geometric invariant function.

It is to be noted that the integrals in Eqs. (2.5) and (2.14) are taken over the entire three-dimensional manifold of space. Correspondingly, the time as defined by Eq. (2.13), refers to the change in configuration of the entire universe. Thus, this time will be called the "global" time. However, given that $d\mathcal{L}^2$ and B are both simple integrals, it is possible to restrict them both to some localized region U of space, defining

$$d\mathcal{L}_U^2 = \int_U G^{ijmn}\, dg_{ij}\, dg_{mn}\, d^3x$$

and

$$B_U = \int_U b\sqrt{g}\, d^3x \qquad (2.15)$$

We can then define the concept of "local" time $d\tau_U$ for the neighbourhood U, by :

$$d\tau_U^2 = \frac{d\mathcal{L}_U^2}{B_U} \qquad (2.16)$$

referring to the change in configuration in that localized region only.

We now postulate that all such local times referring to the same global time are equal. That is to say that $d\mathcal{L}_U^2/B_U$ is the same for all neighbourhoods U, or, in the limit of infinitesimal neighbourhoods, that

$$\frac{d\mathcal{L}^2({}^3x)}{b\sqrt{g}\,({}^3x)} = const.\ (independent\ of\ point\ {}^3x) \qquad (2.17)$$

where we have defined

$$d\mathcal{L}^2({}^3x) = G^{ijmn}({}^3x)\, dg_{ij}({}^3x)\, dg_{mn}({}^3x) \qquad (2.18)$$

This postulate contains the statement that it is not necessary to look at the change in configuration of the entire universe in order to measure time. It is sufficient to measure the change in configuration of only a localized region of the universe, and one is ensured that the local time thus obtained will be equal to that of any other region, and indeed, [as is clear from Eqs. (2.17), (2.16) and (2.13)] equal to the global time.

It will be seen in the following that the postulate expressed by Eq. (2.17) need not be considered as an independent principle but is in fact contained in the relativity principle to be expressed in the next section.

Concluding, we note that the change from the common notion of time to the ideas expressed in this section, was brought about by putting the clocks, namely the instruments which we use to measure time, back into the physical system, letting both be described by the same physical laws. The chronos principle is thus the criterion of consistency of any physical theory with respect to the measurement of time.

3. THE RELATIVITY PRINCIPLE RESTATED

Let us now be given a line in superspace, namely a continuous sequence of three-dimensional Riemannian geometries the points of which are intercorresponded and the concepts of length and of (global) time along the line are defined in the manner described in the previous section. Such a sequence constitutes a four-dimensional Riemannian space-time manifold, if we define the corresponded points of the geometries to be those at the bases and tips of the normals to the hypersurfaces which these geometries represent in space-time, the squares of the lengths of all the normals joining the corresponded points of two adjacent hypersurfaces being equal in absolute value to the square of the global time interval for these two geometries as defined in the previous section. It follows from the above definitions that the four-dimensional geometry of space-time is given by the following expression for the square of the element of distance defined in it :

$$ {}^{(4)}ds^2 = \pm d\tau^2 + g_{ij}({}^3x, \tau)\, dx^i dx^j \tag{3.1} $$

It is then clear that the fourth co-ordinate, which is here the global time τ, has the physical meaning of time for all localized neighbourhoods of space only if the requirement expressed by Eq. (2.17) is satisfied.

Further, as Eq. (3.1) denotes, both ++++ and -+++ signatures of the constructed space-time are consistent with our assumptions. Nevertheless, we postulate that the signature of the physical space-time is -+++.

The four-dimensional element of distance [Eq. (3.1)] and with it the geometry of space-time, has physical meaning and its construction is justified, if not only purely temporal $(-d\tau^2)$ or purely spatial $(g_{ij}\, dx^i\, dx^j)$, but also mixed distances have physical meaning.

Since not the co-ordinate system, but rather the geometry of space-time should be the physical entity, if we make a four-dimensional co-ordinate transformation

$$x^i = x^i({}^3x', \tau')$$
$$\tau = \tau({}^3x', \tau') \tag{3.2}$$

the element of distance of space-time should remain invariant, its square being expressed in the new co-ordinate system in the form

$$\begin{aligned} {}^{(4)}ds^2 &= g'_{\mu\nu}\, dx^\mu dx^\nu \\ &= g'_{44}\, d\tau'^2 + 2g'_{4n}\, d\tau' dx'^n + g'_{mn}\, dx'^m dx'^n \end{aligned} \tag{3.3}$$

where

$$g'_{44} = -\left(\frac{\partial \tau}{\partial \tau'}\right)^2 - g_{ij}\,\frac{\partial x^i}{\partial \tau'}\,\frac{\partial x^j}{\partial \tau'} \tag{3.4}$$

$$g'_{4n} = g_{ij}\,\frac{\partial x^i}{\partial \tau'}\,\frac{\partial x^j}{\partial x'^n} - \frac{\partial \tau}{\partial \tau'}\,\frac{\partial \tau}{\partial x'^n} \tag{3.5}$$

$$g'_{mn} = \frac{\partial x^i}{\partial x'^m}\,\frac{\partial x^j}{\partial x'^n}\, g_{ij} - \frac{\partial \tau}{\partial x'^m}\,\frac{\partial \tau}{\partial x'^n} \tag{3.6}$$

