H. MASAOKA
KODAI MATH. J.
33 (2010), 233–239

THE CLASSES OF BOUNDED HARMONIC FUNCTIONS AND HARMONIC FUNCTIONS WITH FINITE DIRICHLET INTEGRALS ON HYPERBOLIC RIEMANN SURFACES

Hiroaki Masaoka

Dedicated to Professor Yoichi Imayoshi on his sixtieth birthday

Introduction

For an open Riemann surface R, we denote by $HP(R)$, $HB(R)$ and $HD(R)$ the class of differences of positive harmonic functions on R, bounded harmonic functions on R, and harmonic functions with finite Drichlet integrals on R, respectively. And denote by $HP_+(R)$, $HB_+(R)$ and $HD_+(R)$ the class of positive harmonic functions on R, bounded and positive harmonic functions on R, and positive harmonic functions with finite Drichlet integrals on R, respectively. Note that $HX(R) = HX_+(R) - HX_+(R)$, $X = P, B, D$. It is easily seen that $HB(R) \subset HP(R)$ and $HD(R) \subset HP(R)$ (cf. [9]).

We say that an open Riemann surface R is *parabolic* (resp. *hyperbolic*) if R does not admit (resp. admits) Green's functions on R. It is well-known that, if R is parabolic, then $HP(R)$, $HB(R)$ and $HD(R)$ consist of constant functions (cf. [9]).

Hereafter, we consider only hyperbolic Riemann surfaces R. Let $\Delta = \Delta^{R,M}$ and $\Delta_1 = \Delta_1^{R,M}$ the *Martin boundary* of R and the *minimal Martin boundary* of R, respectively. We refer to [2] for details about the Martin boundary.

In [4] (resp. [5]) we gave necessary and sufficient conditions in terms of Martin boundary in order that the converse $HB(R) \supset HP(R)$ (resp. $HD(R) \supset HP(R)$) of the above respective inclusion relations hold.

Though the inclusion relation between $HB(R)$ and $HD(R)$ does not generally hold, it seems to be an interesting problem to give a necessary and sufficient condition in order that $HB(R)$ coincides with $HD(R)$. The purpose of this article is to prove the following

Main Theorem. *Suppose that R is hyperbolic. Then the followings are equivalent by pairs*:

Received May 19, 2009; revised November 10, 2009.

(i) $HB(R) = HD(R)$;

(ii) *there exists a null set* N *of* Δ *with respect to the harmonic measure such that* $\Delta_1 \backslash N$ *consists of finitely many points with positive harmonic measures whose Martin functions have finite Dirichlet integrals*;

(iii) $\dim HB(R) = \dim HD(R) < \infty$,

where $\dim HX(R)$ *is the dimension of the linear space* $HX(R)$, $X = B, D$.

Finally the author would like to express his deepest gratitude to Prof. S. Segawa for his valuable comment and at the same time to a referee for his helpful advice. He told the author that Prof. M. Nakai [8] gave an alternative proof for Main Theorem.

1. Preliminaries

In this section we state several propositions in order to prove Main Theorem in Introduction in the next section.

Let z_0 be a specified point of R, which serves as a reference point. Denote by $\omega_z(\cdot)$ the *harmonic measure* on Δ with respect to $z \in R$. We also denote by $k_\zeta(z)$ $((\zeta, z) \in (R \cup \Delta) \times R)$ the *Martin function* on R with pole at ζ.

First we give a characterization for boundedness of Martin function.

Proposition 1 (cf. [2, Hilfssatz 13.3]). *Let* ζ *belong to* Δ_1. *Then the Martin function* $k_\zeta(\cdot)$ *with pole at* ζ *is bounded on* R *if and only if the harmonic measure* $\omega_.(\{\zeta\})$ *of the singleton* $\{\zeta\}$ *is positive.*

Next we review for fundamental properties concerning $HD(R)$.

