The Circle Criterion and Input-to-State Stability

New Perspectives on a Classical Result

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Feedback interconnections consisting of a linear system L in the forward path and a static sector-bounded nonlinearity f in the negative feedback path are ubiquitous in control theory and practice (see figures 1 and 2). With origins in the classical work [1], such interconnections are referred to as systems of Lur'e type, while the study of their stability properties constitutes absolute stability theory.

Absolute stability theory investigates stability through the interplay of the frequency-domain properties of the linear component L and sector data for the nonlinearity f. In essence, if L and the sector data of f are matched in a sufficiently "nice" manner, then the interconnection is stable. Notwithstanding the simplicity of its formulation, stability analysis of Lur'e systems and closely related topics, such as hyperstability, the Kalman-Popov-Yakubovich lemma, also known as the positive-real lemma, passivity, positive realness, and the S-procedure, embrace subtle features that have generated much attention since the appearance of [1]. This attention relates not only to the early literature on the emerging area of nonlinear control — in [2], it is noted that, by 1968, over 200 papers on absolute stability had appeared — but also to the later literature as evidenced by the survey articles [3]-[5]. Accounts of the classical theory can be found in many textbooks and monographs [6]-[15]. A central theme of the present article is a particular criterion for absolute stability, namely, the circle criterion. We recall this result in the section "The Circle Criterion and Lyapunov Stability". While the circle criterion is well established, we consider it from a perhaps unfamiliar – but nevertheless intriguing – point of view, namely, by relating it to a complexified version of the Aizerman conjecture [16], [17]. With reference to the feedback interconnection of Figure 1, with $L = (A, b, c)$, a linear single-input single-output state space system, and locally Lipschitz sector-bounded f, with $\alpha v^2 \leq v f(v) \leq \beta v^2$ for all v, the Aizerman conjecture postulates a characterization of asymptotic stability of the zero equilibrium of the interconnection in terms of stabilizing gains for L. In particular, it conjectures that the equilibrium of the interconnection is asymptotically stable if and only if $A - kbc^*$ is Hurwitz for all gains $k \in (\alpha, \beta)$. This conjecture is known to be false, but holds true in case of the complexified version alluded to above.

A distinguishing facet of the present article is a treatment of systems of Lur'e type with the additional feature of an exogenous input or disturbance d, as shown in Figure 3, wherein the single-input, single-output linear system L in the forward path has the state-space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad x(0) = x^0, \\ y(t) &= c^* x(t) \end{aligned}$$

with the function u given by the feedback relation

$$u(t) = d(t) - f(c^* x(t)).$$

For a specific example, see "An Example from Circuit Theory". The investigation in this article of Lur'e-type systems with input is predicated on the the concept of input-to-state stability (ISS), which we outline in "The Concept of Input-to-State Stability". In the specialized context of the Lur'e interconnection in Figure 3, ISS pertains to stability of the map from the initial condition and disturbance pair (x^0, d) to the state x. Moreover, ISS of the interconnection implies absolute stability of the interconnection. In the section "The Circle Criterion and ISS", the circle criterion is embedded in an ISS framework. This framework subsumes variants of the classical circle criterion and establishes that the hypotheses of the classical theory not only imply absolute stability but also ensure the stronger ISS property. Applications of this theory to systems with quantization, output disturbances, and hysteresis are described in "Quantization and Output Disturbances" and "Hysteretic Feedback Systems".

The treatment of the circle criterion in this article differs from the classical framework in three fundamental aspects, specifically, i) nonlinearities of greater generality than the standard class of locally Lipschitz functions are permitted in the feedback path; ii) in contrast with most of the existing literature, wherein the focus is on global asymptotic stability and L^2 or L^∞ stability, ISS issues are addressed here, in the spirit of [18], [19]; and iii) the sector conditions of the classical theory are weakened. With reference to i), we develop a framework of sufficient generality to encompass not only time-varying continuous nonlinearities but also discontinuous nonlinearities, such as quantization as well as certain causal operators, in particular, hysteresis, in the feedback path. With reference to ii), we identify conditions on the linear and nonlinear components in the feedback loop under which ISS of the interconnection is guaranteed. With reference to iii), through the concept of a generalized sector condition, the investigation is extended to include nonlinearities that satisfy a sector condition only within the complement of a compact interval, see Figure 4. For a prototype of iii), see "An Example from Circuit Theory". To facilitate the treatment of iii), a theory is developed pertaining to ISS with bias; this concept is outlined in "The Concept of Input-to-State Stability". The underlying approach to ISS with bias can be described as follows. With a given continuous nonlinearity f satisfying a sector condition on the complement $\mathbb{R}\backslash K$ of the compact interval K, as in Figure 4, we associate a continuous function $\tilde{f}$ that satisfies the sector condition on $\mathbb{R}$, as in Figure 2, and coincides with f on $\mathbb{R}\backslash K$. We then exploit the equivalence of the two interconnections shown in Figure 5, wherein $\tilde{d} := d + \tilde{f}(y) - f(y)$. In particular, if the interconnection on the right in Figure 5 is ISS, then the original interconnection on the left in Figure 5 is ISS with bias, where the bounded function $\tilde{f} - f$ is the source of the bias term.

With a view to a broad treatment of i) and iii), we adopt a set-valued standpoint that gives rise to a formulation of the basic problem in terms of a differential inclusion. The theory of differential inclusions mirrors fundamental aspects of the standard theory of differential equations [20]-[22]. In a control context, this theory has ramifications in the study of discontinuous feedback, hybrid systems, systems with quantization, and hysteretic systems. Differential inclusions are prominent in the tutorial articles [23] and [24] on discontinuous dynamical systems and hybrid dynamical systems, respectively.

Against this background and with reference to Figure 6, the focus of the paper is a tutorial overview of absolute stability, ISS, and boundedness properties of the feedback interconnection of a finite-dimensional, linear, single-input, single-output system (A, b, c) and a set-valued nonlinearity Φ. Throughout, we assume that Δ is a set-valued map in which input or disturbance signals are embedded. As a simple example to fix ideas, consider again the interconnection shown in Figure 3 with a sector-bounded nonlinearity as in Figure 2, with $\alpha y^2 \leq y f(y) \leq \beta y^2$ for all y, and disturbance d. This system is subsumed by the system shown in Figure 6, where the set-valued maps Δ and Φ are defined by

$$\Delta(t) := \{d(t)\},$$

$$\Phi(y) := \begin{cases} [\alpha y, \beta y], & y \geq 0, \\ [\beta y, \alpha y], & y < 0\,. \end{cases}$$

Note that Φ satisfies the sector condition

$$\alpha y^2 \leq yw \leq \beta y^2, \quad y \in \mathbb{R},\ w \in \Phi(y)\,,$$

which, for economy of notation and keeping mind that $\Phi(y)$ is a set, we also write as

$$\alpha y^2 \leq y\Phi(y) \leq \beta y^2, \quad y \in \mathbb{R}.$$

Absolute stability results typically depend on the interplay of frequency-domain properties of the linear component and the sector constraints for the nonlinearity, but not on the particular form or shape of the nonlinear component. Therefore, it seems natural to consider set-valued nonlinearities in the context of absolute stability theory. This point of view is becoming more widespread [15], [19], [25], [26].

Of course, if, as in the early classical literature on absolute stability, we restrict attention to interconnections with only static nonlinearities in the feedback path, then there is nothing to be gained by adopting a set-valued formulation; indeed such a formulation would be pedantic. The point to bear in mind here is that we seek an analytical framework of sufficient generality to encompass inter alia feedback systems with causal operators, and hysteresis operators in particular, in the feedback loop. To illustrate this objective, let F be a causal operator acting on scalar-valued functions in the domain dom(F) of F, which is a subset of $C[0,\infty)$. Consider the feedback system, structurally of Lur'e type, with input d, given by the functional differential equation

$$\dot{x}(t) = Ax(t) + b\big[d(t) - (F(c^*x))(t)\big]. \tag{1}$$

By causality of F we mean that, for all $y, z \in \text{dom}(F)$ and all $\tau > 0$, if y and z coincide on the interval $[0, \tau]$, then $F(y)$ and $F(z)$ also coincide on $[0, \tau]$. To associate (1) with the structure of Figure 6, we assume that F can be embedded in a set-valued map Φ in the sense that, for every $y \in \text{dom}(F)$,

$$(F(y))(t) \in \Phi(y(t)), \quad \text{a.a. } t \geq 0. \tag{2}$$

If the input d is such that $d(t) \in \Delta(t)$ for almost every t, then every solution of (1) is necessarily a solution of the feedback interconnection in Figure 6. In this sense, properties of solutions of the feedback interconnection are inherited by solutions of (1). Therefore, if the analysis can establish desirable properties of solutions of the overarching formulation in Figure 6, then these properties also hold for solutions of (1). As a concrete example, consider backlash or mechanical play, illustrated in Figure 7(a) and comprising a link consisting of two components, denoted I and II. The displacements of each part, with respect to a fixed origin, at time $t \geq 0$ are given by $y(t)$ and $z(t)$ with $|y(t) - z(t)| \leq \sigma$ for all $t \geq 0$, and $z(0) = y(0) + \xi$, where $\xi \in [-\sigma, \sigma]$ plays the role of the initial condition. The position $z(t)$ of II remains constant as long as the position $y(t)$ of I remains within the interior of II. For each continuous function y, we describe the evolution of the position of I by denoting the corresponding position of II by $z(t) = (F(y))(t)$. The action of the operator F is captured in Figure 7(b). Observe that, for each $y \in C[0, \infty)$, the embedding (2) holds if we define the set-valued map Φ by $\Phi(s) := [s - \sigma\, ,\, s + \sigma]$ for all $s \in \mathbb{R}$. As shown in this article, the operator F is causal and forms the basic building block of the class of hysteresis operators known as Preisach operators, see ”Hysteretic Feedback

Systems". The relevance of hysteresis within the control community is underlined by the special issue of the *IEEE Control Systems Magazine* [27], see also [28]-[35].

For notation and terminology used throughout this article, see "Notation and Terminology". Formal proofs of the stated results can be found in the section "Proofs".

Feedback Systems with Set-Valued Nonlinearities

The feedback system shown in Figure 6 corresponds to the initial-value problem

$$\dot{x}(t) - Ax(t) \in b\left(\Delta(t) - \Phi(c^*x(t))\right), \quad x(0) = x^0 \in \mathbb{F}^n, \;\; \Delta \in \mathcal{D}_{\mathbb{F}}, \tag{3}$$

where $A \in \mathbb{F}^{n\times n}$, $b, c \in \mathbb{F}^n$, $\Phi \in \mathcal{U}_{\mathbb{F}}$, and $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. For most applications, only the case $\mathbb{F} = \mathbb{R}$ is relevant. However, to investigate the relationship between the classical circle criterion and the complex Aizerman conjecture, it is convenient to develop the theory also for the complex case. As for the set-valued input Δ, the situation most relevant for applications is the singleton-valued case $\Delta(t) = \{d(t)\}$, with $d \in L^\infty_{\mathrm{loc}}[0,\infty)$. However, including set-valued inputs Δ comes at no extra cost and turns out to be convenient in the analysis of ISS with bias, in the context of which the nonlinearity Φ is replaced by another set-valued nonlinearity $\tilde{\Phi}$, and the resulting set-valued difference

$$\tilde{\Phi}(c^*x(t)) - \Phi(c^*x(t)) = \{\tilde{w} - w \colon \tilde{w} \in \tilde{\Phi}(c^*x(t)),\, w \in \Phi(c^*x(t))\}$$

is absorbed into $\Delta(t)$ for all $t \geq 0$. See the proof of Corollary 16 for a detailed elaboration of this idea.

A *solution* of (3) is an absolutely continuous function $x\colon [0,T) \to \mathbb{F}^n$, where $0 < T \leq \infty$, such that $x(0) = x^0$ and the differential inclusion in (3) is satisfied almost everywhere on $[0,T)$.

A solution $x\colon [0,T) \to \mathbb{F}^n$ is *maximal* if it has no right extension that is also a solution, that is, there does not exist a solution $x_{\rm e}\colon [0,T_{\rm e}) \to \mathbb{F}^n$ of (3) such that $T_{\rm e} > T$ and $x_{\rm e}(t) = x(t)$ for all $t \in [0,T)$. A solution $x\colon [0,T) \to \mathbb{F}^n$ is *global* if $T = \infty$, that is, if it exists on $[0,\infty)$.

Before developing a stability theory for systems of the form (3), we state an existence result that is an immediate consequence of [21, Corollary 5.2].

Lemma 1: Let $\Phi \in \mathcal{U}_{\mathbb{F}}$. For each $x^0 \in \mathbb{F}^n$ and each $\Delta \in \mathcal{D}_{\mathbb{F}}$, the initial-value problem (3) has a solution. Moreover, every solution can be extended to a maximal solution. Finally, if a maximal solution is bounded, then it is global.

As noted above, one of the motivations for considering feedback systems given by differential inclusions of the form (3) is that functional differential equations of the form (1) with a dynamic nonlinearity F can be imbedded into the set-valued formulation (3), provided there exists $\Phi \in \mathcal{U}_{\mathbb{F}}$ such that (2) holds for every $y \in \mathrm{dom}(F)$. Another motivation for studying the inclusion (3) is that it allows us to consider discontinuous nonlinearities. To be more specific, we consider the following example of a quantized feedback system [36], [37].

Example 2: Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous static nonlinearity and consider the system

$$\dot{x}(t) = Ax(t) + b(d(t) - f(c^* x(t))), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{4}$$

where $d \in L^\infty_{\rm loc}[0,\infty)$. If the system (4) is subject to quantization of the output $y = c^* x$, we obtain the differential equation with discontinuous righthand side given by

$$\dot{x}(t) = Ax(t) + b\big(d(t) - (f \circ q_\eta)(c^* x(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{5}$$

where $q_\eta : \mathbb{R} \to \mathbb{R}$, parameterized by $\eta > 0$, is the uniform quantizer (see Figure 8) given by

$$q_\eta(v) = 2m\eta, \quad v \in \big((2m-1)\eta\,,\,(2m+1)\eta\big], \quad m \in \mathbb{Z}. \tag{6}$$

We interpret the differential equation in (5), which has discontinuous righthand side, in a set-valued sense as follows. First, we embed the quantizer q_η in the set-valued map $Q_\eta \in \mathcal{U}_\mathbb{R}$ defined by

$$Q_\eta(v) := \begin{cases} \{q_\eta(v)\}, & v \in \big((2m-1)\eta\,,\,(2m+1)\eta\big), \quad m \in \mathbb{Z}, \\ [2m\eta\,,\,2(m+1)\eta], & v = (2m+1)\eta, \quad m \in \mathbb{Z}. \end{cases} \tag{7}$$

This embedding essentially "fills in" the jumps in Figure 8 to yield the graph shown in Figure 9. Now, we subsume (5) in the differential inclusion

$$\dot{x}(t) - Ax(t) - bd(t) \in -b\Phi_\eta(c^* x(t)), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{8}$$

where $\Phi_\eta \in \mathcal{U}_\mathbb{R}$ is given by

$$\Phi_\eta(v) := f(Q_\eta(v)) = \{f(\zeta) \colon \zeta \in Q_\eta(v)\}.$$

With $\Delta \in \mathcal{D}_\mathbb{R}$ defined by $\Delta(t) := \{d(t)\}$, (8) can be rewritten as

$$\dot{x}(t) - Ax(t) \in b\big(\Delta(t) - \Phi_\eta(c^* x(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n,$$

which is of the form (3). We return to this example in the section "Quantization and Output Disturbances". ♢

In the following, for each $x^0 \in \mathbb{F}^n$ and each $\Delta \in \mathcal{D}_\mathbb{F}$, the notation $\mathcal{X}(x^0, \Delta)$ denotes the set of all maximal solutions of (3) corresponding to the initial condition x^0 and the input Δ. It follows from Lemma 1 that $\mathcal{X}(x^0, \Delta) \neq \varnothing$ for each $(x^0, \Delta) \in \mathbb{F}^n \times \mathcal{D}_\mathbb{F}$. We emphasize that maximal solutions of (3) are not necessarily unique, in which case $\mathcal{X}(x^0, \Delta)$ contains more than one element. For convenience, we set $\mathcal{X}(x^0) := \mathcal{X}(x^0, 0)$, wherein, and henceforth, the particular map $\Delta\colon t \mapsto \{0\}$ is denoted by $\Delta = 0$.

Definition 3: Assume that $\Delta = 0$ in (3). System (3) is *stable in the large* if every maximal solution of (3) is global and there exists $\gamma \in \mathcal{K}$ such that, for every $x^0 \in \mathbb{F}^n$ and every $x \in \mathcal{X}(x^0)$,

$$\|x(t)\| \le \gamma(\|x^0\|), \quad t \ge 0. \tag{9}$$

System (3) is *asymptotically stable in the large* if (3) is stable in the large and $\lim_{t\to\infty} x(t) = 0$ for every global solution x of (3). System (3) is *globally exponentially stable* if every maximal solution of (3) is global and there exist constants g and $\varepsilon > 0$ such that, for every $x^0 \in \mathbb{F}^n$ and every $x \in \mathcal{X}(x^0)$,

$$\|x(t)\| \le g e^{-\varepsilon t}\|x^0\|, \quad t \ge 0. \tag{10}$$

Definition 4: System (3) is *input-to-state stable with bias* (ISS with bias) if there exist $\gamma_1 \in \mathcal{KL}$, $\gamma_2 \in \mathcal{K}$, and $\theta \ge 0$ such that, for each $(x^0, \Delta) \in \mathbb{F}^n \times \mathcal{D}_{\mathbb{F}}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ is global and

$$\|x(t)\| \le \max\big\{\gamma_1(t, \|x^0\|), \gamma_2(\|\Delta\|_{L^\infty[0,t]} + \theta)\big\}, \quad t \ge 0. \tag{11}$$

The numbers θ and $\gamma_2(\theta)$ are the *bias parameter* and *bias*, respectively. If $\theta = 0$, then (3) is *input-to-state stable* (ISS).

Definition 4 generalizes the concept of ISS [38] to encompass set-valued nonlinearities and allow for bias. We also remark that, in Definition 4, the assumption that every solution $x \in \mathcal{X}(x^0, \Delta)$ is global is made for presentational purposes only and is, in fact redundant. If $[0, T)$ is the interval of existence of a maximal solution $x \in \mathcal{X}(x^0, \Delta)$ and the estimate in (11) holds for all $t \in [0, T)$, then, by Lemma 1, it follows that $T = \infty$.

The Circle Criterion and Lyapunov Stability

Initially, we consider stability properties of the system (3) with $\Delta = 0$. Let $\mathbf{G}$ denote the transfer function of the linear system (A, b, c), that is, the strictly proper rational function given by

$$\mathbf{G}(s) = c^*(sI - A)^{-1}b. \tag{12}$$

In the context of real systems $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$, the Aizerman conjecture [39], which is known to be false, can be stated as follows.

Aizerman conjecture. If $A - kbc^*$ is Hurwitz for all $k \in (\alpha, \beta)$, then the origin of the system $\dot{x} = Ax - bf(c^*x)$ is globally asymptotically stable for every locally Lipschitz $f\colon \mathbb{R} \to \mathbb{R}$ with the property that $\alpha < f(v)/v < \beta$ for all $v \neq 0$.

The first goal is to state and prove a version of the circle criterion, which we call the Aizerman version of the circle criterion because it shows that the Aizerman conjecture is true in the context of complex systems. We then show how more familiar versions of the circle criterion can be derived from the Aizerman version.

For $(A, b, c) \in \mathbb{C}^{n\times n} \times \mathbb{C}^n \times \mathbb{C}^n$, let $S(A, b, c)$ denote the set of all stabilizing complex gains, that is,

$$S(A, b, c) := \{k \in \mathbb{C}\colon A - kbc^* \text{ is Hurwitz}\}.$$

Theorem 5: **(Aizerman version of the circle criterion)** Assume that $\Delta = 0$, $\Phi \in \mathcal{U}_{\mathbb{C}}$, and $\Phi(0) = \{0\}$. Furthermore, let $z \in \mathbb{C}$ and $r > 0$, and assume that $\mathbb{D}(z, r) \subset S(A, b, c)$. For $v \neq 0$, let $\Phi(v)/v$ denote the set $\{w/v\colon\ w \in \Phi(v)\}$.

(i) If

$$\Phi(v)/v \subset \overline{\mathbb{D}}(z,r), \quad v \in \mathbb{C} \setminus \{0\}, \tag{13}$$

then (3) is stable in the large. Moreover, (9) holds with $\gamma \in \mathcal{K}$ given by $\gamma(s) = gs$, where the constant $g > 0$ depends on (A, b, c), z, and r, but not on Φ.

(ii) If

$$\Phi(v)/v \subset \mathbb{D}(z,r), \quad v \in \mathbb{C} \setminus \{0\}, \tag{14}$$

then (3) is asymptotically stable in the large.

(iii) If there exists $r_1 \in (0, r)$ such that

$$\Phi(v)/v \subset \mathbb{D}(z,r_1), \quad v \in \mathbb{C} \setminus \{0\}, \tag{15}$$

then (3) is globally exponentially stable. Moreover, (10) holds with constants $\varepsilon > 0$ and $g > 0$ depending on (A, b, c), z, r, and r_1, but not on Φ.

To interpret Theorem 5, it is useful to introduce some terminology. The complex number k is a *gain* of Φ if there exist $v \in \mathbb{C}\setminus\{0\}$ and $w \in \Phi(v)$ such that $k = w/v$. With this terminology, Theorem 5 says, roughly speaking, the following. If all linear gains in $\mathbb{D}(z,r)$ stabilize (A, b, c), as illustrated in Figure 10, then every set-valued nonlinearity $\Phi \in \mathcal{U}_{\mathbb{C}}$ that has all its gains in $\mathbb{D}(z,r)$ stabilizes (A, b, c). Consequently, Theorem 5 shows that the complex version of Aizerman's conjecture is true. This fact is in stark contrast with the failure of Aizerman's conjecture over the reals. For more details, including counterexamples, on Aizerman's conjecture over the reals, see [40, Chapter 7]. Furthermore, [16, Example 4.1] analyzes a class of counterexamples given in [40]. The analysis in [40] shows that Aizerman's conjecture over the reals fails "dramatically" in the sense that, for every $\delta \in (0, 1)$, there exist a system (A, b, c) and $\beta > 0$ such that $A - kbc^*$ is

Hurwitz for all $k \in (-\beta, \beta)$, but there exists a globally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ satisfying $-\delta\beta < f(v)/v < \delta\beta$ for all $v \in \mathbb{R}\backslash\{0\}$ and such that the origin of $\dot{x} = Ax - bf(c^*x)$ is not globally asymptotically stable.

Theorem 5 is closely related to stability radius theory. To see this, assume that A is Hurwitz. Then Theorem 5 applies with $r = r_{\mathbb{C}}(A; b, c)$, where

$$r_{\mathbb{C}}(A; b, c) := \inf\{|k| \colon k \in \mathbb{C} \text{ s.t. } A - kbc^* \text{ is not Hurwitz}\}$$

is the structured complex stability radius of A with respect to the "weightings" b and c [17], [41]. Theorem 5 shows that, for every $\Phi \in \mathcal{U}_{\mathbb{C}}$ with $\Phi(0) = \{0\}$ and such that all gains of Φ are bounded by $r_{\mathbb{C}}(A; b, c)$, the nonlinear system (3) remains stable. Moreover, if $\kappa \in \mathbb{C}$ is a destabilizing gain of minimal modulus, that is, $A - \kappa bc^*$ is not Hurwitz and $|\kappa| = r_{\mathbb{C}}(A; b, c)$, then, by statement (i) of Theorem 5, $A - \kappa bc^*$ is still marginally stable, or equivalently, if λ is an eigenvalue of $A - \kappa bc^*$, then $\operatorname{Re} \lambda \leq 0$ and λ is semisimple if $\operatorname{Re} \lambda = 0$. The complex stability radius also plays a role in the proof of Theorem 5. In particular, the proof is based on on a Riccati equation result from stability radius theory combined with Lyapunov techniques; see the section "Proofs".

Discs of stabilizing gains play a pivotal role in Theorem 5, in contrast with classical versions of the circle criterion wherein positive-real and sector conditions are ubiquitous. In many situations, it is more intuitive to think in terms discs of stabilizing gains. This point of view is partially inspired by classical results from the stability theory of linear multi-step methods in numerical analysis, which can be considered as Aizerman versions of the discrete-time circle criterion [42].

We now show how more classical, and perhaps more familiar, versions of the circle criterion can be obtained as corollaries of Theorem 5. To this end, if $\mathbf{H}$ is a rational function and $k \in \mathbb{C}$, we set $\mathbf{H}_k := \mathbf{H}(1 + k\mathbf{H})^{-1}$ and define

$$\mathbf{S}(\mathbf{H}) := \{k \in \mathbb{C} \colon \mathbf{H}_k \in H^\infty\}.$$

Note that if (A, b, c) is stabilizable and detectable, then $S(A, b, c) = \mathbf{S}(\mathbf{G})$, where $\mathbf{G}$ is given by (12).

In the following, we relate the disc conditions of Theorem 5 to positive-real and sector conditions. The next result characterizes the disc condition $\mathbb{D}(z, r) \subset \mathbf{S}(\mathbf{H})$ for a rational function $\mathbf{H}$ in terms of a positive-real property.

Lemma 6: Let $\mathbf{H}$ be a rational function, $r > 0$, and $z \in \mathbb{C}$. Set $\kappa := z - r$ and assume that $\mathbf{H}(s) \not\equiv -1/\kappa$. Then $\mathbb{D}(z, r) \subset \mathbf{S}(\mathbf{H})$ if and only if $1 + 2r\mathbf{H}_\kappa$ is positive real.

Lemma 7 below expresses sector conditions for a set-valued nonlinearity F in the form of conditions requiring all gains of F to be contained in suitable discs. This result is proved by direct algebraic calculation, which is therefore omitted.

Lemma 7: Let $v \mapsto F(v) \subset \mathbb{C}$ be a set-valued map defined on $\mathbb{C}$ and with nonempty values, let $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, and set

$$z := (\alpha + \beta)/2 \in \mathbb{C}, \quad r := |\alpha - \beta|/2 > 0.$$

(i) The map F satisfies the sector condition

$$\operatorname{Re}\big((w - \alpha v)(\overline{w - \beta v})\big) \leq 0, \quad w \in F(v),\ v \in \mathbb{C}$$

if and only if $F(0) = \{0\}$ and $F(v)/v \subset \overline{\mathbb{D}}(z, r)$ for all $v \in \mathbb{C} \setminus \{0\}$.

(ii) The map F satisfies the sector condition

$$\mathrm{Re}\left((w-\alpha v)(\overline{w-\beta v})\right) < 0, \quad w \in F(v),\ v \in \mathbb{C} \setminus \{0\}$$

if and only if $F(v)/v \subset \mathbb{D}(z, r)$ for all $v \in \mathbb{C} \setminus \{0\}$.