The components of the infinitesimal normal vectors [2] $N^\mu d\tau'$ joining the hypersurfaces τ' and $\tau' + d\tau'$ satisfy the equations

$$g'_{\mu n} N^\mu = 0 \tag{3.7}$$

Therefore, since $N^4 = 1$,

$$N^m = -g'^{mn} g'_{4n} \quad \text{and} \quad g'_{4n} = -N_n \tag{3.8}$$

The squares of the lengths, $-N^2 d\tau'^2$, of these vectors are given by

$$-N^2 = g'_{\mu\nu} N^\mu N^\nu = g'_{44} - g'_{mn} N^m N^n \tag{3.9}$$

Now consider the world lines $x'^\mu = x'^\mu(s)$ that satisfy the equations

$$\frac{dx'^\mu}{ds} = \frac{N^\mu}{N} \tag{3.10}$$

the right-hand side of the above equations being the components of the unit normal to the hypersurfaces $\tau' = \text{const}$. One such world line passes through each point in space-time. Let the co-ordinate transformation (3.2) be such that along one of these lines $x'^\mu = x'^\mu_0(s)$ as well as along lines in its immediate vicinity [for any line $x'^\mu = x'^\mu_1(s) = x'^\mu_0(s) + dx'^\mu(s)$ that satisfies Eq. (3.10)]

$$N = 1 \tag{3.11}$$

Then for that infinitesimal family of lines, $ds^2 = -d\tau'^2$ and the same relation holds between the co-ordinate τ' and the length in space-time along these world lines as between the co-ordinate τ and the length along the world lines $x^i = \text{const}$ of the co-ordinate system of Eq. (3.1) Hence the same physical meanings must be given to τ and τ' in relation to the world lines to which they refer. Thus $d\tau'$ must be the time interval as measured by an observer in a laboratory the motion of which the afore-mentioned family of world lines represents. We will then say that a local standard of simultaneity exists in this laboratory, and that the latter is at rest with respect to the co-ordinate system of Eq. (3.3).

If the chronos principle is to hold true for this laboratory also, then it must be true that

$$d\tau'^2 = \frac{d\mathcal{L}'^2({}^3x'_0)}{R'\sqrt{g'}({}^3x'_0)} = \frac{G'^{ijmn}\, dg'_{ij}\, dg'_{mn}({}^3x'_0)}{R'\sqrt{g'}({}^3x'_0)} \tag{3.12}$$

while the intercorrespondence conditions

$$[G'^{ijmn} dg'_{mn}({}^3x'_0)]_{;j'} = 0 \tag{3.13}$$

are satisfied. Here the difference in the metric functions is taken at the points where the world line $x'^{\mu} = x_0^{\mu}(s)$ intersects the hypersurfaces $\tau' = \text{const.}$

We arrive at the conclusion that the square of the length of a world line in space-time is equal to minus the square of the time as measured by an observer whose motion that world line represents. Thus we have reached the sought for physical meaning of a more general element of distance in space-time. It is, however, obvious that this meaning can only be given in the case that $ds^2 \leq 0$ on that world line, or *)

$$g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \leq 1 \tag{3.14}$$

the equations of the world line being expressed in the co-ordinate system of Eq. (3.1), a condition on the motion of physical objects placed by the accepted signature of space-time. Positive squares of elements of distance in space-time can then be interpreted as purely spatial distances in a co-ordinate system with respect to which such moving observers are at rest.

There is no way for an observer in a localized laboratory, the motion of which the afore-mentioned family of world lines represents, to measure directly the motion of the laboratory with respect to the co-ordinate system of Eq. (3.1), by doing experiments confined within this laboratory. It follows as a logical necessity that the laws of physics should have the same form in the co-ordinate system $({}^3x')$, with respect to which the laboratory is at rest, as in the co-ordinate system $({}^3x)$ of Eq. (3.1). In conclusion, we arrive at the following statement which constitutes the relativity principle :

---

*) The ratio of the unit of time to that of spatial distance was chosen so that the speed of light in vacuum $c = 1$.

The laws of physics should retain their form under a co-ordinate transformation of the type of that of Eqs. (3.2), indeed they should have the same form in all co-ordinate systems.

4. GENERAL RELATIVITY DEDUCED

We shall now synthesize the three fundamental principles established in the previous sections, namely the quantum principle, the chronos principle and the relativity principle, and we shall show that only one physical theory is consistent with all three of them.

Let us go back to the quantum principle, Eq. (1.8) and let us call S the phase of the exponential

$$S = \int_{path} A \, d\mathcal{L} \tag{4.1}$$

Now we have shown in the previous sections how, employing the chronos principle, we can construct a four-dimensional space-time geometry out of a path in superspace. Thus, that principle taken into account, we can think of expressing S, which is an as yet arbitrary path integral in superspace, as a space-time integral. In doing this, we must also take into account the following two restrictions motivated from the necessity that the intercorrespondence conditions [Eqs. (2.10)] and the definition of time [Eq. (2.13)] be part of the laws of the physical theory to be derived :

1) that the variation of S with respect to the intercorrespondence of points give back the same intercorrespondence conditions ;
2) that the variation of S with respect to the time interval $d\tau$ give back the definition of time.

It can be seen that restriction 1) tells us that the superspace vector $d\mathcal{G}$ enters in S only in terms of its length $d\mathcal{L}$. Then the most general expansion of S from a superspace path integral to a space-time integral that conforms with this restriction is the following :

$$S = \int_{path} A \, d\mathcal{L} = \frac{1}{\sum_{n=0}^{\infty} c_n} \int \left\{ c_0 B^{1/2} + c_1 \frac{d\mathcal{L}}{d\tau} + \frac{c_2}{B^{1/2}} \left(\frac{d\mathcal{L}}{d\tau} \right)^2 + \cdots \right\} A \, d\tau \tag{4.2}$$

where the coefficients c_n are constants. Restriction 2 then requires us to set

$$C_0 = C_2 + 2C_3 + 3C_4 + \cdots \tag{4.3}$$

Now we take into account the relativity principle and we require that S, in which through the quantum principle the physical laws are contained, be invariant under four-dimensional co-ordinate transformations. Since in Eq. (4.2) only a single integration over τ occurs, it is clear that S must be a simple invariant integral over the four-dimensional space-time in question. Such an integral has in general the form :

$$S = \int I \sqrt{-{}^{(4)}g}\, d^4x \tag{4.4}$$

where I is an invariant function of the four-dimensional metric tensor and its derivatives. Such a function should also contain derivatives of the metric tensor of order greater than or equal to the second, since no invariants can be constructed out of the metric tensor and its first derivatives other than simple constants.

We now draw attention to the fact that in the integrant on the right-hand side of Eq. (4.2) only first time derivatives occur. This therefore contradicts invariance for S, except in the case that this integrant differs from an invariant by a divergence and an integration by parts has taken place. But such integration by parts, eliminating derivatives of order higher than the first, can only be done in the case that this integrant originally contained derivatives with respect to the time only up to the second order, and those of the second order only linearly. It follows that the same must be true for the function I, and, furthermore, as invariance demands, not only with respect to time derivatives but also with respect to derivatives of all dimensions.

