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THE CLASS OF THE AFFINE LINE IS A ZERO DIVISOR IN THE GROTHENDIECK RING

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Abstract

We show that the class of the affine line is a zero divisor in the Grothendieck ring of algebraic varieties over complex numbers. The argument is based on the Pfaffian-Grassmannian double mirror correspondence.

1. Introduction

The Grothendieck ring $K_0(Var/\mathbb{C})$ of complex algebraic varieties is a fundamental object of algebraic geometry. It is defined as the quotient of the group of formal integer linear combinations $\sum_i a_i[Z_i]$ of isomorphism classes of complex algebraic varieties modulo the relations

$$[Z] - [U] - [Z\backslash U]$$

for all open subvarieties $U \subseteq Z$. The product structure is induced from the Cartesian product.

The main result of this paper is the following.

Theorem 2.13. *The class L of the affine line is a zero divisor in the Grothendieck ring of varieties over $\mathbb{C}$.*

The class $L = [\mathbb{C}^1]$ of the affine line plays an important role in the study of $K_0(Var/\mathbb{C})$. For example, it has been proved in [10] that the quotient of $K_0(Var/\mathbb{C})$ by L has a natural basis indexed by the classes of projective algebraic varieties up to stable birational equivalence. In other instances one needs to localize $K_0(Var/\mathbb{C})$ by L (see [4, 12]), so it is important to know whether L is a nonzero divisor. While it has been shown in [14] that $K_0(Var/\mathbb{C})$ is not a domain, there remained a hope that L is nonetheless a nonzero divisor in $K_0(Var/\mathbb{C})$.

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This problem was brought to our attention by an elegant recent preprint of Galkin and Shinder [5] in which the authors prove that if L is a nonzero divisor in $K_0(Var/\mathbb{C})$ (a weaker condition that $L^2a = 0$ implies $a \in \langle L \rangle$ in fact suffices), then a rational smooth cubic fourfold in $\mathbb{P}^5$ must have its Fano variety of lines birational to a symmetric square of a K3 surface. This paper puts a dent in this approach to (ir)rationality of cubic fourfolds.

The consequence of our construction is another important result, which was pointed out to us by Evgeny Shinder. A cut-and-paste conjecture (or question) of Larsen and Lunts [10, Question 1.2] asks whether any two algebraic varieties X and Y with $[X] = [Y]$ in the Grothendieck ring can be cut into disjoint unions of pairwise isomorphic locally closed subvarieties.

Theorem 2.14. *The cut-and-paste conjecture of Larsen and Lunts fails.*

The negative answer to this conjecture is important in view of its potential applications to rationality of motivic zeta functions; see [4], [11].

The main idea of the proof of Theorems 2.13 and 2.14 is to compare the two sides X_W and Y_W of the Pfaffian-Grassmannian double mirror correspondence. These are nonbirational smooth Calabi-Yau threefolds which are derived equivalent. There is a natural variety (a frame bundle over the Cayley hypersurface of X_W) whose class in the Grothendieck ring can be expressed both in terms of $[X_W]$ and in terms of $[Y_W]$. This provides a relation

$$\Big([X_W] - [Y_W]\Big)(L^2 - 1)(L - 1)L^7 = 0$$

in the Grothendieck ring, which then implies that L is a zero divisor.

After this preprint appeared, the result was improved to

$$\Big([X_W] - [Y_W]\Big)(L + 1)L^6 = 0$$

by Kuznetsov [9] and then later to

$$\Big([X_W] - [Y_W]\Big)L^6 = 0$$

independently by Chambert-Loir and Martin [3, 13].

2. The construction

2.1. Pfaffian and Grassmannian double mirror Calabi-Yau varieties. Let V be a 7-dimensional complex vector space. Let $W \subset \Lambda^2 V^\vee$ be a generic 7-dimensional space of skew forms on V. These data encode two smooth Calabi-Yau varieties X_W and Y_W as follows.

Definition 2.1. We define X_W as a subvariety of the Grassmannian $G(2, V)$ of dimension two subspaces $T_2 \subset V$, which is the locus of all $T_2 \in G(2, V)$

with $w\Big|_{T_2} = 0$ for all $w \in W$. We define Y_W as a subvariety of the Pfaffian variety $Pf(V) \subset \mathbb{P}\Lambda^2 V$ of skew forms on V whose rank is less than 6. It is defined as the intersection of $Pf(V)$ with $\mathbb{P}W \subset \mathbb{P}\Lambda^2 V$.

The following proposition summarizes the properties of X_W and Y_W that will be used later.

Proposition 2.2. *The following statements hold for a general choice of W.*

- *The varieties X_W and Y_W are smooth Calabi-Yau threefolds.*
- *The varieties X_W and Y_W are not isomorphic, or even birational, to each other.*
- *All forms $\mathbb{C}w \in Y_W$ have rank 4. All forms $\mathbb{C}w \in \mathbb{P}W \backslash Y_W$ have rank 6.*

Proof. Smoothness of X_W and Y_W has been shown by Rødland [15]. They are not isomorphic to each other because the ample generators D_X and D_Y of their respective Picard groups have $D_X^3 = 42$ and $D_Y^3 = 14$. The statement that X_W and Y_W are not birational follows from the fact that they are non-isomorphic Calabi-Yau threefolds with Picard number one; see [2].

