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THE CLASSICAL BANACH SPACES ℓ_φ/h_φ

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Abstract. In this paper we study some structural and geometric properties of the quotient Banach spaces $\ell_\varphi(I)/h_\varphi(\mathcal{S})$, where I is an arbitrary set, φ is an Orlicz function, $\ell_\varphi(I)$ is the corresponding Orlicz space on I and $h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall\lambda > 0,\ \exists s \in \mathcal{S}$ such that $I_\varphi(\frac{x-s}{\lambda}) < \infty\}$, $\mathcal{S}$ being the ideal of elements with finite support. The results we obtain here extend and complete the ones obtained by Leonard and Whitfield (Rocky Mountain J. Math. **13** (1983), 531–539). We show that $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is not a dual space, that $Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}) = \emptyset$, if $\varphi(t) > 0$ for every $t > 0$, that $S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ has no smooth points, that it cannot be renormed equivalently with a strictly convex or smooth norm, that $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is a Grothendieck space, etc.

1. Notation and preliminaries

Let $\varphi : \mathbb{R} \to [0, +\infty]$ denote an Orlicz function, i.e. a function which is even, nondecreasing, left continuous for $x \geq 0$, $\varphi(0) = 0$ and $\varphi(x) \to \infty$ as $x \to \infty$. Define $a(\varphi) = \sup\{t \geq 0 : \varphi(t) = 0\}$, $\tau(\varphi) = \sup\{t \geq 0 : \varphi(t) < \infty\}$ and assume that $\tau(\varphi) > 0$. Fix an arbitrary set I and, for $x \in \mathbb{R}^I$, define $I_\varphi(x) = \sum_{i\in I}\varphi(x_i)$. Let $\ell_\varphi(I)$ be the corresponding Orlicz space, i.e. $\ell_\varphi(I) = \{x \in \mathbb{R}^I : \exists\lambda > 0$ such that $I_\varphi(x/\lambda) < \infty\}$. Consider in $\ell_\varphi(I)$ the F-norm $|x|_\varphi := \inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq \lambda\}$, $\forall x \in \ell_\varphi(I)$, and the associated distance $d(x, y) = |x - y|_\varphi$. It is known that $(\ell_\varphi(I), d)$ is a complete F-space.

Let $\mathcal{S} \subseteq \ell_\varphi(I)$ be the ideal of elements of finite support. Define $h_\varphi(\mathcal{S})$ by:

$$h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall\lambda > 0,\ \exists s \in \mathcal{S} \text{ such that } I_\varphi(\frac{x-s}{\lambda}) < \infty\},$$

and $\delta(x)$ by:

$$\delta(x) = \inf\{\lambda > 0 : \exists s \in \mathcal{S} \text{ such that } I_\varphi(\frac{x-s}{\lambda}) < \infty\}, x \in \ell_\varphi(I).$$

Clearly, $h_\varphi(\mathcal{S})$ is a closed ideal of $\ell_\varphi(I)$ such that $h_\varphi(\mathcal{S}) = \{x \in \ell_\varphi(I) : \forall\lambda > 0,\ I_\varphi(\lambda x) < \infty\}$, if φ is finite, and $\overline{\mathcal{S}} = h_\varphi(\mathcal{S})$, where $\overline{\mathcal{S}}$ is the closure of $\mathcal{S}$ in $\ell_\varphi(I)$.

We are interested in the quotient space $\ell_\varphi(I)/h_\varphi(\mathcal{S})$. Hence we must impose the condition $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Note that this happens if and only if I is infinite and $\varphi \notin \Delta_2^0$, i.e. φ doesn't satisfy the Δ_2 condition at 0.

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If φ is convex we can consider the Luxemburg norm $\|\cdot\|_L$ and the Luxemburg distance d_L:

$$\|x\|_L = \inf\{\lambda > 0 : I_\varphi(x/\lambda) \le 1\}, \qquad d_L(x,y) = \|x-y\|_L, \qquad x,y \in \ell_\varphi(I),$$

as well as the Amemiya-Orlicz norm $\|\cdot\|_o$ and the Amemiya-Orlicz distance d_o:

$$\|x\|_o = \inf_{k>0}\{\tfrac{1}{k}(1+I_\varphi(kx))\}, \qquad d_o(x-y) = \|x-y\|_o, \qquad x,y \in \ell_\varphi(I).$$

It is known that, $\forall x \in \ell_\varphi(I), \|x\|_L \le \|x\|_o \le 2\|x\|_L$ and that these norms define on $\ell_\varphi(I)$ the same topology as $|\cdot|_\varphi$. Denote by B^L_φ (resp. B^o_φ) and S^L_φ (resp. S^o_φ) the closed unit ball and unit sphere of $(\ell_\varphi(I), \|\cdot\|_L)$ (resp. $(\ell_\varphi(I), \|\cdot\|_o)$). Recall that a Banach *M-space* is a Banach lattice $(X, \|\cdot\|)$ such that $\|x \vee y\| = \|x\| \vee \|y\|$, whenever $x, y \in X^+$.