It has long been shown [10] that the only geometrical invariant function that satisfies these conditions is

$$I = k\,({}^{(4)}R + \lambda) \tag{4.5}$$

where ${}^{(4)}R$ is the four-dimensional scalar of curvature and k and λ are constants. The constant k may be chosen to be any specified number by a suitable choice of the unit of spatial distance. It turns out that if we choose the unit of spatial distance so that the product of the gravitational constant G and Planck's constant $\hbar$, $G\hbar = 1$, the constant k becomes $(1/16\pi)$ expressed in those units. The constant λ is the so-called "cosmological" constant and it shall be assumed to be zero in the following, although generalization of the results to follow to the case of a non-vanishing λ, is trivial.

Expressing the invariant I of Eq. (4.4) in the co-ordinate system of Eq. (3.1) we get :

$$I\sqrt{g} = \frac{1}{16\pi}\left\{-2g_{ij}\frac{d}{d\tau}\left(G^{ijmn}\frac{dg_{mn}}{d\tau}\right) - G^{ijmn}\frac{dg_{ij}}{d\tau}\frac{dg_{mn}}{d\tau} + R\sqrt{g}\right\} \tag{4.6}$$

where R is here the three-dimensional scalar of curvature and

$$G^{ijmn} = \frac{\sqrt{g}}{4}\left[\frac{(g^{im}g^{jn} + g^{in}g^{jm})}{2} - g^{ij}g^{mn}\right] \tag{4.7}$$

After performing the afore-mentioned integration by parts we obtain for S the expression

$$S = \frac{1}{16\pi}\int d\tau\left\{\int d^3x\left[G^{ijmn}\frac{dg_{ij}}{d\tau}\frac{dg_{mn}}{d\tau} + R\sqrt{g}\right]\right\} \tag{4.8}$$

Comparing the above expression with that of Eq. (4.2), we conclude that $c_3 = c_4 = \dots = 0$, and from Eq. (4.3) $c_0 = c_2$. Then keeping in mind the requirement, deduced in Section 3, that $d\mathcal{L}^2$ be a simple space integral, we must set $\frac{1}{2}(A/B^{\frac{1}{2}}) = \text{const.}$, which can be chosen to be $1/16\pi$ without loss of generality. Then we conclude that :

$$d\mathcal{L}^2 = \int G^{ijmn} \, dg_{ij} \, dg_{mn} \, d^3x$$

and

$$8\pi A B^{1/2} = B = \int R \sqrt{g} \, d^3x \tag{4.9}$$

Thus we have deduced the previously undetermined functionals A and B as well as the tensor G^{ijmn} [the latter being that of Eq. (4.7)]. Furthermore, since S was deduced to have a unique form [Eq. (4.8)], unique must be the set of physical laws contained in it.

In the limit of large S, the so-called "classical" limit only the paths in superspace for which S is extremal contribute in the sum of Eq. (1.8). This extremization refers of course to the variation of the sequence of three-dimensional space geometries that constitute a superspace path. However, since the afore-mentioned requirements 1 and 2 have been satisfied, S is also varied with respect to the inter-correspondence of points of adjacent three-geometries joined by normal vectors, as well as with respect to the length of those vectors. As a result, the entire four-dimensional space-time geometry that corresponds to the superspace path is varied, and we must classically admit only those four-dimensional geometries which extremize S. The extremization equations are

$$\frac{\delta S}{\delta g_{\mu\nu}} = -G^{\mu\nu} \sqrt{-{}^{(4)}g} = 0 \tag{4.10}$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \, {}^{(4)}R \tag{4.11}$$

($R^{\mu\nu}$ being the four-dimensional Ricci tensor) Einstein's equations of general relativity theory. These equations, expressed in the co-ordinate system of Eq. (3.1) assume the following form [7] :

$$G^{ij}\sqrt{g} = 2\left[G^{ijmn}\frac{d^2 g_{mn}}{d\tau^2} + C^{rskl,ij}\frac{dg_{rs}}{d\tau}\frac{dg_{kl}}{d\tau}\right] + \left(R^{ij} - \frac{1}{2}g^{ij}R\right)\sqrt{g} = 0 \tag{4.12}$$

where R^{ij} is the three-dimensional Ricci tensor and $C^{k\ell rs,ij}$ is the affine connection of superspace :

$$C^{klrs,ij} = \frac{1}{2}\left(\frac{\partial G^{klij}}{\partial g_{rs}} + \frac{\partial G^{rsij}}{\partial g_{kl}} - \frac{\partial G^{klrs}}{\partial g_{ij}} \right) \tag{4.13}$$

and

$$G^{i4}\sqrt{g} = -2\, G^{ijmn}\left(\frac{dg_{mn}}{d\tau}\right)_{;j} = 0 \tag{4.14}$$

$$G^{44}\sqrt{g} = -\frac{1}{2}\left(G^{ijmn}\frac{dg_{ij}}{d\tau}\frac{dg_{mn}}{d\tau} - R\sqrt{g}\right) = 0 \tag{4.15}$$

It is evident that, as required Eqs. (4.13) which were obtained from the variation of S with respect to the g_{i4} are nothing else but the inter-correspondence conditions Eqs. (2.10), and Eq. (4.14) which was obtained from the variation of S with respect to g_{44} contains both the definition of global time Eq. (2.13) and the equivalence of all local times Eq. (2.17).

