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THE CLASS NUMBER OF THE FIELD OF 5^nTH ROOTS OF UNITY

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ABSTRACT. Let h_n^- be the relative class number of the field of 5^nth roots of unity. If l is any prime number, then the l-part of h_n^- is bounded independent of n.

Let k be a number field, K/k the cyclotomic $\mathbf{Z}_p$-extension of k, and k_n the unique intermediate field of degree p^n over k. In [4] it was conjectured that if $l \neq p$ is any prime number then the l-part of the class number of k_n is bounded independent of n. This conjecture arose from analogy with the case of function fields over finite fields, where $\mathbf{Z}_p$-extensions can be obtained by extending the field of constants. In this case it is not difficult to show that the l-part of the order of the group of divisor classes of degree zero is bounded independent of n [4].

For number fields, the conjecture has been proved when $k/\mathbf{Q}$ is abelian and $p = 2$ or 3. The main obstacle for larger primes p is the existence of p-adic $(p-1)$st roots of unity, which are, of course, harder to handle when $p \geqslant 5$. In this note we attack the case of the simplest $\mathbf{Z}_5$-extension, namely the one obtained by adjoining all 5^nth roots of unity, for all $n \geqslant 1$, to the field of 5th roots of unity.

Recall that for any imaginary abelian number field K there is a maximal real subfield K^+, and since K/K^+ is totally ramified (at ∞) the class number h^+ of K^+ divides the class number h of K. The quotient h/h^+ is called the relative class number h^-.

THEOREM. *Let h_n^- be the relative class number of $\mathbf{Q}(\zeta_{5^n})$, where ζ_{5^n} is a primitive 5^nth root of unity. Let l be any prime number and let $l^{e_n^-} \| h_n^-$. Then e_n^- is bounded as $n \to \infty$. In fact, if m_l is determined by $5^{m_l} \| l^4 - 1$ and $n \geqslant 2m_l + 2$, then $l \nmid h_n^-/h_{n-1}^-$. ($l^a \| b$ means $l^a | b$, $l^{a+1} \nmid b$).*

PROOF. Since 5 is a regular prime, $5 \nmid h_n^-$ for any n [2]. Therefore, we assume $l \neq 5$.

The set of odd Dirichlet characters of $\mathbf{Q}(\zeta_{5^n})$ is obtained as follows: Let χ_1, χ_2 be the odd characters for $\mathbf{Q}(\zeta_5)/\mathbf{Q}$ and let ψ_n be a character of conductor 5^n such that $\psi_n(a)$ depends only on a^4 mod 5^n (therefore ψ_n generates the

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characters of the subfield of $\mathbf{Q}(\zeta_{5^n})$ of degree 5^{n-1} over $\mathbf{Q}$). The odd characters of $\mathbf{Q}(\zeta_{5^n})$ are then $\{\chi_i\psi_n^j\}$, where $i = 1, 2$ and $0 \leqslant j \leqslant 5^{n-1} - 1$. Since $\chi_i\psi_n^j$ is also a character for $\mathbf{Q}(\zeta_{5^{n-1}})$ if $5|j$, we obtain from the analytic class number formula [1]:

$$\frac{h_n^-}{h_{n-1}^-} = 5 \cdot \prod_{i=1}^{2} \prod_{j=1;5\nmid j}^{5^{n-1}-1} \left(-\tfrac{1}{2} B_{1,\chi_i\psi_n^j}\right),$$

where

$$B_{1,\chi_i\psi_n^j} = \frac{1}{5^n} \sum_{0<a<5^n} a\chi_i\psi_n^j(a).$$

Let $F_m = \mathbf{Q}(\sqrt{-1}, \zeta_{5^m})$. Note that $B_{1,\chi_i\psi_n^j} \in F_n$. For positive integers n, m with $n \geqslant 2m$, let Tr be the trace function from F_n to F_m. We shall calculate $\mathrm{Tr}(\frac{1}{2} B_{1,\chi\psi_n})$, where $\chi = \chi_1$ or χ_2. Note that $\mathrm{Tr}(\zeta_{5^c}) = 0$ if $c > m$. Therefore,

$$\mathrm{Tr}(\psi_n(a)) \neq 0 \Leftrightarrow \psi_n(a)^{5^m} = 1 \Leftrightarrow \psi_n(a^{5^m}) = 1$$
$$\Leftrightarrow a^{4\cdot 5^m} \equiv 1 \bmod 5^n \Leftrightarrow a^4 \equiv 1 \bmod 5^{n-m},$$

in which case $\mathrm{Tr}(\psi_n(a)) = 5^{n-m}\psi_n(a)$. Therefore,

$$\mathrm{Tr}\left(\tfrac{1}{2} B_{1,\chi\psi_n}\right) = \frac{1}{2 \cdot 5^m} \sum_{0<a<5^n;\, a^4\equiv 1\ (5^{n-m})} a\chi\psi_n(a).$$

Let b satisfy $b^4 \equiv 1 \bmod 5^{n-m}$, $1 < b < \frac{1}{2}5^{n-m}$ (so $b \equiv 2$ or $3 \bmod 5$). Then the sum becomes

$$\frac{1}{2 \cdot 5^m} \sum_{a\equiv \pm 1\ (5^{n-m})} a\chi\psi_n(a) + \frac{1}{2 \cdot 5^m} \sum_{a\equiv \pm b\ (5^{n-m})} a\chi\psi_n(a).$$