Proposition 1.1. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then:*

1. *For each $x \in \ell_\varphi(I)$ we have $\delta(x) = d(x, h_\varphi(\mathcal{S}))$ and, if φ is convex, also $\delta(x) = d_L(x, h_\varphi(\mathcal{S})) = d_o(x, h_\varphi(\mathcal{S}))$.*
2. *δ is a monotone seminorm on $\ell_\varphi(I)$ such that $ker(\delta) = h_\varphi(\mathcal{S})$.*
3. *Let $\|\cdot\|$ be the quotient F-norm on $\ell_\varphi(I)/h_\varphi(\mathcal{S})$. Then $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), \|\cdot\|)$ is a Banach M-space.*
4. *If φ is convex, the space $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ equipped with the quotient norms corresponding to the Luxemburg norm as well as to the Orlicz norm is order isomorphic and isometric to $(\ell_\varphi(I)/h_\varphi(\mathcal{S}), \|\cdot\|)$.*

Proof. (1) Let $x \in \ell_\varphi(I)$ and fix $\epsilon > 0$. Then $\exists s \in \mathcal{S}$ such that $I_\varphi\left(\frac{x-s}{\delta(x)+\epsilon}\right) < +\infty$ and $0 \le s^+ \le x^+, 0 \le s^- \le x^-$. Pick $\{y_\alpha\}_{\alpha\in A}, \{z_\alpha\}_{\alpha\in A}$ in $h_\varphi(\mathcal{S})^+$ with $y_\alpha \uparrow x^+ - s^+, z_\alpha \uparrow x^- - s^-$. Since I_φ is o-continuous, we get:

$$I_\varphi\left(\frac{x-s-y_\alpha+z_\alpha}{\delta(x)+\epsilon}\right) = I_\varphi\left(\frac{x^+-s^+-y_\alpha+x^--s^--z_\alpha}{\delta(x)+\epsilon}\right) \to 0$$

with respect to (for short, wrt) $\alpha \in A$. Hence $d(x, h_\varphi(\mathcal{S})) \le \delta(x)$, since $\epsilon > 0$ is arbitrary. If φ is convex, the above also proves that $d_L(x, h_\varphi(\mathcal{S})) \le \delta(x)$. Concerning the Amemiya-Orlicz norm, since $I_\varphi\left(\frac{x-s-y_\alpha+z_\alpha}{\delta(x)+\epsilon}\right) \to 0$ wrt $\alpha \in A$, we have:

$$\|x-s-y_\alpha+z_\alpha\|_o \le (\delta(x)+\epsilon)\left[1 + I_\varphi\left(\frac{x-s-y_\alpha+z_\alpha}{\delta(x)+\epsilon}\right)\right]$$
$$\to \delta(x)+\epsilon \text{ wrt } \alpha \in A,$$

whence, ϵ being arbitrary, it follows that $d_o(x, h_\varphi(\mathcal{S})) \le \delta(x)$.

For the contrary inequality, if $\delta(x) = 0$, the above proves that $0 = \delta(x) = d(x, h_\varphi(\mathcal{S})) = d_L(x, h_\varphi(\mathcal{S})) = d_o(x, h_\varphi(\mathcal{S}))$. Assume that $\delta(x) > 0$ and pick a fixed $y \in h_\varphi(\mathcal{S})$. Suppose that there exists $0 < \lambda < \delta(x)$ such that $I_\varphi\left(\frac{x-y}{\lambda}\right) < +\infty$. Take $\lambda < t < \delta(x)$ and denote $r = \lambda/t$. Then $0 < r < 1$ and $\exists s \in \mathcal{S}$ such that $I_\varphi\left(\frac{y-s}{(1-r)t}\right) < +\infty$. Since $\frac{x-s}{t} = r\frac{x-y}{rt} + (1-r)\frac{y-s}{(1-r)t}$, we have:

$$I_\varphi\left(\frac{x-s}{t}\right) \le I_\varphi\left(\frac{x-y}{\lambda}\right) + I_\varphi\left(\frac{y-s}{(1-r)t}\right) < +\infty,$$

a contradiction. Hence $\forall 0<\lambda<\delta(x),\ \forall y\in h_\varphi(\mathcal{S}),\ I_\varphi\left(\frac{x-y}{\lambda}\right)=+\infty$, which implies $d(x,h_\varphi(\mathcal{S}))\geq\delta(x)\leq d_L(x,h_\varphi(\mathcal{S}))$. As $\|\cdot\|_o\geq\|\cdot\|_L$, we also get $d_o(x,h_\varphi(\mathcal{S}))\geq\delta(x)$.

(2) and (3) were proved in [15] and (4) follows easily from the above. □

In the sequel $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ will be the Banach M-space $(\ell_\varphi(I)/h_\varphi(\mathcal{S}),\|\cdot\|)$ and Q the quotient map $Q:\ell_\varphi(I)\to\ell_\varphi(I)/h_\varphi(\mathcal{S})$. Let βI denote the Stone-Weierstrass compactification of I, when we consider in I the discrete topology. Denote by $\mathfrak{F}(I)$ the class of finite subsets of I. If $x\in\mathbb{R}^I$ and $A\subseteq I$, define $x_A=x\cdot\mathbf{1}_A$ and $x^A=x\cdot\mathbf{1}_{I\setminus A}$.

Proposition 1.2. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I)\neq h_\varphi(\mathcal{S})$. If $a(\varphi)>0$, then*

$$\ell_\varphi(I)/h_\varphi(\mathcal{S})\cong(\ell_\infty(I)/c_o(I),\|\cdot\|_\infty)\cong(C(\beta I\backslash I),\|\cdot\|_\infty)$$

(order isomorphism and isometry).