If we make a general four-dimensional co-ordinate transformation [of the type of Eqs. (3.2)] these equations become

$$G'^{i4}\sqrt{g'} = -2\, G'^{ijmn}\left[\frac{\left(\frac{dg'_{mn}}{d\tau'} + N_{m;n'} + N_{n;m'}\right)}{N}\right]_{;j'} = 0$$

(4.16)

$$G'^{44}\sqrt{g'} = -\frac{1}{2}\Big[G'^{ijmn} \frac{\left(\frac{dg'_{ij}}{d\tau'} + N_{i;j'} + N_{j;i'}\right)}{N} \frac{\left(\frac{dg'_{mn}}{d\tau'} + N_{m;n'} + N_{n;m'}\right)}{N} - R'\sqrt{g'}\Big] = 0 \tag{4.17}$$

It can then be seen from Eqs. (2.9) that for the infinitesimal family of world lines satisfying Eqs. (3.10), for which condition (3.11) is satisfied, the above equations are identical to Eqs. (3.13) and (3.12). This proves the consistency of the chronos principle with the relativity principle.

It was seen that the quantity S as expressed in Eq. (4.8) plays the rôle of an action in geometrodynamics. Thus the quantity in curls in the same equation can be considered to be the geometrodynamical Lagrangian :

$$L = \frac{1}{16\pi}\int \Big[G^{ijmn} \frac{dg_{ij}}{d\tau} \frac{dg_{mn}}{d\tau} + R\sqrt{g} \Big] d^3x \tag{4.18}$$

Then we can define the dynamical variables canonically conjugated to the metric functions g_{ij}, in other words the geometrodynamical canonical momenta π^{ij} by [1),2)] :

$$\pi^{ij} = \frac{\delta L}{\delta(dg_{ij}/d\tau)} = \frac{1}{8\pi} G^{ijmn} \frac{dg_{mn}}{d\tau} \tag{4.19}$$

They are evidently $1/8\pi$ times the covariant components of the supervector $d\mathcal{G}/d\tau$. We can finally define a geometrodynamical Hamiltonian [1),2)]

$$\mathcal{H} = \int \pi^{ij} \frac{dg_{ij}}{d\tau} d^3x - L$$
$$= \int \left(4\pi G^{-1}_{ijmn} \pi^{ij}\pi^{mn} - R\sqrt{g}/16\pi\right) d^3x \tag{4.20}$$

where

$$G^{-1}_{ijmn} = \frac{1}{\sqrt{g}} \left(g_{im}g_{jn} + g_{in}g_{jm} - g_{ij}g_{mn} \right) \tag{4.21}$$

are the components of the matrix inverse to that with components $G^{mnk\ell}$, namely

$$G^{-1}_{ijmn}\, G^{mnk\ell} = \frac{1}{2}\left(\delta_i^k \delta_j^\ell + \delta_j^k \delta_i^\ell\right)$$

It is seen from Eq. (4.15) that

$$\mathcal{H} = -\frac{1}{8\pi}\int G^{44}\sqrt{g}\, d^3x \tag{4.22}$$

Then Hamilton's equations *)

$$\frac{dg_{ij}}{d\tau} = \left\{ g_{ij}, \mathcal{H} \right\} \tag{4.23}$$

$$\frac{d\pi^{ij}}{d\tau} = \left\{ \pi^{ij}, \mathcal{H} \right\} \tag{4.24}$$

are satisfied, Eq. (4.22) giving back the definition of the geometrodynamical canonical momenta, and Eq. (4.24) giving the equations $G^{ij}\sqrt{g} = 0$ [Eqs. (4.12)]. The only laws which must thus be added to these to get the full set of laws of general relativity theory are thus Eqs. (4.14) and (4.15), or, in other words, the constraints [1),2)]

$$\pi^{ij}{}_{;j} = 0 \tag{4.25}$$

$$\mathcal{H}(^3x) = 0 \tag{4.26}$$

where we have defined the Hamiltonian density

---

*) The Poisson bracket $\{A,B\}$ of two functionals A and B of $g_{ij}(^3x)$ and $\pi^{ij}(^3x)$ is defined as follows :

$$\{A, B\} = \int d^3x \left(\frac{\delta A}{\delta g_{ij}} \frac{\delta B}{\delta \pi^{ij}} - \frac{\delta B}{\delta g_{ij}} \frac{\delta A}{\delta \pi^{ij}} \right)$$

$$\mathcal{H}(^3x) = 4\pi G^{-1}{}_{ijmn}\,\pi^{ij}\pi^{mn} - R\sqrt{g}/16\pi$$
$$= -G^{44}\sqrt{g}/8\pi \tag{4.27}$$

It has long been shown [11] that the tensor $G^{\mu\nu}$, and therefore the equations of general relativity theory [Eq. (4.10)], satisfies the identities

$$G^{\mu\nu}{}_{;(4)\nu} = 0 \tag{4.28}$$

As a result, only six out of the ten equations (4.10) are independent, the same number as that of the independent degrees of freedom of the four-dimensional geometry of space-time. These identities, written in the coordinate system of Eq. (3.1) assume the form :

$$G_k{}^{\nu}{}_{;(4)\nu}\sqrt{g} = \sqrt{g}\,G_k{}^{j}{}_{;j} + \frac{d}{d\tau}(\sqrt{g}\,G_k{}^{4}) = 0 \tag{4.29}$$

$$G^{4\nu}{}_{;(4)\nu}\sqrt{g} = \frac{1}{2}\frac{dg_{ij}}{d\tau}G^{ij}\sqrt{g} + \sqrt{g}\,G^{4j}{}_{;j} + \frac{d}{d\tau}(\sqrt{g}\,G^{44}) = 0 \tag{4.30}$$

where four-dimensional covariant derivatives appear on the left and three-dimensional ones on the right. It can be seen from the above identities that if Eqs. (4.14) and Eq. (4.15) are satisfied at the initial instant of time, Eqs. (4.12) guarantee that they are satisfied at all later times. Thus, Eqs. (4.14) and Eq. (4.15) are constraints on the initial value data [12].