(iii) Let $\eta \in (0, r^2)$. The map F satisfies the sector condition

$$\mathrm{Re}\left((w-\alpha v)(\overline{w-\beta v})\right) \leq -\eta |v|^2, \quad w \in F(v),\ v \in \mathbb{C}$$

if and only if $F(0) = \{0\}$ and $F(v)/v \subset \overline{\mathbb{D}}(z, \sqrt{r^2 - \eta})$ for all $v \in \mathbb{C} \setminus \{0\}$.

We now formulate a result that generalizes the classical circle criterion to differential inclusions of the form (3) with $\mathbb{F} = \mathbb{C}$.

Theorem 8: **(Classical circle criterion – the complex case)** Assume that $\Delta = 0$, $(A, b, c) \in \mathbb{C}^{n\times n} \times \mathbb{C}^n \times \mathbb{C}^n$ is stabilizable and detectable, and $\Phi \in \mathcal{U}_{\mathbb{C}}$. Furthermore, let $\alpha, \beta \in \mathbb{C}$ and assume that $(1 + \beta \mathbf{G})(1 + \alpha \mathbf{G})^{-1}$ is positive real.

(i) If

$$\mathrm{Re}\left((w-\alpha v)(\overline{w-\beta v})\right) \leq 0, \quad w \in \Phi(v),\ v \in \mathbb{C}, \tag{16}$$

then (3) is stable in the large. Moreover, (9) holds with $\gamma \in \mathcal{K}$ given by $\gamma(s) = gs$, where the constant $g > 0$ depends on (A, b, c), α, and β, but not on Φ.

(ii) If $\Phi(0) = \{0\}$ and

$$\mathrm{Re}\left((w-\alpha v)(\overline{w-\beta v})\right) < 0, \quad w \in \Phi(v), \quad v \in \mathbb{C} \setminus \{0\}, \tag{17}$$

then (3) is asymptotically stable in the large.

(iii) If there exists $\eta > 0$ such that

$$\mathrm{Re}\left((w-\alpha v)(\overline{w-\beta v})\right) \leq -\eta |v|^2, \quad w \in \Phi(v),\ v \in \mathbb{C}, \tag{18}$$

then (3) is globally exponentially stable. Moreover, (10) holds with constants $\varepsilon > 0$ and $g > 0$ depending on (A, b, c), α, β, and η, but not on Φ.

Note that the linear system (A, b, c) is assumed to be only stabilizable and detectable, in contrast with the presentation of the circle criterion in the textbook literature [11], [14], [25], wherein controllability and observability are assumed.

We show how Theorem 5, Lemma 6, and Lemma 7 can be used to prove Theorem 8. We consider the derivation of only statement (i); statements (ii) and (iii) can be dealt with in an analogous way. To this end, let $\psi \in [0, 2\pi)$ be the argument of $\beta - \alpha$, so that $\beta - \alpha = |\beta - \alpha| e^{i\psi}$. Set $\tilde{A} := A - \alpha b c^*$ and $\tilde{b} := e^{i\psi} b$ and define $\tilde{\Phi} \in \mathcal{U}_{\mathbb{C}}$ by

$$\tilde{\Phi}(v) := e^{-i\psi}(\Phi(v) - \alpha v), \quad v \in \mathbb{C}.$$

By positive realness of $(1 + \beta \mathbf{G})(1 + \alpha \mathbf{G})^{-1}$ it follows that $1 + |\beta - \alpha| \tilde{\mathbf{G}}$ is positive real, where

$$\tilde{\mathbf{G}}(s) := e^{i\psi} \mathbf{G}_\alpha(s) = e^{i\psi} \mathbf{G}(s) \big(1 + \alpha \mathbf{G}(s)\big)^{-1} = c^*(sI - \tilde{A})^{-1} \tilde{b}.$$

Setting $r := |\beta - \alpha|/2$, it follows from Lemma 6 that $\mathbb{D}(r, r) \subset \mathbf{S}(\tilde{\mathbf{G}})$. Since (A, b, c) is stabilizable and detectable, it follows that $(\tilde{A}, \tilde{b}, c)$ is stabilizable and detectable, and we conclude that

$$\mathbb{D}(r, r) \subset S(\tilde{A}, \tilde{b}, c). \tag{19}$$

By (16), $\Phi(0) = \{0\}$ and, moreover, by Lemma 7, $\Phi(v)/v \subset \overline{\mathbb{D}}(z, r)$ for all $v \in \mathbb{C} \setminus \{0\}$, where $z := (\alpha + \beta)/2$. Observe that $\tilde{\Phi}(0) = \{0\}$ and $e^{-i\psi}(\overline{\mathbb{D}}(z, r) - \alpha) = \overline{\mathbb{D}}(r, r)$. Therefore, $\tilde{\Phi}(v)/v \subset \overline{\mathbb{D}}(r, r)$ for all $v \in \mathbb{C} \setminus \{0\}$, which, in conjunction with (19) and an application of statement (i) of Theorem 5 to the system

$$\dot{x} - \tilde{A} x \in -\tilde{b} \tilde{\Phi}(c^* x), \quad x(0) = x^0 \in \mathbb{C}^n, \tag{20}$$

shows that (20) is stable in the large. Since (3) and (20) have the same solutions, it follows that (3) is stable in the large, establishing statement (i) of Theorem 8.

As a corollary of Theorem 8, we obtain the following real version of the circle criterion.

Corollary 9: **(Classical circle criterion – the real case)** Assume that $\Delta = 0$, $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is stabilizable and detectable, and $\Phi \in \mathcal{U}_{\mathbb{R}}$ with $\Phi(0) = \{0\}$. Furthermore, let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real.

(i) If

$$\alpha v^2 \le \Phi(v)v \le \beta v^2, \quad v \in \mathbb{R}, \tag{21}$$

then (3) is stable in the large. Moreover, (9) holds with $\gamma \in \mathcal{K}$ given by $\gamma(s) = gs$, where the constant $g > 0$ depends on (A, b, c), α, and β, but not on Φ.

(ii) If

$$\alpha v^2 < \Phi(v)v < \beta v^2, \quad v \in \mathbb{R}\backslash\{0\}, \tag{22}$$

then (3) is asymptotically stable in the large.

(iii) If there exists $\delta > 0$ such that

$$(\alpha + \delta)v^2 \le \Phi(v)v \le (\beta - \delta)v^2, \quad v \in \mathbb{R}, \tag{23}$$

then (3) is globally exponentially stable. Moreover, (10) holds with constants $\varepsilon > 0$ and $g > 0$ depending on (A, b, c), α, β, and δ, but not on Φ.

To derive Corollary 9 from Theorem 8, it is convenient to complexify the real map $\Phi \in \mathcal{U}_{\mathbb{R}}$ by defining

$$\Phi_c(v) := \Phi(\operatorname{Re} v) + i\Phi(\operatorname{Im} v) = \{w_1 + iw_2 \colon w_1 \in \Phi(\operatorname{Re} v),\, w_2 \in \Phi(\operatorname{Im} v)\}, \quad v \in \mathbb{C}.$$

Observe that $\Phi_c \in \mathcal{U}_{\mathbb{C}}$ and, if $\Phi(0) = \{0\}$, then $\Phi_c(v) = \Phi(v)$ for all $v \in \mathbb{R}$. Furthermore, if $\Phi(0) = \{0\}$ and Φ satisfies (21), then

$$\mathrm{Re}\,\big((w - \alpha v)(\overline{w - \beta v})\big) \leq 0, \quad w \in \Phi_c(v),\ v \in \mathbb{C},$$

that is, Φ_c satisfies the complex sector condition (16). Part (i) of Corollary 9 follows now from part (i) of Theorem 8. Parts (ii) and (iii) of Corollary 9 can be proved in a similar way.

While the set-valued quantization map Q_η, defined by (7) and illustrated in Figure 9, satisfies the sector condition (21) with $\alpha = 0$ and $\beta = 2$, there are many set-valued nonlinearities of interest, in particular, set-valued nonlinearities relevant to the description of hysteretic and friction phenomena, that satisfy one of the sector conditions (21), (22), or (23) not for all $v \in \mathbb{R}$, or not for all $v \in \mathbb{R}\backslash\{0\}$ in the case of (22), but only for all v with $|v|$ sufficiently large. Stability results for the Lur'e-type system (3) with set-valued nonlinearities Φ of this type, that is, sector bounded outside a compact interval, are presented in an ISS context in the section "The Circle Criterion and ISS", see corollaries 16, 20, and 21.

Corollary 9 can be used to derive stability properties of time-varying Lur'e-type systems of the form

$$\dot{x}(t) = Ax(t) + b\big(d(t) - f(t, c^* x(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{24}$$

provided that $f\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies a suitable sector condition uniformly in t. Here we assume that f is sufficiently regular to guarantee well-posedness of (24). In particular, it is assumed that f is continuous in its second argument. If, for example, there exists $\delta > 0$ such that f satisfies the sector condition

$$(\alpha + \delta)v^2 \leq f(t, v) \leq (\beta - \delta)v^2, \quad (t, v) \in [0, \infty) \times \mathbb{R},$$

then define $\Phi \in \mathcal{U}_{\mathbb{R}}$ by

$$\Phi(v) = \begin{cases} [(\alpha+\delta)v, (\beta-\delta)v], & v \geq 0, \\ [(\beta-\delta)v, (\alpha+\delta)v], & v < 0\,. \end{cases} \tag{25}$$

Note that $\Phi(0) = \{0\}$ and Φ satisfies the sector condition (23). Furthermore, for each $v \in \mathbb{R}$, $f(t, v) \in \Phi(v)$ for all $t \geq 0$, and every solution of the time-varying system (24) is also a solution of (3) with Φ given by (25). Consequently, if $(1+\beta\mathbf{G})(1+\alpha\mathbf{G})^{-1}$ is positive real, statement (iii) of Corollary 9 guarantees that all solutions of the time-varying system (24) decay exponentially fast.

We give an example that shows that, in statement (iii) of Corollary 9, the constant $\delta > 0$ is essential for exponential stability. Consider the integrator $\dot{x} = u$ and apply negative feedback $u = -f(x)$ to obtain the initial-value problem

$$\dot{x} = -f(x), \quad x(0) = x^0, \tag{26}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is the saturating nonlinearity given by

$$f(v) = \begin{cases} v^3, & v \in [-1, 1], \\ +1, & v > 1, \\ -1, & v < -1\,, \end{cases}$$

see Figure 11. Setting $\Phi(v) := \{f(v)\}$, we see that the sector condition (22) holds if and only if $\alpha \leq 0$ and $\beta > 1$. We also note that there exists $\delta > 0$ such that (23) is satisfied if and only if $\alpha < 0$ and $\beta > 1$. The transfer function $\mathbf{G}$ in this example is given by $\mathbf{G}(s) = 1/s$, and

$$\frac{1+\beta\mathbf{G}(s)}{1+\alpha\mathbf{G}(s)} = \frac{s+\beta}{s+\alpha}$$

is positive real if and only if $\alpha \geq 0$ and $\beta \geq 0$. Therefore, if $(s+\beta)/(s+\alpha)$ is positive real, then there is no value $\delta > 0$ for which the sector condition (23) on Φ holds. On the other hand, both the positive real condition on $(s+\beta)/(s+\alpha)$ and the sector condition (22) hold if and only if $\alpha = 0$ and $\beta > 1$. Consequently, by statement (ii) of Corollary 9, we can conclude that (26) is asymptotically stable in the large. While the sufficient conditions associated with statement (iii) of Corollary 9 fail to hold in this example, this failure does not by itself rule out the possibility of global exponential stability. However, the conclusion that (26) is not globally exponentially stable can be arrived at by computing the solution of (26). For example, if $x^0 > 1$, the solution x of (26) is given by

$$x(t) = \begin{cases} x^0 - t, & t \in [0, x^0 - 1], \\ 1/\sqrt{1 + 2(t + 1 - x^0)}, & t > x^0 - 1. \end{cases} \tag{27}$$

The formula (27) implies in particular that (26) is not globally exponentially stable. Hence, this example shows that, in statement (iii) of Corollary 9, the existence of a positive constant $\delta > 0$ is essential for global exponential stability; in fact, the weaker sector condition (22) does not suffice.

The following lemma, which gives graphical characterizations of the positive realness of $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ in terms of the Nyquist diagram of $\mathbf{G}$, shows why Corollary 9 is called the circle criterion. Recall that, if $\mathbf{G}$ does not have any poles on the imaginary axis, then the *Nyquist diagram* of $\mathbf{G}$ is defined to be the closure of the set $\mathbf{G}(i\mathbb{R}) = \{\mathbf{G}(i\omega) \colon \omega \in \mathbb{R}\}$ regarded as an oriented curve, whose orientation is induced by increasing ω.

Lemma 10: For $\alpha < \beta$ with $\alpha\beta \neq 0$, let $D(\alpha, \beta)$ denote the open disc in the complex plane with center in $\mathbb{R}$ and such that $-1/\alpha$ and $-1/\beta$ belong to the boundary of $D(\alpha, \beta)$. The following statements hold.

(i) If $\alpha\beta > 0$ and $\mathbf{G}$ does not have any poles on the imaginary axis, then $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$

is positive real if and only if the Nyquist diagram of $\mathbf{G}$ does not intersect the disc $D(\alpha, \beta)$ and encircles it p times in the counterclockwise sense, where p denotes the number of poles in $\mathbb{C}_+$.

(ii) If $\alpha\beta < 0$, then $(1+\beta\mathbf{G})(1+\alpha\mathbf{G})^{-1}$ is positive real if and only if $\mathbf{G} \in H^\infty$ and the Nyquist diagram of $\mathbf{G}$ is contained in $\overline{D}(\alpha, \beta)$.

For convenience, in Lemma 10 we use the notation $D(\alpha, \beta)$. This disc is identical to $\mathbb{D}(z, r)$, where $z = -(\alpha + \beta)/(2\alpha\beta)$ and $r = (\beta - \alpha)/(2\alpha\beta)$.

The following example illustrates Lemma 10.

Example 11: Assume that $\mathbf{G}$ is given by $\mathbf{G}(s) = 10/(s^3 + 5s^2 + 4s - 10)$, which has one pole in $\mathbb{C}_+$ at $s = 1$. The remaining poles are located at $s = -2 \pm i$. With reference to Figure 12, we see that, for $\alpha = 1.07$ and $\beta = 1.5$, the Nyquist diagram of $\mathbf{G}$ does not intersect the disc $D(\alpha, \beta)$ and encircles it once in the counterclockwise sense. Therefore, by statement (i) of Lemma 10, $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real.

Now assume that $\mathbf{G}$ is given by $\mathbf{G}(s) = 10/(s^3 + 7s^2 + 16s + 10)$, whose poles are $s = -1$ and $s = -2 \pm i$. With reference to Figure 13, we see that the Nyquist diagram of $\mathbf{G}$ is contained in the closed disc $\overline{D}(-1, 1)$ and thus, by statement (ii) of Lemma 10, $(1-\mathbf{G})(1+\mathbf{G})^{-1}$ is positive real. $\diamondsuit$

The following result shows that if, in Corollary 9, the assumption of positive realness is replaced by the stronger assumption of strict positive realness, then the value of the constant δ in statement (iii) of Corollary 9 can be taken to be 0. In this context, see also [43, Theorem 5.1] and [11, Theorem 7.1].

Corollary 12: Assume that $\Delta = 0$, $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is stabilizable and detectable, and $\Phi \in \mathcal{U}_{\mathbb{R}}$, where $\Phi(0) = \{0\}$. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. If $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is strictly positive real and

$$\alpha v^2 \leq \Phi(v)v \leq \beta v^2, \quad v \in \mathbb{R},$$

then (3) is globally exponentially stable. Moreover, (10) holds with constants $\varepsilon > 0$ and $g > 0$ depending on (A, b, c), α, β, and δ, but not on Φ.

The next result extends statements (i) and (ii) of Corollary 9 to the case $\beta = \infty$; note, however, that the assumption of stabilizability is replaced by controllability.

Theorem 13: Assume that $\Delta = 0$, $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is controllable and detectable, and $\Phi \in \mathcal{U}_{\mathbb{R}}$, where $\Phi(0) = \{0\}$. Furthermore, let $\alpha \in \mathbb{R}$ and assume that $\mathbf{G}(1+\alpha\mathbf{G})^{-1}$ is positive real.

(i) If $\Phi(0) = \{0\}$ and

$$\alpha v^2 \leq \Phi(v)v, \quad v \in \mathbb{R}, \tag{28}$$

then (3) is stable in the large. If, in addition, (A, b, c) is observable, then there exists $g > 0$ such that

$$\|x(t)\| \leq g\|x^0\|, \quad t \geq 0,\ x \in \mathcal{X}(x^0),$$

where g depends on (A, b, c) and α, but not on Φ.

(ii) If

$$\alpha v^2 < \Phi(v)v, \quad v \in \mathbb{R} \setminus \{0\}, \tag{29}$$

then (3) is asymptotically stable in the large.

Theorem 13 can be used to extend statements (i) and (ii) of Corollary 9 to the case $\alpha = -\infty$ and $\beta < \infty$.

We close this section with a result that is in the spirit of the real Aizerman conjecture in the sense that a condition on the linear component of the feedback system is identified that together with the assumption $(\alpha, \beta) \subset S(A, b, c)$ guarantees that (3) is asymptotically stable in the large for all $\Phi \in \mathcal{U}_{\mathbb{R}}$ with $\Phi(0) = \{0\}$ and such that (22) holds. To this end, recall the notation $\mathbf{G}_k = \mathbf{G}(1 + k\mathbf{G})^{-1}$.

Corollary 14: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is stabilizable and detectable. Let $\alpha < \beta$ and set $k := (\alpha + \beta)/2$. If $(\alpha, \beta) \subset S(A, b, c)$ and

$$\max\{|\mathbf{G}_k(i\omega)| \colon \omega \in \mathbb{R} \text{ s.t. } \mathbf{G}_k(i\omega) \in \mathbb{R}\} = \|\mathbf{G}_k\|_{H^\infty}, \tag{30}$$

then (3) is asymptotically stable in the large for all $\Phi \in \mathcal{U}_{\mathbb{R}}$ with $\Phi(0) = \{0\}$ and such that $\alpha v^2 < \Phi(v)v < \beta v^2$ for all $v \in \mathbb{R}\backslash\{0\}$.

Note that (30) says that the maximal distance from the Nyquist diagram of $\mathbf{G}_k$ to the origin is attained when the Nyquist diagram intersects with the real axis. The transfer function $\mathbf{G}$ given by $\mathbf{G}(s) = 10/(s^3 + 7s^2 + 16s + 10)$, which is considered in Example 11, satisfies (30) with $k = 0$, see Figure 13.

To see how Corollary 14 can be derived from Corollary 9, it is convenient to define

$$l := \frac{\beta - \alpha}{2}, \quad A_k := A - kbc^*.$$

Then $(\alpha, \beta) = (k - l, k + l)$ and, since $(\alpha, \beta) \subset S(A, b, c)$, we have

$$(-l, l) \subset S(A_k, b, c). \tag{31}$$

By (30), there exists $\omega_0 \in \mathbb{R}$ such that $\mathbf{G}_k(i\omega_0) \in \mathbb{R}$ and $|\mathbf{G}_k(i\omega_0)| = \|\mathbf{G}_k\|_{H^\infty}$. Setting $r := 1/|\mathbf{G}_k(i\omega_0)| = 1/\|\mathbf{G}_k\|_{H^\infty}$, it follows from a small-gain argument that

$$\mathbb{D}(0, r) \subset \mathbf{S}(\mathbf{G}_k) = S(A_k, b, c). \tag{32}$$

Furthermore, the real output feedback gain $\kappa := -1/\mathbf{G}_k(i\omega_0)$, if applied to $\mathbf{G}_k$, is destabilizing in the sense that $\mathbf{G}_k(1+\kappa\mathbf{G}_k)^{-1}$ has a pole at $i\omega_0$. Consequently, the matrix $A_k - \kappa bc^*$ is not Hurwitz. Now $\kappa = r$ or $\kappa = -r$ and thus, by (31), $l \leq r$. Invoking (32) yields $\mathbb{D}(0, l) \subset S(A_k, b, c)$, which is equivalent to $\mathbb{D}(k, l) \subset S(A, b, c)$. Therefore, by Lemma 6, $1 + 2l\mathbf{G}_\alpha = (1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real, and Corollary 14 follows from Corollary 9.

The Circle Criterion and ISS

We now arrive at one of the main concerns, namely, ISS properties of feedback interconnections of Lur'e type. The following theorem is the first of the two main results on input-to-state stability.

Theorem 15: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is stabilizable and detectable, and $\Phi \in \mathcal{U}_\mathbb{R}$, where $\Phi(0) = \{0\}$. Furthermore, let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real and (23) holds for some $\delta > 0$. Then there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on Φ, such that, for each $x^0 \in \mathbb{R}^n$ and each $\Delta \in \mathcal{D}_\mathbb{R}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ of (3) is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2\|\Delta\|_{L^\infty[0,t]}, \quad t \geq 0. \tag{33}$$

In particular, the system (3) is ISS

Theorem 15 is a refinement of a version of the classical circle criterion [11], [14]. In particular, Theorem 15 shows that, under the standard assumptions of the circle criterion, ISS is guaranteed. We emphasize that proof of Theorem 15 is based on small-gain and exponential weighting techniques but not on Lyapunov methods, see “Proofs” for details. This technique is used in [8, Section V.3] to prove classical stability results of input-output type as well as in [44] to derive a version of the circle criterion that guarantees exponential stability for a class of infinite-dimensional state-space systems. However, its application here is in an ISS context, with origins in [19]. In particular, while the standard textbook version of the circle criterion for state-space systems is usually proved using Lyapunov techniques combined with the positive-real lemma [9, pp. 375], [11, Theorem 7.1], [14, p. 227], or [45, pp. 587], the proof of Theorem 15 given in the section “Proofs” provides an alternative, more elementary, approach. Moreover, the methodology can be extended to an infinite-dimensional setting [29].

In the following corollary of Theorem 15, we consider not only nonlinearities satisfying (23) for all arguments $v \in \mathbb{R}$, but also nonlinearities $\Phi \in \mathcal{U}_{\mathbb{R}}$ with the property that there exists a compact interval $K \subset \mathbb{R}$ such that (23) holds for all arguments $v \in \mathbb{R}\backslash K$, that is,

$$(\alpha + \delta)v^2 \leq \Phi(v)v \leq (\beta - \delta)v^2, \quad v \in \mathbb{R}\backslash K, \tag{34}$$

see Figure 14. For example, single-input, single-output hysteretic elements can be subsumed by this set-valued formulation provided that the characteristic diagram of the hysteresis is contained in the graph of some $\Phi \in \mathcal{U}_{\mathbb{R}}$, see Theorem S3 in “Hysteretic Feedback Systems”.

Corollary 16: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is stabilizable and detectable, and $\Phi \in \mathcal{U}_{\mathbb{R}}$. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real. Furthermore, assume that there exist $\delta > 0$ and a compact interval $K \subset \mathbb{R}$, with $0 \in K$, such that (34) holds. Define

$$\theta := \sup_{v\in K}\sup_{w\in\Phi(v)} \operatorname{dist}(w, I_v), \tag{35}$$

where

$$I_v := \begin{cases} [(\alpha+\delta)v, (\beta-\delta)v], & v \geq 0, \\ [(\beta-\delta)v, (\alpha+\delta)v], & v < 0. \end{cases}$$

Then there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on Φ and K, such that, for each $x^0 \in \mathbb{R}^n$ and each $\Delta \in \mathcal{D}_{\mathbb{R}}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ of (3) is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2\big(\|\Delta\|_{L^\infty[0,t]} + \theta\big), \quad t \geq 0. \tag{36}$$

In particular, the system (3) is ISS with bias $g_2\theta$.

The bias parameter θ defined by (35) provides a natural measure of the extent of the violation of the sector condition $(\alpha+\delta)v^2 \leq \Phi(v)v \leq (\beta-\delta)v^2$ for v in the interval K. The assumption that the interval K contains 0 is imposed for convenience. This assumption is not essential for ISS with bias. Indeed, an inspection of the proof of Corollary 16 shows that, if 0 is not contained in K, then the assertion of Corollary 16 remains valid provided that, on the right-hand side of (36), the term θ is replaced by $\max\{(|\Phi(0)|, \theta)\}$.

Note that, even if the feedback system under investigation is not subject to external inputs or disturbances, Corollary 16 is still of interest because, although the sector condition is not required to hold globally but holds only outside a compact interval, boundedness of all solutions is guaranteed and, moreover, $\limsup_{t\to\infty}\|x(t)\| \leq g_2\theta$.

Next we consider situations that are not covered by Theorem 15. In particular, such situations involve the consideration of feedback nonlinearities with not necessarily linear sector boundaries, as typified, in the case of singleton-valued maps Φ, by figures 15 and 16. For example, the latter figure encompasses nonlinearities with logarithmic growth as well as nonlinearities with exponential growth.

The following two hypotheses involve nonlinear counterparts of the sector conditions (23) and (28).

Hypothesis (H1) $\Phi(0) = \{0\}$, and there exist $\varphi \in \mathcal{K}_\infty$ and $\beta, \delta > 0$ such that

$$\varphi(|v|)|v| \leq \Phi(v)v \leq (\beta - \delta)v^2, \qquad v \in \mathbb{R}, \tag{37}$$

and $1 + \beta\mathbf{G}$ is positive real.

Hypothesis (H2) $\Phi(0) = \{0\}$, and there exists $\varphi \in \mathcal{K}_\infty$ such that

$$\varphi(|v|)|v| \leq \Phi(v)v, \qquad v \in \mathbb{R}, \tag{38}$$

and $\mathbf{G}$ is positive real.

In both **(H1)** and **(H2)**, the assumption that φ is unbounded is essential for ISS. If $\mathcal{K}_\infty$ is replaced by $\mathcal{K}$ in either case, then the ISS property does not necessarily hold. For example, let $\varphi \in \mathcal{K}$ be bounded and choose a bounded nonlinearity $\Phi \in \mathcal{U}_\mathbb{R}$ satisfying either (37) for some $\beta, \delta > 0$ or (38). Consider the one-dimensional case wherein $(A, b, c) = (0, 1, 1)$ and thus $\mathbf{G}$ is given by $\mathbf{G}(s) = 1/s$. Evidently, both $\mathbf{G}$ and $1 + \beta\mathbf{G}$ are positive real. Therefore, **(H1)** or **(H2)**, as appropriate, holds with $\mathcal{K}_\infty$ replaced by $\mathcal{K}$. In either case, and with constant input $\Delta(t) = \{d\}$, we have

$$\dot{x}(t) - d \in -\Phi(x(t)), \quad x(0) = x^0,$$

which, for $d > \sup_{v \in \mathbb{R}} |\Phi(v)|$, has an unbounded solution, and thus the ISS property fails to hold.

Theorem 17: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is controllable and observable, $\Phi \in \mathcal{U}_{\mathbb{R}}$, and either **(H1)** or **(H2)** holds.

(i) There exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $(x^0, \Delta) \in \mathbb{R}^n \times \mathcal{D}_{\mathbb{R}}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ of (3) is global and

$$\|x(t)\| \le \max\big\{\gamma_1(t, \|x^0\|), \gamma_2(\|\Delta\|_{L^\infty[0,t]})\big\}, \quad t \ge 0.$$

In particular, the system (3) is ISS.