Proof. First of all, it is clear that $\ell_\varphi(I)=\ell_\infty(I)$ and $h_\varphi(\mathcal{S})=c_o(I)$, as sets and algebraically. Consider the map $i:\ell_\infty(I)\to\ell_\varphi(I)$ such that $i(x)=a(\varphi)\cdot x$ and the quotient map $q:\ell_\infty(I)\to\ell_\infty(I)/c_o(I)$. Note that $|i(x)|_\varphi\leq\|x\|_\infty$ and that:

$$\forall x\in\ell_\infty(I),\ \|q(x)\|=\inf_{A\in\mathfrak{F}(I)}\|x^A\|_\infty,$$
$$\|Q(i(x))\|=d(i(x),h_\varphi(\mathcal{S}))=\inf_{A\in\mathfrak{F}(I)}|i(x^A)|_\varphi.$$

Clearly, $\|Q(i(x))\|\leq\|q(x)\|$, whence, if $\|q(x)\|=0$, we get $\|Q(i(x))\|=\|q(x)\|=0$. Assume that $\|q(x)\|=:a>0$ and take $0<\epsilon<a$. Find sequences, $\{A_n\}_{n\geq1}$ in $\mathfrak{F}(I)$ and $\{i_n\}_{n\geq1}$ in I, such that $A_n\subseteq A_{n+1},\ i_n\in A_{n+1}\setminus A_n$ and $|x_{i_n}|>a-\epsilon/2$. Then:

$$\forall n\geq1,\ I_\varphi\left(\frac{i(x^{A_n})}{a-\epsilon}\right)=I_\varphi\left(\frac{a(\varphi)\cdot x^{A_n}}{a-\epsilon}\right)\geq\sum_{k>n}\varphi\left(\frac{a(\varphi)\cdot x_{i_k}}{a-\epsilon}\right)=\infty,$$

which implies $|i(x^{A_n})|_\varphi\geq a-\epsilon,\ \forall n\geq1$, whence $\|Q(i(x))\|\geq a-\epsilon$. Since $\epsilon>0$ is arbitrary, we get $\|Q(i(x))\|\geq a$ and finally $\|Q(i(x))\|=a$. □

2. Proximinality

Let (X,D) be a metric linear space with a distance D and $M\subseteq X$ a subspace of X. Consider the distance $D(x,M)=\inf\{D(x,m):m\in M\},\ x\in X$, and say that $x\in X$ is *M-approximable* if $\exists m\in M$ such that $D(x,M)=D(x,m)$. Denote by $Ap(M,X)$ the subset of M-approximable elements of X. If $Ap(M,X)=X$, M is said to be *proximinal* in X. If M is proximinal in X then, obviously, M is closed in X.

Let $(X,\|\cdot\|)$ be a normed space and $M\subseteq X$ a closed subspace. Denote by $B_X,\ S_X$ its closed unit ball and unit sphere, respectively, and by X^* its topological dual. Define $Top(M,X)=\{x\in S_X:\ \text{distance }(x,M)=1\}$. Clearly, $Top(M,X)\subseteq Ap(M,X)\setminus M$ and $x\in Top(M,X)$ iff $x\in S_X$ and $q(x)\in S_{X/M}$, where q is the canonical quotient map $q:X\to X/M$. In normed spaces, the proximinality has been characterized by Godini as follows:

Theorem 2.1 (Godini). *If X is a normed space and $M \subseteq X$ a closed subspace, then the following are equivalent: (1) $q(B_X) = B_{X/M}$; (2) $q(B_X)$ is closed in X/M; (3) M is proximinal in X.*

Proof. See [7]. $\square$

Proposition 2.2. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then:*

(a) *$h_\varphi(\mathcal{S})$ is proximinal in $(\ell_\varphi(I), |\cdot|_\varphi)$ and, if φ is convex, also in $(\ell_\varphi(I), \|\cdot\|_L)$.*
(b) *Assume that φ is convex. Then:*
  (1) *$x \in Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_z))$ iff $|x| \in Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_z))$, for $z = L$ or $z = o$.*
  (2) *$Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_o)) = Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_L)) \cap S^o_\varphi$.*
  (3) *$Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_L)) = \{x \in \ell_\varphi(I) : I_\varphi(x) \leq 1, I_\varphi(\lambda x^A) = \infty,\ \forall \lambda > 1,\ \forall A \in \mathfrak{F}(I)\}$.*
  (4) *If $a(\varphi) = 0$, then*

$$Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_o)) = \emptyset.$$

If $a(\varphi) > 0$, then

$$Top(h_\varphi(\mathcal{S}), (\ell_\varphi(I), \|\cdot\|_o)) = \{x \in \ell_\varphi(I) : |x_i| \leq a(\varphi),\ \forall i \in I,$$

and $\forall \epsilon > 0,\ card\{i \in I : |x_i| \geq a(\varphi) - \epsilon\} = \infty\}$.
  (5) *$h_\varphi(\mathcal{S})$ is proximinal in $(\ell_\varphi(I), \|\cdot\|_o)$ iff $a(\varphi) > 0$.*

Proof. (a) Pick $x \in \ell_\varphi(I)$. If $\delta(x) = 0$, by Proposition 1.1 we get that $d(x, h_\varphi(\mathcal{S})) = 0$. Hence $x \in h_\varphi(\mathcal{S})$ since $h_\varphi(\mathcal{S})$ is closed in $(\ell_\varphi(I), |\cdot|_\varphi)$.