The initial value problem of general relativity theory is formulated as follows : give two three-dimensional space geometries $\mathcal{G}$ and $\mathcal{G}' = \mathcal{G} + d\mathcal{G}$ infinitesimally different from each other. Use Eqs. (4.14) (multiplied by $d\tau$, so that the time does not enter in them) to determine the intercorrespondence of their points. Then, if the condition (equivalence of local times)

$$\frac{d\mathcal{L}^2({}^3x)}{R\sqrt{g}\,({}^3x)} = \frac{G^{ijmn}\, dg_{ij}\, dg_{mn}({}^3x)}{R\sqrt{g}\,({}^3x)} = const.\ (\text{independent of } {}^3x) \tag{4.31}$$

is satisfied, and furthermore

$$\frac{d\mathcal{L}^2}{\mathcal{R}} > 0 \tag{4.32}$$

where we have defined

$$\mathcal{R} = \int R\sqrt{g}\, d^3x \tag{4.33}$$

then a classical time exists, defined by

$$d\tau^2 = d\mathcal{L}^2 / \mathcal{R} \tag{4.34}$$

conditions (4.31) and (4.34) being together equivalent to the single condition Eq. (4.15). In that case, the metric functions $g_{ij}({}^3x)$ and their first time derivatives $dg_{ij}({}^3x)/d\tau$ are consistently determined initially (of course up to a co-ordinate transformation involving the three spatial dimensions only), and we have the necessary and sufficient data to integrate Eqs. (4.12), determining uniquely all following members of the sequence of spatial three-dimensional geometries, assembled together in a four-dimensional space-time geometry in consistency with Eqs. (4.14) and (4.15).

In the case that for the afore-mentioned first two three-dimensional geometries $d\mathcal{L}^2/\mathcal{R} < 0$ time cannot classically be defined, and no solution of the equations of general relativity theory passing through those geometries exists.

If at a certain point in the geometrical evolution the denominator $\mathcal{R}$ of Eq. (4.34) vanishes, then the evolution stops, since for any finite time interval there is no change in the geometry (moment of maximum expansion of the universe). On the other hand, if to two infinitesimally different geometries there corresponds $d\mathcal{L}^2 = 0$ while the geometries are different, which is certainly possible since the metric in superspace is indefinite, as can be seen from Eq. (4.7), then a singularity exists in the geometrical evolution, since in that case, $d\tau$ being zero, $dg_{ij}/d\tau$ diverges (cosmological singularity).

5. THE QUANTUM THEORY

It was proved in the previous paper [7] that if the form of S of Eq. (4.8) is substituted in the formulation of the quantum principle, then that principle leads us to the following equation for the probability amplitude functional $\varphi[\mathscr{G},\tau]$ (the sequential parameter is here τ) :

$$\mathcal{H}\varphi = i\,\frac{\partial\varphi}{\partial\tau} \tag{5.1}$$

the Hamiltonian operator $\mathcal{H}$ given by [cf. Eq. (A.12) in the Appendix] :

$$\mathcal{H} = \int d^3x \left\{ -\frac{4\pi}{G^{1/2}}\,\frac{\delta}{\delta g_{ij}}\, G^{1/2} G^{-1}_{ijmn}\,\frac{\delta}{\delta g_{mn}} - \frac{R\sqrt{g}}{16\pi} \right\} \tag{5.2}$$

where G is the determinant of the matrix G^{ijmn}. In the classical limit Eq. (5.1) will give the dynamical equations of general relativity theory, Eqs. (4.12)

.It can be proved that the operator $\mathcal{H}$ is Hermitian, so that the condition expressed by Eq. (1.7) is satisfied, however, it is of course not of a definite sign. Since this operator does not depend explicitly on τ, Eq. (5.1) may be solved by separation of the variables $\mathscr{G}$ and τ. The general solution is of the form

$$\varphi[\mathscr{G},\tau] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda\tau}\,\psi_\lambda[\mathscr{G}]\,d\lambda \tag{5.3}$$

where the ψ_λ's are eigenfunctions of $\mathcal{H}$ with eigenvalues λ. It then follows from the above equation and the definition of the physical probability amplitude $\psi[\mathscr{G}]$, given by Eq. (1.9), that the latter is the eigenfunction (s) of $\mathcal{H}$ with eigenvalue zero. Thus the physical probability amplitude satisfies the equation [7] :

$$\mathcal{H}\psi = 0 \tag{5.4}$$

It is seen that the above equation represents the quantum form of the definition of global time, since that definition is manifested in general relativity theory by the vanishing of the total classical Hamiltonian. It remains to be seen what are the constraints on the probability amplitude functional ψ imposed by the quantum form of Eqs. (4.13) and (4.15) of general relativity, or, equivalently, by the constraints expressed by Eqs. (4.25) and (4.26).

In order to find the quantum analogue of the constraints (4.25), it is necessary to know what are the quantum operators that correspond to the classical dynamical variables $g_{ij}({}^3x)$ and $\pi^{ij}({}^3x)$. Postulating a correspondence between classical Poisson brackets and quantum operators, we require those operators to fulfil the following relation [1),2)] :

$$\left[g_{ij}({}^3x),\ \pi^{mn}({}^3x')\right] = \frac{i}{2}\left(\delta_i^m\delta_j^n + \delta_j^m\delta_i^n\right)\delta^3(x-x') \tag{5.5}$$

From the above relation it follows that if the operator $g_{ij}({}^3x)$ represents multiplication by the functions $g_{ij}({}^3x)$, then the operator $\pi^{ij}({}^3x)$ is given by

$$\pi^{ij}({}^3x) = \frac{1}{i}\,\frac{\delta}{\delta g_{ij}({}^3x)} \tag{5.6}$$

The quantum form of the constraints (4.24) is therefore

$$\left(\frac{\delta\psi}{\delta g_{ij}}\right)_{;j} = 0 \tag{5.7}$$

It is well known that the above equations are identities for any functional ψ that is independent of the co-ordinate system used to describe the geometry.

Consequently, the only constraint remaining to be imposed is the quantum form of Eq. (4.26), namely of the vanishing of the Hamiltonian density, which, as we saw, contains in addition to the definition of global time, the equivalence of local times. The Hamiltonian density operator $\mathcal{H}({}^3x)$ must be the integrant of the expression on the right-hand side of Eq. (5.2), namely :

$$\mathcal{H}(^3x) = -\frac{4\pi}{G^{1/2}}\frac{\delta}{\delta g_{ij}}G^{1/2}G^{-1}_{ijmn}\frac{\delta}{\delta g_{mn}} - \frac{R\sqrt{g}}{16\pi} \tag{5.8}$$

The quantum constraint to be imposed is therefore :

$$\mathcal{H}(^3x)\,\psi = 0 \tag{5.9}$$

It will be shown in the following that any invariant functional ψ which satisfies Eq. (5.4), identically satisfies the above constraint for all space points ^3x.