(ii) In the case wherein **(H1)** holds, γ_1 and γ_2 depend on (A, b, c), φ, β, and δ, but not on Φ.

In contrast with the small-gain and exponential weighting technique, which is crucial in the proof of Theorem 15, the proof of Theorem 17 is based on a Lyapunov argument. The key step in this argument is to establish the existence of a ISS Lyapunov function, which is a Lyapunov function with special properties. More precisely, we have the following lemma.

Lemma 18: Under the hypotheses of Theorem 17, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that

$$\alpha_1(\|\xi\|) \le V(\xi) \le \alpha_2(\|\xi\|), \quad \xi \in \mathbb{R}^n, \tag{39}$$

$$\max_{w\in\Phi(c^*\xi)} \langle \nabla V(\xi), A\xi + b(d - w)\rangle \le -\alpha_3(\|\xi\|) + \alpha_4(|d|), \quad (\xi, d) \in \mathbb{R}^n \times \mathbb{R}. \tag{40}$$

Moreover, in the case wherein **(H1)** holds, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and V depend on (A, b, c), φ, β, and δ, but not on Φ.

The proof of Lemma 18 is rather technical, see "Proofs" for details. The approach is akin to that of [18] insofar as parts of the argument adopted in the proof of Lemma 18 are variants of arguments used in [18]. Lemma 18 plays a central role in the proof of Theorem 17. In the extensive literature on ISS in the context of differential equations, the fact that the existence of a C^∞ ISS Lyapunov function is both necessary and sufficient for ISS is well established [38], [46]. See also "The Concept of Input-to-State Stability". For the present purposes, we require a suitable variant of the arguments establishing sufficiency of the ISS-Lyapunov function condition, wherein we impose only C^1 smoothness on the function. Again, details can be found in "Proofs".

Example 19: Consider the circuit example in "An Example from Circuit Theory", that is, the system given by (S1) and (S2), where, in (S4), strict inequality holds for every $v \neq 0$ and, moreover, $\lim_{v\to\pm\infty} |h(v)| = \infty$. Define $\varphi \in \mathcal{K}_\infty$ by

$$\varphi(s) = \varphi_0(s) \inf_{|\sigma|\geq s} |h(\sigma)|, \quad s \geq 0,$$

where $\varphi_0 : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, and such that $0 < \varphi_0(s) < 1$ for all $s > 0$; the functions given by $\varphi_0(s) = 1 - 1/(s+1)$ and $\varphi_0(s) = 1 - e^{-s}$ are typical examples. By construction,

$$\varphi(|v|)|v| \leq h(v)v, \quad v \in \mathbb{R}.$$

Combining this inequality with the positive realness of the transfer function (S3), it follows that **(H2)** holds, and thus, by Theorem 17, we conclude that the system (S1) is ISS. $\diamondsuit$

In the next result, we consider nonlinearities for which the inequality (37) is required to hold only for values v outside some nonempty compact interval K, thereby relaxing hypotheses **(H1)** and **(H2)**. The price paid for this added generality is that the ISS property is lost and replaced by ISS with bias.

Corollary 20: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is controllable and observable, let $\Phi \in \mathcal{U}_{\mathbb{R}}$, let $\beta > 0$, and assume that $1 + \beta\mathbf{G}$ is positive real. Furthermore, assume that there exist $\varphi \in \mathcal{K}_\infty$, $\delta > 0$, and a compact interval $K \subset \mathbb{R}$, with $0 \in K$, such that $\varphi(s) \leq (\beta - \delta)s$ for all $s \geq 0$ and

$$\varphi(|v|)|v| \leq \Phi(v)v \leq (\beta - \delta)v^2, \quad v \in \mathbb{R}\backslash K. \tag{41}$$

Define

$$\theta := \sup_{v \in K} \sup_{w \in \Phi(v)} \operatorname{dist}(w, I_v),$$

where

$$I_v := \begin{cases} [\varphi(v), (\beta - \delta)v], & v \geq 0, \\ [(\beta - \delta)v, -\varphi(|v|)], & v < 0. \end{cases}$$

Then there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$, depending on (A, b, c), φ, β, and δ, but not on Φ and K, such that, for each $x^0 \in \mathbb{R}^n$ and each $\Delta \in \mathcal{D}_{\mathbb{R}}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ of (3) is global and

$$\|x(t)\| \leq \max\left\{\gamma_1(t, \|x^0\|)\,,\, \gamma_2(\|\Delta\|_{L^\infty[0,t]} + \theta)\right\}, \quad t \geq 0. \tag{42}$$

In particular, system (3) is ISS with bias $\gamma_2(\theta)$.

Corollary 21: Assume that $(A, b, c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n$ is controllable and observable, $\Phi \in \mathcal{U}_{\mathbb{R}}$, and $\mathbf{G}$ is positive real. Furthermore, assume that there exist $\varphi, \psi \in \mathcal{K}_\infty$ and a compact interval $K \subset \mathbb{R}$, with $0 \in K$, such that $\varphi(s) \leq \psi(s)$ for all $s \geq 0$ and

$$\varphi(|v|)|v| \leq \Phi(v)v \leq \psi(|v|)|v|, \quad v \in \mathbb{R}\backslash K. \tag{43}$$

Define

$$\theta := \sup_{v \in K} \sup_{w \in \Phi(v)} \operatorname{dist}(w, I_v), \tag{44}$$

where

$$I_v := \begin{cases} [\varphi(v), \psi(v)], & \text{if } v \geq 0, \\ [-\psi(|v|), -\varphi(|v|)], & \text{if } v < 0. \end{cases}$$

Then there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$, depending on (A, b, c), φ, and ψ, but not on Φ and K, such that, for each $x^0 \in \mathbb{R}^n$ and each $\Delta \in \mathcal{D}_{\mathbb{R}}$, every solution $x \in \mathcal{X}(x^0, \Delta)$ of (3) is global and

$$\|x(t)\| \leq \max\left\{\gamma_1(t, \|x^0\|)\,,\, \gamma_2(\|\Delta\|_{L^\infty[0,t]} + \theta)\right\}, \quad t \geq 0. \tag{45}$$

In particular, the system (3) is ISS with bias $\gamma_2(\theta)$.

The proofs of corollaries 20 and 21 are similar to that of Corollary 16 and are therefore left to the reader.

Example 22: Consider again the circuit example, that is, the system given by (S1) and (S2), where h now describes a negative resistance element, that is, $h(0) = 0$, $h'(0) < 0$, $h(v) \to \infty$ as $v \to \infty$, and $h(v) \to -\infty$ as $v \to -\infty$. As in Example 19, let $\varphi_0\colon [0, \infty) \to [0, \infty)$ be continuous, strictly increasing, and such that $0 < \varphi_0(s) < 1$ for all $s > 0$. Let $k > \max\{|v| : h(v) = 0\}$ and define $\varphi \in \mathcal{K}_\infty$ by setting

$$\varphi(s) = \varphi_0(s) \inf_{|\sigma| \geq s} |h(\sigma)|, \;\; s \geq k,$$

and

$$\varphi(s) = s\varphi(k)/k, \;\; 0 \leq s < k.$$

Furthermore, let $\psi_0\colon [0, \infty) \to [0, \infty)$ be continuous, strictly increasing, and such that $\psi_0(s) > 1$ for all $s > 0$. Define $\psi \in \mathcal{K}_\infty$ by

$$\psi(s) = \psi_0(s) \sup_{k \leq |\sigma| \leq s} |h(\sigma)|, \;\; s \geq k,$$

and

$$\psi(s) = s\psi(k)/k, \;\; 0 \le s < k.$$

Then $\varphi(s) \le \psi(s)$ for all $s \ge 0$ and

$$\varphi(|v|)|v| \le h(v)v \le \psi(|v|)|v|, \quad v \in \mathbb{R}\backslash[-k,k].$$

Combining this fact with the positive realness of the transfer function (S3), it follows from Corollary 21 that the system (S1) is ISS with bias. The bias parameter θ is given by

$$\theta = \sup_{v\in[-k,k]} \mathrm{dist}(h(v), I_v),$$

where I_v is defined as in (44). $\diamondsuit$

Quantization and Output Disturbances

Let $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous static nonlinearity, and consider the system

$$\dot{x}(t) = Ax(t) + b(d(t) - f(c^*x(t))), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{46}$$

where $d \in L^\infty_{\mathrm{loc}}[0,\infty)$. As before, we denote the transfer function of the linear system (A, b, c) by $\mathbf{G}$, that is, $\mathbf{G}(s) = c^*(sI - A)^{-1}b$. In the following, we want to analyze asymptotic properties of system (46) subject to two classes of disturbances, namely, output disturbances, that is, in (46) the term $f(c^*x(t))$ is replaced by $f(c^*x(t) + d_{\mathrm{o}}(t))$, where $d_{\mathrm{o}} \in L^\infty_{\mathrm{loc}}[0,\infty)$, and output quantization, that is, in (46) the term $f(c^*x(t))$ is replaced by $(f \circ q_\eta)(c^*x(t))$, where the uniform output quantizer q_η is given by (6).

To this end, it is useful to state two auxiliary robustness results. Let $\varrho = (\varrho_1, \varrho_2) \in [0,\infty) \times [0,\infty)$ and define $F_\varrho \in \mathcal{U}_{\mathbb{R}}$ by

$$F_\varrho(v) = \{f(v+r) : r \in [-\varrho_1, \varrho_1]\} + [-\varrho_2, \varrho_2], \quad v \in \mathbb{R}. \tag{47}$$

The following lemma is a consequence of Corollary 16. A detailed proof can be found in the section "Proofs".

Lemma 23: Assume that (A, b, c) is stabilizable and detectable. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real and there exists $\delta > 0$ such that

$$(\alpha + \delta)v^2 \leq f(v)v \leq (\beta - \delta)v^2, \quad v \in \mathbb{R}. \tag{48}$$

Then there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on f, such that, for each $\varrho \in [0,\infty) \times [0,\infty)$, each $F \in \mathcal{U}_{\mathbb{R}}$ satisfying $F(v) \subset F_\varrho(v)$ for all $v \in \mathbb{R}$, each $x^0 \in \mathbb{R}^n$, and each $d \in L^\infty_{\text{loc}}[0,\infty)$, every maximal solution x of

$$\dot{x}(t) - Ax(t) - bd(t) \in -bF(c^*x(t)), \quad x(0) = x^0 \tag{49}$$

is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \|\varrho\|), \quad t \geq 0.$$

In particular, the system (49) is ISS with bias $g_2\|\varrho\|$.

Lemma 23, in the context of the special case $\rho = 0$, shows that under the assumptions imposed on (A, b, c), $\mathbf{G}$, and f, there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$ such that, for every $x^0 \in \mathbb{R}^n$ and every $d \in L^\infty_{\text{loc}}[0,\infty)$, every maximal solution x of (46) is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2\|d\|_{L^\infty[0,t]}, \quad t \geq 0, \tag{50}$$

which can also be obtained as a consequence of Theorem 15. Lemma 23 also guarantees that if, in (46), the nonlinearity f is subjected to a set-valued perturbation such that the resulting nonlinearity F is in $\mathcal{U}_{\mathbb{R}}$ and contained in the ϱ-neighbourhood F_ϱ of f, then, by adding the constant $g_2\|\varrho\|$ to the right-hand side of (50), we obtain an estimate for the solutions of the perturbed system.

The next lemma is a consequence of corollaries 20 and 21. The proof is given in the section "Proofs".

Lemma 24: Assume that (A, b, c) is controllable and observable, and either **(H1)** or **(H2)** holds with $\Phi(v) = \{f(v)\}$ for all $v \in \mathbb{R}$. Then there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $\varrho \in [0, \infty) \times [0, \infty)$, each $F \in \mathcal{U}_{\mathbb{R}}$ satisfying $\Phi(v) \subset F_\varrho(v)$ for all $v \in \mathbb{R}$, each $x^0 \in \mathbb{R}^n$, and each $d \in L^\infty_{\mathrm{loc}}[0, \infty)$, every maximal solution x of (49) is global and

$$\|x(t)\| \le \max\left\{\gamma_1(t, \|x^0\|), \gamma_2(\|d\|_{L^\infty[0,t]} + \|\varrho\|)\right\}, \quad t \ge 0.$$

In particular, the system (49) is ISS with bias $\gamma_2(\|\varrho\|)$.

The comment after the statement of Lemma 23 applies mutatis mutandis to Lemma 24.

PID control in the presence of quantization

With reference to Figure 17, we consider the double integrator with a static nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ in the input channel and subject to input quantization given by

$$\ddot{\xi}(t) = (f \circ q_\eta)(u(t)), \quad \xi(0) = \xi^0, \quad \dot{\xi}(0) = \xi^1, \tag{51}$$

where $q_\eta : \mathbb{R} \to \mathbb{R}$, parameterized by $\eta > 0$, is the uniform quantizer described in Example 2, see Figure 8. The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and sector bounded in the sense that there exist $\alpha > 0$ and $\varphi \in \mathcal{K}_\infty$ such that

$$\alpha v^2 + \varphi(|v|)|v| \leq v f(v), \quad v \in \mathbb{R}. \tag{52}$$

Figure 18 illustrates the case in which φ is linear, that is, there exists $\varepsilon > 0$ such that $\varphi(s) = \varepsilon s$ for all $s \geq 0$.

Adopting the PID control structure

$$u(t) = -\Big(k_p(\xi(t) - r) + k_d\dot{\xi}(t) + k_i \int_0^t (\xi(\tau) - r)\mathrm{d}\tau + k_i z^0\Big), \quad z^0 \in \mathbb{R}, \tag{53}$$

with gains $k_p, k_d, k_i > 0$, the control objective is to asymptotically track an arbitrary constant reference signal $r \in \mathbb{R}$, that is, $e(t) \to 0$ as $t \to \infty$, where $e(t) := \xi(t) - r$.

Writing $z(t) := \int_0^t e(\tau)\mathrm{d}\tau + z^0$, $x(t) := \big[e(t),\, \dot{e}(t),\, z(t)\big]^*$, $x^0 := \big[\xi^0 - r,\, \xi^1,\, z^0\big]^*$ and

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} -k_p \\ -k_d \\ -k_i \end{pmatrix}, \tag{54}$$

with transfer function $\mathbf{G}$ given by

$$\mathbf{G}(s) = c^*(sI - A)^{-1}b = \frac{k_d s^2 + k_p s + k_i}{s^3},$$

we see that the closed-loop initial-value problem (51)-(53) can be expressed in the form

$$\dot{x}(t) = Ax(t) - b(f \circ q_\eta)(c^* x(t)), \quad x(0) = x^0. \tag{55}$$

Note that the linear system (A, b, c) is controllable and observable, and its transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(s) = c^*(sI - A)^{-1}b = \frac{k_d s^2 + k_p s + k_i}{s^3}.$$

As in Example 2, we interpret the differential equation (55) with discontinuous righthand side in a set-valued sense by embedding the quantizer q_η in the set-valued map $Q_\eta \in \mathcal{U}_\mathbb{R}$, see (6) and (7), and also figures 8 and 9. We now subsume (55) in the differential inclusion

$$\dot{x}(t) - Ax(t) \in -b\Phi_\eta(c^* x(t)), \quad x(0) = x^0 \in \mathbb{R}^3, \tag{56}$$

where $\Phi_\eta \in \mathcal{U}$ is given by

$$\Phi_\eta(v) := f(Q_\eta(v)) = \{f(\zeta) \colon \zeta \in Q_\eta(v)\}. \tag{57}$$

Set $\tilde{f}(v) := f(v) - \alpha v$ and $\tilde{\Phi}_\eta(v) := \Phi_\eta(v) - \alpha v$ for all $v \in \mathbb{R}$ and $\tilde{A} := A - \alpha bc^*$. Note that x is a solution of (56) if and only if x is solution of

$$\dot{x}(t) - \tilde{A}x(t) \in -b\tilde{\Phi}_\eta(c^* x(t)), \quad x(0) = x^0 \in \mathbb{R}^3. \tag{58}$$

Note further that, for all $v \in \mathbb{R}$,

$$\tilde{\Phi}_\eta(v) \subset \tilde{f}(Q_\eta(v)) + \alpha Q_\eta(v) - \alpha v \subset \{\tilde{f}(v + r) : r \in [-\eta, \eta]\} + \alpha[-\eta, \eta].$$

Therefore, in order to apply Lemma 24 to (58), it is sufficient to check that, in the context of the linear system $(\tilde{A}, b, c)$ and the nonlinearity $\tilde{f}$, the hypotheses of Lemma 24 are satisfied. It follows from (52) that

$$\varphi(|v|)|v| \le \tilde{f}(v)v, \quad v \in \mathbb{R}. \tag{59}$$

Next, we choose the controller gains to ensure that the transfer function $\mathbf{G}(1 + \alpha\mathbf{G})^{-1}$ of the linear system given by $(\tilde{A}, b, c)$ is positive real. Let $k_p > 0$. Choose $k_d > 0$ sufficiently large and $k_i > 0$ sufficiently small so that

$$\alpha k_d^2 > k_p, \quad k_i < \min\left\{\alpha k_d k_p\,,\, k_p^2/(2k_d)\right\}.$$

With these choices, we have $\mathbf{G}(1 + \alpha\mathbf{G})^{-1} \in H^\infty$ and

$$\mathrm{Re}\left(\mathbf{G}(i\omega)\big(1 + \alpha\mathbf{G}(i\omega)\big)^{-1}\right) \ge 0, \quad \omega \in \mathbb{R},$$

showing that $\mathbf{G}(1+\alpha\mathbf{G})^{-1}$ is positive real. Using (59), it follows that, in the context of the linear system $(\tilde{A}, b, c)$ and the single-valued nonlinearity $\tilde{f}$, hypothesis **(H2)** holds.

Therefore, Lemma 24 can be applied to (58) and thus we can conclude that there exist $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for all $\eta > 0$ and all $x^0 \in \mathbb{R}^3$, every maximal solution x of (58), and hence of (56), is global and satisfies

$$\|x(t)\| \leq \max\left\{\gamma_1(t, \|x^0\|)\,,\, \gamma_2(\eta)\right\}, \quad t \geq 0,$$

In particular, for each fixed $\eta > 0$, Lemma 24 guarantees tracking with asymptotic accuracy $\gamma_2(\eta)$. Moreover, we see that the quantized PID-controlled system is such that exact asymptotic tracking is achieved in the limit as $\eta \downarrow 0$.

For numerical simulation, let $f(v) = v(1+v^2)$, which satisfies (52) with $\alpha = 1/2$ and φ given by $\varphi(s) = \varepsilon s$, where $\varepsilon \in (0, 1/2)$. For the reference value $r = 1$ and the controller gains $k_p = 1, k_d = 4$, and $k_i = 0.1$, Figure 19 shows MATLAB-generated simulations for three values of the quantization parameter η, illustrating the property that asymptotic tracking is recovered as η tends to zero.

Lur'e systems subject to output quantization

Consider again the quantized feedback system described in Example 2. Recall that this system, with input $d \in L^\infty_{\text{loc}}[0,\infty)$ and continuous static nonlinearity f, is expressed in the form

$$\dot{x}(t) - Ax(t) - bd(t) \in -b\Phi_\eta(c^* x(t)), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{60}$$

where $\Phi_\eta \in \mathcal{U}_\mathbb{R}$ is given by

$$\Phi_\eta(v) := f(Q_\eta(v)) = \{f(\zeta) \colon \zeta \in Q_\eta(v)\}.$$

Note that (60) is of the form (3) with $\Phi = \Phi_\eta$ and $\Delta(t) = \{d(t)\}$ for all $t \geq 0$.

Corollary 25: Assume that (A, b, c) is stabilizable and detectable. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real and there exists $\delta > 0$ such that

$$(\alpha + \delta)v^2 \leq f(v)v \leq (\beta - \delta)v^2, \quad v \in \mathbb{R}. \tag{61}$$

Then there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on f, such that, for each $x^0 \in \mathbb{R}^n$, each $\eta > 0$, and each $d \in L^\infty_{\text{loc}}[0, \infty)$, every maximal solution x of (60) is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \eta), \quad t \geq 0.$$

In particular, system (60) is ISS with bias $g_2\eta$.

To show how Corollary 25 follows from Lemma 23, let x be a maximal solution of (60) and let $F_{(\eta,0)} \in \mathcal{U}_\mathbb{R}$ be defined by (47). Then $\Phi_\eta(v) \subset F_{(\eta,0)}(v)$ for all $v \in \mathbb{R}$, and, therefore, x is also a maximal solution of

$$\dot{x}(t) - Ax(t) - bd(t) \in -bF_{(\eta,0)}(c^*x(t)), \quad x(0) = x^0.$$

It follows from Lemma 23 that there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on f, such that, for each $x^0 \in \mathbb{R}^n$, each $d \in L^\infty_{\text{loc}}[0, \infty)$, and each $\eta > 0$, x is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \eta), \quad t \geq 0,$$

establishing Corollary 25.

Invoking Lemma 24 instead of Lemma 23, an argument similar to the one above yields the following corollary.

Corollary 26: Assume that (A, b, c) is controllable and observable, and either **(H1)** or **(H2)** holds with $\Phi(v) = \{f(v)\}$ for all $v \in \mathbb{R}$. Then there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $x^0 \in \mathbb{R}^n$, each $\eta > 0$, and each $d \in L^\infty_{\text{loc}}[0, \infty)$, every maximal solution x of (60) is global and

$$\|x(t)\| \le \max\left\{\gamma_1(t, \|x^0\|), \gamma_2(\|d\|_{L^\infty[0,t]} + \eta)\right\}, \quad t \ge 0.$$

In particular, system (60) is ISS with bias $\gamma_2(\eta)$.

Lur'e systems subject to output disturbances

Consider the system

$$\dot{x}(t) = Ax(t) + b\big(d(t) - f(c^* x(t) + d_{\text{o}}(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n, \tag{62}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $d, d_{\text{o}} \in L^\infty_{\text{loc}}[0, \infty)$, see also Figure 20. The following result shows that, under the standard assumptions of the classical circle criterion, the system (62) is ISS with respect to d and d_{o}.

Corollary 27: Assume that (A, b, c) is stabilizable and detectable. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and assume that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real and there exists $\delta > 0$ such that

$$(\alpha + \delta)v^2 \le f(v)v \le (\beta - \delta)v^2, \quad v \in \mathbb{R}. \tag{63}$$

Then there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A, b, c), α, β, and δ, but not on f, such that, for all $x^0 \in \mathbb{R}^n$ and all $d, d_{\text{o}} \in L^\infty_{\text{loc}}[0, \infty)$, every maximal solution x of (62) is global and

$$\|x(t)\| \le g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \|d_{\text{o}}\|_{L^\infty[0,t]}), \quad t \ge 0. \tag{64}$$

In particular, the system (62) is ISS with respect to d and d_{o}.

If either **(H1)** or **(H2)** holds, then we have the following result.

Corollary 28: Assume that (A, b, c) is controllable and observable, and either **(H1)** or **(H2)** holds with $\Phi(v) = \{f(v)\}$ for all $v \in \mathbb{R}$. Then there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for all $x^0 \in \mathbb{R}^n$ and all $d, d_{\mathrm{o}} \in L^\infty_{\mathrm{loc}}[0,\infty)$, every maximal solution x of (62) is global and

$$\|x(t)\| \le \max\left\{\gamma_1(t, \|x^0\|), \gamma_2(\|d\|_{L^\infty[0,t]} + \|d_{\mathrm{o}}\|_{L^\infty[0,t]})\right\}, \quad t \ge 0.$$

In particular, system (62) is ISS with respect to d and d_{o}.

The proof of Corollary 27 can be found in the section "Proofs". The proof of Corollary 28 is similar and is therefore not included.

Proofs

Proof of Theorem 5. Let $z \in \mathbb{C}$, $r > 0$ and assume that $\mathbb{D}(z, r) \subset S(A, b, c)$. Assume further that $\Delta = 0$. Let $x^0 \in \mathbb{C}^n$ and $x \in \mathcal{X}(x^0)$. Setting $\tilde{A} := A - zbc^*$ and defining $\tilde{\Phi} \in \mathcal{U}_{\mathbb{C}}$ by $\tilde{\Phi}(v) := \Phi(v) - zv$, it follows that x is also a maximal solution of

$$\dot{x}(t) - \tilde{A}x(t) \in -b\tilde{\Phi}(c^*x(t))\,, \quad x(0) = x^0. \tag{65}$$

The proof of statement (i) makes essential use of arguments from [17, pp. 703]. Note that the complex stability radius

$$r_{\mathbb{C}}(\tilde{A}; b, c) := \inf\{|k| \colon k \in \mathbb{C} \text{ s.t. } \tilde{A} + kbc^* \text{ is not Hurwitz}\}$$

satisfies $r_{\mathbb{C}}(\tilde{A}; b, c) \geq r$. By [41] or [47, Theorem 23.3.1], there exists a matrix $P = P^* \geq 0$ solving the Riccati equation

$$P\tilde{A} + \tilde{A}^* P + r^2 cc^* + Pbb^* P = 0. \tag{66}$$

Note that, as an immediate consequence of (66), we have

$$\ker P \subset \ker c^*. \tag{67}$$

For all $\xi \in \mathbb{C}^n$, define $V(\xi) := \langle \xi, P\xi \rangle$ and

$$V_{\mathrm{d}}(\xi) := \{2\mathrm{Re}\,\langle \tilde{A}\xi - bw, P\xi \rangle \colon w \in \tilde{\Phi}(c^*\xi)\},$$

so that

$$(V \circ x)'(t) \in V_{\mathrm{d}}(x(t)), \quad \text{a.a. } t \in [0, T), \tag{68}$$

where $[0, T)$ is the maximal interval of existence of x. Invoking (66), we have

$$V_{\mathrm{d}}(\xi) = \{-|w + b^* P\xi|^2 - r^2|c^*\xi|^2 + |w|^2 \colon w \in \tilde{\Phi}(c^*\xi)\}, \quad \xi \in \mathbb{C}^n. \tag{69}$$

Assume now that (13) holds. Then,

$$|\tilde{\Phi}(c^*\xi)| \leq r|c^*\xi|, \quad \xi \in \mathbb{C}^n, \tag{70}$$

and therefore, by (69),

$$\max V_{\mathrm{d}}(\xi) \leq 0, \quad \xi \in \mathbb{C}^n. \tag{71}$$

Consequently, by (68),

$$(V \circ x)'(t) \leq 0 \quad \text{a.a. } t \in [0, T). \tag{72}$$

Let Π be the orthogonal projection of $\mathbb{C}^n$ onto $(\ker P)^\perp$ and define the function $x_\perp$ by setting $x_\perp(t) = \Pi x(t)$ for all $t \in [0, T)$. The restriction of the quadratic form V to $(\ker P)^\perp$

is positive definite, so that there exists $\varepsilon > 0$ such that $V(\xi) \geq \varepsilon \|\xi\|^2$ for all $\xi \in (\ker P)^\perp$. Moreover, $V(x(t)) = V(x_\perp(t))$ for all $t \in [0, T)$, and thus, invoking (72), we conclude that

$$\varepsilon \|x_\perp(t)\|^2 \leq V(x_\perp(t)) \leq V(x_\perp(0)) = V(x^0) \leq \|P\| \|x^0\|^2, \quad t \in [0, T). \tag{73}$$

Now, by (67), $c^* x(t) = c^* x_\perp(t)$ for all $t \in [0, T)$, and therefore, by (73),

$$|c^* x(t)| \leq g_0 \|x^0\|, \quad t \in [0, T), \tag{74}$$

where $g_0 := \|c\| \sqrt{\|P\|/\varepsilon}$. Furthermore, applying Filippov's selection theorem shows that there exists a measurable function $u : [0, T) \to \mathbb{R}$ such that $u(t) \in -\tilde{\Phi}(c^* x(t))$ for a.a. $t \in [0, T)$ and

$$\dot{x}(t) = \tilde{A} x(t) + b u(t), \quad \text{a.a. } t \in [0, T). \tag{75}$$

See "Filippov's Selection Theorem" for details. By (70),

$$|u(t)| \leq r |c^* x(t)|, \quad \text{a.a. } t \in [0, T), \tag{76}$$

which, combined with (74), yields

$$|u(t)| \leq r g_0 \|x^0\|, \quad \text{a.a. } t \in [0, T). \tag{77}$$

Since $\tilde{A}$ is Hurwitz an application of the variation-of-parameters formula to (75) shows that there exist positive constants g_1 and g_2, depending only on $\tilde{A}$ and b, such that

$$\|x(t)\| \leq g_1 \|x^0\| + g_2 \|u\|_{L^\infty(0,T)}, \quad t \in [0, T)\,.$$

This argument shows that x is bounded and thus, by Lemma 1, $T = \infty$, that is, the solution x is global. Moreover, using (77) and setting $g := g_1 + r g_0 g_2$, we obtain

$$\|x(t)\| \leq g \|x^0\|, \quad t \geq 0\,,$$

completing the proof of statement (i).