Definition 1. Fix $z_0 \in R$. For $h, u \in HD(R)$, set

$$D(h, u) := \int_R (\operatorname{grad} h(z), \operatorname{grad} u(z))\, dv(z),$$

where $\operatorname{grad} h(z)$ is the gradient of h at z, $(\operatorname{grad} h(z), \operatorname{grad} u(z))$ is the usual inner product of two vectors $\operatorname{grad} h(z)$ and $\operatorname{grad} u(z)$ in $\mathbf{R}^2$ and v is the area element on R.

For $h \in HD(R)$, denote by $D(h) := D(h, h)$. We call it the Dirichlet integral of h. By the discussion in [1, p. 400] we have the following

Proposition 2 (cf. [1]). *$HD(R)$ is a Hilbert space with the inner product $D(\cdot\,,\cdot)$ in the above definition, two functions being identified if their difference is a constant function.*

Proposition 3 (cf. [3]). *$h \in HD(R)$ if and only if h has the minimal fine limit $h^*(\zeta)$ at almost every point $\zeta (\in \Delta_1)$ with respect to the harmonic measure ω_{z_0} such* *that* $h(z) = \int_{\Delta_1} h^*(\zeta)\, d\omega_z(\zeta)$, $\int_{\Delta_1} |h^*(\zeta)|^2\, d\omega_{z_0}(\zeta) < \infty$, *and the following property holds*:

$$\int_{\Delta_1}\int_{\Delta_1} (h^*(\zeta) - h^*(\xi))^2 \theta_{z_0}(\zeta,\xi)\, d\omega_{z_0}(\zeta)\, d\omega_{z_0}(\xi) < \infty,$$

where $\theta_{z_0}(\zeta,\xi)$ *is the Naïm kernel on* $(R \cup \Delta\backslash\{z_0\}) \times (R \cup \Delta\backslash\{z_0\})$ (cf. [6]).

Then, moreover,

$$D(h) = q\int_{\Delta_1}\int_{\Delta_1} (h^*(\zeta) - h^*(\xi))^2 \theta_{z_0}(\zeta,\xi)\, d\omega_{z_0}(\zeta)\, d\omega_{z_0}(\xi),$$

where q *is the absolute constasnt.*

Denote by $MHB_+(R)$ (resp. $MHD_+(R)$) the class of all finite limit functions of monotone increasing sequences of $HB_+(R)$ (resp. $HD_+(R)$). Set $MHX(R) = MHX_+(R) - MHX_+(R)$ $(X = B, D)$. The class $MHB(R)$ is called the class of quasi-bounded functions on R. By [2, Folgesatz 13.1, Satz 13.4 and Satz 14.2] we have the following

Lemma 1 (cf. [2]). *It holds that*

$$MHB(R) = \Big\{ h \,|\, h \text{ has the minimal fine limit } h^*(\zeta) \text{ at almost every point } \zeta(\in \Delta_1) \text{ with respect to } \omega_{z_0} \text{ with } u(z) = \int_{\Delta_1^M} u^*(\zeta)\, d\omega_z(\zeta) \Big\}.$$

From Proposition 3 and Lemma 1 the next lemma is easily deduced.

Lemma 2. $MHD(R) \subset MHB(R)$.

By the above lemma we have the following

Proposition 4. *Suppose that* $\dim HB(R) < \infty$. *Then* $HD(R) \subset HB(R)$.

Proof. Suppose that $\dim HB(R) < \infty$. By [5, Theorem 2], $HB(R) = MHB(R)$. Hence, it follows from Lemma 2 that $HD(R) \subset MHD(R) \subset MHB(R) = HB(R)$.