Assume that $\delta(x) > 0$ and $x \geq 0$. Let $\epsilon_k \downarrow 1$ be such that $1 - \frac{1}{\epsilon_k} =: \eta_k \leq 2^{-k}, k \geq 1$. Since $I_\varphi\left(\frac{x}{\delta(x)\epsilon_1}\right) < \infty$ and I_φ is o-continuous, there exists a finite subset $A_1 \subseteq I$ such that $I_\varphi\left(\frac{x-u_1}{\delta(x)\epsilon_1}\right) \leq 2^{-2}a$, where $u_1 := x \cdot \mathbf{1}_{A_1}$ and $0 < a \leq \inf\{1, \delta(x)\}$ is arbitrary. Let $x_2 := x - u_1$. Then there exists a finite subset $A_2 \subseteq I \backslash A_1$ such that $I_\varphi\left(\frac{x_2-u_2}{\delta(x)\epsilon_2}\right) \leq 2^{-3}a$, where $u_2 := x \cdot \mathbf{1}_{A_2}$. By reiteration we obtain a family of pairwise disjoint elements $\{u_n\}_{n\geq 1}$ in $\mathcal{S}^+$ such that, if $x_n = x - \sum_{k=0}^{n-1} u_k, n \geq 1, u_0 = 0$, then $u_n \leq x_n$ and $I_\varphi\left(\frac{x_{n+1}}{\delta(x)\epsilon_n}\right) \leq 2^{-n-1}a$.

Let $g_r = \sum_{k=0}^r \eta_k u_{k+1},\ \eta_o = 1$. We claim that $\{g_r\}_{r\geq 0}$ is a Cauchy sequence in $(\ell_\varphi(I), |\cdot|_\varphi)$. Indeed, fix $\epsilon > 0$ and take $r_o \in \mathbb{N}$ such that, $\forall r > r_o,\ \eta_r/\epsilon \leq \frac{1}{\delta(x)\epsilon_r}$ and $\sum_{k\geq r_o} 2^{-(k+1)} \leq \epsilon/a$. Then, $\forall s \geq r > r_o$, we have:

$$I_\varphi\left(\frac{g_s - g_r}{\epsilon}\right) = \sum_{k=r+1}^{s} I_\varphi\left(\frac{\eta_k u_{k+1}}{\epsilon}\right) \leq \sum_{k=r+1}^{s} I_\varphi\left(\frac{u_{k+1}}{\delta(x)\epsilon_k}\right) \leq (\epsilon/a)a = \epsilon.$$

Hence $\sum_{k\geq 0} \eta_k u_{k+1} =: g \in h_\varphi(\mathcal{S})$. Note also that $\sum_{k\geq 0} u_{k+1} =: f \in \ell_\varphi(I)$, because $\ell_\varphi(I)$ is σ-o-complete and $0 \leq f \leq x$. Let $z = x - f$. Then $f \wedge z = 0$ and $0 \leq z \leq x_{k+1}, \forall k \geq 0$. So $I_\varphi\left(\frac{z}{\delta(x)\epsilon_k}\right) \leq 2^{-(k+1)}a, \forall k \geq 1$. Since I_φ is left-continuous, we get $I_\varphi\left(\frac{z}{\delta(x)}\right)=0$. Hence:

$$I_\varphi\left(\frac{x-g}{\delta(x)}\right)=I_\varphi\left(\frac{x-z-g+z}{\delta(x)}\right)=I_\varphi\left(\frac{\sum_{k\geq 0}(1-\eta_k)u_{k+1}+z}{\delta(x)}\right)$$
$$=\left[\sum_{k\geq 0}I_\varphi\left(\frac{u_{k+1}}{\delta(x)\epsilon_k}\right)+I_\varphi\left(\frac{z}{\delta(x)}\right)\right]\leq a\sum_{k\geq 0}2^{-(k+1)}\leq a.$$

Thus $D(x,g)\leq\delta(x)$ with $D=d$ or $D=d_L$ and $d_L(x,y)=\|x-y\|_L$. Since $D(x,g)\geq\delta(x)$, we get $D(x,g)=\delta(x)$.

In the general case (i.e. $x^+>0,x^->0$), if $\delta(x)>0$ (i.e. $x\notin h_\varphi(\mathcal{S})$), by the above it is possible to find $g_1,g_2\in h_\varphi(\mathcal{S})$ such that $0\leq g_1\leq x^+,0\leq g_2\leq x^-$ and $I_\varphi\left(\frac{x^+-g_1}{\delta(x)}\right)\leq\frac{a}{2}\geq I_\varphi\left(\frac{x^--g_2}{\delta(x)}\right)$. Thus, if $g=g_1-g_2$, we get $I_\varphi\left(\frac{x-g}{\delta(x)}\right)=\left[I_\varphi\left(\frac{x^+-g_1}{\delta(x)}\right)+I_\varphi\left(\frac{x^--g_2}{\delta(x)}\right)\right]\leq a$. Hence $D(x,g)=\delta(x)$.

(b)(1) Observe that, for $z=L$ or $z=o$, we have $\|x\|_z=\|\,|x|\,\|_z$ and $d_z(x,h_\varphi(\mathcal{S}))=\inf\{\|x-y\|_z : y\in h_\varphi(\mathcal{S})\}=\inf\{\|\,|x|-y\|_z : y\in h_\varphi(\mathcal{S})\}=d_z(|x|,h_\varphi(\mathcal{S}))$.

(b)(2) If $f\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))$, then $1=d_o(f,h_\varphi(\mathcal{S}))=d_L(f,h_\varphi(\mathcal{S}))\leq\|f\|_L\leq\|f\|_o=1$. Hence $f\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_L))\cap S^o_\varphi$.