We proceed to prove statement (ii). Note that, by (72) and the fact that $T = \infty$, the limit of $V(x(t))$ as $t \to \infty$ exists and is finite. Let Ω denote the omega-limit set of x. We claim that

$$\Omega \subset \ker c^*. \tag{78}$$

Seeking a contradiction, suppose the claim is not true. Then there exists $\zeta \in \Omega$ such that $c^*\zeta \neq 0$. Choose $\varepsilon > 0$ such that $c^*\xi \neq 0$ for all $\xi \in B_\varepsilon$, where $B_\varepsilon := \{\xi \in \mathbb{C}^n \colon \|\xi - \zeta\| \leq \varepsilon\}$. Since (14) holds, it follows that

$$|\tilde{\Phi}(c^*\xi)| - r|c^*\xi| < 0, \qquad \xi \in B_\varepsilon. \tag{79}$$

Next, we assert that a stronger property holds, namely, that there exists $\delta > 0$ such that

$$|\tilde{\Phi}(c^*\xi)| - r|c^*\xi| < -\delta, \qquad \xi \in B_\varepsilon. \tag{80}$$

Suppose otherwise. Then there exists a sequence (ξ_j, w_j) with $\xi_j \in B_\varepsilon$ and $w_j \in \tilde{\Phi}(c^*\xi_j)$ for all positive integers j, and

$$\lim_{j\to\infty} \big(|w_j| - r|c^*\xi_j|\big) = 0.$$

This sequence is bounded and thus has a convergent subsequence, the limit of which we denote by (ξ_∞, w_∞). By compactness of B_ε, it follows that $\xi_\infty \in B_\varepsilon$. By upper semicontinuity of the map $\tilde{\Phi}$ and compactness of its values, $w_\infty \in \tilde{\Phi}(c^*\xi_\infty)$. Hence, $|w_\infty| - r|c^*\xi_\infty| = 0$ and thus $|\tilde{\Phi}(c^*\xi_\infty)| - r|c^*\xi_\infty| \geq 0$, contradicting (79). Therefore, (80) holds, which, in conjunction with (69), gives

$$\max V_{\mathrm{d}}(\xi) \leq -\delta, \quad \xi \in B_\varepsilon. \tag{81}$$

Let (t_j) be a sequence in $[0, \infty)$ such that $t_j \to \infty$ and $x(t_j) \to \zeta$ as $j \to \infty$. Since x is bounded, it follows that $\dot{x}$ is essentially bounded and thus x is uniformly continuous. Therefore, there exists $\tau > 0$ such that

$$\|x(t_j + t) - x(t_j)\| \leq \varepsilon/2, \quad t \in [0, \tau], \quad j \in \mathbb{N}.$$

Choosing $j_0 \in \mathbb{N}$ such that $\|x(t_j) - \zeta\| \leq \varepsilon/2$ for all $j \geq j_0$, it follows that

$$x(t) \in B_\varepsilon, \quad t \in \cup_{j \geq j_0}[t_j, t_j + \tau].$$

Combining this fact with (68) and (81), we conclude that

$$(V \circ x)'(t) \leq -\delta, \quad \text{a.a. } t \in \cup_{j \geq j_0}[t_j, t_j + \tau].$$

Integrating from t_j to $t_j + \tau$, $j \geq j_0$, yields

$$V(x(t_j + \tau)) \leq V(x(t_j)) - \delta\tau, \quad j \geq j_0,$$

contradicting the convergence of $V(x(t))$ as $t \to \infty$. Consequently, (78) is true and thus, $\lim_{t\to\infty} c^* x(t) = 0$. Invoking (75), (76), the fact that $T = \infty$, and the Hurwitz property of $\tilde{A}$, we obtain that $x(t) \to 0$ as $t \to \infty$, completing the proof of statement (ii).

To prove statement (iii), assume that there exists $r_1 \in (0, r)$ such that (15) holds. Since $r_{\mathbb{C}}(\tilde{A}; b, c) \geq r$, there exists $\kappa > 0$ such that $r_{\mathbb{C}}(\tilde{A} + \kappa I; b, c) > r_1$. Again, by [41] or [47, Theorem 23.3.1], there exists a matrix $P_\kappa = P_\kappa^* \geq 0$ solving the Riccati equation

$$P_\kappa(\tilde{A} + \kappa I) + (\tilde{A}^* + \kappa I)P_\kappa + r_1^2 cc^* + P_\kappa bb^* P_\kappa = 0,$$

and hence

$$P_\kappa \tilde{A} + \tilde{A}^* P_\kappa + r_1^2 cc^* + P_\kappa bb^* P_\kappa = -2\kappa P_\kappa. \tag{82}$$

As an immediate consequence of (82) we have that $\ker P_\kappa \subset \ker c^*$. Defining V and V_{d} as before, but with P_κ replacing P, and invoking (82), we have

$$V_{\mathrm{d}}(\xi) = \{-|w + b^* P_\kappa \xi|^2 - r_1^2 |c^* \xi|^2 + |w|^2 - 2\kappa V(\xi) \colon w \in \tilde{\Phi}(c^* \xi)\}, \quad \xi \in \mathbb{C}^n.$$

Since

$$|w| \leq r_1 |c^* \xi|, \qquad w \in \tilde{\Phi}(c^* \xi), \quad \xi \in \mathbb{C}^n, \tag{83}$$

we conclude that

$$\max V_{\mathrm{d}}(\xi) \leq -2\kappa V(\xi), \quad \xi \in \mathbb{C}^n.$$

Consequently, by (68) with $T = \infty$,

$$(V \circ x)'(t) \leq -2\kappa (V \circ x)(t), \quad \text{a.a. } t \geq 0,$$

and thus,

$$V(x(t)) \leq e^{-2\kappa t} V(x^0), \quad t \geq 0.$$

An argument similar to that used to obtain (73) shows that there exists a constant $g_\kappa > 0$, depending only on (A, b, c), z, r, and r_1, such that

$$|c^* x(t)| \leq g_\kappa e^{-\kappa t} \|x^0\|, \quad t \geq 0. \tag{84}$$

As above, Filippov's selection theorem guarantees the existence of a measurable function $u : \mathbb{R} \to \mathbb{R}$ such $u(t) \in -\tilde{\Phi}(c^* x(t))$ for a.a. $t \geq 0$ and

$$\dot{x}(t) = \tilde{A} x(t) + b u(t), \quad \text{a.a. } t \geq 0. \tag{85}$$

By (83) and (84),

$$|u(t)| \leq r_1 g_\kappa e^{-\kappa t} \|x^0\|, \quad \text{a.a. } t \geq 0. \tag{86}$$

Since $\tilde{A}$ is Hurwitz, the conjunction of (85) and (86) imply the existence of constants $g > 0$ and $\varepsilon > 0$ such that

$$\|x(t)\| \leq g e^{-\varepsilon t} \|x^0\|, \quad t \geq 0.$$

Hence statement (iii) holds. □

Proof of Lemma 6. We proceed in two steps.

Step 1. In this step, we first prove the assertion in the specific case of $z = r$. The more general case $z \in \mathbb{C}$ is treated in Step 2. If $z = r$, then $\kappa = 0$ and $\mathbf{H}_\kappa = \mathbf{H}_0 = \mathbf{H}$. Furthermore, note that $\mathbb{D}(r, r) \subset \mathbf{S}(\mathbf{H})$ if and only if $-1/\mathbf{H}(s) \notin \mathbb{D}(r, r)$ for all $s \in \overline{\mathbb{C}}_+$. Now, for every $s \in \mathbb{C}$, the condition $-1/\mathbf{H}(s) \notin \mathbb{D}(r, r)$ is equivalent to $|1 + r\mathbf{H}(s)|^2 \geq r^2|\mathbf{H}(s)|^2$, which, in turn, is equivalent to $1 + 2r\mathrm{Re}\,\mathbf{H}(s) \geq 0$. Hence $\mathbb{D}(r, r) \subset \mathbf{S}(\mathbf{H})$ is equivalent to the positive realness of $1 + 2r\mathbf{H}$.

Step 2. Let $z \in \mathbb{C}$ and note that

$$\mathbf{S}(\mathbf{H}_\kappa) = \mathbf{S}(\mathbf{H}) - \kappa.$$

Therefore, since $\mathbb{D}(z, r) \subset \mathbf{S}(\mathbf{H})$ is equivalent to $\mathbb{D}(r, r) = \mathbb{D}(z, r) - \kappa \subset \mathbf{S}(\mathbf{H}) - \kappa$, it follows that $\mathbb{D}(z, r) \subset \mathbf{S}(\mathbf{H})$ if and only if $\mathbb{D}(r, r) \subset \mathbf{S}(\mathbf{H}_\kappa)$. By Step 1, the last inclusion is equivalent to the positive realness of $1 + 2r\mathbf{H}_\kappa$, completing the proof. $\square$

Proof of Lemma 10. The positive realness of $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is equivalent to the positive realness of $1 + (\beta - \alpha)\mathbf{G}(1 + \alpha\mathbf{G})^{-1}$, which in turn, by Lemma 6, is equivalent to

$$\mathbb{D}(z, r) \subset \mathbf{S}(\mathbf{G}), \tag{87}$$

where $r := (\beta - \alpha)/2$, $z := (\alpha + \beta)/2$, and $\mathbf{S}(\mathbf{G}) := \{k \in \mathbb{C} : \mathbf{G}(1 + k\mathbf{G})^{-1} \in H^\infty\}$.

To prove statement (i), assume that $\alpha\beta > 0$ and note that in this case the function $s \mapsto -1/s$ maps $\mathbb{D}(z, r)$ onto $D(\alpha, \beta)$. It now follows from the Nyquist criterion that (87) is equivalent to the statement that the Nyquist diagram of $\mathbf{G}$ does not intersect the disc $D(\alpha, \beta)$ and encircles it p times in the counterclockwise sense.

To prove statement (ii), assume that $\alpha\beta < 0$ and note that, in this case, $0 \in \mathbb{D}(z, r)$ and the function $s \mapsto -1/s$ maps $\mathbb{D}(z, r)$ onto $(\mathbb{C} \setminus \overline{D}(\alpha, \beta)) \cup \{\infty\}$. Consequently, if (87) holds, then $\mathbf{G} \in H^\infty$ and

$$\mathbf{G}(i\omega) \in \overline{D}(\alpha, \beta), \quad \omega \in \mathbb{R}. \tag{88}$$

Conversely, if $\mathbf{G} \in H^\infty$ and (88) is satisfied, then it follows from the Nyquist criterion that (87) holds. □

Proof of Corollary 12. By statement (iii) of Corollary 9, it suffices to show that strict positive realness of $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ implies positive realness of $(1 + (\beta + \delta)\mathbf{G})(1 + (\alpha - \delta)\mathbf{G})^{-1}$ for all sufficiently small $\delta > 0$. Recalling that $\mathbf{G}_\alpha := \mathbf{G}(1 + \alpha\mathbf{G})^{-1}$ and noting that $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1} = 1 + (\beta - \alpha)\mathbf{G}_\alpha$, it follows from strict positive realness that there exists $\eta > 0$ such that

$$1 + (\beta - \alpha)\mathrm{Re}\,\mathbf{G}_\alpha(s - \eta) \geq 0, \quad s \in \mathbb{C}_+. \tag{89}$$

We claim that

$$1 + (\beta - \alpha) \inf_{s \in \mathbb{C}_+} \mathrm{Re}\,\mathbf{G}_\alpha(s) > 0. \tag{90}$$

Seeking a contradiction, suppose that (90) is not true. Then, since $\mathbf{G}_\alpha$ is strictly proper, there exists $s_0 \in \overline{\mathbb{C}}_+$ such that $1 + (\beta - \alpha)\mathrm{Re}\,\mathbf{G}_\alpha(s_0) = 0$. By (89), $\mathbf{G}_\alpha$ is analytic in the half plane $\mathrm{Re}\, s > \eta$, and, consequently, $1 + (\beta - \alpha)\mathrm{Re}\,\mathbf{G}_\alpha$ is harmonic in the half plane $\mathrm{Re}\, s > \eta$. The minimum principle for harmonic functions shows that $1 + (\beta - \alpha)\mathrm{Re}\,\mathbf{G}_\alpha(s) = 0$ for all s with $\mathrm{Re}\, s > \eta$. On the other hand, by strict properness of $\mathbf{G}_\alpha$,

$$\lim_{|s| \to \infty} \big(1 + (\beta - \alpha)\mathrm{Re}\,\mathbf{G}_\alpha(s)\big) = 1,$$

yielding the desired contradiction. Therefore, (90) holds. Since

$$\lim_{\delta\downarrow 0}\|\mathbf{G}_{\alpha-\delta}-\mathbf{G}_\alpha\|_{H^\infty}=0,$$

we conclude from (90) that, for all sufficiently small $\delta>0$,

$$1+\big(\beta+\delta-(\alpha-\delta)\big)\operatorname{Re}\mathbf{G}_{\alpha-\delta}(s)\geq 0,\quad s\in\mathbb{C}_+.$$

Therefore,

$$\frac{1+(\beta+\delta)\mathbf{G}}{1+(\alpha-\delta)\mathbf{G}}=1+\big(\beta+\delta-(\alpha-\delta)\big)\mathbf{G}_{\alpha-\delta}$$

is positive real for all sufficiently small $\delta>0$. □

Proof of Theorem 13. Let $x^0\in\mathbb{R}^n$ and $x\in\mathcal{X}(x^0)$. Defining $\tilde{A}:=A-\alpha bc^*$ and $\tilde{\Phi}\in\mathcal{U}_{\mathbb{R}}$ by $\tilde{\Phi}(v):=\Phi(v)-\alpha v$, it follows that x is also a maximal solution of

$$\dot{x}(t)-\tilde{A}x(t)\in -b\tilde{\Phi}(c^*x(t))\,,\quad x(0)=x^0.$$

Note that $(\tilde{A},b,c)$ is a controllable and detectable realization of $\mathbf{G}(1+\alpha\mathbf{G})^{-1}$. A variant of the positive-real lemma, see [48, Problem 5.2.2], guarantees the existence of a real matrix $P=P^*\geq 0$ such that

$$P\tilde{A}+\tilde{A}^*P\leq 0,\quad Pb=c. \tag{91}$$

For all $\xi\in\mathbb{R}^n$, define $V(\xi):=\langle\xi,P\xi\rangle$ and

$$V_{\mathrm{d}}(\xi):=\{2\langle\tilde{A}\xi-bw,P\xi\rangle\colon w\in\tilde{\Phi}(c^*\xi)\}.$$

Then we have

$$(V\circ x)'(t)\in V_{\mathrm{d}}(x(t)),\quad \text{a.a. } t\in[0,T), \tag{92}$$

where $[0,T)$ is the maximal interval of existence of x. The second identity in (91) yields

$$V_{\mathrm{d}}(\xi)=\{2\langle P\tilde{A}\xi,\xi\rangle-2w(c^*\xi)\colon w\in\tilde{\Phi}(c^*\xi)\},\quad \xi\in\mathbb{R}^n. \tag{93}$$

Assume that (28) holds. Then

$$0 \leq w(c^*\xi), \quad w \in \tilde{\Phi}(c^*\xi), \quad \xi \in \mathbb{R}^n. \tag{94}$$

Combining this inequality with (93) and with the fact that, by (91), $\langle P\tilde{A}\xi, \xi\rangle \leq 0$ for all $\xi \in \mathbb{R}^n$, it follows that

$$\max V_{\mathrm{d}}(\xi) \leq -2\min\{w(c^*\xi) : w \in \tilde{\Phi}(c^*\xi)\} \leq 0, \quad \xi \in \mathbb{R}^n. \tag{95}$$

Consequently, by (92),

$$(V \circ x)'(t) \leq 0, \quad \text{a.a. } t \in [0, T). \tag{96}$$

Let Π be the orthogonal projection of $\mathbb{R}^n$ onto $(\ker P)^\perp$ and define the function $x_\perp$ by setting $x_\perp(t) = \Pi x(t)$ for all $t \in [0, T)$. As in the proof of Theorem 5, it can be shown that there exists $g_0 > 0$, depending only on P and c, such that

$$\|c^* x(t)\| \leq g_0 \|x^0\|, \quad t \in [0, T). \tag{97}$$

Let $\eta > 0$ and note that positive realness of $\mathbf{G}(1 + \alpha\mathbf{G})^{-1}$ implies that $\mathbf{G}(1 + (\alpha + \eta)\mathbf{G})^{-1} \in H^\infty$. Consequently, $\hat{A} := A - (\alpha + \eta)bc^*$ is Hurwitz. Defining $\hat{\Phi} \in \mathcal{U}_{\mathbb{R}}$ by $\hat{\Phi}(v) := \Phi(v) - (\alpha + \eta)v$, it follows that x is also a maximal solution of

$$\dot{x}(t) - \hat{A}x(t) \in -b\hat{\Phi}(c^* x(t))\,, \quad x(0) = x^0. \tag{98}$$

Furthermore, an application of Filippov's selection theorem shows that there exists a measurable function $u : [0, T) \to \mathbb{R}$ such that $u(t) \in -\hat{\Phi}(c^* x(t))$ for a.a. $t \in [0, T)$ and

$$\dot{x}(t) = \hat{A}x(t) + bu(t), \quad \text{a.a. } t \in [0, T). \tag{99}$$

Define a nondecreasing function $\gamma_0\colon [0,\infty) \to [0,\infty)$ by $\gamma_0(s) := \max\{|\hat{\Phi}(v)|\colon |v| \le s\}$. Then the function $\gamma_1\colon [0,\infty) \to [0,\infty)$ defined by

$$\gamma_1(0) = 0, \quad \gamma_1(s) = \frac{1}{s}\int_s^{2s} \gamma_0(\sigma)\mathrm{d}\sigma, \quad s > 0,$$

is in $\mathcal{K}_\infty$ and satisfies $\gamma_0(s) \le \gamma_1(s)$ for all $s \ge 0$. It follows that

$$|u(t)| \le \gamma_1(|c^* x(t)|), \quad \text{a.a. } t \in [0,T), \tag{100}$$

which, combined with (97), yields

$$|u(t)| \le \gamma_1(g_0\|x^0\|), \quad \text{a.a. } t \in [0,T). \tag{101}$$

Since $\hat{A}$ is Hurwitz, an application of the variation-of-parameters formula to (99) shows that there exist positive constants g_1 and g_2, depending only on $\hat{A}$ and b, such that

$$\|x(t)\| \le g_1\|x^0\| + g_2\|u\|_{L^\infty(0,T)}, \quad t \in [0,T)\,.$$

It follows that x is bounded and thus, by Lemma 1, $T = \infty$, that is, the solution x is global. Moreover, using (101) and defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(s) := g_1 s + g_2\gamma_1(g_0 s)$, we obtain

$$\|x(t)\| \le \gamma(\|x^0\|), \quad t \ge 0,$$

completing the proof of stability in the large.

If (A, b, c) is observable, then the positive-real lemma guarantees the existence of a positive-definite solution $P = P^* > 0$ of (91). Consequently, (96) leads to

$$\|x(t)\| \le \sqrt{\|P\|\|P^{-1}\|}\|x^0\|, \quad t \in [0,T)\,,$$

which, together with Lemma 1, implies that $T = \infty$. Hence the above inequality is valid for $T = \infty$, showing that (9) holds with $\gamma(s) = gs$, where $g := \sqrt{\|P\|\|P^{-1}\|}$.

Finally, the proof of statement (ii) is similar to the proof of statement (ii) of Theorem 5. □

Proof of Theorem 15. Let $x^0 \in \mathbb{R}^n$, $\Delta \in \mathcal{D}_{\mathbb{R}}$, and $x \in \mathcal{X}(x^0, \Delta)$. Let $[0, T)$ be the maximal interval of existence of x, where $0 < T \leq \infty$. An application of Filippov's selection theorem shows that there exists a measurable function $u : [0, T) \to \mathbb{R}$ such that $u(t) \in \Delta(t) - \Phi(c^* x(t))$ for a.a. $t \in [0, T)$ and

$$\dot{x}(t) = Ax(t) + bu(t), \quad \text{a.a. } t \in [0, T).$$

With $k := (\alpha + \beta)/2$ and $A_k := A - kbc^*$, we have

$$x(t) = e^{A_k t} x^0 + \int_0^t e^{A_k(t-\tau)} b\big(u(\tau) + kc^* x(\tau)\big) \mathrm{d}\tau, \quad t \in [0, T)\,. \tag{102}$$

Since $u(t) \in \Delta(t) - \Phi(c^* x(t))$ for a.a. $t \in [0, T)$, there exist functions $d, w : [0, T) \to \mathbb{R}^m$, not necessarily measurable, such that $u(t) = d(t) - w(t)$, $d(t) \in \Delta(t)$ and $w(t) \in \Phi(c^* x(t))$ for a.a. $t \in [0, T)$. By assumption, there exists $\delta > 0$ such that (23) holds, and thus

$$(\alpha + \delta - k)\big(c^* x(\tau)\big)^2 \leq w(\tau)\big(c^* x(\tau)\big) - k\big(c^* x(\tau)\big)^2 \leq (\beta - \delta - k)\big(c^* x(\tau)\big)^2, \quad \text{a.a. } \tau \in [0, T).$$

Since $\Phi(0) = \{0\}$, it follows that

$$|w(\tau) - kc^* x(\tau)| \leq (l - \delta)|c^* x(\tau)|, \quad \text{a.a. } \tau \in [0, T)\,,$$

where $l := (\beta - \alpha)/2$. Consequently,

$$|u(\tau) + kc^* x(\tau)| \leq |\Delta(\tau)| + (l - \delta)|c^* x(\tau)|, \quad \text{a.a. } \tau \in [0, T). \tag{103}$$

Using the estimate (103) in (102) leads to

$$\|x(t)\| \leq \|e^{A_k t} x^0\| + \|b\| \int_0^t \|e^{A_k(t-\tau)}\| |\Delta(\tau)| \mathrm{d}\tau + (l - \delta)\|b\|\|c\| \int_0^t \|e^{A_k(t-\tau)}\| \|x(\tau)\| \mathrm{d}\tau, \quad t \in [0, T)\,. \tag{104}$$

We now show that $T = \infty$. Seeking a contradiction, suppose that $T < \infty$. Then, by the inequality (104), there exists a constant $a > 0$ such that

$$\|x(t)\| \le a \left(1 + \int_0^t \|x(\tau)\| \mathrm{d}\tau\right), \quad t \in [0, T).$$

By Gronwall's lemma, it follows that the maximal solution x of (102) is bounded on $[0, T)$, which, in conjunction with Lemma 1, contradicts the supposition that $T < \infty$. Consequently, $T = \infty$.

The positive-real assumption implies that

$$\|\mathbf{G}_k\|_{H^\infty} \le \frac{1}{l}, \tag{105}$$

as is shown at the end of the proof. Since $\mathbf{G}_k \in H^\infty$ and $\mathbf{G}_k(s) = \mathbf{G}(I + k\mathbf{G}(s))^{-1} = c^*(sI - A_k)^{-1}b$, the stabilizability and detectability assumptions guarantee that A_k is Hurwitz. Let $\varepsilon > 0$ be sufficiently small so that $A_k + \varepsilon I$ is Hurwitz and

$$g := \sup_{\mathrm{Re}\, s \ge -\varepsilon} |\mathbf{G}_k(s)| < 1/(l - \delta). \tag{106}$$

Set $y := c^*x$ and, for all $t \ge 0$, define $y_\varepsilon(t) := e^{\varepsilon t}y(t)$ and $u_\varepsilon(t) := e^{\varepsilon t}u(t)$. It follows from (102) that

$$y_\varepsilon(t) = c^* e^{(A_k + \varepsilon I)t} x^0 + \int_0^t c^* e^{(A_k + \varepsilon I)(t-\tau)} b(u_\varepsilon(\tau) + k y_\varepsilon(\tau)) \mathrm{d}\tau, \quad t \ge 0.$$

Setting $k_0 := \left(\int_0^\infty \|c^* e^{(A_k+\varepsilon I)\tau}\|^2 \mathrm{d}\tau\right)^{1/2} < \infty$, we obtain

$$\|y_\varepsilon\|_{L^2[0,t]} \le k_0 \|x^0\| + g\|u_\varepsilon + k y_\varepsilon\|_{L^2[0,t]}, \quad t \ge 0. \tag{107}$$

By (103),

$$|u_\varepsilon(\tau) + k y_\varepsilon(\tau)| \le |\Delta_\varepsilon(\tau)| + (l - \delta)|y_\varepsilon(\tau)|, \quad \text{a.a. } \tau \ge 0, \tag{108}$$

where $\Delta_\varepsilon(\tau) := e^{\varepsilon\tau}\Delta(\tau)$ for all $\tau \geq 0$. From (106), we see that $g(l-\delta) < 1$. Hence, setting $k_1 := 1/(1-g(l-\delta))$ and invoking (107) and (108), we have

$$\|y_\varepsilon\|_{L^2[0,t]} \leq k_1 \left(k_0\|x^0\| + g\|\Delta_\varepsilon\|_{L^2[0,t]}\right), \quad t \geq 0. \tag{109}$$

By (102),

$$e^{\varepsilon t}x(t) = e^{(A_k+\varepsilon I)t}x^0 + \int_0^t e^{(A_k+\varepsilon I)(t-\tau)}b(u_\varepsilon(\tau) + ky_\varepsilon(\tau)))\mathrm{d}\tau, \quad t \geq 0,$$

which, together with (108), yields

$$\|x(t)\|e^{\varepsilon t} \leq k_2\|x^0\| + \|b\|\int_0^t \|e^{(A_k+\varepsilon I)(t-\tau)}\|\big(|\Delta_\varepsilon(\tau)| + (l-\delta)|y_\varepsilon(\tau)|\big)\mathrm{d}\tau, \quad t \geq 0, \tag{110}$$

where $k_2 := \sup_{t\geq 0}\|e^{(A_k+\varepsilon I)t}\|$. Invoking Hölder's inequality to estimate the integral on the right-hand side of (110), we conclude that there exists a constant $k_3 > 0$ such that

$$\|x(t)\|e^{\varepsilon t} \leq k_2\|x^0\| + k_3(\|\Delta_\varepsilon\|_{L^2[0,t]} + (l-\delta)\|y_\varepsilon\|_{L^2[0,t]}), \quad t \geq 0. \tag{111}$$

Combining (109) with (111), we conclude that there exist constants k_4 and k_5 such that

$$\|x(t)\|e^{\varepsilon t} \leq k_4\|x^0\| + k_5\|\Delta_\varepsilon\|_{L^2[0,t]}, \quad t \geq 0\,.$$

Finally, noting that $\|\Delta_\varepsilon\|_{L^2[0,t]} \leq (e^{\varepsilon t}/\sqrt{2\varepsilon})\|\Delta\|_{L^\infty[0,t]}$ for all $t \geq 0$, we conclude that there exist constants $g_1 \geq 1$ and $g_2 > 0$ such that

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2\|\Delta\|_{L^\infty[0,t]}, \quad t \geq 0,$$

which is (33).