2. Proof of Main Theorem

Suppose that (i) holds. Further we suppose that there exists a point $\zeta \in \Delta$ such that $\omega_{z_0}(U_\rho(\zeta)) > 0$ for any positive ρ and $\omega_{z_0}(\{\zeta\}) = 0$, where $U_\rho(\zeta)$ is the disc with center ζ and radius ρ with respect to the standard metric on $R \cup \Delta$. Hence, there exists a monotone decreasing sequence $\{\rho_n\}_{n=1}^{\infty}$ with

$\lim_{n\to\infty}\rho_n = 0$, $\omega_{z_0}(U_{\rho_n}(\zeta)\backslash U_{\rho_{n+1}}(\zeta)) > 0$ $(n \in \mathbf{N})$ and $\lim_{n\to\infty}\omega_{z_0}(U_{\rho_n}(\zeta)) = 0$. Set $u_n(z) = \omega_z(U_{\rho_n}(\zeta))$ $(z \in R,\ n \in \mathbf{N})$. Since $HB(R) = HD(R)$, $D(u_n) < +\infty$. First we show that $\{D(u_n)\}_{n=1}^{\infty}$ is bounded. Suppose that $\{D(u_n)\}_{n=1}^{\infty}$ is unbounded. Set $V_n = U_{\rho_n}(\zeta)\backslash U_{\rho_{n+1}}(\zeta)$ $(n \in \mathbf{N})$. Then $U_{\rho_n}(\zeta) = \bigcup_{\tau=n}^{\infty} V_\tau$ $(n \in \mathbf{N})$ and $V_0 = \Delta\backslash U_{\rho_1}(\zeta)$. By Proposition 3 we have

$$\begin{aligned}\frac{1}{2}D(u_n) &= q\int_{U_{\rho_n}(\zeta)}\int_{\Delta\backslash U_{\rho_n}(\zeta)} \theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &= q\sum_{\tau=n}^{\infty}\sum_{\sigma=0}^{n-1}\int_{V_\tau}\int_{V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi).\end{aligned}$$

Hence there exists a subsequence $\{D(u_{n_\nu})\}_{\nu=1}^{\infty}$ of $\{D(u_n)\}_{n=1}^{\infty}$ with

$$\begin{aligned}&\int_{\bigcup_{\tau=n_\nu}^{n_{\nu+1}-1}V_\tau}\int_{\bigcup_{\sigma=0}^{n_\nu-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\quad= \sum_{\tau=n_\nu}^{n_{\nu+1}-1}\sum_{\sigma=0}^{n_\nu-1}\int_{V_\tau}\int_{V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi) \geq \nu^5.\end{aligned}$$

Set $u = \sum_{\nu=1}^{\infty} u_{n_\nu}/\nu^2$. It is easily seen that $u \in HB(R)$. On the other hand, we have

$$\begin{aligned}\frac{1}{2}D(u) &= q\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\mu-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_\mu}^{n_{\mu+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi) \quad (n_0 = 0)\\ &\geq q\sum_{\mu=1}^{l}\sum_{\nu=0}^{\mu-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_\mu}^{n_{\mu+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\geq q\sum_{\nu=0}^{l-1}\left(\sum_{j=\nu+1}^{l}\frac{1}{j^2}\right)^2\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\geq \frac{q}{l^4}\sum_{\nu=0}^{l-1}\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=n_\nu}^{n_{\nu+1}-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &= \frac{q}{l^4}\int_{\bigcup_{\tau=n_l}^{n_{l+1}-1}V_\tau}\int_{\bigcup_{\sigma=0}^{n_l-1}V_\sigma}\theta_{z_0}(\eta,\xi)\,d\omega_{z_0}(\eta)\,d\omega_{z_0}(\xi)\\ &\geq \frac{q}{l^4}l^5 = ql\end{aligned}$$

for every $l \in \mathbf{N}$. Hence $u \notin HD(R)$. This is a contradiction.