If $f\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_L))\cap S^o_\varphi$, then $1=d_L(f,h_\varphi(\mathcal{S}))=d_o(f,h_\varphi(\mathcal{S}))\leq\|f\|_o=1$. Hence $f\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))$.

(b)(3) It is enough to remark that $x\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_L))$ iff $\|x\|_L\leq 1$ and $\delta(x)\geq 1$. But these conditions are equivalent to $I_\varphi(x)\leq 1$ and, $\forall\lambda>1$, $\forall A\in\mathfrak{F}(I)$, $I_\varphi(\lambda x^A)=\infty$.

(b)(4) First of all, note that if $x\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))$, then $|x_i|\in[0,a(\varphi)],\forall i\in I$. Indeed, we have that $\delta(x)\geq 1$, i.e.:

$$(*)\qquad \forall\lambda>1,\ \forall A\in\mathfrak{F}(I),\ I_\varphi(\lambda x^A)=\infty.$$

Since $1=\|x\|_o=\inf_{k>0}\{\frac{1}{k}(1+I_\varphi(kx))\}$, we get that $1=1+I_\varphi(x)$, whence $I_\varphi(x)=0$ and $|x_i|\in[0,a(\varphi)]$, $\forall i\in I$.

Therefore, if $a(\varphi)=0$, it is clear that $Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))=\emptyset$. Assume that $a(\varphi)>0$ and that $x\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))$. Then, by the above, $|x_i|\leq a(\varphi)$, $\forall i\in I$. By (*) it follows that ,$\forall\epsilon>0$, $\operatorname{card}\{i\in I:|x_i|\geq a(\varphi)-\epsilon\}=\infty$. Finally if $x\in\ell_\varphi(I)$ satisfies $|x_i|\leq a(\varphi)$, $\forall i\in I$, and $\operatorname{card}\{i\in I:|x_i|\geq a(\varphi)-\epsilon\}=\infty$, $\forall\epsilon>0$, we easily conclude that $\|x\|_o=\inf_{k>0}\{\frac{1}{k}(1+I_\varphi(kx))\}=1$ and that $\delta(x)\geq 1$, i.e. $x\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_o))$.

(b)(5) If $a(\varphi)=0$ it is clear, by the above, that $h_\varphi(\mathcal{S})$ is not proximinal in $(\ell_\varphi(I),\|\cdot\|_o)$. Assume that $a(\varphi)>0$. By Proposition 2.1, it is enough to prove that, if $x\in Top(h_\varphi(\mathcal{S}),(\ell_\varphi(I),\|\cdot\|_L))^+$, then there exists $f\in h_\varphi(\mathcal{S})$, $0\leq f\leq x$, such that $\|x-f\|_o=1$. Denote $h:=(x-a(\varphi))\vee 0$ and observe that $h\in h_\varphi(\mathcal{S})$ (because, $\forall\lambda>0$, $\operatorname{card}\{i\in I:\lambda h_i>a(\varphi)\}<\aleph_0$). Clearly $I_\varphi(x-h)=0$ and, $\forall\lambda>1$, $I_\varphi(\lambda(x-h))=\infty$ (because $d_L(x,h_\varphi(\mathcal{S}))=d_L(x-h,h_\varphi(\mathcal{S}))=1$). Hence:

$$\|x-h\|_o=\inf_{k>0}\frac{1}{k}(1+I_\varphi(k(x-h)))=1+I_\varphi(x-h)=1.$$

□

3. Extremal structures

Denote by $Ext(C)$ the set of *extreme points* of a convex set C. If $a(\varphi) > 0$, we have, by Proposition 1.2 and [10, Theorem 4.1], that the ball $B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ has an abundance of extreme points. In fact, we get

$$Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}) = Ext(B_{\ell_\infty(I)/c_0(I)}) = q(Ext(B_{\ell_\infty(I)}))$$

and

$$B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})} = \overline{co}(Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})})).$$

If $a(\varphi) = 0$ the situation is completely different.

Proposition 3.1. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$ and $a(\varphi) = 0$. Then $Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}) = \emptyset$.*

Proof. Assume that $e \in Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})})$. Pick $w \in \ell_\varphi(I)$ such that $Q(w) = e$. Then $d(w, h_\varphi(\mathcal{S})) = 1$ and there exists $g \in h_\varphi(\mathcal{S})$ such that $1 = d(w, h_\varphi(\mathcal{S})) = d(w, g) = d(w - g, 0)$, whence, $\forall\lambda > 1$, $I_\varphi\left(\frac{w-g}{\lambda}\right) \leq \lambda$. By the left-continuity of I_φ we get that $I_\varphi(w - g) \leq \lambda$, $\forall\lambda > 1$, i.e. $I_\varphi(w - g) \leq 1$. Let $u = w - g$ and suppose, without loss of generality, that $I_\varphi(u) \leq 1/2$ (if not, put $u_i = 0$ for $i \in A$ and some $A \in \mathfrak{F}(I)$). Since $a(\varphi) = 0$, we can choose a countable subset $B = \{i_n\}_{n\geq 1}$ of I such that $u_{i_n} \to 0$, as $n \to \infty$, and, if $h = u \cdot \mathbf{1}_B$, then $h \in h_\varphi(\mathcal{S})$ and $Q(u - h) = e$. Since $a(\varphi) = 0$ we have that $\text{card}(\text{supp}(u)) = \aleph_0$. Let $\text{supp}(u) = \{j_r\}_{r\geq 1}$ and define $x, y \in \ell_\varphi(I)$ as follows:

$$x_i = \begin{cases} u_i, & \text{if } i \notin B \\ u_{j_k}, & \text{if } i = i_k,\ k \geq 1 \end{cases}, \qquad y_i = \begin{cases} u_i, & \text{if } i \notin B \\ -u_{j_k}, & \text{if } i = i_k,\ k \geq 1 \end{cases}.$$