The statement about the rank of the forms follows from the fact that W is generic, since the locus of rank 2 forms in $\mathbb{P}\Lambda^2 V^\vee$ is of codimension 10. Alternatively, if $\mathbb{C}w \in Y_W$ has rank 2, then Y_W is automatically singular at $\mathbb{C}w$. □

Remark 2.3. The varieties X_W and Y_W are double-mirror to each other, in the sense that they have the same mirror family. This is just a heuristic statement, but it does indicate that geometry of X_W is intimately connected to that of Y_W. For example, it was shown independently in [2] and [8] that X_W and Y_W have equivalent derived categories.

2.2. Cayley hypersurface and its frame bundle. The main technical tool of this paper is the so-called Cayley hypersurface of X_W. It is the hypersurface in $G(2, V) \times \mathbb{P}W$ which consists of pairs $(T_2, \mathbb{C}w)$ with the property $w\Big|_{T_2} = 0$. The class of X_W in the Grothendieck ring of varieties over $\mathbb{C}$ is related to that of H as follows.

Proposition 2.4. *The following equality holds in the Grothendieck ring:*

$$[H] = [G(2,7)][\mathbb{P}^5] + [X_W]L^6.$$

Proof. Consider the projection of H onto $G(2, V)$. The restriction of this map to the preimage of X_W is a trivial fibration with fiber $\mathbb{P}W = \mathbb{P}^6$. The restriction of it to the complement of X_W is a Zariski locally trivial fibration with fiber $\mathbb{P}^5$. Indeed, the hyperplanes of w that vanish on a given T_2 can be Zariski locally identified with a fixed $\mathbb{P}^5$ by projecting from a fixed point in $\mathbb{P}W$. This gives

$$[H]=[X_W][\mathbb{P}^6]+([G(2,7)]-[X_W])[\mathbb{P}^5]=[G(2,7)][P^5]+[X_W]([\mathbb{P}^6]-[\mathbb{P}^5]),$$

which proves the claim. □

Remark 2.5. In the proof of Proposition 2.4 we used the statement that for a Zariski locally trivial fibration $Z\to B$ with fiber F there holds $[Z]=[B][F]$ in $K_0(Var/\mathbb{C})$. We will use this statement repeatedly in the subsequent arguments.

We can project the Cayley hypersurface H onto the second factor $\pi: H\to\mathbb{P}W$. We will have different fibers depending on whether the image lies in Y_W or not. While we would like to say that the restriction of π to the preimages of Y_W and its complement are Zariski locally trivial, we do not know if this is true or not. So instead of using H itself we will pass to the frame bundle $\tilde{H}$ over H.

Definition 2.6. We denote by $\tilde{H}$ the frame bundle of H, i.e. the space of triples (v_1,v_2,w) where v_1 and v_2 are linearly independent vectors in V and w is an element of $\mathbb{P}W$ such that $w(v_1,v_2)=0$.

Remark 2.7. Since $\tilde{H}$ is the frame bundle of the Zariski locally trivial vector bundle (pullback of the tautological subbundle on $G(2,V)$) on H, the fibration $\tilde{H}\to H$ is Zariski locally trivial. An easy calculation shows that

$$[\tilde{H}]=[H](L^2-1)(L^2-L) \tag{1}$$

in the Grothendieck ring.

We now consider the projection $\tilde{H}\to\mathbb{P}W$. Notice that we have

$$\tilde{H}=\tilde{H}_1\sqcup\tilde{H}_2 \tag{2}$$

where $\tilde{H}_1$ is the preimage of Y_W and $\tilde{H}_2$ is the preimage of its complement in $\mathbb{P}W$.

Proposition 2.8. *The following equality holds in the Grothendieck ring:*

$$[\tilde{H}_1]=[Y_W]\Big((L^3-1)(L^7-L)+(L^7-L^3)(L^6-L)\Big).$$

Proof. There is a subvariety $\tilde{H}_{1,1}$ in $\tilde{H}_1$ given by the condition $v_1\in Ker(w)$. Forgetting v_2 realizes $\tilde{H}_{1,1}$ as a Zariski locally trivial fibration with fiber $\mathbb{C}^7-\mathbb{C}$ over the space of pairs (v_1,w) with $v_1\in Ker(w),\ v_1\neq 0$. This in turn is a Zariski locally trivial fibration over Y_W with fiber $(\mathbb{C}^3-\mathrm{pt})$, since all $\mathbb{C}w\in Y_W$ have rank 4. Putting all this together, we have

$$[\tilde{H}_{1,1}]=[Y_W](L^3-1)(L^7-L)$$

in the Grothendieck ring. Similarly, the complement $\tilde{H}_{1,2}$ of $\tilde{H}_{1,1}$ in $\tilde{H}_1$ satisfies

$$[\tilde{H}_{1,2}]=[Y_W](L^7-L^3)(L^6-L).$$

Indeed, $\tilde{H}_{1,2}$ forms a vector bundle of rank 6 over the space of pairs (v_1, w), since the condition $w(v_1, v_2) = 0$ is now nontrivial. The result of the proposition now follows from $[\tilde{H}_1] = [\tilde{H}_{1,1}] + [\tilde{H}_{1,2}]$. □