We now evaluate $\sum_{a\equiv\pm c\ (5^{n-m})} a\chi\psi_n(a)$, where $0 < c < 5^{n-m}$ and $5 \nmid c$. First, note that

$$\sum_{0\leqslant k<5^m} \psi_n(c + k5^{n-m}) = \psi_n(c)\sum \psi_n(1 + c^{-1}k5^{n-m}) = 0,$$

since the latter sum is over the subgroup $(1 + 5^{n-m}\mathbf{Z})/(1 + 5^n\mathbf{Z})$ of $(\mathbf{Z}/5^n\mathbf{Z})^\times$. Therefore, using the fact that χ is an odd character of conductor 5 and ψ_n is an even character, we obtain

$$\sum_{t=0}^{5^m-1} (c + 5^{n-m}t)\chi\psi_n(c + 5^{n-m}t) + (5^n - c - 5^{n-m}t)\chi\psi_n(5^n - c - 5^{n-m}t)$$
$$= 2\chi\psi_n(c) \cdot 5^{n-m} \sum_{t=0}^{5^m-1} t\psi_n(1 + c^{-1}t5^{n-m}).$$

Let $\zeta = \psi_n(1 + 5^{n-m})$, so ζ is a primitive 5^mth root of unity. Then, since $n \geqslant 2m$, we have

$$(1 + 5^{n-m})^{\tau} \equiv 1 + \tau 5^{n-m} \bmod 5^n,$$

so we obtain

$$\sum_{a \equiv \pm c\ (5^{n-m})} a\chi\psi_n(a) = 2 \cdot \chi\psi_n(c) 5^{n-m} \sum_{t=0}^{5^m - 1} t\zeta^{c^{-1}t}$$

$$= 2\chi\psi_n(c) \cdot 5^n \cdot 1/\left(\zeta^{c^{-1}} - 1\right).$$

Therefore,

$$5^{n-m}\mathrm{Tr}\left(\tfrac{1}{2} B_{1,\chi\psi_n}\right) = \frac{1}{\zeta - 1} + \frac{\chi\psi_n(b)}{\zeta^{b^{-1}} - 1} = \frac{1}{\zeta - 1} + \frac{\chi(b)\zeta^a}{\zeta^{b^{-1}} - 1}$$

for some integer a, since $b^4 \equiv 1 \bmod 5^{n-m}$ implies that $\psi_n(b)^{5^m} = 1$.

Now, suppose $n \geqslant 2m_l + 2$ and that $l | h_n^- / h_{n-1}^-$. Then there is a prime $\tilde{l}$ over l such that $\tilde{l} | \frac{1}{2} B_{1,\chi\psi_n^j}$ for some $\chi = \chi_1$ or χ_2 and some j with $5 \nmid j$. By changing the original choice of ψ_n if necessary, we may assume that $\tilde{l} | \frac{1}{2} B_{1,\chi\psi_n}$. The definition of m_l implies that all primes lying above l, in particular $\tilde{l}$, are inert for F_n / F_{m_l}. If we let $m = m_l + 1$ (therefore $n \geqslant 2m$), we find that $\tilde{l} | 5^{m-n}\mathrm{Tr}(\frac{1}{2} B_{1,\chi\psi_n})$. Now let $\sigma \in \mathrm{Gal}(F_m / F_{m_l})$ be defined by $\sigma(\zeta_{5^m}) = \zeta_{5^m}^j$, where $j \equiv 1 \bmod 5^{m_l}$. Since $\tilde{l}$ is inert for F_m / F_{m_l} it follows that

$$\tilde{l}\, |\sigma\left(5^{m-n}\mathrm{Tr}\left(\tfrac{1}{2} B_{1,\chi\psi_n}\right)\right) = \frac{1}{\zeta^j - 1} + \frac{\chi(b)\zeta^{aj}}{\zeta^{jb^{-1}} - 1}.$$

Let $\alpha = \zeta^{5^{m_l}}$. Then we obtain (let $j = 1 + u5^{m_l}$)

$$\zeta^{b^{-1}}\alpha^{ub^{-1}} - 1 + \chi(b)\zeta^{a+1}\alpha^{u(a+1)} - \chi(b)\zeta^a\alpha^{ua} \equiv 0 \bmod \tilde{l}$$

for all integers u. By linear independence of characters [3, p. 209], or Vanermonde determinants, we see that the coefficients of each power of α must vanish mod $\tilde{l}$. The only way this can happen is to have b^{-1}, 0, $a + 1$, and a satisfy either $0 \equiv a + 1$, $b^{-1} \equiv a \bmod 5$ or $0 \equiv a$, $b^{-1} \equiv a + 1 \bmod 5$ (since $\alpha^5 = 1$; note that $b^{-1} \equiv 0$ cannot happen because of the choice of b). Therefore, either $b^{-1} \equiv -1 \bmod 5$ or $b^{-1} \equiv 1 \bmod 5$. But neither of these can happen since $b \equiv 2$ or $3 \bmod 5$. We have therefore obtained a contradiction; consequently we must have $l \nmid h_n^- / h_{n-1}^-$. Q.E.D.

We remark that using the action of the Galois group plus linear independence of characters is essentially equivalent to multiplying the expression by an arbitrary root of unity and applying the trace function. This latter method was employed in [4, p. 180]. The calculations are the same in both cases. In fact, the trace function could have been used in the last step of the above proof, with the same result.

Finally, we note that the above theorem unfortunately does not seem to imply the corresponding result for the full class number h_n, since the usual techniques for using h_n^- to obtain results about h_n^+ require that the lth roots of unity be in the field being considered (see [4]).

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