It remains to be shown that (105) holds. To this end note that, by positive realness of the transfer function $(1+\beta\mathbf{G})(1+\alpha\mathbf{G})^{-1}$,

$$0 \leq \frac{1+\beta\mathbf{G}(s)}{1+\alpha\mathbf{G}(s)} + \frac{1+\beta\overline{\mathbf{G}}(s)}{1+\alpha\overline{\mathbf{G}}(s)}, \quad s \in \mathbb{C}_+.$$

Multiplying by $|1+\alpha\mathbf{G}(s)|^2/2$ and rearranging, we obtain

$$-\frac{\alpha\beta}{2}|\mathbf{G}(s)|^2 \leq 1 + k\big(\mathbf{G}(s)+\overline{\mathbf{G}}(s)\big) + \frac{\alpha\beta}{2}|\mathbf{G}(s)|^2, \quad s \in \mathbb{C}_+.$$

Adding $(\alpha^2+\beta^2)|\mathbf{G}(s)|^2/4$ to both sides shows that

$$l^2|\mathbf{G}(s)|^2 \leq \big(1+k\mathbf{G}(s)\big)\big(1+k\overline{\mathbf{G}}(s)\big), \quad s \in \mathbb{C}_+.$$

Consequently,

$$\left|\frac{\mathbf{G}(s)}{1+k\mathbf{G}(s)}\right|^2 \leq \frac{1}{l^2}, \quad s \in \mathbb{C}_+,$$

from which (105) follows. □

Proof of Corollary 16. First, it follows from the upper semicontinuity of the set-valued nonlinearity Φ together with the compactness of its values and the compactness of K that θ is finite. Let $x^0 \in \mathbb{R}^n$, $\Delta \in \mathcal{D}_\mathbb{R}$, and $x \in \mathcal{X}(x^0,\Delta)$. Let $[0,T)$, $0 < T \leq \infty$, be the maximal interval of existence of x and write $y := c^*x$. Define $z \in L^1_{\text{loc}}([0,T),\mathbb{R}^n)$ by $z := \dot{x} - Ax$. Since $z(t) \in b\big(\Delta(t) - \Phi(y(t))\big)$ for almost every $t \in [0,T)$, there exist functions $d, w : [0,T) \to \mathbb{R}$ such that

$$(d(t), w(t)) \in \Delta(t) \times \Phi(y(t)), \qquad t \in [0,T),$$

and $z(t) = b\big(d(t) - w(t)\big)$ for a.a. $t \in [0,T)$. Define a set-valued function $\tilde{\Phi} \in \mathcal{U}_\mathbb{R}$ by setting $\tilde{\Phi}(v) := I_v$ for all $v \in \mathbb{R}$. Then $\tilde{\Phi}(0) = \{0\}$ and

$$(\alpha+\delta)v^2 \leq \tilde{\Phi}(v)v \leq (\beta-\delta)v^2, \quad v \in \mathbb{R}.$$

For each $t \in [0,T)$, let $\tilde{w}(t) \in \tilde{\Phi}(y(t))$ be the unique point of the compact interval $\tilde{\Phi}(y(t))$ closest to $w(t) \in \Phi(y(t))$. Then

$$|w(t) - \tilde{w}(t)| = \begin{cases} \text{dist}(w(t), I_{y(t)}), & \text{if } y(t) \in K, \\ 0, & \text{if } y(t) \in \mathbb{R}\backslash K\,, \end{cases}$$

so that $|w(t) - \tilde{w}(t)| \leq \theta$ for all $t \in [0, T)$.

Define $\tilde{\Delta} \in \mathcal{D}_{\mathbb{R}}$ by $\tilde{\Delta}(t) := \Delta(t) + [-\theta, \theta]$ and $\tilde{d} : [0, T) \to \mathbb{R}$ by $\tilde{d}(t) := d(t) - w(t) + \tilde{w}(t)$. Then, for a.a. $t \in [0, T)$,

$$z(t) = b(\tilde{d}(t) - \tilde{w}(t)), \quad \tilde{d}(t) \in \tilde{\Delta}(t), \quad \tilde{w}(t) \in \tilde{\Phi}(y(t)),$$

and thus the solution x of (3) is also a solution of

$$\dot{x}(t) - Ax(t) \in b\big(\tilde{\Delta}(t) - \tilde{\Phi}(c^* x(t))\big), \quad x(0) = x^0. \tag{112}$$

Applying Theorem 15 to (112) completes the proof. □

Proof of Theorem 17. By Lemma 18, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and a continuously differentiable function $V \colon \mathbb{R}^n \to [0, \infty)$ such that (39) and (40) hold. Let $x^0 \in \mathbb{R}^n$ and $\Delta \in \mathcal{D}_{\mathbb{R}}$ be arbitrary. By Lemma 1, (3) has a solution and every solution can be maximally extended. Let $x \colon [0, T) \to \mathbb{R}^n$ be a maximal solution of (3). By (40), we have

$$\big(V \circ x\big)'(t) \leq -\alpha_3(\|x(t)\|) + \alpha_4(|\Delta(t)|), \quad \text{a.a. } t \in [0, T). \tag{113}$$

We first show that $T = \infty$. Seeking a contradiction, suppose $T < \infty$. Then, by local essential boundedness of Δ and continuity of α_4, there exists $c_0 > 0$ such that $\alpha_4(|\Delta(t)|) \leq c_0$ for all $t \in [0, T)$. By the final assertion of Lemma 1, x is unbounded, contradicting the fact that, by (113), $\alpha_1(\|x(t)\|) \leq V(x(t)) \leq V(x^0) + c_0 T$ for all $t \in [0, T)$. Therefore, every maximal solution of (3) is global.

Write $\alpha_5 := \alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}_\infty$ and let $\alpha_6 : [0, \infty) \to [0, \infty)$ be a locally Lipschitz function such that $\alpha_6 \leq \alpha_5(s)$ for all $s \geq 0$ and $\alpha_6(s) > 0$ for all $s > 0$. The existence of such a function α_6, which is intuitively reasonable, is established at the end of this proof. Define the locally

Lipschitz function

$$Z\colon \mathbb{R}\to\mathbb{R},\quad \zeta\mapsto Z(\zeta) := \begin{cases} -\alpha_6(\zeta)/2, & \zeta\geq 0,\\ \\ 0, & \zeta<0,\end{cases}$$

consider the scalar system

$$\dot{z}(t) = Z(z(t)),$$

and let $\gamma : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ denote the corresponding flow. Observe that 0 is an equilibrium of his system and $Z(s)s < 0$ for all $s > 0$. It follows that the restriction γ_0 of γ to $[0,\infty)\times[0,\infty)$ is in $\mathcal{KL}$. Now define $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ by

$$\gamma_1(t,s) := \alpha_1^{-1}(\gamma_0(t,\alpha_2(s))),\qquad \gamma_2(s) := \big(\alpha_1^{-1}\circ\alpha_2\circ\alpha_3^{-1}\big)(2\alpha_4(s)).$$

For simplicity of notation, write $k(t) := \alpha_4\big(\|\Delta\|_{L^\infty[0,t]}\big)$, where $t\geq 0$, and define the sets

$$T_1 := \big\{t\geq 0\colon\ V(x(t)) \leq \big(\alpha_2\circ\alpha_3^{-1}\big)\big(2k(t)\big)\big\},$$

$$T_2 := [0,\infty)\backslash T_1 = \big\{t\geq 0\colon\ V(x(t)) > \big(\alpha_2\circ\alpha_3^{-1}\big)\big(2k(t)\big)\big\}.$$

Observe that

$$\|x(t)\| \leq \gamma_2(\|\Delta\|_{L^\infty[0,t]}),\quad t\in T_1, \tag{114}$$

and, moreover,

$$\alpha_3(\|x(t)\|) > 2k(t),\quad t\in T_2.$$

Invoking (40), we obtain

$$(V\circ x)'(t) \leq -\alpha_3(\|x(t)\|) + k(t),\quad \text{a.a. } t\geq 0.$$

Combining the last two inequalities gives

$$(V\circ x)'(t) \leq -\frac{\alpha_3(\|x(t)\|)}{2},\quad \text{a.a. } t\in T_2.$$

By (39), $\alpha_3(\|x(t)\|) \geq \alpha_5(V(x(t))$, whence $\alpha_3(\|x(t)\|) \geq \alpha_6\big(V(x(t))\big) = -2Z\big(V(x(t))\big)$ and thus

$$(V \circ x)'(t) \leq Z\big((V \circ x)(t)\big) < 0, \quad \text{a.a. } t \in T_2. \tag{115}$$

We claim that, if $t \in T_2$, then $[0,t] \subset T_2$. Let $t \in T_2$. Since k is nondecreasing, it follows from the definition of T_2, that, to establish the claim, it is sufficient to prove that

$$V(x(s)) \geq V(x(t))\,, \quad s \in [0,t]. \tag{116}$$

Let $\tau \in (0,t]$ be such that $V(x(\tau)) \geq V(x(t))$. Then $\tau \in T_2$ and, appealing to the continuity of $V \circ x$ and the fact that k is nondecreasing, we can conclude that there exists $\sigma \in [0,\tau)$ such that $[\sigma,\tau] \subset T_2$. Therefore, by (115), $(V \circ x)' < 0$ almost everywhere on $[\sigma,\tau]$, which shows $V(x(s)) > V(x(\tau)) \geq V(x(t))$ for all $s \in [\sigma,\tau)$. Consequently, (116) holds, and thus $[0,t] \subset T_2$.

Let $t \in T_2$. Then, $[0,t] \subset T_2$, and hence, by (115),

$$(V \circ x)'(s) \leq Z\big((V \circ x)(s)\big), \quad \text{a.a. } s \in [0,t].$$

Therefore, $(V \circ x)(t) \leq \gamma_0(t, V(x^0))$, and, since $t \in T_2$ is arbitrary,

$$V(x(t)) \leq \gamma_0(t, V(x^0)) \leq \gamma_0(t, \alpha_2(\|x^0\|)), \quad t \in T_2.$$

Invoking (39), we conclude that

$$\|x(t)\| \leq \gamma_1(t, \|x^0\|), \quad t \in T_2,$$

which, in conjunction with (114), yields

$$\|x(t)\| \leq \max\{\gamma_1(t, \|x^0\|), \gamma_2(\|\Delta\|_{L^\infty[0,t]})\}, \quad t \geq 0,$$

completing the proof of (i).

Now assume that **(H1)** holds. Then, by Lemma 18, the functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and V depend on (A, b, c), φ, β, and δ, but not on Φ. Therefore, the functions γ_1 and γ_2, constructed in the above argument, also depend only on (A, b, c), φ, β, and δ, but not on Φ.

Finally, it remains to show that there exists a locally Lipschitz function $\alpha_6 \colon [0, \infty) \to [0, \infty)$ such that $\alpha_6(s) \le \alpha_5(s)$ for all $s \ge 0$ and $\alpha_6(s) > 0$ for all $s > 0$. Define $\alpha_6 \colon [0, \infty) \to [0, \infty)$ by $\alpha_6(0) := 0$ and

$$\alpha_6(s) := \frac{2\beta(s)}{s} \int_{s/2}^{s} \alpha_5(t) \mathrm{d}t, \quad s > 0.$$

where $\beta \colon [0, \infty) \to [0, 1]$ is given by $\beta(s) = s(2 - s)$ for $s \in [0, 1]$ and $\beta(s) = 1$ for $s > 1$. Then $\alpha_6(s) \le \alpha_5(s)$ for all $s \ge 0$ and $\alpha_6(s) > 0$ for all $s > 0$. Moreover, α_6 is continuously differentiable and hence locally Lipschitz. □

Proof of Lemma 18. For brevity, we present the argument only in the case for which hypothesis **(H1)** holds. The case in which **(H2)** holds is structurally similar and we refer the reader to the proof of [19, Lemma 5.1] for full details.

Let $(A, b, c) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \times \mathbb{R}^n$ be controllable and observable, let $\Phi \in \mathcal{U}_{\mathbb{R}}$, and assume that **(H1)** holds. Then, $1 + \beta \mathbf{G}$ is positive real and thus, by the positive-real lemma, there exists $l \in \mathbb{R}^n$ and a symmetric, positive-definite $P \in \mathbb{R}^{n \times n}$ such that

$$PA + A^* P = -ll^*, \quad Pb = c - \sqrt{2/\beta}\, l\,. \tag{117}$$

Define $V_0 \colon \mathbb{R}^n \to [0,\infty)$, $\xi \mapsto \langle \xi, P\xi \rangle$. Then, for $\xi \in \mathbb{R}^n$ and $(d,w) \in \mathbb{R} \times \Phi(c^*\xi)$,

$$\begin{aligned}\langle \nabla V_0(\xi), A\xi + b(d-w)\rangle &= \langle \xi, (PA + A^*P)\xi\rangle + 2\langle \xi, Pb(d-w)\rangle \\ &= -\big(l^*\xi\big)^2 + 2\big(c^*\xi\big)(d-w) - 2\sqrt{2/\beta}\,\big(l^*\xi\big)(d-w) \\ &= -\big(l^*\xi + \sqrt{2/\beta}\,(d-w)\big)^2 + (2/\beta)(d-w)^2 + 2\big(c^*\xi\big)(d-w) \\ &\le 2d^2/\beta + (4/\beta)|d||w| + 2w^2/\beta + 2|c^*\xi||d| - 2\big(c^*\xi\big)w. \qquad (118)\end{aligned}$$

Note that, by (37) and the fact that $\Phi(0) = \{0\}$,

$$|w| \le (\beta - \delta)|c^*\xi|, \quad w^2 \le (\beta-\delta)\big(c^*\xi\big)w, \quad \xi \in \mathbb{R}^n,\ w \in \Phi(c^*\xi),$$

which, when combined with (118), gives

$$\begin{aligned}\langle \nabla V_0(\xi), A\xi + b(d-w)\rangle &\le 2d^2/\beta + \big(2 + 4(\beta-\delta)/\beta\big)|d||c^*\xi| - (2\delta/\beta)\big(c^*\xi\big)w \\ &= 2d^2/\beta + 2(3-2k_0)|d||c^*\xi| - 2k_0\big(c^*\xi\big)w, \quad \xi \in \mathbb{R}^n,\ (d,w) \in \mathbb{R} \times \Phi(c^*\xi), \quad (119)\end{aligned}$$

wherein, for notational convenience, we have set $k_0 := \delta/\beta$.

For $\xi \in \mathbb{R}^n$ and $d \in \mathbb{R}$, we consider two exhaustive cases.

Case 1. If $2(3-2k_0)|d| \le k_0\varphi(|c^*\xi|)$, then

$$2(3-2k_0)|d||c^*\xi| \le k_0\varphi(|c^*\xi|)|c^*\xi| \le k_0\big(c^*\xi\big)w, \quad w \in \Phi(c^*\xi).$$

Case 2. If $2(3-2k_0)|d| > k_0\varphi(|c^*\xi|)$, then $\varphi^{-1}\big(2(3-2k_0)|d|/k_0\big) > |c^*\xi|$, and thus

$$2(3-2k_0)|d||c^*\xi| < 2(3-2k_0)|d|\varphi^{-1}\big(2(3-2k_0)|d|/k_0\big) = \gamma(|d|),$$

where the function $\gamma \in \mathcal{K}_\infty$ is defined by

$$\gamma(s) := 2(3-2k_0)s\varphi^{-1}\big(2(3-2k_0)s/k_0\big), \quad s \ge 0.$$

Therefore, we conclude that

$$2(3-2k_0)|d||c^*\xi| \leq k_0\big(c^*\xi\big)w + \gamma(|d|), \quad \xi \in \mathbb{R}^n,\ (d,w) \in \mathbb{R} \times \Phi(c^*\xi),$$

which, together with (119), yields

$$\langle \nabla V_0(\xi), A\xi + b(d-w)\rangle \leq -k_0(c^*\xi)w + \gamma(|d|) + 2d^2/\beta,$$

$$\xi \in \mathbb{R}^n, \quad (d,w) \in \mathbb{R} \times \Phi(c^*\xi). \quad (120)$$

Next, by observability, there exists $h \in \mathbb{R}^n$ such that $A - hc^*$ is Hurwitz. Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric, positive-definite matrix such that

$$Q(A-hc^*) + (A-hc^*)^*Q = -3I,$$

and define $W\colon \mathbb{R}^n \to [0,\infty)$ by $W(\xi) := \langle \xi, Q\xi\rangle$. Then, we have

$$\langle \nabla W(\xi), A\xi + b(d-w)\rangle \leq -2\|\xi\|^2 + k_1\|\xi\|\big(|c^*\xi| + |w|\big) + k_1 d^2,$$

$$\xi \in \mathbb{R}^n, \quad (d,w) \in \mathbb{R} \times \Phi(c^*\xi), \quad (121)$$

with $k_1 := \max\big\{2\|Qb\|, 2\|Qh\|, \|Qb\|^2\big\}$.

For notational convenience, define $f_0\colon [0,\infty) \to [0,\infty)$ by $f_0(s) = (1+\beta-\delta)s$ and define the continuous, nondecreasing function $f_1\colon (0,1] \to (0,\infty)$ by

$$f_1(s) := \min_{t\in[s,1]} \frac{t\varphi(t)}{(f_0(t))^2}\,.$$

Moreover, define $f_2\colon [0,\infty) \to [0,\infty)$ by

$$f_2(s) := \begin{cases} 0, & s = 0, \\ \min\{s, f_1(s)\}, & s \in (0,1], \\ f_1(1) + s - 1, & s > 1. \end{cases}$$

It can be verified that f_2 is continuous, nondecreasing, and unbounded. Writing $f_3 := f_2 \circ f_0^{-1}$, we see that f_3 is continuous, nondecreasing, and unbounded, with $f_3(0) = 0$ and, for later use, we record that

$$f_3((1+\beta-\delta)|v|)((1+\beta-\delta)|v|)^2 = (f_3 \circ f_0)(|v|)\big(f_0(|v|)\big)^2$$
$$= f_2(|v|)\big(f_0(|v|)\big)^2 \leq f_1(|v|)\big(f_0(|v|)\big)^2 \leq |v|\varphi(|v|), \quad |v| \leq 1. \quad (122)$$

Next, define $\eta \in \mathcal{K}_\infty$ by

$$\eta(s) := \frac{1}{k_1}\sqrt{\frac{s}{\|Q\|}}\,, \quad s \geq 0,$$

and define the continuous, nondecreasing, and unbounded function $\sigma := f_3 \circ \eta$. Let s^* be the unique point in $(0,\infty)$ with the property $\eta(s^*)\sigma(s^*) = 1$. Define the continuous function $\rho\colon [0,\infty) \to [0,\infty)$ and the continuously differentiable function $V_1\colon \mathbb{R}^n \to [0,\infty)$ by

$$\rho(s) := \begin{cases} \sigma(s), & 0 \leq s \leq s^*, \\ 1/\eta(s), & s > s^*, \end{cases}$$

and

$$V_1(\xi) := \int_0^{W(\xi)} \rho(s)\,\mathrm{d}s\,,$$

respectively. Note that

$$\rho(s) \leq \sigma(s^*) = 1/\eta(s^*) =: k_2, \quad s \geq 0\,, \quad (123)$$

$$\rho(W(\xi))\|\xi\| \leq k_1\sqrt{\|Q\|\|Q^{-1}\|} =: k_3, \quad \xi \in \mathbb{R}^n\,, \quad (124)$$

$$\rho(W(\xi))\|\xi\|^2 \geq \|\xi\| \min\big\{\|\xi\| f_3\big(\|\xi\|/k_3\big)\,,\, k_1\big\}\,, \quad \xi \in \mathbb{R}^n\,. \quad (125)$$

Equation (123) is an immediate consequence of the definition of ρ. To confirm that (124) and (125) hold, we introduce the sets

$$S_1 := \{\xi \in \mathbb{R}^n :\ W(\xi) > s^*\},$$

$$S_2 := \mathbb{R}^n \backslash S_1 = \{\xi \in \mathbb{R}^n :\ W(\xi) \leq s^*\}.$$

Then we have

$$\rho(W(\xi))\|\xi\| = \frac{\|\xi\|}{\eta(W(\xi))} = \frac{\|\xi\| k_1 \sqrt{\|Q\|}}{\sqrt{\langle \xi, Q\xi \rangle}} \leq k_3, \quad \xi \in S_1,$$

and

$$\rho(W(\xi))\|\xi\| \leq \frac{\sqrt{\|Q^{-1}\| s^*}}{\eta(s^*)} = k_3, \quad \xi \in S_2,$$

and thus (124) holds. To see that (125) also holds, simply note that

$$\rho(W(\xi)) = \frac{1}{\eta(W(\xi))} = \frac{k_1 \sqrt{\|Q\|}}{\sqrt{\langle \xi, Q\xi \rangle}} \geq \frac{k_1}{\|\xi\|}, \quad \xi \in S_1$$

and

$$\rho(W(\xi)) = \sigma(W(\xi)) = f_3 \left(\frac{\sqrt{W(\xi)}}{k_1 \sqrt{\|Q\|}} \right) \geq f_3\big(\|\xi\|/k_3\big), \quad \xi \in S_2.$$

The conjunction of (121) and (123) now gives

$$\langle \nabla V_1(\xi), A\xi + b(d-w) \rangle \leq -2\rho(W(\xi))\|\xi\|^2 + k_1 \rho(W(\xi))\|\xi\| \big(|c^*\xi| + |w|\big) + k_1 k_2 d^2,$$

$$\xi \in \mathbb{R}^n, \ \ (d, w) \in \mathbb{R} \times \Phi(c^*\xi). \quad (126)$$

We proceed to obtain a convenient estimate of the term $k_1 \rho(W(\xi))\|\xi\| \big(|c^*\xi| + |w|\big)$. Observing that

$$vw = |v||w| \geq |w|, \ \ w \in \Phi(v), \quad |v| \geq 1,$$

writing $k_4 := \min\{1, \varphi(1)\}/2$, and invoking (37), we can conclude that

$$2vw \geq |v|\varphi(|v|) + |w| \geq |v|\varphi(1) + |w| \geq 2k_4\big(|v| + |w|\big), \quad w \in \Phi(v), \quad |v| \geq 1,$$

which, together with (124), gives

$$\rho(W(\xi))\|\xi\|\big(|c^*\xi| + |w|\big) \leq \frac{k_3}{k_4}\,(c^*\xi)w, \quad w \in \Phi(c^*\xi), \;\; |c^*\xi| \geq 1. \tag{127}$$

Moreover, by (37) and (122),

$$f_3(|v| + |\Phi(v)|)\big(|v| + |\Phi(v)|\big)^2 \leq f_3(|v| + \psi(|v|))\big(|v| + \psi(|v|)\big)^2 \leq |v|\varphi(|v|), \quad |v| < 1.$$

Note that, if $w \in \Phi(c^*\xi)$ and $k_1(|c^*\xi| + |w|) \geq \|\xi\|$, then

$$\rho(W(\xi)) \leq \sigma(W(\xi)) \leq \sigma(\|Q\|\|\xi\|^2) \leq \sigma\big(k_1^2\|Q\|(|c^*\xi| + |w|)^2\big) = f_3(|c^*\xi| + |w|).$$

Therefore, if $w \in \Phi(c^*\xi), \; k_1(|c^*\xi| + |w|) \geq \|\xi\|$ and $|c^*\xi| < 1$, then

$$\begin{aligned} k_1\rho(W(\xi))\|\xi\|\big(|c^*\xi| + |w|\big) &\leq \rho(W(\xi))\|\xi\|^2 + \frac{k_1^2}{4}\,\rho(W(\xi))\big(|c^*\xi| + |w|\big)^2 \\ &\leq \rho(W(\xi))\|\xi\|^2 + \frac{k_1^2}{4}\,(c^*\xi)w. \end{aligned}$$

On the other hand, if $w \in \Phi(c^*\xi), \; k_1(|c^*\xi| + |w|) \leq \|\xi\|$ and $|c^*\xi| < 1$, then

$$k_1\rho(W(\xi))\|\xi\|\big(\|c^*\xi| + |w|\big) \leq \rho(W(\xi))\|\xi\|^2.$$

Using the fact that $(c^*\xi)w \geq 0$ for all $w \in \Phi(c^*\xi)$ and all $\xi \in \mathbb{R}^n$, it follows that

$$k_1\rho(W(\xi))\|\xi\|\big(|c^*\xi| + |w|\big) \leq \rho(W(\xi))\|\xi\|^2 + \frac{k_1^2}{4}\,(c^*\xi)w, \quad w \in \Phi(c^*\xi), \quad |c^*\xi| < 1. \tag{128}$$

Writing $k_5 := \max\{k_1k_3/k_4\,,\, k_1^2/4\}$, the conjunction of (127) and (128) gives

$$k_1\rho\big(W(\xi)\big)\|\xi\|\big(|c^*\xi| + |w|\big) \leq \rho\big(W(\xi)\big)\|\xi\|^2 + k_5\big(c^*\xi\big)w, \quad \xi \in \mathbb{R}^n, \; w \in \Phi(c^*\xi).$$

The latter, together with (126), implies

$$\langle \nabla V_1(\xi), A\xi + b(d-w)\rangle \le -\rho(W(\xi))\|\xi\|^2 + k_5(c^*\xi)w + k_1k_2d^2,$$

$$\xi \in \mathbb{R}^n, \quad (d,w) \in \mathbb{R}\times\Phi(c^*\xi). \quad (129)$$