Proposition 2.9. *The following equality holds in the Grothendieck ring:*

$$[\tilde{H}_2] = \Big([\mathbb{P}^6] - [Y_W]\Big)\Big((L-1)(L^7 - L) + (L^7 - L)(L^6 - L)\Big).$$

Proof. The argument is completely analogous to that of Proposition 2.8. The only difference is that a form $\mathbb{C}w \notin Y_W$ has rank 6 and thus a 1-dimensional kernel. □

As a corollary of Propositions 2.8 and 2.9 we get the formula for $[\tilde{H}]$.

Proposition 2.10. *The following equality holds in the Grothendieck ring:*

$$[\tilde{H}] = [\mathbb{P}^6](L^7 - L)(L^6 - 1) + [Y_W](L^2 - 1)(L - 1)L^7.$$

Proof. This follows immediately from (2) and Propositions 2.8 and 2.9. □

2.3. Main theorem. We are now ready to prove our main result. We start with the following formula derived from the calculations of the previous subsection.

Proposition 2.11. *The following equality holds in the Grothendieck ring:*

$$\Big([X_W] - [Y_W]\Big)(L^2 - 1)(L - 1)L^7 = 0.$$

Proof. We use Proposition 2.10 and Proposition 2.4 with equation (1) to get expressions for $[\tilde{H}]$, in terms of $[Y_W]$ and $[X_W]$ respectively. By subtracting one from the other we get

$$\begin{aligned}&\Big([X_W] - [Y_W]\Big)(L^2 - 1)(L - 1)L^7\\ &\qquad = [\mathbb{P}^6](L^7 - L)(L^6 - 1) - [G(2,7)][\mathbb{P}^5](L^2 - 1)(L^2 - L),\end{aligned}$$

which then equals zero in view of $[G(2,7)](L^2 - 1)(L^2 - L) = (L^7 - 1)(L^7 - L)$ and $[\mathbb{P}^6](L^6 - 1) = [\mathbb{P}^5](L^7 - 1)$. □

Remark 2.12. It was communicated to us by Kuznetsov [9] that the factor $(L^2 - 1)(L - 1)L^7$ in the statement of Proposition 2.11 can be replaced by $(L + 1)L^6$ by considering the projectivization of the tautological subbundle instead of the frame bundle. Later, Chambert-Loir and Martin [3, 13] independently showed that

$$\Big([X_W] - [Y_W]\Big)L^6 = 0.$$

Their argument relies on the fact that a skew-symmetric form over any field has a standard symplectic basis.

Theorem 2.13. *The class L of the affine line is a zero divisor in the Grothendieck ring of varieties over $\mathbb{C}$.*

Proof. In view of Proposition 2.11, it suffices to show that

$$\Big([X_W]-[Y_W]\Big)(L^2-1)(L-1)$$

is a nonzero element of the Grothendieck ring. In fact, we can argue that it is a nonzero element modulo L. Indeed, if it were zero modulo L, this would mean that $[X_W]=[Y_W] \mod L$. This implies that X_W is stably birational to Y_W, by [10]. This means that for some $k\geq 0$ the varieties $X_W\times\mathbb{P}^k$ and $Y_W\times\mathbb{P}^k$ are birational to each other. We now consider the MRC fibration [7], which is a birational invariant of an algebraic variety. Importantly, if X is not uniruled (for example a Calabi-Yau variety), then the base of the MRC fibration of $X\times\mathbb{P}^k$ is X. Thus, birationality of $X_W\times\mathbb{P}^k$ and $Y_W\times\mathbb{P}^k$ implies birationality of X_W and Y_W, which is known to be false; see Proposition 2.2. □

It was observed by Evgeny Shinder that the construction of this paper provides a negative answer to the cut-and-paste question of Larsen and Lunts [10, Question 1.2], which asks whether any two varieties with equal classes in the Grothendieck ring can be cut up into isomorphic pieces.[1]

Theorem 2.14. *The cut-and-paste conjecture of Larsen and Lunts fails.*

Proof. The equality

$$[X_W](L^2-1)(L-1)L^7=[Y_W](L^2-1)(L-1)L^7$$

implies that trivial $GL(2,\mathbb{C})\times\mathbb{C}^6$ bundles over X_W and Y_W have the same class in the Grothendieck ring. If it were possible to cut them into unions of isomorphic varieties, then $X_W\times GL(2,\mathbb{C})\times\mathbb{C}^6$ would be birational to $Y_W\times GL(2,\mathbb{C})\times\mathbb{C}^6$. This implies that X_W and Y_W are stably birational, and thus birational, in contradiction to Proposition 2.2. □

Remark 2.15. Our method works over any field of characteristic zero. It does not appear to work in positive characteristics, since results of [10] are based on [1], which in turn relies on the resolution of singularities.

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[1] Another counterexample to the question was recently announced by Ilya Karzhemanov in [6].

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