First, we draw attention to the classical identity expressed by Eq. (4.30), and we note that if the dynamical equations [Eqs. (4.12)] are satisfied and in addition the constraints (4.25) are satisfied then [cf. Eq. (4.27)]

$$\frac{d\mathcal{H}(^3x)}{d\tau} = 0 \tag{5.10}$$

For every classical quantity which is a constant of the motion, there should correspond a quantum operator which commutes with the total Hamiltonian.

We therefore compute the commutator of the Hamiltonian density $\mathcal{H}(^3x)$ with the total Hamiltonian $\mathcal{H}$:

$$\begin{aligned}[\mathcal{H}(^3x), \mathcal{H}]\,\psi = \frac{1}{2}\Big\{ & G^{-1}_{ijmn}(^3x)\frac{\delta \mathcal{R}}{\delta g_{ij}(^3x)}\frac{\delta\psi}{\delta g_{mn}(^3x)} \\ & - \int G^{-1}_{ijmn}(^3x')\frac{\delta(R(^3x)\sqrt{g}(^3x))}{\delta g_{ij}(^3x')}\frac{\delta\psi}{\delta g_{mn}(^3x')}\,d^3x'\Big\} \\ & + \frac{1}{4}\Big[\frac{\mathcal{D}^2\mathcal{R}}{\mathcal{D}\mathcal{G}(^3x)^2} - \frac{\mathcal{D}^2(R(^3x)\sqrt{g}(^3x))}{\mathcal{D}\mathcal{G}^2}\Big]\psi\end{aligned} \tag{5.11}$$

where the functional $\mathcal{R}$ is that of Eq. (4.32), while the operators $\mathcal{D}^2/\mathcal{D}\mathcal{G}^2$ and $\mathcal{D}^2/\mathcal{D}\mathcal{G}(^3x)^2$ are defined by Eqs. (A.12) and (A.13) in the Appendix. Now

$$\frac{\delta \mathcal{R}}{\delta g_{ij}(^3x)} = -(R^{ij} - \tfrac{1}{2} g^{ij} R)\sqrt{g} \tag{5.12}$$

while expressing $R(^3x)\sqrt{g}(^3x)$ as $\int R(^3x')\sqrt{g}(^3x')\delta^3(x'-x)d^3x'$, and using the result of Eq. (A.4) in the Appendix [noting Eq. (A.5)] we obtain

$$\frac{\delta(R(^3x)\sqrt{g}(^3x))}{\delta g_{ij}(^3x')} = -(R^{ij}(^3x') - \tfrac{1}{2} g^{ij}(^3x')R(^3x'))\sqrt{g}(^3x')\delta^3(x'-x) + 4G^{ijmn}(^3x')(\delta^3(x'-x))_{;mn} \tag{5.13}$$

The covariant derivatives in the second term on the right are taken at the point $^3x'$. Substituting the results of Eqs. (5.12) and (5.13) in the term in curls on the right-hand side of Eq. (5-11), and doing the necessary integrations by parts, we obtain for that term

$$-2\left(\frac{\delta\psi}{\delta g_{mn}}\right)_{;mn}$$

It can further be seen that the term in square brackets on the right-hand side of the above equation vanishes, the invariant R being linear in the curvature tensor R_{imjn} and not containing any of its derivatives. Hence, it follows that

$$[\mathcal{H}(^3x), \mathcal{H}]\,\psi = -2\left(\frac{\delta\psi}{\delta g_{mn}}\right)_{;mn} = 0 \tag{5.14}$$

the last step due to the fact that Eqs. (5.7) are identically satisfied.

Since $\mathcal{H}(^3x)$ and $\mathcal{H}$ commute, it follows that they possess the same set of eigenfunctions. Thus, if there is a functional ψ satisfying the equation

$$\mathcal{H}\psi = \lambda\psi \tag{5.15}$$

the same functional would also satisfy equations of the type

$$\mathcal{H}(^3x)\psi = \rho(^3x)\psi \tag{5.16}$$

where

$$\int \rho(^3x)\, d^3x = \lambda \tag{5.17}$$

Now it has been shown [4] that the entity T_ϵ defined by :

$$T_\epsilon = \int G^{4i}\epsilon_i \sqrt{g}\, d^3x = -\int 2\pi^{ij}{}_{;j}\, \epsilon_i\, d^3x$$

where ϵ_i are the covariant components of an infinitesimal vector field, the contravariant components of which are the same at all points, generates in both its classical and its quantum form, infinitesimal translations by the vector ϵ^i. In other words, for a classical variable $A(g_{ij}(^3x), \pi^{ij}(^3x))$

$$-\{T_\epsilon, A(^3x)\} = A_{,i}\,\epsilon^i \tag{5.18}$$

while for a quantum operator $A(g_{ij}(^3x), \delta/\delta g_{ij}(^3x))$

$$i[T_\epsilon, A(^3x)] = A_{,i}\,\epsilon^i \tag{5.19}$$

Hence we can express the Hamiltonian density operator at the point $({}^3x+{}^3\epsilon)$ in terms of the same operator at the point $({}^3x)$:

$$\mathcal{H}({}^3x+{}^3\epsilon)\psi = \left\{\mathcal{H}({}^3x) + i\left[T_\epsilon , \mathcal{H}({}^3x)\right]\right\}\psi \tag{5.20}$$

or, taking into account Eqs. (5.16),

$$\rho({}^3x+{}^3\epsilon)\psi = \rho({}^3x)\psi - i\left(\mathcal{H}({}^3x)-\rho({}^3x)\right)T_\epsilon\psi \tag{5.21}$$

It is obvious, however, from Eqs. (5.7) that

$$T_\epsilon\,\psi = 0 \tag{5.22}$$

identically. It then follows from Eq. (5.21) that $\rho({}^3x+{}^3\epsilon)=\rho({}^3x)$, or in other words

$$\rho = \text{const. (independent of } {}^3x) \tag{5.23}$$

Then, Eq. (5.17) gives $\rho\times(\text{total volume of space})=\lambda$. Hence, if a functional ψ is found which satisfies Eq. (5.15) with $\lambda=0$, the same functional will satisfy Eqs. (5.16) with $\rho=0$. This is the result we had sought to prove *).