Now define $V := k_5V_0 + k_0V_1$. Then, by (120) and (129),

$$\langle \nabla V(\xi), A\xi + b(d-w)\rangle \le -k_0\rho(W(\xi))\|\xi\|^2 + \big(k_0k_1k_2 + 2k_5/\beta\big)d^2 + k_5\gamma(|d|),$$

$$\xi \in \mathbb{R}^n, \quad (d,w) \in \mathbb{R}\times\Phi(c^*\xi). \quad (130)$$

Finally, defining $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ by

$$\alpha_1(s) := k_5 s^2/\|P^{-1}\|, \qquad \alpha_2(s) := k_5\|P\|s^2 + k_0\int_0^{\|Q\|s^2}\rho(\tau)\,\mathrm{d}\tau,$$

$$\alpha_3(s) := k_0\, s\, \min\{sf_3(s/k_3),\, k_1\}, \quad \alpha_4(s) := \big(k_0k_1k_2 + 2k_5/\beta\big)s^2 + k_5\gamma(s),$$

we have

$$\begin{aligned}
\alpha_1(\|\xi\|) &= k_5\|P^{-1}\|^{-1}\|\xi\|^2 \\
&\le k_5\langle \xi, P\xi\rangle \\
&= k_5V_0(\xi) \\
&\le V(\xi) \\
&\le k_5\|P\|\|\xi\|^2 + k_0\int_0^{\|Q\|\|\xi\|^2}\rho(\tau)\,\mathrm{d}\tau \\
&= \alpha_2(\|\xi\|), \qquad \xi \in \mathbb{R}^n
\end{aligned}$$

and, invoking (125) and (130),

$$\langle \nabla V(\xi), A\xi + b(d-w)\rangle \le -\alpha_3(\|\xi\|) + \alpha_4(|d|), \qquad (\xi,d)\in\mathbb{R}^n\times\mathbb{R}.$$

$\square$

Proof of Lemma 23. The sector condition (48) implies

$$(\alpha+\delta)(v+r)^2 - f(v+r)r \le f(v+r)v \le (\beta-\delta)(v+r)^2 - f(v+r)r, \quad (r,v)\in\mathbb{R}^2.$$

Setting $\kappa := \max\{|\alpha+\delta|, |\beta-\delta|\}$ and again invoking (48) shows that

$$|f(v+r)r| \le \kappa(|vr| + r^2), \quad (r,v)\in\mathbb{R}^2. \tag{131}$$

Therefore,

$$(\alpha+\delta)v^2 - 3\kappa(|vr|+r^2) \le f(v+r)v \le (\beta-\delta)v^2 + 3\kappa(|vr|+r^2), \quad (r,v)\in\mathbb{R}^2.$$

Defining $\lambda_1 := \max\{1, 12\kappa/\delta\}$, it follows that

$$\kappa(|v|\varrho_1 + \varrho_1^2) \le \delta v^2/6, \quad v\in\mathbb{R}\backslash[-\lambda_1\varrho_1, \lambda_1\varrho_1],$$

and thus,

$$(\alpha+\delta/2)v^2 \le f(v+r)v \le (\beta-\delta/2)v^2, \quad r\in[-\varrho_1,\varrho_1],\ v\in\mathbb{R}\backslash[-\lambda_1\varrho_1,\lambda_1\varrho_1].$$

Consequently,

$$\big(\alpha+\delta/2-\varrho_2/|v|\big)v^2 \le F_\varrho(v)v \le \big(\beta-\delta/2+\varrho_2/|v|\big)v^2, \quad v\in\mathbb{R}\backslash[-\lambda_1\varrho_1,\lambda_1\varrho_1].$$

Setting $\lambda_2 := 4/\delta$ and noting that $\varrho_2/|v| \le \delta/4$ for all $v\in\mathbb{R}\backslash[-\lambda_2\varrho_2,\lambda_2\varrho_2]$, we obtain

$$(\alpha+\delta/4)v^2 \le F_\varrho(v)v \le (\beta-\delta/4)v^2, \quad v\in\mathbb{R}\backslash[-\lambda\|\varrho\|, \lambda\|\varrho\|],$$

where $\lambda := \max\{\lambda_1,\lambda_2\}$. Therefore, we can apply Corollary 16, with $\Phi = F_\varrho$ and $K = [-\lambda\|\varrho\|, \lambda\|\varrho\|]$, to conclude that there exist constants $k_1 > 0$, $k_2 > 0$, and $\varepsilon > 0$, depending only on (A,b,c), α, β, and δ, such that, for each $\varrho\in[0,\infty)\times[0,\infty)$, each $x^0\in\mathbb{R}^n$, and each $d\in L^\infty_{\mathrm{loc}}[0,\infty)$, every maximal solution x of

$$\dot{x}(t) - Ax(t) - bd(t) \in -bF_\varrho(c^*x(t)), \quad x(0)=x^0 \tag{132}$$

is global and

$$\|x(t)\| \leq k_1 e^{-\varepsilon t}\|x^0\| + k_2(\|d\|_{L^\infty[0,t]} + \theta_\varrho), \quad t \geq 0, \tag{133}$$

where

$$\theta_\varrho := \sup_{|v| \leq \lambda \|\varrho\|} \sup_{w \in F_\varrho(v)} \operatorname{dist}(w, I_v),$$

with

$$I_v := \begin{cases} [(\alpha + \delta/4)v, (\beta - \delta/4)v], & v \geq 0, \\ [(\beta - \delta/4)v, (\alpha + \delta/4)v], & v < 0. \end{cases}$$

From (131), it follows that

$$|f(v+r)| \leq \kappa(\lambda\|\varrho\| + \|\varrho\|) \leq 2\kappa\lambda\|\varrho\|, \quad v \in [-\lambda\|\varrho\|, \lambda\|\varrho\|],\ r \in [-\varrho_1, \varrho_1],$$

and thus,

$$|F_\varrho(v)| \leq (2\kappa\lambda + 1)\|\varrho\|,\ v \in [-\lambda\|\varrho\|, \lambda\|\varrho\|]. \tag{134}$$

Setting

$$k_3 := \max\left\{2\kappa\lambda + 1\,,\ \lambda|\alpha + \delta/4|\,,\ \lambda|\beta - \delta/4|\right\},$$

we have that, for all $v \in [-\lambda\|\varrho\|, \lambda\|\varrho\|]$,

$$F_\varrho(v) \subset [-k_3\|\varrho\|, k_3\|\varrho\|], \qquad I_v \subset [-k_3\|\varrho\|, k_3\|\varrho\|].$$

Consequently, $\theta_\varrho \leq 2k_3\|\varrho\|$. Setting $g_1 := k_1$ and $g_2 := 2k_2k_3$, and invoking (133), it follows that every maximal solution x of (132) is global and

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \|\varrho\|), \quad t \geq 0.$$

The assertion of the lemma now follows, since, for each $F \in \mathcal{U}_\mathbb{R}$ satisfying $F(v) \subset F_\varrho(v)$ for all $v \in \mathbb{R}$, every maximal solution of (49) is also a maximal solution of (132). $\square$

Proof of Lemma 24. Assume first that **(H2)** holds with $\Phi(v) = \{f(v)\}$. Defining $\psi \in \mathcal{K}_\infty$ by

$$\psi(s) = s + \max_{|\sigma| \leq s} |f(\sigma)|, \quad s \geq 0,$$

it follows that

$$\varphi(|v|)|v| \leq f(v)v \leq \psi(|v|)|v|, \quad v \in \mathbb{R}. \tag{135}$$

For $|v| \geq 2\rho_1$ and $|r| \leq \rho_1$, we have $|v|/2 \leq |v| - |r| \leq |v + r| \leq |v| + |r| \leq 2|v|$. Therefore,

$$\varphi(|v|/2) \leq \varphi(|v + r|), \quad r \in [-\varrho_1, \varrho_1],\, v \in \mathbb{R}\backslash[-2\varrho_1, 2\varrho_1]$$

and

$$\psi(2|v|) \geq \psi(|v + r|), \quad r \in [-\varrho_1, \varrho_1],\, v \in \mathbb{R}\backslash[-\varrho_1, \varrho_1].$$

Invoking (135), it follows that

$$\varphi(|v|/2) \leq |f(v + r)| \leq \psi(2|v|), \quad r \in [-\varrho_1, \varrho_1],\, v \in \mathbb{R}\backslash[-2\varrho_1, 2\varrho_1].$$

Since $f(v + r)v \geq 0$ for all $r \in [-\varrho_1, \varrho_1]$ and all $v \in \mathbb{R}\backslash[-2\varrho_1, 2\varrho_1]$, we conclude that

$$\varphi(|v|/2)|v| \leq f(v + r)v \leq \psi(2|v|)|v|, \quad r \in [-\varrho_1, \varrho_1],\, v \in \mathbb{R}\backslash[-2\varrho_1, 2\varrho_1].$$

Hence,

$$\varphi(|v|/2)|v| - \varrho_2|v| \leq F_\varrho(v)v \leq \psi(2|v|)|v| + \varrho_2|v|, \quad v \in \mathbb{R}\backslash[-2\varrho_1, 2\varrho_1]. \tag{136}$$

Defining $\mu \in \mathcal{K}_\infty$ by

$$\mu(s) := \max\{2\varphi^{-1}(2s), \psi^{-1}(s)/2\}, \quad s \geq 0,$$

we have that, for every $s \geq 0$ and every $t \geq \mu(s)$, $\varphi(t/2) \geq 2s$ and $\psi(2t) \geq s$. Consequently, defining $\varphi_1, \psi_1 \in \mathcal{K}_\infty$ by

$$\varphi_1(s) := \varphi(s/2)/2, \quad \psi_1(s) := 2\psi(2s), \quad s \geq 0,$$

and setting $a(\varrho) := \max\{2\varrho_1, \mu(\varrho_2)\}$, we have that

$$\varphi_1(|v|)|v| \leq F_\varrho(v)v \leq \psi_1(|v|)|v|, \quad v \in \mathbb{R}\backslash[-a(\varrho), a(\varrho)].$$

Therefore, it follows from Corollary 21, with $\Phi = F_\varrho$ and $K = [-a(\varrho), a(\varrho)]$, that there exist $\kappa_1 \in \mathcal{KL}$ and $\kappa_2 \in \mathcal{K}$ such that, for each $\varrho \in [0,\infty) \times [0,\infty)$, each $x^0 \in \mathbb{R}^n$, and each $d \in L^\infty_{\text{loc}}[0,\infty)$, every maximal solution x of

$$\dot{x}(t) - Ax(t) - bd(t) \in -bF_\varrho(c^*x(t)), \quad x(0) = x^0 \tag{137}$$

is global and

$$\|x(t)\| \leq \max\left\{\kappa_1(t, \|x^0\|), \kappa_2(\|d\|_{L^\infty[0,t]} + \theta_\varrho)\right\}, \quad t \geq 0, \tag{138}$$

where

$$\theta_\varrho := \sup_{|v| \leq a(\varrho)} \sup_{w \in F_\varrho(v)} \text{dist}(w, I_v),$$

with

$$I_v := \begin{cases} [\varphi_1(v), \psi_1(v)], & v \geq 0 \\ [-\psi_1(|v|), -\varphi_1(|v|)], & v < 0. \end{cases}$$

Moreover, note that, for all $r \in [-\varrho_1, \varrho_1]$ and all $v \in [-a(\varrho), a(\varrho)]$,

$$|f(v+r)| \leq \psi(|v+r|) \leq \psi(a(\varrho) + \varrho_1) \leq \psi_1(a(\varrho) + \varrho_1).$$

Consequently,

$$|F_\varrho(v)| \leq \psi_1(a(\varrho) + \varrho_1) + \varrho_2, \quad v \in [-a(\varrho), a(\varrho)].$$

Setting $b(\varrho) := \psi_1(a(\varrho) + \varrho_1) + \varrho_2$, it follows that $F_\varrho(v) \subset [-b(\varrho), b(\varrho)]$ and $I_v \subset [-b(\varrho), b(\varrho)]$ for all $v \in [-a(\varrho), a(\varrho)]$, implying

$$\theta_\varrho \leq 2b(\varrho), \quad \varrho \in [0,\infty) \times [0,\infty). \tag{139}$$

Also, since $a(\varrho) \leq \mu(\|\varrho\|) + 2\|\varrho\|$ for all $\varrho \in [0,\infty) \times [0,\infty)$, we have

$$b(\varrho) \leq \psi_1(\mu(\|\varrho\|) + 3\|\varrho\|) + \|\varrho\|, \quad \varrho \in [0,\infty) \times [0,\infty). \tag{140}$$

The function $\psi_2 \colon [0,\infty) \to [0,\infty)$ defined by

$$\psi_2(s) := 2(\psi_1(\mu(s) + 3s) + s), \quad s \geq 0,$$

is in $\mathcal{K}_\infty$. Inequalities (139) and (140) now yield

$$\theta_\varrho \leq \psi_2(\|\varrho\|), \quad \varrho \in [0,\infty) \times [0,\infty).$$

Setting $\gamma_1 := \kappa_1$ and $\gamma_2 = \kappa_2 \circ (\mathrm{id} + \psi_2)$, it follows, invoking (138), that every maximal solution x of (137) is global and

$$\|x(t)\| \leq \max\left\{\gamma_1(t, \|x^0\|), \gamma_2(\|d\|_{L^\infty[0,t]} + \|\varrho\|)\right\}, \quad t \geq 0.$$

Since, for each $F \in \mathcal{U}_\mathbb{R}$ satisfying $F(v) \subset F_\varrho(v)$ for all $v \in \mathbb{R}$, every maximal solution of (49) is also a maximal solution of (137), we can conclude that the assertion of the lemma is valid under the assumption that **(H2)** holds.

Under the assumption that **(H1)** holds, proof of the assertion of the lemma is similar to the above proof and is therefore omitted. □

Proof of Corollary 27. We proceed in two steps.

Step 1. In this step, we assume that $d_{\rm o} \in L^\infty[0,\infty)$. Set $\varrho := (\|d_{\rm o}\|_{L^\infty[0,\infty)}, 0)$. Let $F_\varrho \in \mathcal{U}_\mathbb{R}$ be defined by (47), and let x be a maximal solution x of (62). Every maximal solution x of (62) is also a maximal solution of

$$\dot{x}(t) - Ax(t) - bd(t) \in -bF_\varrho(c^*x(t)), \quad x(0) = x^0. \tag{141}$$

Applying Lemma 23 shows that there exist constants $g_1 > 0$, $g_2 > 0$, and $\varepsilon > 0$, depending on (A,b,c), α, β, and δ, but not on f, such that, for each $x^0 \in \mathbb{R}^n$, each $d \in L^\infty_{\rm loc}[0,\infty)$ and each $d_{\rm o} \in L^\infty[0,\infty)$, every maximal solution x of (141) is global and

$$\|x(t)\| \le g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \|d_{\rm o}\|_{L^\infty[0,\infty)}), \quad t \ge 0.$$

Step 2. Now, let $d_{\rm o} \in L^\infty_{\rm loc}[0,\infty)$. Let x be a maximal solution of (62). Seeking a contradiction, suppose that the maximal interval of existence of x is of the form $[0,T)$, where $T < \infty$. By Lemma 1, $\limsup_{t\to T}\|x(t)\| = \infty$. Define $\tilde{d}_{\rm o} \in L^\infty[0,\infty)$ by

$$\tilde{d}_{\rm o}(t) := \begin{cases} d_{\rm o}(t), & 0 \le t \le T, \\ 0, & t > T, \end{cases}$$

and note that x is also a maximal solution of

$$\dot{x}(t) = Ax(t) + b\big(d(t) - f(c^*x(t) + \tilde{d}_{\rm o}(t))\big), \quad x(0) = x^0. \tag{142}$$

By Step 1, every maximal solution of (142) is global, yielding a contradiction. Therefore, the solution x is global.

It remains to show that (64) holds. To this end, let $\tau > 0$ be fixed, but arbitrary, define $\hat{d}_{\rm o} \in L^\infty[0,\infty)$ by

$$\hat{d}_{\rm o}(t) := \begin{cases} d_{\rm o}(t), & 0 \le t \le \tau, \\ 0, & t > \tau, \end{cases}$$

and consider the initial-value problem

$$\dot{x}(t) = Ax(t) + b\big(d(t) - f(c^* x(t) + \hat{d}_{\mathrm{o}}(t))\big), \quad x(0) = x^0. \tag{143}$$

Let x be a maximal solution of (62). We know that x is global and $x|_{[0,\tau]}$ is a solution of (143) on the interval $[0,\tau]$. Let $\hat{x}$ be a maximal solution of (143) extending $x|_{[0,\tau]}$. By Step 1, $\hat{x}$ is global and

$$\|\hat{x}(t)\| \le g_1 e^{-\varepsilon t}\|x^0\| + g_2(\|d\|_{L^\infty[0,t]} + \|\hat{d}_{\mathrm{o}}\|_{L^\infty[0,\infty)}), \quad t \ge 0.$$

Finally, since $\|\hat{d}_{\mathrm{o}}\|_{L^\infty[0,\infty)} = \|d_{\mathrm{o}}\|_{L^\infty[0,\tau]}$ and $x(\tau) = \hat{x}(\tau)$, we conclude that

$$\|x(\tau)\| \le g_1 e^{-\varepsilon\tau}\|x^0\| + g_2(\|d\|_{L^\infty[0,\tau]} + \|d_{\mathrm{o}}\|_{L^\infty[0,\tau]}).$$

Since τ is arbitrary, (64) now follows. □

Proof of Theorem S3. Let $y \in C[0,\infty)$ and $t \ge 0$ be arbitrary. Note initially that, by the definition of the backlash operator,

$$\big(\mathcal{B}_{\sigma,\,\xi(\sigma)}(y)\big)(t) \in [y(t) - \sigma, y(t) + \sigma], \quad t \ge 0.$$

Case 1. Assume $y(t) \ge 0$. Writing $E_1 := [0, y(t)]$ and $E_2 := (y(t), \infty)$, we have

$$\begin{aligned}
\big(\mathcal{P}_\xi(y)\big)(t) &\ge \left(\int_{E_1} + \int_{E_2}\right)\int_0^{y(t)-\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) - |w_0| \\
&\ge b_1\int_{E_1}(y(t)-\sigma)\mu(\mathrm{d}\sigma) + b_2\int_{E_2}(y(t)-\sigma)\mu(\mathrm{d}\sigma) - |w_0| \\
&= \big(b_1\mu(E_1) + b_2\mu(E_2)\big)y(t) - b_1\int_{E_1}\sigma\,\mu(\mathrm{d}\sigma) - b_2\int_{E_2}\sigma\,\mu(\mathrm{d}\sigma) - |w_0| \\
&\ge a_1 b_1 y(t) - a_2 b_2 - |w_0| = a_{\mathcal{P}} y(t) - \theta_{\mathcal{P}}\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}\big(\mathcal{P}_\xi(y)\big)(t) &\le \int_0^\infty \int_0^{y(t)+\sigma} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + |w_0| \\ &\le b_2 \int_0^\infty (y(t)+\sigma)\mu(\mathrm{d}\sigma) + |w_0| \\ &\le a_1 b_2 y(t) + a_2 b_2 + |w^0| \\ &= b_{\mathcal{P}} y(t) + \theta_{\mathcal{P}},\end{aligned}$$

which establishes (S14).

Case 2. Now assume $y(t) \le 0$. The argument used in Case 1 applies mutatis mutandis to conclude (S15).

Finally, the inequality (S16) is a consequence of (S14) and (S15). □

Conclusions

Adopting a tutorial style of presentation, this article provides an overview of the circle criterion and its connection with ISS. Classical absolute stability theory, and the circle criterion in particular, is concerned with the analysis of a feedback interconnection of Lur'e type, which consists of a linear system in the forward path and a sector-bounded nonlinearity in the negative feedback path. The classical methodology seeks to conclude stability of the interconnected system through the interplay of frequency-domain properties of the linear component and sector data for the nonlinearity. This article adopts a similar standpoint, but with several features that distinguish it from the classical approach. Firstly, classical absolute stability results are revisited in the context of systems described by differential inclusions and within a framework based on the complex Aizerman conjecture. This methodology provides new perspectives on classical results. Secondly, nonlinearities of greater generality, including hysteresis and quantization operators, are permitted in the feedback path. To accommodate this generality, an analytic framework of set-valued maps and differential inclusions is adopted. Thirdly, in contrast with the classical literature which is focussed mainly on asymptotic stability of the feedback interconnection, ISS issues are addressed and resolved. Fourthly, the sector conditions of the classical theory are significantly weakened. In particular, through the interaction of the notions of ISS with bias and generalized sector conditions, results pertaining to feedback nonlinearities satisfying a sector condition only in the complement of a compact set are obtained. These results facilitate applications to hysteretic and quantized feedback systems.

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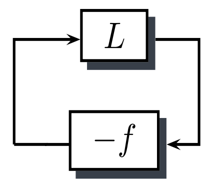

Figure 1. A classical Lur'e system. The negative feedback interconnection consists of a linear system L in the forward path and a static, sector-bounded nonlinearity f in the feedback path.

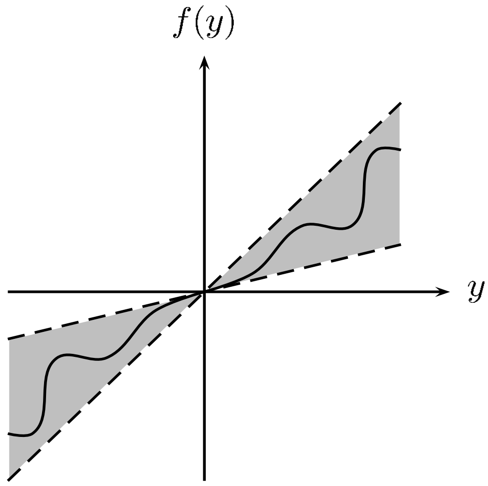

Figure 2. Sector-bounded nonlinearity f. The graph of f is contained in the shaded sector determined by two lines through the origin.

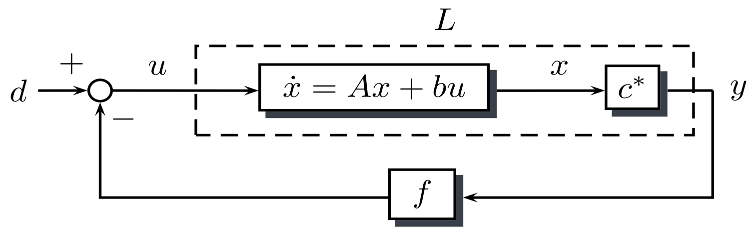

Figure 3. Lur'e system. This system consists of a linear system L and a static nonlinearity f with exogenous input d, representing either a reference or disturbance signal.

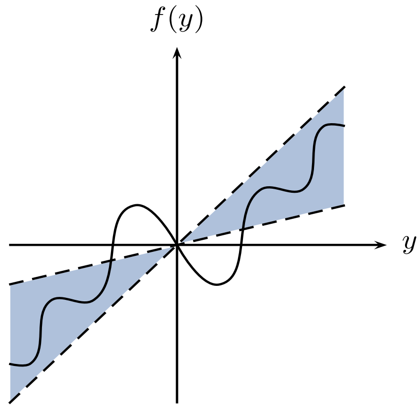

Figure 4. Nonlinearity f satisfying a generalized sector condition. The points $(y, f(y))$ of the graph of f are contained in the shaded area, for all $|y|$ sufficiently large.

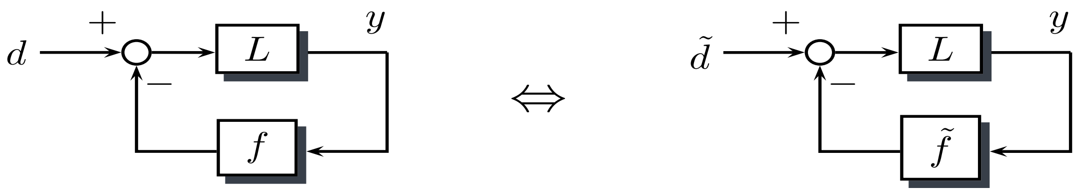

Figure 5. Equivalent interconnections. The nonlinearity f satisfies a sector condition on the complement $\mathbb{R}\backslash K$ of the compact interval K. The continuous function $\tilde{f}$ coincides with f on $\mathbb{R}\backslash K$ and satisfies the same sector condition, but on the whole real line. Then $\tilde{f} - f$ is bounded, and $\tilde{d} := d + \tilde{f}(y) - f(y)$ is locally bounded. The term $\tilde{f}(y) - f(y)$ is the source of the ISS bias.

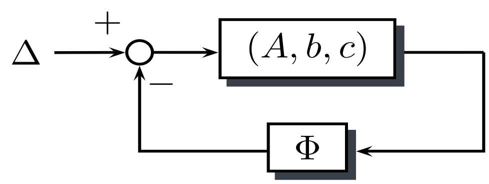

Figure 6. A system of Lur'e type in a set-valued setting. The linear system (A, b, c) is interconnected with the set-valued nonlinearity Φ, and the resulting feedback system is subjected to a set-valued exogenous input Δ.

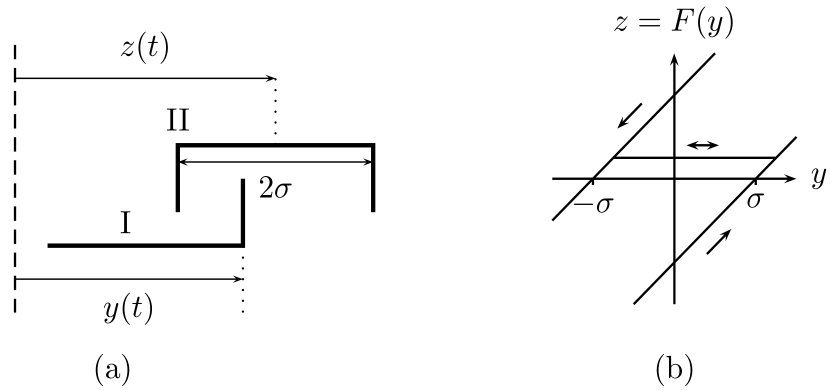

Figure 7. Backlash or play hysteresis. (a) depicts a mechanical play consisting of two components, denoted I and II. The displacements of each part at time $t \geq 0$, denoted by $y(t)$ and $z(t)$, satisfy $|y(t) - z(t)| \leq \sigma$ for all $t \geq 0$, where $z(0) = y(0) + \xi$ for the initial condition $\xi \in [-\sigma, \sigma]$. In particular, the position $z(t)$ of II remains constant as long as the position $y(t)$ of I remains within the interior of II. Denoting the corresponding operator by F, (b) illustrates the action of F. If, for example, component I makes contact with the right end of component II at $t_0 \geq 0$ and $y(t)$ is nondecreasing on the interval $[t_0, t_1]$, where $t_1 > t_0$, then $z(t) = y(t) - \sigma$ for all $t \in [t_0, t_1]$.