By definition of u_n we find that $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R. Taking sufficiently large intger m_0 and replacing $\{u_n\}_{n=1}^{\infty}$ with $\{u_n\}_{n=m_0}^{\infty}$, we may suppose that $u_1 < 1$ on R. This implies that $\omega_{z_0}(V_0) > 0$. Since $\{u_n\}_{n=1}^{\infty}$

converges to 0 locally uniformlly on R and $\{D(u_n)\}_{n=1}^{\infty}$ is bounded, by [9, the discussion in the proof of Theorem in p. 149] we find that, for every $v \in HD(R)$,

$$D(u_n, v) \to 0 \quad (n \to \infty).$$

By Mazur's Theorem (cf. [10, Theorem 2 (p. 120)]), for every v, there exist an integer n_v and non-negative sequences $\{\alpha_{v,j}\}_{j=1}^{n_v}$ such that $\sum_{j=1}^{n_v} \alpha_{v,j} = 1$ and $D(\sum_{j=1}^{n_v} \alpha_{v,j} u_j) < v^{-2}$. On the other hand, since $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ is bounded, we can take a subsequence of $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ such that $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j(z_0)\}_{v=1}^{\infty}$ converges to a constant α. Hence, by [3, Theorems 4.1 and 4.2], $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ converges to α in $L^2(\Delta, \omega_{z_0})$, where $L^2(\Delta, \omega_{z_0})$ is the set of square integrable functions on Δ with respect to ω_{z_0} and hence, by [3, the result in the first paragraph of section 12], $\{\sum_{j=1}^{n_v} \alpha_{v,j} u_j\}_{v=1}^{\infty}$ converges to α locally uniformlly on R. Hence, by [3, Theorem 4.3], and the facts that $\sum_{j=1}^{n_v} \alpha_{v,j} = 1$ and that $\omega_{z_0}(V_0) > 0$, we find that $\alpha = 0$.

Set $w_v = \sum_{j=1}^{n_v} \alpha_{v,j} u_j$. Take a subsequence $\{w_{v_\lambda}\}_{\lambda=1}^{\infty}$ of $\{w_v\}_{v=1}^{\infty}$ with $w_{v_\lambda}(z_0) < 1/\lambda^2$. Set $s = \sum_{\lambda=1}^{\infty} w_{v_\lambda}$. By [3, Theorem 4.2] s is well-defined. We find that $s \in HD(R)$ and that s is unbounded on any neigborhood of ζ, that is, $s \in HD(R) \backslash HB(R)$. This is a conradiction.

Hence, if $\zeta (\in \Delta)$ satisfies that $\omega_{z_0}(U_\rho(\zeta)) > 0$ for every positive $\rho, \omega_{z_0}(\{\zeta\}) > 0$. It follows from this fact that there exists a subset N of Δ such that $\omega_{z_0}(N) = 0$ and that $\Delta_1 \backslash N$ consists of at most countably many points with positive harmonic measure. To see this set

$$N = \{\zeta \in \Delta \,|\, \text{there exists a positive } \rho_\zeta \text{ with } \omega_{z_0}(U_{\rho_\zeta}(\zeta)) = 0\}$$

and set $F = \Delta \backslash N$. Clearly $F \cup N = \Delta$, $F \cap N = \emptyset$ and $\omega_{z_0}(\{\zeta\}) > 0$ for every $\zeta \in F$. Hence F is an at most countable subset of Δ_1 because $\omega_{z_0}(\Delta) = 1$ and $\omega_{z_0}(\Delta \backslash \Delta_1) = 0$. Hence it is sufficient to prove that $\omega_{z_0}(N) = 0$. Set $O = \bigcup_{\zeta \in N} U_{\rho_\zeta}(\zeta)$. Clearly O is an open subset of $R \cup \Delta$ and $O \cap \Delta = N$. By the Lindelöf theorem there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ of N with $O = \bigcup_{n=1}^{\infty} U_{\rho_{\xi_n}}(\xi_n)$. Hence $\omega_{z_0}(N) \le \omega_{z_0}(O) \le \sum_{n=1}^{\infty} \omega_{z_0}(U_{\rho_{\xi_n}}(\xi_n)) = 0$, and hence, $\omega_{z_0}(N) = 0$.

Suppose that $\sharp(\Delta_1 \backslash N) = \aleph_0$, where $\sharp(\Delta_1 \backslash N)$ is the cardinal number of $\Delta_1 \backslash N$. Set $\Delta_1 \backslash N = \{\zeta_n\}_{n=1}^{\infty}$.