Then $Q(x) \neq Q(y)$ (because $x - y \notin h_\varphi(\mathcal{S})$), $Q(x), Q(y) \in B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ (because $I_\varphi(x), I_\varphi(y) \leq 1$) and $\frac{1}{2}(Q(x) + Q(y)) = Q(u - h) = e$, a contradiction. Hence $Ext(B_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}) = \emptyset$. □

If X is a normed space and $x \in S_X$, denote $Grad(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$. We say that $x \in S_X$ is *smooth* iff $\text{card}(Grad(x)) = 1$.

Proposition 3.2. *Let I be an infinite set and φ an Orlicz function such that $h_\varphi(\mathcal{S}) \neq \ell_\varphi(I)$. Then $S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ has no smooth points.*

Proof. Let $e \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$. Pick $x \in \ell_\varphi(I)$ such that $I_\varphi(x) \leq 1$ and $Q(x) = e$. Then $I_\varphi(\lambda x) = \infty$, $\forall\lambda > 1$. We claim that there exists $C \subseteq I$ such that, if $y = x_C$ and $z = x^C$, then $Q(y), Q(z) \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$. Indeed, since $I_\varphi((1 + 2^{-n})x) = \infty$, we can choose two sequences of nonempty and finite subsets $\{A_n\}_{n\geq 1}$, $\{B_n\}_{n\geq 1}$ of I such that: (i) $\sum_{i\in A_n} \varphi((1 + 2^{-n})x_i) \geq 2^n \leq \sum_{i\in B_n} \varphi((1 + 2^{-n})x_i)$; (ii) $A_n \cap B_n = \emptyset = (A_n \cup B_n) \cap (A_m \cup B_m)$, $n \neq m$. Now, take $C = \cup_{n\geq 1} A_n$. Note that $I_\varphi(y \pm z) = I_\varphi(x) \leq 1$, $Q(y \pm z) \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ and $y + z = x$.

There exists $y^* \in Grad(Q(y))$ and $z^* \in Grad(Q(z))$ such that:

$$1 \geq y^*(Q(y) \pm Q(z)) = y^*(Q(y)) \pm y^*(Q(z)) = 1 \pm y^*(Q(z)),$$

whence we get $y^*(Q(z)) = 0$. In a similar way, we get $z^*(Q(y)) = 0$. This means that $y^* \neq z^*$. We have:

$$y^*(Q(x)) = y^*(Q(y) + Q(z)) = y^*(Q(y)) + y^*(Q(z)) = 1 + 0 = 1,$$
$$z^*(Q(x)) = z^*(Q(y) + Q(z)) = z^*(Q(y)) + z^*(Q(z)) = 0 + 1 = 1,$$

which means that $y^*, z^* \in Grad(e)$, so e is not smooth. □

4. Order completeness and order continuity

In [15] it is proved that every $x \in (\ell_\varphi(I)/h_\varphi(\mathcal{S})) \setminus \{0\}$ is σ-o-continuous and not σ-o-complete. Recall that a vector x of a Banach lattice X is: (i) σ-o-continuous if for every decreasing sequence $\{x_n\}_{n\geq 1}$ in X^+ such that $x_n \leq |x|$ and $\inf_{n\geq 1} x_n = 0$, we have $\|x_n\| \downarrow 0$; (ii) σ-o-complete if for every increasing sequence $\{x_n\}_{n\geq 1}$ in X^+ such that $x_n \leq |x|$, there exists $\sup_{n\geq 1} x_n$. In particular, an increasing sequence in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ has supremum if and only if it is a Cauchy sequence.

As a consequence, we get the following known fact: if I is an infinite set and $\{A_n\}_{n\geq 1}$ a sequence of closed-and-open (clopen) subsets of $\beta I\backslash I$ such that $A_n \subseteq A_{n+1}$ and $A_n \neq A_{n+1}$, then $\overline{A}$ is not open in $\beta I\backslash I$, with $A := \bigcup_{n\geq 1} A_n$. Indeed, let φ be the convex Orlicz function such that $\varphi(t) = 0$ if $|t| \leq 1$, but $\varphi(t) = \infty$ whenever $|t| > 1$. Then $\ell_\varphi(I)/h_\varphi(\mathcal{S}) \cong (C(\beta I\backslash I), \|\cdot\|_\infty)$ (order isomorphism and isometry). Consider in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ the sequence $\{\mathbf{1}_{A_n}\}_{n\geq 1}$, which is increasing and bounded by $\mathbf{1}_{\beta I\backslash I}$. Since $\|\mathbf{1}_{A_{n+1}\backslash A_n}\| = 1$, we get that $\{\mathbf{1}_{A_n}\}_{n\geq 1}$ is not Cauchy, whence this sequence has no supremum. But, if $\overline{A}$ were open, $\mathbf{1}_{\overline{A}}$ should be the supremum of this sequence. Hence $\overline{A}$ is not open and $\beta I\backslash I$ is not basically disconnected. Recall that a compact Hausdorff space K is basically disconnected if the closure of every open F_σ-set (i.e. a countable union of closed sets) in K is open (see [9, pg.4]).