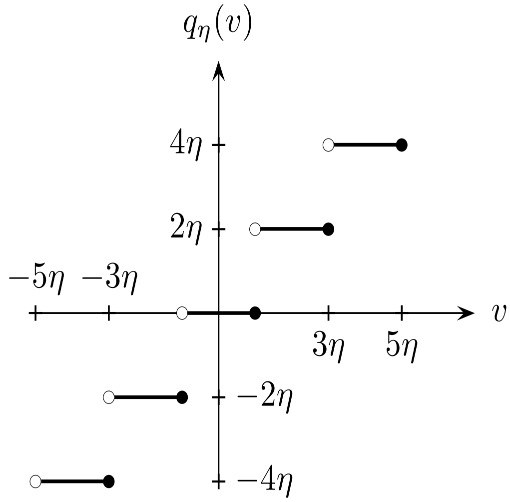

Figure 8. Uniform quantizer q_η. For every $v \in \mathbb{R}$ there exists a unique integer $m \in \mathbb{Z}$ such that $v \in \big((2m-1)\eta\,,\,(2m+1)\eta\big]$ and the quantizer q_η maps v to $2m\eta$.

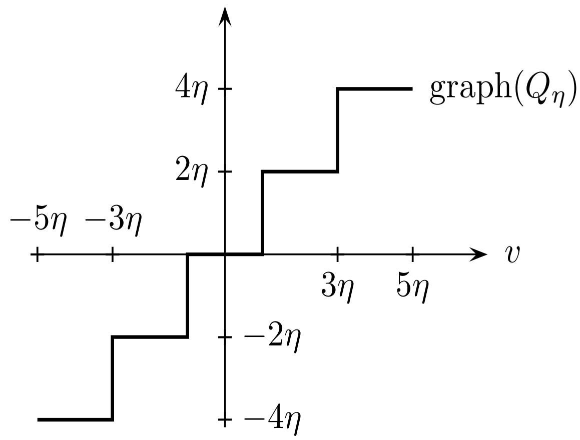

Figure 9. The graph of the set-valued map $Q_\eta \in \mathcal{U}_\mathbb{R}$. This map is the natural set-valued version of the single-valued uniform quantizer q_η. For each $v \in \mathbb{R}$, the set $Q_\eta(v)$ is the smallest convex set containing $\lim_{w\uparrow v} q_\eta(w)$ and $\lim_{w\downarrow v} q_\eta(w)$. In particular, for $m \in \mathbb{Z}$, $Q_\eta(v) = \{2m\eta\}$ for all $v \in \big((2m-1)\eta, (2m+1)\eta\big)$ and $Q_\eta(v) = [2m\eta, 2(m+1)\eta]$ for all $v = (2m+1)\eta$.

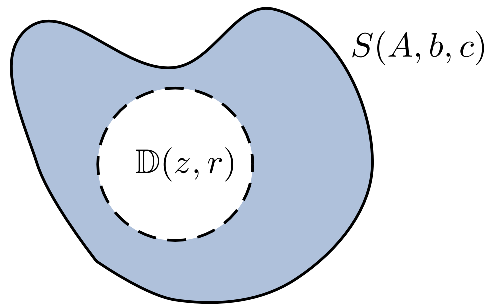

Figure 10. A disc $\mathbb{D}(z,r)$ of stabilizing complex gains. If (A,b,c) is stabilizable and detectable, then, by Lemma 6, the disc $\mathbb{D}(z,r)$ is contained in the set $S(A,b,c)$ of stabilizing complex gains if and only if the rational function $1+2rc^*\big(sI-(A-\kappa bc^*)\big)^{-1}b$ is positive real, where $\kappa := z-r$.

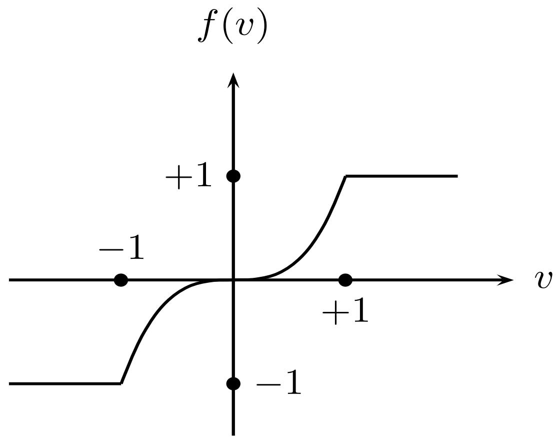

Figure 11. Saturating nonlinearity f. The feedback $u = -f(x)$ applied to the integrator $\dot{x} = u$ yields asymptotic stability in the large, but not global exponential stability.

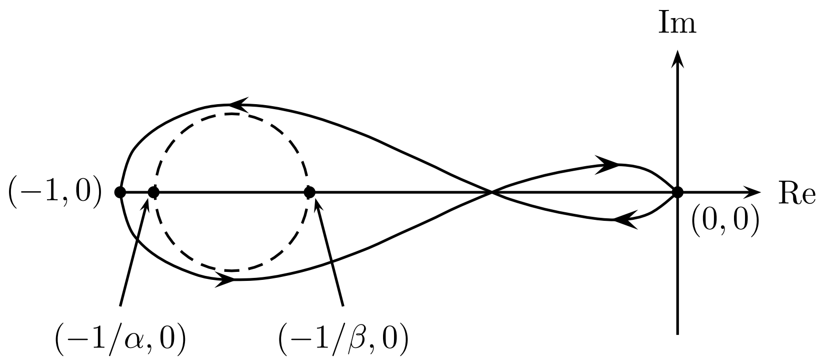

Figure 12. Nyquist diagram of $\mathbf{G}(s) = 10/(s^3 + 5s^2 + 4s - 10)$ and the disc $D(\alpha, \beta)$ with $\alpha = 1.07$ and $\beta = 1.5$. The Nyquist diagram of $\mathbf{G}$ does not intersect the disc $D(\alpha, \beta)$ and encircles it once in the counterclockwise sense. Therefore, by statement (i) of Lemma 10, $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real.

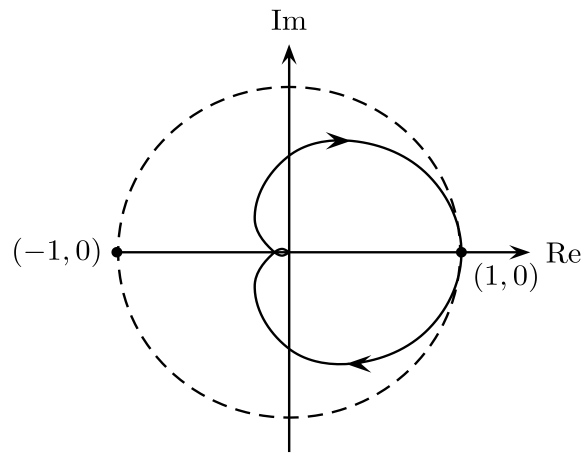

Figure 13. The Nyquist diagram of $\mathbf{G}(s) = 10/(s^3 + 7s^2 + 16s + 10)$ and the closed unit disc $\overline{D}(-1, 1)$. The Nyquist diagram of $\mathbf{G}$ is contained in the closed disc $\overline{D}(-1, 1)$ and thus, by statement (ii) of Lemma 10, $(1 - \mathbf{G})(1 + \mathbf{G})^{-1}$ is positive real.

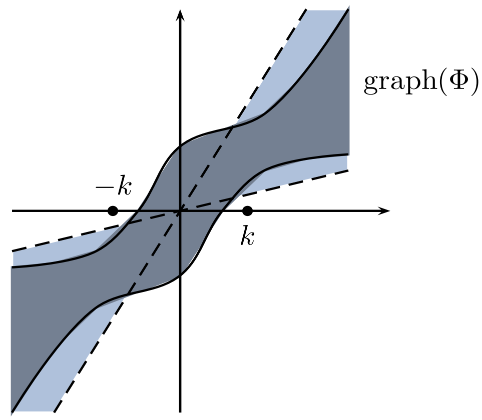

Figure 14. Set-valued Φ satisfying the sector condition (34) with $K = [-k, k]$. For every $v \in \mathbb{R}$ such that $|v| > k$ and every $w \in \Phi(v)$, the point (v, w) lies in the sector given by the shaded area bounded by the two dashed lines passing through the origin.

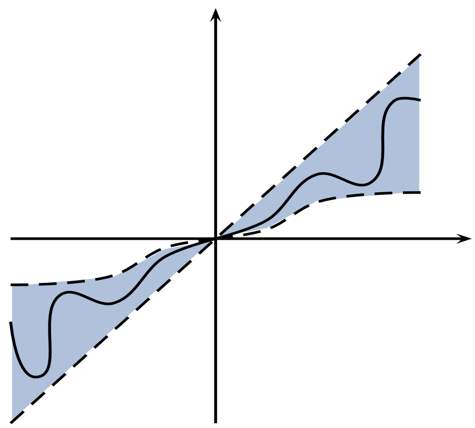

Figure 15. **(H1)**-type sector. Inequalities (37) hold if and only if the graph of the nonlinearity Φ, illustrated here in the case of a singleton-valued map for simplicity, lies in the shaded region bounded by the line of slope $\beta - \delta$ through the origin and the curve given by the graph of φ and its reflection through the origin.

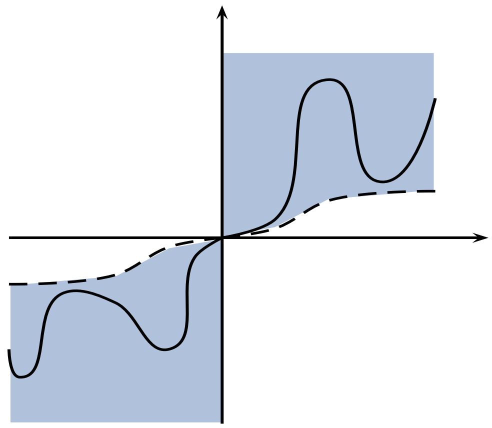

Figure 16. **(H2)**-type sector. Inequality (38) holds if and only if the graph of the nonlinearity Φ, illustrated here in the case of a singleton-valued map for simplicity, lies in the shaded region bounded by the vertical axis and the curve given by the graph of φ and its reflection through the origin.

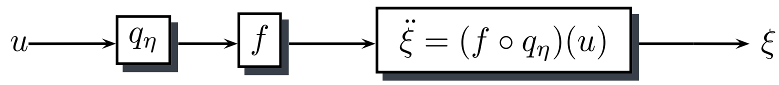

Figure 17. PID control application. The controlled system is the double integrator with input nonlinearity f and a uniform quantizer q_η parameterized by $\eta > 0$.

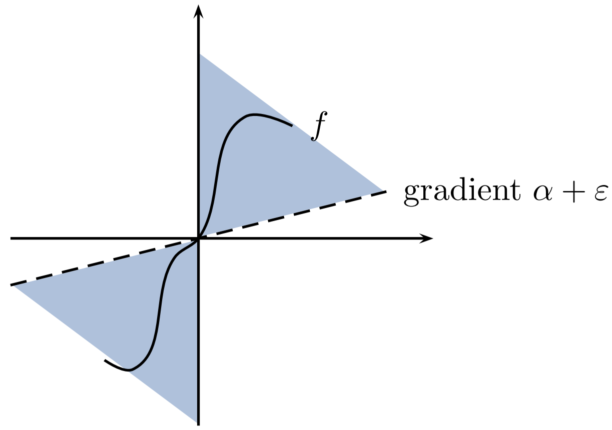

Figure 18. PID control application. Sector-bounded static nonlinearity. The graph of f is required to lie in the shaded region bounded by the vertical axis and the line of slope $\alpha + \varepsilon$ through the origin.

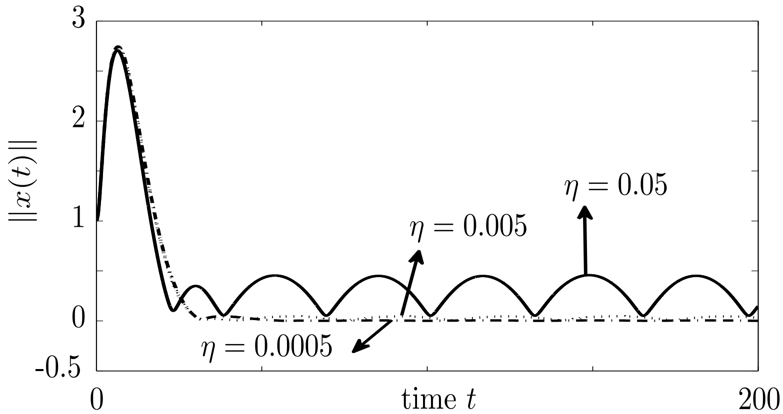

Figure 19. PID controlled system (56). This plot shows the behavior of the system (56) for three values of the quantization parameter η. In (56), (A, b, c) and Φ_η are given by (54) and (57), respectively, with nonlinearity $f\colon v \mapsto v(1 + v^2)$, controller gains $k_p = 1$, $k_d = 4$, and $k_i = 0.1$, and reference signal $r = 1$. The objective of asymptotic tracking of the reference signal, equivalently, convergence to 0 as $t \to \infty$ of the first component $e(t)$ of the solution, is attained in the limit as $\eta \to 0$.

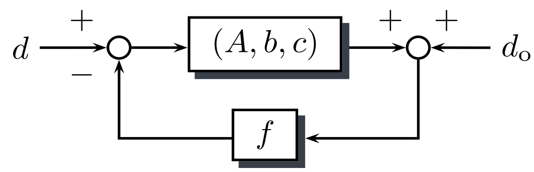

Figure 20. Lur'e system. The linear system (A, b, c) is in the forward path, the nonlinearity f is in the negative feedback path, the exogenous input is d and the output disturbance is d_{o}.

Sidebar 1: Notation and Terminology

For $M \in \mathbb{C}^{p\times q}$, $M^* \in \mathbb{C}^{q\times p}$ denotes the conjugate transpose of M. If all entries of M are real, then M^* is the transpose of M. For $z \in \mathbb{C}$ and $r > 0$, let $\mathbb{D}(z,r)$ denote the open disc in $\mathbb{C}$ of radius r and with center z. The open right-half complex plane is denoted by $\mathbb{C}_+$. The space of bounded analytic functions on $\mathbb{C}_+$ is denoted by $H^\infty = H^\infty(\mathbb{C}_+)$. If $\mathbf{H} \in H^\infty$, then $\|\mathbf{H}\|_{H^\infty} := \sup_{s\in\mathbb{C}_+}|\mathbf{H}(s)|$.

Positive real functions. Let $\mathbf{H}$ be a real or complex rational function. The function $\mathbf{H}$ is *positive real* if $\operatorname{Re}\mathbf{H}(s) \geq 0$ for all $s \in \mathbb{C}_+$ such that s not a pole of $\mathbf{H}$. It can be shown that positive realness of $\mathbf{H}$ implies that $\mathbf{H}$ does not have any poles in $\mathbb{C}_+$. The function $\mathbf{H}$ is *strictly positive real* if there exists $\varepsilon > 0$ such that the shifted rational function $s \mapsto \mathbf{H}(s-\varepsilon)$ is positive real.

Absolutely continuous functions. The importance of absolute continuity stems from the fact that absolutely continuous functions are precisely those functions for which the fundamental theorem of calculus in the context of Lebesgue integration is valid. Let $I \subset \mathbb{R}$ be an interval and $\mathbb{F}$ either $\mathbb{R}$ or $\mathbb{C}$. A function $x\colon I \to \mathbb{F}^n$ is absolutely continuous if, and only if, x is differentiable at almost all (a.a.) $t \in I$, $\dot{x} \in L^1_{\mathrm{loc}}(I,\mathbb{F}^n)$, the space of locally Lebesgue integrable functions $I \to \mathbb{F}^n$, and, for every fixed $a \in I$, $x(t) = x(a) + \int_a^t \dot{x}(s)\mathrm{d}s$ for all $t \in I$.

Function classes $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$. Let $\mathcal{K}$ denote the set of continuous and strictly increasing functions $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$. The set of all functions $f \in \mathcal{K}$ with the property that $f(s) \to \infty$ as $s \to \infty$ is denoted by $f \in \mathcal{K}_\infty$. Finally, $\mathcal{KL}$ denotes the class of all functions $f\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ such that, for each $r \in [0,\infty)$, the function $s \mapsto f(r,s)$

is in $\mathcal{K}$ and, for each $s \in [0, \infty)$, the function $r \mapsto f(r, s)$ is nonincreasing with $f(r, s) \to 0$ as $r \to \infty$. Functions in $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ are sometimes referred to as *comparison functions*.

Set-valued maps. In the following, $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. A set-valued map $v \mapsto \Phi(v) \subset \mathbb{F}$, with nonempty values and defined on $\mathbb{F}$, is *upper semicontinuous at* $v_0 \in \mathbb{F}$ if, for every open set W containing $\Phi(v_0)$, there exists an open set V containing v_0 such that, for all $v \in V$, $\Phi(v) \subset W$, see Figure S1. The map Φ is *upper semicontinuous* if it is upper semicontinuous at every point in $\mathbb{F}$.

Let $\mathcal{U}_\mathbb{F}$ denote the set of all upper semicontinuous maps $v \mapsto \Phi(v) \subset \mathbb{F}$ such that, for all $v \in \mathbb{F}$, the set $\Phi(v)$ is compact and convex. In the real case, $\Phi \in \mathcal{U}_\mathbb{R}$ if and only if Φ is upper semicontinuous and, for all $v \in \mathbb{R}$, $\Phi(v)$ is of the form $[w_1, w_2]$ for $w_1, w_2 \in \mathbb{R}$ with $w_1 \leq w_2$.

Let D be a set-valued map defined on an interval $I \subset \mathbb{R}$ and with nonempty values contained in $\mathbb{F}^m$. The map D is *measurable* if the preimage $D^{-1}(W) := \{t \in I \colon D(t) \cap W \neq \varnothing\}$ of every open set $W \subset \mathbb{F}^m$ is Lebesgue measurable. Moreover, for nonempty $S \subset \mathbb{F}$, we define $|S| := \sup\{|s| \colon s \in S\}$. A set-valued map Δ defined on $[0, \infty)$ with nonempty values contained in $\mathbb{F}$ is *locally essentially bounded* if Δ is measurable and the function $t \mapsto |\Delta(t)|$ is in $L^\infty_{\text{loc}}[0, \infty)$, the space of measurable locally essentially bounded functions $[0, \infty) \to \mathbb{R}$. The set of all locally essentially bounded set-valued maps defined on $[0, \infty)$ and with compact and convex values contained in $\mathbb{F}$ is denoted by $\mathcal{D}_\mathbb{F}$. Finally, for $\Delta \in \mathcal{D}_\mathbb{F}$ and a bounded interval $I \subset [0, \infty)$, we define

$$\|\Delta\|_{L^p(I)} := \left(\int_I |\Delta(t)|^p \mathrm{d}t \right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|\Delta\|_{L^\infty(I)} := \operatorname{ess\,sup}_{t \in I} |\Delta(t)|.$$

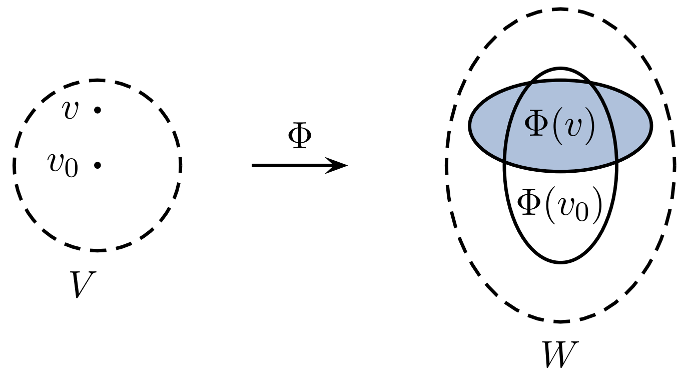

Figure S1. Upper semicontinuity of the set-valued map Φ. For every v_0 in $\mathbb{F}$, every open neighborhood W of $\Phi(v_0)$ contains the image under Φ of some open neighborhood V of v_0, that is, $\Phi(v) \subset W$ for all $v \in V$.

Sidebar 2: An Example from Circuit Theory

Consider the circuit in Figure S2, consisting of a capacitor with capacitance $C > 0$, an inductor with inductance $L > 0$, a current source $\imath$, and a nonlinear resistive element with current-voltage characteristic given by the continuously differentiable function $h : \mathbb{R} \to \mathbb{R}$. Adopting the current through the inductor L and the voltage across the capacitor C as the state variables x_1 and x_2, respectively, elementary circuit analysis gives

$$L\dot{x}_1(t) = x_2(t), \quad C\dot{x}_2(t) = -x_1(t) - h(x_2(t)) + \imath(t).$$

We thus arrive at the equivalent representation

$$\dot{x}(t) = Ax(t) + bu(t), \quad u(t) = d(t) - f(c^* x(t)), \tag{S1}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1/L \\ -1/C & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c, \quad f(v) = \frac{h(v)}{C}, \quad d(t) = \frac{\imath(t)}{C}, \tag{S2}$$

and c^* denotes the transpose of the column vector c. This structure forms a prototype for the general class of systems investigated in the paper. Note that the transfer function $\mathbf{G}$ of the linear system (A, b, c), given by

$$\mathbf{G}(s) = c^*(sI - A)^{-1}b = \frac{s}{s^2 + 1/(CL)}, \tag{S3}$$

is positive real.

Nonnegative resistance element. We assume that h satisfies the condition

$$0 \le h(v)v, \quad v \in \mathbb{R}. \tag{S4}$$

Consider first the unforced system, that is, $\imath = 0$. Then a suitable version of the classical circle criterion, given in Theorem 13, guarantees that there exists $g > 0$ such that every solution x of (S1) is defined on $[0, \infty)$ and

$$\|x(t)\| \leq g\|x(0)\|, \quad t \geq 0.$$

If, in (S4), strict inequality holds for every $v \neq 0$, then, by Theorem 13, $\lim_{t\to\infty} x(t) = 0$, that is, 0 is globally attractive.

Now consider the system with forcing, that is, $\imath \neq 0$. If, in (S4), strict inequality holds for every $v \neq 0$ and if $\lim_{v\to\pm\infty} |h(v)| = \infty$, then Theorem 17 can be used to show that the system given by (S1) and (S2) is ISS, see Example 19.

Negative resistance element. Finally, let h describe a negative resistance element, that is, $h(0) = 0$, $h'(0) < 0$, $h(v) \to \infty$ as $v \to \infty$, and $h(v) \to -\infty$ as $v \to -\infty$; an example is shown in Figure S3. Such a characteristic typically occurs if the resistive element is given by a twin-tunnel-diode circuit. In the case of negative resistance, the condition (S4) does not hold for all $v \in \mathbb{R}$, but only for all $v \in \mathbb{R}\backslash K$ for some suitable compact interval K. This situation is addressed in Example 22.

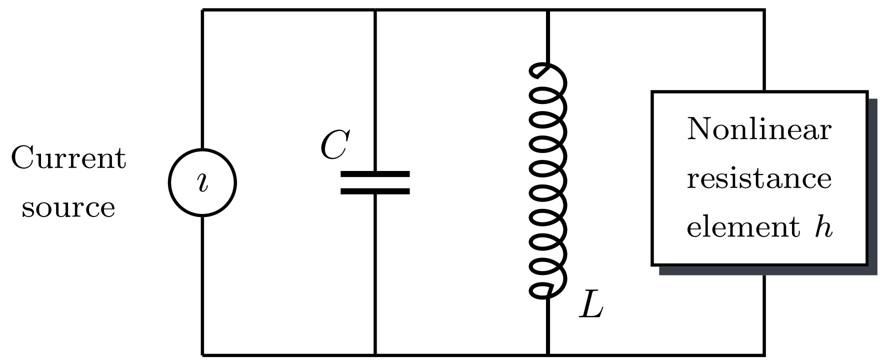

Figure S2. Example from circuit theory. A parallel connection of a current source, capacitor, inductor, and nonlinear resistive element h.

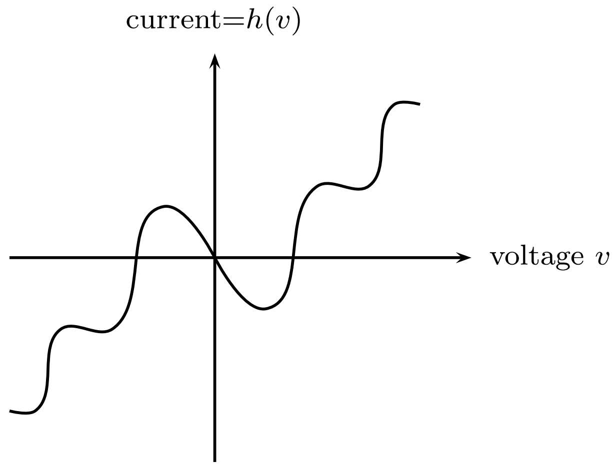

Figure S3. Negative resistance element with characteristic h. The function h satisfies $h(0) = 0$, $h'(0) < 0$, $h(v) \to \infty$ as $v \to \infty$, and $h(v) \to -\infty$ as $v \to \infty$.

Sidebar 3: The Concept of Input-to-State Stability

Since its inception in the 1980s, the concept of input-to-state stability (ISS) has generated a rich body of results relating to stability properties of nonlinear systems with inputs. A succinct description of the area can be found in [38]. Here, we provide a brief overview and, for simplicity of presentation, we restrict attention to single-input systems.

ISS concerns stability-type questions pertaining to systems with input u, which, on the one hand, might be an exogenous disturbance/perturbation or, on the other hand, might be a control open to choice. These systems are of the form

$$\dot{x}(t) = g(x(t), u(t)), \quad x(0) = x^0, \tag{S5}$$

where, typically, it is assumed that $g\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is sufficiently regular to ensure that, for each initial condition $x^0 \in \mathbb{R}^n$ and every locally essentially bounded input $u \in L^\infty_{\text{loc}}[0, \infty)$, the system (S5) has a unique solution $x\colon [0, \infty) \to \mathbb{R}^n$. ISS investigates properties of the map

$$(x^0, u(\cdot)) \mapsto x(\cdot)$$

using a concept that encompasses two desirable modes of dynamic behavior.

(i) *Bounded-input bounded-state (BIBS) property*: for every $x^0 \in \mathbb{R}^n$ and every essentially bounded input u, the solution x of (S5) is bounded.

(ii) *Convergent-input convergent-state (CICS) property:* for every $x^0 \in \mathbb{R}^n$ and every input u with $u(t) \to 0$ as $t \to \infty$, the solution x of (S5) is such that $x(t) \to 0$ as $t \to \infty$.

By way of motivation, consider the single-input, linear initial-value problem

$$\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x^0, \quad A \in \mathbb{R}^{n \times n},\ b \in \mathbb{R}^n \tag{S6}$$

with unique solution $x\colon [0,\infty) \to \mathbb{R}^n$ given by

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}bu(s)\,\mathrm{d}s, \quad t \geq 0.$$

If we assume that A is Hurwitz, then there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|e^{At}\| \leq Me^{-\alpha t}, \quad t \geq 0.$$

Therefore,

$$\|x(t)\| \leq Me^{-\alpha t}\|x^0\| + M\|b\| \sup_{s\in[0,t]} \|u(s)\| \int_0^t e^{-\alpha(t-s)}\mathrm{d}s, \quad t \geq 0$$

and hence, with $\gamma := M\|b\|/\alpha$, we have

$$\|x(t)\| \leq Me^{-\alpha t}\|x^0\| + \gamma \sup_{s\in[0,t]} \|u(s)\|, \quad t \geq 0. \tag{S7}$$

Thus, for the linear system (S6), the Hurwitz condition on A leads to the estimate (S7), which, in turn, implies both the BIBS property and the CICS property. Conversely, if there exist constants $M, a, \gamma > 0$ such that (S7) holds for all solutions of (S6), then A is Hurwitz.