Set $u_n(z) = \omega_z(\{\zeta_j\}_{j=n}^{\infty})$ $(z \in R)$. Since $HB(R) = HD(R)$, $D(u_n) < \infty$. First we show that $\{D(u_n)\}_{n=1}^{\infty}$ is bounded. Suppose that $\{D(u_n)\}_{n=1}^{\infty}$ is unbounded. By Proposition 3 we have

$$\frac{1}{2} D(u_n) = q \sum_{\tau=n}^{\infty} \sum_{\sigma=1}^{n-1} \theta_{z_0}(\zeta_\tau, \zeta_\sigma) \omega_{z_0}(\{\zeta_\tau\}) \omega_{z_0}(\{\zeta_\sigma\}).$$

Hence there exists a subsequence $\{D(u_{n_v})\}_{v=1}^{\infty}$ of $\{D(u_n)\}_{n=1}^{\infty}$ with

$$\sum_{\tau=n_v}^{n_{v+1}-1} \sum_{\sigma=1}^{n_v-1} \theta_{z_0}(\zeta_\tau, \zeta_\sigma) \omega_{z_0}(\{\zeta_\tau\}) \omega_{z_0}(\{\zeta_\sigma\}) \ge v^5.$$

Set $u = \sum_{\nu=1}^{\infty} u_{n_\nu}/\nu^2$. It is easily seen that $u \in HB(R)$. On the other hand, for any integer $l(\geq 2)$, we have

$$\begin{aligned}
\frac{1}{2}D(u) &= q\sum_{\mu=2}^{\infty}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\}) \\
&\geq q\sum_{\mu=2}^{l}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\left(\sum_{j=\nu+1}^{\mu}\frac{1}{j^2}\right)^2\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\}) \\
&\geq \frac{q}{l^4}\sum_{\mu=2}^{l}\sum_{\nu=1}^{\mu-1}\sum_{\tau=n_\mu}^{n_{\mu+1}-1}\sum_{\sigma=n_\nu}^{n_{\nu+1}-1}\theta_{z_0}(\zeta_\tau,\zeta_\sigma)\omega_{z_0}(\{\zeta_\tau\})\omega_{z_0}(\{\zeta_\sigma\}) \\
&\geq \frac{q}{l^4}l^5 = ql.
\end{aligned}$$

Hence $u \notin HD(R)$. This is a contradiction.

By definition of u_n we find that $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R. Replacing $\{u_n\}_{n=1}^{\infty}$ with $\{u_n\}_{n=2}^{\infty}$, we may suppose that $u_1 < 1$ on R. This implies that $\omega_{z_0}(\Delta\backslash\{\zeta_n\}_{n=1}^{\infty}) > 0$. Since $\{u_n\}_{n=1}^{\infty}$ converges to 0 locally uniformlly on R and $\{D(u_n)\}_{n=1}^{\infty}$ is bounded, by [9, the discussion in the proof of Theorem in p. 149] we find that, for every $v \in HD(R)$,

$$D(u_n, v) \to 0 \quad (n \to \infty).$$

By Mazur's Theorem (cf. [10, Theorem 2 (p. 120)]), for every ν, there exist an integer n_ν and non-negative sequences $\{\alpha_{\nu,j}\}_{j=1}^{n_\nu}$ such that $\sum_{j=1}^{n_\nu}\alpha_{\nu,j} = 1$ and $D(\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j) < \nu^{-2}$. On the other hand, since $\{\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j\}_{\nu=1}^{\infty}$ is bounded, we can take a subsequence of $\{\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j\}_{\nu=1}^{\infty}$ such that $\{\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j(z_0)\}_{\nu=1}^{\infty}$ converges to a constant α. Hence, by [3, Theorems 4.1 and 4.2], $\{\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j\}_{\nu=1}^{\infty}$ converges to α in $L^2(\Delta, \omega_{z_0})$, where $L^2(\Delta, \omega_{z_0})$ is the set of square integrable functions on Δ with respect to ω_{z_0} and hence, by [3, the result in the first paragraph of section 12], $\{\sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j\}_{\nu=1}^{\infty}$ converges to α locally uniformlly on R. Hence, by [3, Theorem 4.3], and the facts that $\sum_{j=1}^{n_\nu}\alpha_{\nu,j} = 1$ and that $\omega_{z_0}(\Delta\backslash\{\zeta_n\}_{n=1}^{\infty}) > 0$, we find that $\alpha = 0$.