5. Rotundity and smoothness

Proposition 5.1. *If I is an infinite set and φ is an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$, then there exists an order isomorphic isometric copy of $C(\beta\mathbb{N}\backslash\mathbb{N})$ in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$.*

Proof. Pick $x \in \ell_\varphi(I)^+$ such that $I_\varphi(x) \leq 1$, $Q(x) \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ and, if $A := \mathrm{supp}(x)$, then $\mathrm{card}(A) = \aleph_0$. Let $\{\lambda_n\}_{n\geq 1}$ be a sequence in $\mathbb{R}^+$ such that $\lambda_n \downarrow 1$. Note that $I_\varphi(\lambda_n(x - s)) = \infty$, $\forall n \geq 1$, $\forall s \in \mathcal{S}$. Choose a sequence $\{A_n\}_{n\geq 1}$ of pairwise disjoint finite subsets of A such that $A = \cup_{n\geq 1} A_n$ and $I_\varphi(\lambda_n \cdot x \cdot \mathbf{1}_{A_n}) > 1$, $n \geq 1$. If $a = (a_n)_{n\geq 1} \in \ell_\infty$, put $a^k = (0, \dots, 0, a_{k+1}, a_{k+2} \dots)$ and define $T : \ell_\infty \to \ell_\varphi(I)$ by $Ta = \sum_{n\geq 1} a_n \cdot x \cdot \mathbf{1}_{A_n}$. Clearly, T is continuous and we have $\frac{1}{\lambda_k}\|a^k\|_\infty \leq \|Ta^k\|_L \leq \|a^k\|_\infty$. Observe that, if $a = (a_1, a_2, \dots, a_k, 0, 0, \dots)$, then $Ta \in h_\varphi(\mathcal{S})$, whence, by $h_\varphi(\mathcal{S})$ being closed in $\ell_\varphi(I)$, we get that $T(c_0) \subseteq h_\varphi(\mathcal{S})$. Hence, if q is the quotient map $q : \ell_\infty \to \ell_\infty/c_0$, we have the map $i : \ell_\infty/c_0 \to \ell_\varphi(I)/h_\varphi(\mathcal{S})$ such that $i(q(a)) = QT(a)$, $\forall a \in \ell_\infty$. Clearly, this map preserves the order and satisfies $\|q(a)\| = \lim_{k\to\infty} \|a^k\|_\infty = \lim_{k\to\infty} \|Ta^k\|_L = \|QT(a)\|$. Therefore i is an order isomorphic isometry between ℓ_∞/c_o and $i(\ell_\infty/c_o)$. □

Corollary 5.2. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then:*

1. *$\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is not realcompact and cannot be renormed equivalently in order to be rotund or smooth.*
2. *$\ell_\varphi(I)/h_\varphi(\mathcal{S})$ does not have property (C), it is not WCD, it is not w-Lindelöf and $(\ell_\varphi(I)/h_\varphi(\mathcal{S}))^* = h_\varphi(\mathcal{S})^\perp$ is not w^*-angelic.*

Proof. (1) This follows from the fact that $C(\beta\mathbb{N}\backslash\mathbb{N})$ is not realcompact (see [13, p. 146], [3]) and cannot be renormed in order to be rotund or smooth (see [2], [10]).

(2) This is a consequence of (1) (see [6]). □

6. $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is not a dual space

Let I be an infinite set, $\mathfrak{m}$ =card(I) and $P_\omega(I) = \{A \subseteq I : \text{card}(A) = \aleph_0\}$. Then, clearly, card$(P_\omega(I)) = \mathfrak{m}^{\aleph_0} =: \mathfrak{n}$. Note that $\mathfrak{n} \geq \mathfrak{c}$, where $\mathfrak{c}$ =card$(\mathbb{R})$. Also there exists a family $\{A_t\}_{t\in\mathfrak{n}}$ in $P_\omega(I)$ such that card$(A_t \cap A_s) < \aleph_0$, for $t \neq s$. Indeed, let $\{I_t\}_{t\in\mathfrak{m}}$ be a family of pairwise disjoint subsets of I such that card$(I_t) = \mathfrak{m}$, $\forall t \in \mathfrak{m}$. Pick $i_t \in I_t$, $t \in \mathfrak{m}$, and choose a pairwise disjoint family $\{I_{ts}\}_{s\in\mathfrak{m}}$ of subsets of $I_t\backslash\{i_t\}$ such that card$(I_{ts}) = \mathfrak{m}$, $s \in \mathfrak{m}$. Pick $i_{ts} \in I_{ts}$ and choose a pairwise disjoint family $\{I_{tsr}\}_{r\in\mathfrak{m}}$ of subsets of $I_{ts}\backslash\{i_{ts}\}$ such that card$(I_{tsr}) = \mathfrak{m}$, $r \in \mathfrak{m}$. Pick $i_{tsr} \in I_{tsr}$, $r \in \mathfrak{m}$. By reiteration we obtain families of elements $\{i_t\}_{t\in\mathfrak{m}}$, $\{i_{ts}\}_{t,s\in\mathfrak{m}}$, etc., of I. Now, consider the family $\mathfrak{T}$ of sequences of the form $(i_{t_1}, i_{t_1t_2}, i_{t_1t_2t_3}, \dots)$, $t_j \in \mathfrak{m}, j \geq 1$. It is clear that card$(\mathfrak{T}) = \mathfrak{m}^{\aleph_0} = \mathfrak{n}$, card$(T) = \aleph_0$, $\forall T \in \mathfrak{T}$, and that, if $T, S \in \mathfrak{T}$, $T \neq S$, then card$(T \cap S) < \aleph_0$.