In the context of the nonlinear system (S5), the natural counterpart of the Hurwitz condition on A is the property that, with zero input $u = 0$, the origin $0 \in \mathbb{R}^n$ is an equilibrium of the system $\dot{x} = g(x, 0)$, that is, $g(0, 0) = 0$, and this equilibrium is globally asymptotically stable (GAS). In contrast with the linear system, the GAS property implies neither the BIBS nor the CICS property. For example, the scalar system $\dot{x} = -x + x^2u$ has the GAS property; however, with initial data $x(0) = 1$ and bounded and convergent input $u\colon t \mapsto 2e^{-t}$, the system has the unbounded solution $x\colon t \mapsto e^t$, and thus both the BIBS and CICS properties fail to hold. In the nonlinear case, it is therefore natural to seek a counterpart to (S7) that implies the GAS property, the BIBS property, and the CICS property. This goal forms the basis of the definition of input-to-state stability. In the following, comparison functions of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ play a key role; these function classes are defined in "Notation and Terminology".

Definition S1: System (S5) is *input-to-state stable* (ISS) if there exist $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for all $(x^0, u) \in \mathbb{R}^n \times L^\infty_{\mathrm{loc}}[0,\infty)$, the unique solution $x\colon [0,\infty) \to \mathbb{R}^n$ is such that

$$\|x(t)\| \le \gamma_1(t, \|x^0\|) + \gamma_2\big(\sup_{s\in[0,t]}\|u(s)\|\big), \quad t \ge 0. \tag{S8}$$

The concept of ISS has an equivalent definition.

Definition S2: System (S5) is *input-to-state stable* (ISS) if there exist $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for all $(x^0, u) \in \mathbb{R}^n \times L^\infty_{\mathrm{loc}}[0,\infty)$, the unique solution $x\colon [0,\infty) \to \mathbb{R}^n$ is such that

$$\|x(t)\| \le \max\big\{\gamma_1(t, \|x^0\|), \gamma_2\big(\sup_{s\in[0,t]}\|u(s)\|\big)\big\}, \quad t \ge 0. \tag{S9}$$

If system (S5) is ISS, then it has the GAS, BIBS, and CICS properties. ISS admits a characterization in terms of a Lyapunov-like function. Specifically, the system (S5) is ISS if and only if there exists a smooth function $V\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ such that $\alpha_1(\|z\|) \le V(z) \le \alpha_2(\|z\|)$ and $\langle \nabla V(z), g(z,v)\rangle \le -\alpha_3(\|z\|) + \alpha_4(|v|)$ for all $z \in \mathbb{R}^n$ and all $v \in \mathbb{R}$.

A variant of the ISS estimate (S9), namely,

$$\|x(t)\| \le \max\big\{\gamma_1(\|x^0\|, t), \gamma_2\big(\sup_{s\in[0,t]}\|u(s)\| + \theta\big)\big\}, \quad t \ge 0, \tag{S10}$$

where $\theta \ge 0$ is a constant, plays a role in the investigations in this article.

If $\theta = 0$ in (S10), then (S9) is recovered. If $\theta > 0$ and there exist $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ such that (S10) holds for all $(x^0, u) \in \mathbb{R}^n \times L^\infty_{\text{loc}}[0, \infty)$, then we say that (S5) is *ISS with bias* $\gamma_2(\theta) > 0$. In this case, the BIBS property continues to hold, but the CICS property fails to hold. However, with a converging input $u(t) \to 0$ as $t \to \infty$, a particular asymptotic property of solutions is guaranteed, namely,

$$\limsup_{t \to \infty} \|x(t)\| \leq \gamma_2(\theta),$$

and therefore, while the state might fail to approach zero asymptotically, it must approach the ball of radius $\gamma_2(\theta)$ centered at 0. In other words, the asymptotic behavior of it cannot deviate from zero by more than the bias term $\gamma_2(\theta) > 0$. Note that, since $\gamma_2 \in \mathcal{K}_\infty$, if the bias parameter θ tends to zero, then $\gamma_2(\theta)$ also tends to zero and thus ISS, and its attendant properties of GAS and CICS are guaranteed in the limit as $\theta \downarrow 0$. The concept of ISS with bias is equivalent to that of input-to-state practical stability discussed in [49, Definition 2.2 and Remark 1].

Sidebar 4: Hysteretic Feedback Systems

Here, we show how Corollary 16 can be used to analyze stability properties of hysteretic feedback systems.

Consider the feedback interconnection shown in Figure S4, with a hysteresis operator F in the feedback path and a single-valued input d. In the context of hysteretic feedback systems, absolute stability and ISS are discussed in [19], [28], [29], [30], [32], [34], [50], [51]. In the following, we focus on the class of Preisach hysteresis operators. The Preisach operator encompasses both backlash and Prandtl operators. The Preisach operator can model complex hysteresis effects, for example, nested loops in input-output characteristics. A basic building block for these operators is the backlash operator, shown in Figure 7. The backlash operator, also called the play operator, is discussed in [31], [52], [53] and [54].

Let $\sigma \geq 0$ and define $b_\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$b_\sigma(v_1, v_2) := \max\big\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\big\} = \begin{cases} v_1 - \sigma, & \text{if } v_2 < v_1 - \sigma, \\ v_2, & \text{if } v_2 \in [v_1 - \sigma, v_1 + \sigma], \\ v_1 + \sigma, & \text{if } v_2 > v_1 + \sigma\,. \end{cases}$$

Let $C_{\rm pm}[0,\infty)$ denote the space of continuous piecewise monotone functions defined on $[0,\infty)$. For all $\sigma \geq 0$ and $\zeta \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\zeta} : C_{\rm pm}[0,\infty) \to C[0,\infty)$ by

$$(\mathcal{B}_{\sigma,\zeta}(y))(t) = \begin{cases} b_\sigma(y(0), \zeta) & \text{for } t = 0\,, \\ b_\sigma(y(t), (\mathcal{B}_{\sigma,\zeta}(y))(t_i)) & \text{for } t_i < t \leq t_{i+1},\ i = 0, 1, 2, \dots\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \ldots$, $\lim_{n\to\infty} t_n = \infty$, and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ζ plays the role of an initial state. It can be shown that the definition is independent of the choice of the partition (t_i). Figure S5 illustrates how $\mathcal{B}_{\sigma,\zeta}$ acts. The operator $\mathcal{B}_{\sigma,\zeta}$ extends to a Lipschitz continuous hysteresis operator on $C[0,\infty)$, with Lipschitz constant $L = 1$, which is called the *backlash operator* and is denoted by the same symbol $\mathcal{B}_{\sigma,\zeta}$.

Let $\xi\colon [0,\infty) \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a signed Borel measure on $[0,\infty)$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset [0,\infty)$, where $|\mu|$ denotes the total variation of μ. Denoting the Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function, and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi\colon C[0,\infty) \to C[0,\infty)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0, \quad y \in C[0,\infty), \quad t \geq 0\,, \tag{S11}$$

is called a *Preisach* operator. This definition is equivalent to that adopted in [53, Section 2.4], where it is shown that $\mathcal{P}_\xi$ is causal and rate independent. Here *rate independence* means that $\mathcal{P}_\xi(y \circ h) = \mathcal{P}_\xi(y) \circ h$ for every continuous, nondecreasing, and surjective function $h : [0,\infty) \to [0,\infty)$ and all $y \in C[0,\infty)$.

Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|([0,\infty))\|w\|_\infty$ in the sense that

$$\sup_{t\geq 0} |(\mathcal{P}_\xi(y_1))(t) - (\mathcal{P}_\xi(y_2))(t)| \leq L \sup_{t\geq 0} |y_1(t) - y_2(t)|, \quad y_1, y_2 \in C[0,\infty).$$

See [31] for details. This property ensures well-posedness of the feedback interconnection shown in Figure S4 with $F = \mathcal{P}_\xi$.

Setting $w(\cdot,\cdot) = 1$ and $w_0 = 0$ in (S11) yields the *Prandtl* operator $\mathcal{P}_\xi : C[0,\infty) \to C[0,\infty)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)\mu(\mathrm{d}\sigma), \quad y \in C[0,\infty)\,, \quad t \geq 0\,. \tag{S12}$$

Roughly speaking, a Prandtl operator is a weighted sum of backlash operators. For $\xi \equiv 0$ and μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$, where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0,5]$, the Prandtl operator is illustrated in Figure S6.

The next theorem identifies conditions under which the Preisach operator (S11) satisfies a generalized sector bound. For simplicity, we assume that the measure μ and the function w are nonnegative, although the theorem can be extended to signed measures μ and sign-indefinite functions w.

Theorem S3: Let $\mathcal{P}_\xi$ be the Preisach operator defined in (S11). Assume that the measure μ is nonnegative, $a_1 := \mu([0,\infty)) < \infty$, $a_2 := \int_0^\infty \sigma\mu(\mathrm{d}\sigma) < \infty$, $b_1 := \operatorname{ess\,inf}_{(s,\sigma)\in\mathbb{R}\times[0,\infty)} w(s,\sigma) \geq 0$, $b_2 := \operatorname{ess\,sup}_{(s,\sigma)\in\mathbb{R}\times[0,\infty)} w(s,\sigma) < \infty$, and set

$$\alpha_\mathcal{P} := a_1 b_1, \quad \beta_\mathcal{P} := a_1 b_2\,, \quad \theta_\mathcal{P} := a_2 b_2 + |w_0|\,. \tag{S13}$$

Then, for all $y \in C[0,\infty)$ and all $t \geq 0$,

$$\alpha_\mathcal{P} y(t) - \theta_\mathcal{P} \leq (\mathcal{P}_\xi(y))(t) \leq \beta_\mathcal{P} y(t) + \theta_\mathcal{P}, \quad y(t) \geq 0, \tag{S14}$$

and

$$\beta_\mathcal{P} y(t) - \theta_\mathcal{P} \leq (\mathcal{P}_\xi(y))(t) \leq \alpha_\mathcal{P} y(t) + \theta_\mathcal{P}, \quad y(t) \leq 0. \tag{S15}$$

Furthermore, for every $\delta > 0$,

$$(\alpha_\mathcal{P} - \delta) y^2(t) \leq (\mathcal{P}_\xi(y))(t) y(t) \leq (\beta_\mathcal{P} + \delta) y^2(t), \quad |y(t)| \geq \theta_\mathcal{P}/\delta. \tag{S16}$$

For example, the Prandtl operator illustrated in Figure S6 satisfies the hypotheses of Theorem S3. The proof of Theorem S3 can be found in the section "Proofs".

Let $\mathcal{P}_\xi$ be a Preisach operator satisfying the hypotheses of Theorem S3. Let $\alpha_\mathcal{P}$, $\beta_\mathcal{P}$ and $\theta_\mathcal{P}$ be given by (S13) and define $\Phi \in \mathcal{U}_\mathbb{R}$ by

$$\Phi(v) := \begin{cases} [\alpha_\mathcal{P} v - \theta_\mathcal{P}, \beta_\mathcal{P} v + \theta_\mathcal{P}], & v \geq 0, \\ [\beta_\mathcal{P} v - \theta_\mathcal{P}, \alpha_\mathcal{P} v + \theta_\mathcal{P}], & v < 0. \end{cases} \tag{S17}$$

In view of (S14) and (S15),

$$(\mathcal{P}_\xi(y))(t) \in \Phi(y(t)), \qquad y \in C[0,\infty),\ t \geq 0. \tag{S18}$$

We note that, for $\delta > 0$ and $K := [-\theta_\mathcal{P}/\delta\,,\,\theta_\mathcal{P}/\delta]$,

$$(\alpha_\mathcal{P} - \delta)v^2 \leq \Phi(v)v \leq (\beta_\mathcal{P} + \delta)v^2, \quad v \in \mathbb{R}\backslash K,$$

Let the linear system (A, b, c), with transfer function $\mathbf{G}$, be stabilizable and detectable. Write

$$\alpha := \alpha_\mathcal{P} - 2\delta, \quad \beta := \beta_\mathcal{P} + 2\delta \tag{S19}$$

and assume that $(1+\beta\mathbf{G})(1+\alpha\mathbf{G})^{-1}$ is positive real. Then the hypotheses of Corollary 16 hold with Φ given by (S17). Moreover, it can be shown that the bias parameter θ, defined by (35), is given by $\theta = \theta_\mathcal{P}$. Therefore, we can invoke Corollary 16 to conclude properties of solutions of the functional differential equation

$$\dot{x}(t) = Ax(t) + b\big[d(t) - (\mathcal{P}_\xi(c^*x))(t)\big], \quad x(0) = x^0. \tag{S20}$$

By arguments similar to those adopted in [32], it can be shown that, for each $x^0 \in \mathbb{R}^n$ and $d \in L^\infty_{\text{loc}}[0,\infty)$, (S20) has a unique global solution x. By (S18), x also satisfies

$$\dot{x}(t) - Ax(t) \in b\big(\Delta(t) - \Phi(c^*x(t))\big), \quad x(0) = x^0,$$

where $\Delta(t) = \{d(t)\}$. Now an application of Corollary 16 yields the existence of constants $\varepsilon, g_1, g_2 > 0$ such that, for every $x^0 \in \mathbb{R}^n$,

$$\|x(t)\| \leq g_1 e^{-\varepsilon t}\|x^0\| + g_2\left(\|d\|_{L^\infty[0,t]} + \theta_{\mathcal{P}}\right), \qquad t \geq 0. \tag{S21}$$

Example S4: Consider the mechanical system with damping coefficient $\gamma > 0$ and hysteretic restoring force in the form of backlash, with real parameters $\sigma > 0$ and ζ, given by

$$\ddot{y}(t) + \gamma\dot{y}(t) + (\mathcal{B}_{\sigma,\zeta}(y))(t) = d(t). \tag{S22}$$

Since $(\mathcal{B}_{\sigma,\zeta}(y))(t) \in [y(t) - \sigma, y(t) + \sigma]$ for every $y \in C[0,\infty)$ and every $t \in [0,\infty)$, it follows that, for every $\delta > 0$ and every $(t, y) \in [0,\infty) \times C[0,\infty)$ such that $|y(t)| \geq \sigma/\delta$,

$$(1-\delta)y^2(t) \leq \big(\mathcal{B}_{\sigma,\zeta}(y)\big)(t)y(t) \leq (1+\delta)y^2(t).$$

Of course, this fact is also a consequence of Theorem S3, since the backlash operator $\mathcal{B}_{\sigma,\zeta}$ is a special case of the Preisach operator with $\alpha_{\mathcal{P}} = \beta_{\mathcal{P}} = 1$ and $\theta_{\mathcal{P}} = \sigma$, in the notation of Theorem S3.

As in (S19), set $\alpha := \alpha_{\mathcal{P}} - 2\delta = 1 - 2\delta$ and $\beta := \beta_{\mathcal{P}} + 2\delta = 1 + 2\delta$. The transfer function $\mathbf{G}$ is given by $\mathbf{G}(s) = 1/(s^2 + \gamma s)$, and thus,

$$\frac{1 + \beta\mathbf{G}}{1 + \alpha\mathbf{G}} = 1 + \frac{4\delta}{s^2 + \gamma s + 1 - 2\delta}.$$

For all $\delta > 0$ sufficiently small, $(1 + \beta\mathbf{G})(1 + \alpha\mathbf{G})^{-1}$ is positive real. Setting $x := (y, \dot{y})$, it follows that there exist constants $\varepsilon, g_1, g_2 > 0$ such that, for every $x^0 := (y(0), \dot{y}(0)) \in \mathbb{R}^2$, (S21) holds with $\theta_{\mathcal{P}} = \sigma$.

For numerical simulation, assume the data

$$\gamma = 5, \quad \sigma = 1, \quad \zeta = 0, \quad y(0) = 10, \quad \dot{y}(0) = 0.$$

The evolution of the norm $\|x(t)\|$ of the solution is depicted in Figure S7 in the case of zero forcing $d = 0$, and in Figure S8 in the case of sinusoidal forcing $d(t) = \sin t$. $\diamondsuit$

Returning to the non-specific setting given by (S20), we emphasize that estimate (S21) does not guarantee that $d(t) \to 0$ as $t \to \infty$ implies convergence of $x(t)$ as $t \to \infty$. To see this, consider again the mechanical example (S22). Then, for each $\gamma > 0$, there exist constants $\varepsilon, g_1, g_2 > 0$ such that (S21) holds with $x(t) = (y(t), \dot{y}(t))$ and $\theta_{\mathcal{P}} = \sigma$. However, we know from [34, Example 4.8] that, if $d = 0$ and $\gamma \in (1, 2)$, then, for all initial conditions, $\limsup_{t\to\infty} y(t) = \sigma$ and $\liminf_{t\to\infty} y(t) = -\sigma$, equivalently, y has omega-limit set $[-\sigma, \sigma]$, and so $x(t) = (y(t), \dot{y}(t))$ does not converge as $t \to \infty$.

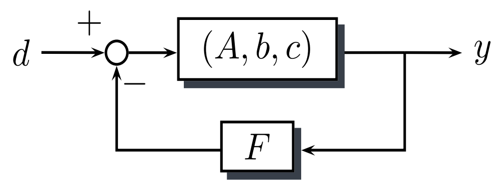

Figure S4. Hysteretic Lur'e system. Feedback interconnection of the linear system (A, b, c) in the forward path, a hysteresis operator F in the negative feedback path, and exogenous input d.

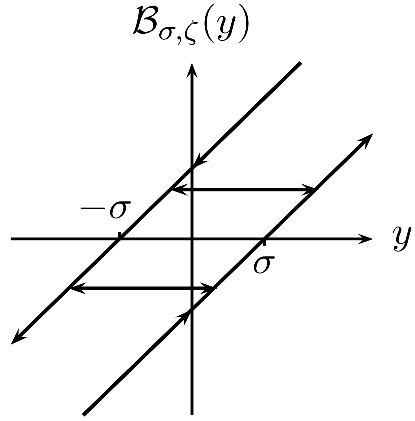

Figure S5. Backlash hysteresis revisited. This diagram shows how the backlash operator $\mathcal{B}_{\sigma,\zeta}$ acts. If, for example, $\zeta = \sigma/2$, $y(0) = 0$ and y is strictly increasing with $\lim_{t\to\infty} y(t) > 3\sigma/2$, then $\big(\mathcal{B}_{\sigma,\zeta}(y)\big)(t) = \zeta = \sigma/2$ for $0 \le t \le t_\sigma$ and $\big(\mathcal{B}_{\sigma,\zeta}(y)\big)(t) = y(t) - \sigma$ for $t > t_\sigma$, where t_σ is the unique positive number such that $y(t_\sigma) = 3\sigma/2$.

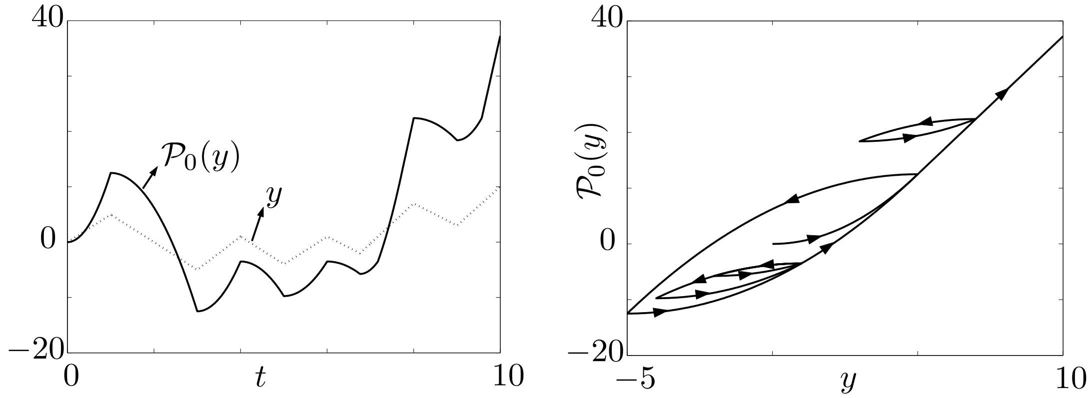

Figure S6. Example of Prandtl hysteresis. Consider the Prandtl operator $\mathcal{P}_\xi$ defined in (S12) with $\xi = 0$ and measure μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$, where $\chi_{[0,5]}$ is the indicator function of the interval $[0, 5]$. The plots depict the response $\mathcal{P}_0(y)$ to a continuous, piecewise linear input y.

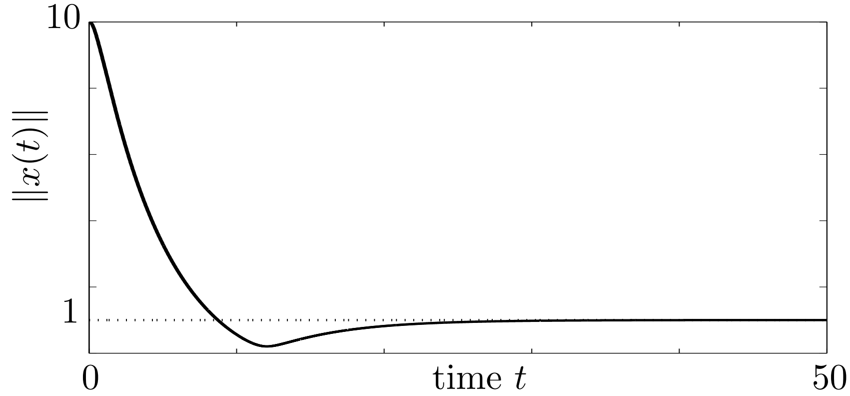

Figure S7. System response for Example S4. Consider Example S4 with parameter values $\gamma = 5$, $\sigma = 1$, $\zeta = 0$, initial data $y(0) = x_1(0) = 10$, $\dot{y}(0) = x_2(0) = 0$, and zero disturbance $d = 0$. This plot shows the evolution of the norm $\|x(t)\|$, and suggests that $\lim_{t\to\infty} \|x(t)\| = \theta_{\mathcal{P}} = \sigma = 1$. However, the theory predicts only the existence of a positive constant g_2 such that $\limsup_{t\to\infty} \|x(t)\| \leq g_2\theta_{\mathcal{P}} = g_2\sigma = g_2$, see (S21).

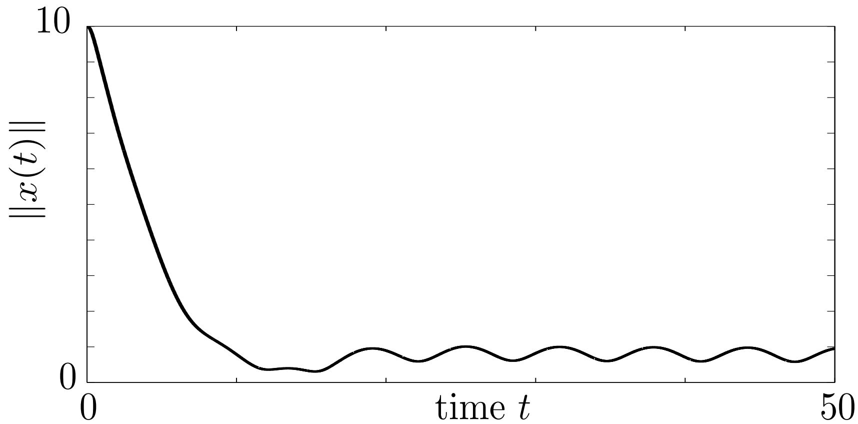

Figure S8. System response for Example S4. Consider Example S4 with parameter values $\gamma = 5$, $\sigma = 1$, $\zeta = 0$, initial data $y(0) = x_1(0) = 10$, $\dot{y}(0) = x_2(0) = 0$, and sinusoidal disturbance $d\colon t \mapsto \sin t$. This plot shows the evolution of the norm $\|x(t)\|$, and suggests that $\limsup_{t\to\infty} \|x(t)\| < 2$. However, the theory predicts only the existence of a positive constant g_2 such that $\limsup_{t\to\infty} \|x(t)\| \leq g_2(\|d\|_{L^\infty} + \theta_{\mathcal{P}}) = 2g_2$, see (S21).

Sidebar 5: Filippov's Selection Theorem

Let I be an interval and let U be a set-valued function defined on I with nonempty values contained in $\mathbb{F}^m$. A function $u : I \to \mathbb{F}^m$ is a *measurable selection* of U if u is measurable and $u(t) \in U(t)$ for a.a. $t \in I$.

Of particular significance in applications to control theory is Theorem S5 below, a measurable selection result involving the composition of a function and a set-valued function. This theorem is frequently referred to as Filippov's selection theorem. For a proof of Theorem S5, see [55, p. 72].

Theorem S5: Let I be an interval, let U be a measurable set-valued function defined on I with nonempty closed values contained in $\mathbb{F}^m$, and let $g : I \times \mathbb{F}^m \to \mathbb{F}^p$ be a function such that, for each $t \in I$, the function $v \mapsto g(t, v)$ is continuous and, for each $v \in \mathbb{F}^m$, the function $t \mapsto g(t, v)$ is measurable. If $z : I \to \mathbb{F}^p$ is a measurable selection of the set-valued function $t \mapsto \{g(t, v) : v \in U(t)\}$, then there exists a measurable selection $u : I \to \mathbb{F}^m$ of U such that $g(t, u(t)) = z(t)$ for a.a. $t \in I$.

In the proofs of theorems 5, 13 and 15, Theorem S5 is used as follows. Let $x : [0, T) \to \mathbb{F}^n$ be a maximal solution of the differential inclusion (3) with $\Phi \in \mathcal{U}_{\mathbb{F}}$ and $\Delta \in \mathcal{D}_{\mathbb{F}}$. Defining $U(t) := \Delta(t) - b\Phi(c^* x(t))$ for all $t \in [0, T)$ and $g(t, v) := Ax(t) + bv$ for all $(t, v) \in [0, T) \times \mathbb{F}$, the functions U and g satisfy the assumptions imposed in Theorem S5 with $m = 1$ and $p = n$. Furthermore, $\dot{x}$ is a measurable selection of the set-valued function

$$t \mapsto \{g(t, v) : v \in U(t)\} = Ax(t) + b\Delta(t) - b\Phi(c^* x(t)).$$

Consequently, by Theorem S5, there exists a measurable selection $u : I \to \mathbb{F}$ of U such that

$g(t, u(t)) = \dot{x}(t)$ for a.a. $t \in I$, or, equivalently, $\dot{x}(t) = Ax(t) + bu(t)$ for a.a. $t \in I$.

Acknowledgment

This work was supported by the UK Engineering & Physical Sciences Research Council (Grant Ref: GR/S94582/01).

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