Set $w_\nu = \sum_{j=1}^{n_\nu}\alpha_{\nu,j}u_j$. Take a subsequence $\{w_{\nu_\lambda}\}_{\lambda=1}^{\infty}$ of $\{w_\nu\}_{\nu=1}^{\infty}$ with $w_{\nu_\lambda}(z_0) < 1/\lambda^2$. Set $s = \sum_{\lambda=1}^{\infty} w_{\nu_\lambda}$. By [3, Theorem 4.2] s is well-defined. Clearly $s \in HD(R)$. Let ξ_0 be an accumlating point of $\{\zeta_j\}_{j=1}^{\infty}$. We find that s is unbounded on any neighborhood of ξ_0. Hence $s \in HD(R)\backslash HB(R)$. This is a contradiction. Hence $\sharp\{\zeta_n\}_{n\geq 1} < \infty$.

Hence, setting $N = \Delta_1\backslash\{\zeta \in \Delta_1 : \omega_{z_0}(\zeta) > 0\}$, by Proposition 1, we find that $\omega_{z_0}(N) = 0$, $\sharp(\Delta_1\backslash N) < \infty$, and $k_\zeta \in HB(R) \cap HD(R)$ for all $\zeta \in \Delta_1\backslash N$. Therefore we have (ii).

Suppose that (ii) holds. Hence, there exists a null set N of Δ with respect to the harmonic measure such that $\Delta_1\backslash N$ consists of finitely many points and the Martin function k_ζ on R with pole at a point ζ of $\Delta_1\backslash N$ is a bounded and positive harmonic function with a finite Dirichlet integral. Put $\sharp(\Delta_1\backslash N) = m$.

$\Delta_1 \backslash N = \{\zeta_1, \dots, \zeta_m\}$. Take any $h \in HB(R)$ (resp. $h \in HD(R)$). Then there exist $h_\imath \in HP_+(R)$ $(\imath = 1, 2)$ with $h = h_1 - h_2$ on R. By the Martin reprensentation theorem there exist the positive measures μ_1 and μ_2 such that

$$h_\imath(z) = \int_{\Delta_1} k_\zeta(z)\, d\mu_\imath(\zeta) = \sum_{j=1}^{m} k_{\zeta_j}(z)\mu_\imath(\{\zeta_j\}) \quad (\imath = 1, 2).$$

Hence $h(z) = h_1(z) - h_2(z) = \sum_{j=1}^{m} k_{\zeta_j}(z)(\mu_1(\{\zeta_j\}) - \mu_2(\{\zeta_j\}))$. Since $k_{\zeta_j} \in HD(R)$ (resp. $k_{\zeta_j} \in HB(R)$) $(j = 1, \dots, m)$, $h \in HD(R)$ (resp. $h \in HB(R)$). Hence, $HB(R) \subset HD(R)$ (resp. $HD(R) \subset HB(R)$), and hence, $HB(R) = HD(R)$. Hence $\dim HB(R) = \dim HD(R) < \infty$. Therefore we have (iii).

Suppose that (iii) holds. Since $\dim HB(R) < \infty$, by Proposition 4, we find that $HD(R) \subset HB(R)$. Since $HD(R)$ is a linear subspace of the linear space $HB(R)$, by the assertion (iii), we find that $HB(R) = HD(R)$. Therefore we have (i).

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Hiroaki Masaoka
Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Kamigamo–Motoyama, Kitaku, Kyoto 603-8555
Japan
E-mail: masaoka@cc.kyoto-su.ac.jp