In conclusion, Eq. (5.4) contains the complete set of laws of quantum geometrodynamics. The physical interpretation of the functional ψ is the following.

---

*) In Ref. 13) it was shown that if there exists a functional $\psi[\mathcal{G}]$, independent from the co-ordinate system, that satisfies the equation $\mathcal{H}({}^3x_0)\psi=0$ for some particular space point $({}^3x_0)$ then the same functional ψ will satisfy the equations $\mathcal{H}({}^3x)\psi=0$ for any other point $({}^3x)$. The assumption on which this proof is based is non-trivial and it can be proved only after the commutation of the Hamiltonian densities at different space points, implied by Eq. (5.14), has been demonstrated.

Let a measurement of the geometry of the universe be made. The quantity

$$P_V = \int_V |\Psi[\mathcal{G}]|^2 \mathcal{D}\mathcal{G} \tag{5.24}$$

where

$$\mathcal{D}\mathcal{G} = \prod_{\text{all } {}^3x} G^{1/2}({}^3x) \prod_{i \geq j} dg_{ij}({}^3x) \tag{5.25}$$

(volume element of superspace), is the probability that the outcome belongs to the region V or superspace. If two measurements of the geometry give outcomes differing by $d\mathcal{G}$, then if $d\mathcal{L}^2/\mathcal{R} \geq 0$, where $d\mathcal{L}$ is the length of the vector $d\mathcal{G}$ in superspace, there is a classical time-interval $d\tau$ between the two measurements defined by

$$d\tau = d\mathcal{L}/\mathcal{R}^{1/2}$$

APPENDIX

In the previous paper a functional $\varphi[\mathcal{G}]$ was called differentiable at the point in superspace representing the geometry $\mathcal{G}$, for the approach $d\mathcal{G}$, if the limit

$$\frac{\mathcal{D}\varphi}{\mathcal{D}\mathcal{G}, d\mathcal{G}} = \lim_{\epsilon \to 0} \left\{ \frac{1}{\epsilon} \left(\varphi[\mathcal{G} + \epsilon d\mathcal{G}] - \varphi[\mathcal{G}] \right) \right\} = \chi[\mathcal{G}, d\mathcal{G}] \tag{A.1}$$

exists. The above expression was named "partial derivative of φ with respect to the approach $d\mathcal{G}$". It was further noted that it is a functional $\chi[\mathcal{G}, d\mathcal{G}]$ which is linear in the supervector $d\mathcal{G}$.

It is well known that a linear functional has in general the form of a scalar product. If the three-dimensional geometry of space is described by giving the metric functions at all space points, it follows from the form of the definition of the element of distance in superspace Eq. (2.5), that such a scalar product has the form

$$\chi[\mathcal{G}, d\mathcal{G}] = \int \frac{\delta\varphi}{\delta g_{ij}} \, dg_{ij} \, d^3x \tag{A.2}$$

The above equation is to be considered as the definition of the functional derivative $\delta\varphi/\delta g_{ij}$, a contravariant supervector which behaves like a tensor density in ordinary space.

Let us now have a functional φ that has the form of a simple integral

$$\varphi = \int F \sqrt{g} \, d^3x \tag{A.3}$$

The invariant F may be considered to be a function of the metric tensor g_{ij}, the curvature tensor R_{imjn}, and the covariant derivative of the latter. Then

$$\frac{1}{\sqrt{g}}\frac{\delta\varphi}{\delta g_{ij}} = \left(\frac{\partial F}{\partial g_{ij}} + \frac{1}{2}g^{ij}F\right) + \left\{\left(\frac{\partial F}{\partial R_{imjn}} + \frac{\partial F}{\partial R_{injm}}\right)_{;mn} + \frac{1}{2}\left(R^{i}{}_{nkm}\frac{\partial F}{\partial R_{kmjn}} + R^{j}{}_{nkm}\frac{\partial F}{\partial R_{kmin}}\right)\right\} \tag{A.4}$$

to which, in the case that F contains covariant derivatives of R_{imjn}, must be added a term obtained from that in curls, by replacing $\partial F/\partial R_{imjn}$ by $-(\partial F/R_{imjn;k})_{;k}$, etc. In using the above result, it should be noted that, in order to properly take into account the symmetries of the curvature tensor R_{imjn}, $\partial R_{acbd}/\partial R_{imjn}$ should be taken to be :

$$\frac{\partial R_{acbd}}{\partial R_{imjn}} = \frac{1}{4}\left(\delta^i_a\delta^m_c\delta^j_b\delta^n_d + \delta^i_c\delta^m_a\delta^j_d\delta^n_b - \delta^i_c\delta^m_a\delta^j_b\delta^n_d - \delta^i_a\delta^m_c\delta^j_d\delta^n_b\right) \tag{A.5}$$

The result of Eq. (A.4) may easily be generalized to the case that φ is a multiple integral.

The partial derivative of a functional $\chi[\mathcal{G}, d\mathcal{G}]$ of the geometry $\mathcal{G}$ and the supervector $d\mathcal{G}$, with respect to the approach $d\mathcal{G}$, was defined in the previous paper to be :

$$\frac{\mathcal{D}\chi[\mathcal{G},d\mathcal{G}]}{\mathcal{D}\mathcal{G},d\mathcal{G}} = \lim_{\epsilon\to 0}\left\{\frac{1}{\epsilon}\left(\chi[\mathcal{G}+\epsilon d\mathcal{G}, (d\mathcal{G})_{\mathcal{G}+\epsilon d\mathcal{G}}] - \chi[\mathcal{G},(d\mathcal{G})_{\mathcal{G}}]\right)\right\} \tag{A.6}$$

Here $(d\mathcal{G})_{\mathcal{G}+\epsilon d\mathcal{G}}$ is the supervector defined at the point $\mathcal{G}+\epsilon d\mathcal{G}$ in superspace and constructed parallel to the original supervector $(d\mathcal{G})_{\mathcal{G}}$ defined at the point $\mathcal{G}$. Since the affine connection in superspace is that given by Eq. (4.12), the components $(dg_{ij})_{\mathcal{G}+\epsilon d\mathcal{G}}$ of the supervector $(d\mathcal{G})_{\mathcal{G}+\epsilon d\mathcal{G}}$ are given by

$$(dg_{ij})_{\mathcal{G}+\epsilon d\mathcal{G}} = dg_{ij} - C_{ij}^{rs\,kl}\, dg_{rs}\, dg_{kl} \tag{A.7}$$

in terms of the components dg_{ij} of the supervector $(d\mathcal{G})_{\mathcal{G}}$. Here

$$C_{ij}^{rs\,kl} = G^{-1}_{ijmn}\, C^{rs\,kl,\,mn} \tag{A.8}$$

If the functional $\chi[\mathcal{G},d\mathcal{G}]$ is that of Eq. (A.1), then the second partial derivative of $\varphi[\mathcal{G}]$ with respect to the approach $d\mathcal{G}$ is defined

$$\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G},d\mathcal{G}^2} = \frac{\mathcal{D}\chi[\mathcal{G},d\mathcal{G}]}{\mathcal{D}\mathcal{G},d\mathcal{G}}$$

Substituting χ from Eq. (A.2), we obtain

$$\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G},d\mathcal{G}^2} = \int \frac{\mathcal{D}}{\mathcal{D}\mathcal{G},d\mathcal{G}}\left(\frac{\delta\varphi}{\delta g_{ij}({}^3x_1)}\right) dg_{ij}({}^3x_1)\, d^3x_1$$
$$- \int C_{ij}^{rs\,kl}\,\frac{\delta\varphi}{\delta g_{ij}}\, dg_{rs}\, dg_{kl}\, d^3x \tag{A.9}$$

Taking into account the result expressed by Eq. (A.2), we can write the first term on the right-hand side of the above equation as

$$\iint \frac{\delta^2 \varphi}{\delta g_{ij}({}^3x_1)\, \delta g_{mn}({}^3x_2)}\, dg_{ij}({}^3x_1)\, dg_{mn}({}^3x_2)\, d^3x_1\, d^3x_2$$

thus defining the second functional derivative.

The second total derivative of the functional $\varphi[\mathcal{G}]$ was then defined in the previous paper to be

$$\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}^2} = \left(\frac{2}{i}\right)\int e^{\,i d\mathcal{L}^2}\, \frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}, d\mathcal{G}^2}\, \mathcal{D}(d\mathcal{G}) \tag{A.10}$$

where the integration is over all supervectors $d\mathcal{G}$, the differential volume element $(d\mathcal{G})$ given by :

$$\mathcal{D}(d\mathcal{G}) = \prod_{\text{all }^3x} G^{1/2}({}^3x) \prod_{i\geqslant j} d(dg_{ij}({}^3x)) \tag{A.11}$$

Substituting for the second partial derivative the expression already obtained, and for $d\mathcal{L}^2$ the expression of Eq. (2.5), we see that the terms for which ${}^3x_1 \neq {}^3x_2$ contribute nothing, since, being odd, they vanish when integrated over all supervectors. Performing the integration over the remaining terms with ${}^3x_1 = {}^3x_2$ we obtain :

$$\begin{aligned}\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}^2} &= \int G^{-1}_{ijmn}\left[\frac{\delta^2\varphi}{\delta g_{ij}({}^3x)\,\delta g_{mn}({}^3x)} - \delta^3(x-x)\, C^{ijmn}_{ab}\, \frac{\delta\varphi}{\delta g_{ab}}\right] d^3x \\ &= \int \frac{1}{G^{1/2}}\, \frac{\delta}{\delta g_{ij}}\left(G^{1/2} G^{-1}_{ijmn}\, \frac{\delta\varphi}{\delta g_{mn}}\right) d^3x\end{aligned} \tag{A.12}$$

Finally the density operator $\mathcal{D}^2/\mathcal{D}\mathcal{G}(^3x)^2$ is defined by :

$$\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}^2} = \int \frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}(^3x)^2}\, d^3x$$

or, in other words

$$\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G}(^3x)^2} = \frac{1}{G^{1/2}}\frac{\delta}{\delta g_{ij}}\left(G^{1/2}G^{-1}_{ijmn}\frac{\delta\varphi}{\delta g_{mn}}\right) \tag{A.13}$$

REFERENCES

1) P.A.M. Dirac - Proc.Roy.Soc.(London) A246, 333 (1958) ; Phys.Rev. 144, 924 (1959).

2) R. Arnowitt, S. Deser and C.W. Misner - "The Dynamics of General Relativity", Chapter in "Gravitation : An Introduction to Current Research", edited by Louis Witten, John Wiley and Sons (1962).

3) J.A. Wheeler - "Geometrodynamics and the Issue of the Final State", Chapter in "Relativity, Groups and Topology", Les Houches (1963).

4) B.S. DeWitt - Phys.Rev. 160, 1113 (1967).

5) C.W. Misner - Phys.Rev. 186, 1319 (1969).

6) K. Kuchar - Phys.Rev. D4, 955 (1971).

7) D. Christodoulou - Proceedings of the Academy of Athens, 47 (20 January 1972).

8) R.P. Feynman - Ph.D. Thesis, Princeton University (1942), Revs. Modern Phys. 20, 367 (1948).

9) A. Einstein - "Die Grundlagen der Allgemeinen Relativitätstheorie", Ann.Phys.Lpz. 49,(1916).

10) H. Weyl - "Space-Time Matter" (Appendix II), Dover (1922).

11) L. Bianchi - "Lezioni di Geometria Differenziale", p. 351, 2nd Edition, Vol. 2 (1902).

12) A. Lichnerowicz - J.Math.Pure Appl. 23, 37 (1944).

13) V. Moncrief - Phys.Rev. 5, 277 (1972).