Lemma 6.1. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. If $\mathfrak{n} = \mathfrak{m}^{\aleph_0}$ and $\mathfrak{m}$ =card(I), there exists an order isomorphic isometric copy of $(c_o(\mathfrak{n}), \|\cdot\|_\infty)$ in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$.*

Proof. Let $\{A_t\}_{t\in\mathfrak{n}}$ be a family of subsets of I such that card$(A_t) = \aleph_0$ and card$(A_t \cap A_s) < \aleph_0$, when $t \neq s$. Pick $x \in \ell_\varphi(I)^+$ such that $I_\varphi(x) \leq 1$, $Q(x) \in S_{\ell_\varphi(I)/h_\varphi(\mathcal{S})}$ and card(supp$(x)) = \aleph_0$. Let supp$(x) = \{j_r\}_{r\geq1}$. If $t \in \mathfrak{n}$ and $A_t = \{i_k\}_{k\geq1}$, define e^t such that $\forall i \in I$, $e^t_i = 0$, if $i \notin A_t$, and $e^t_i = x_{j_r}$, if $i = i_r$, $r \geq 1$. Then clearly, $\forall t_1, t_2, \dots, t_n \in \mathfrak{n}$, $\forall a_1, \dots, a_n \in \mathbb{R}$, we have $\|\sum_{k=1}^n a_k Q(e^{t_k})\| = \sup\{|a_k| : k = 1, \dots, n\}$, i.e. $\{Q(e^t)\}_{t\in\mathfrak{n}}$ is order isomorphically and isometrically equivalent to the unit basis of $c_0(\mathfrak{n})$. □

Proposition 6.2. *If I is an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$, then $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is not a dual space.*

Proof. If $a(\varphi) > 0$, we have by Proposition 1.2 that $\ell_\varphi(I)/h_\varphi(\mathcal{S}) \cong C(\beta I\backslash I)$. Grothendieck (see [8]) has shown that, for a compact Hausdorff space T, T must be hyperstonian in order for $C(T)$ to be a dual space (see [11, p. 95]). But $\beta I\backslash I$ is not hyperstonian because it is not basically disconnected.

Assume that $a(\varphi) = 0$. Then card(supp$(x)) \leq \aleph_0$ for each $x \in \ell_\varphi(I)$. Hence card$(\ell_\varphi(I)) \leq \mathfrak{n} := \mathfrak{m}^{\aleph_0}$, with $\mathfrak{m}$ =card(I). By Lemma 6.1, there exists a copy of $c_o(\mathfrak{n})$ in $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ and, by a classical Rosenthal's result ([12, Cor. 1.2]), if $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ were a dual space, it should contain a copy of $\ell_\infty(\mathfrak{n})$. But this is a contradiction because card$(\ell_\infty(\mathfrak{n})) = 2^{\mathfrak{n}} > \mathfrak{n} \geq$card$(\ell_\varphi(I)/h_\varphi(\mathcal{S}))$. □

7. $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is a Grothendieck space

If I is an infinite set, denote by $\mathfrak{M}(I)$ the Banach lattice of finitely additive signed measures on I (see [14]). It is known that this space is order isomorphic and isometric to $C(\beta I)^*$ (i.e. the space of Radon measures on βI). Let T be this isometry. Then:

(1) If $\nu \in \mathfrak{M}(I)$ and $T(\nu) = \mu \in C(\beta I)^*$, we have, $\forall A \subseteq I$, $\nu(A) = \mu(\overline{A})$, where $\overline{A}$ is the closure of A in βI.
(2) $T(\{\nu \in \mathfrak{M}(I) : \nu(\{i\}) = 0, \ \forall i \in I\}) = C(\beta I \setminus I)^*$ (=Radon measures of $C(\beta I)^*$ supported on $\beta I \setminus I$).

If $a(\varphi) > 0$, let $M = \{\nu \in \mathfrak{M}(I) : \nu(\{i\}) = 0, \ \forall i \in I\} = T^{-1}(C(\beta I \setminus I)^*)$. If $a(\varphi) = 0$, define $M \subseteq \mathfrak{M}(I)$ as the subspace such that $\nu \in M$ iff $\nu(\{i\}) = 0$, $\forall i \in I$, and there exists a sequence $\{G_k\}_{k\geq1}$ of pairwise disjoint subsets of I satisfying:

(1) $|\nu|(I \setminus \bigcup_{k\geq 1} G_k) = 0$;
(2) $\sum_{k\geq 1} \varphi(1/k) \cdot |G_k| < \infty$, where $|G_k|$ =card(G_k);
(3) $\sum_{k\geq 1} \varphi\left(\frac{1}{k}[1+\frac{1}{n}]\right) \cdot |G_k \cap E| = \infty,\ \forall n \geq 1,\ \forall E \subseteq I$ such that $|\nu|(E) > 0$.

Proposition 7.1. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then $(\ell_\varphi(I)/h_\varphi(\mathcal{S}))^*$ is order isomorphic and isometric to M and M is 1-complemented in $C(\beta I)^*$.*

Proof. The proof is essentially the one given by Ando [1]. Let $X = \ell_\varphi(I)/h_\varphi(\mathcal{S})$ and pick $x^* \in X^{*+}$. If $E \subseteq I$, define x^*_E as $x^*_E(Q(h)) = x^*(Q(h_E)), \forall h \in \ell_\varphi(I)$, with $h_E = h \cdot \mathbf{1}_E$. Then $x^*_E \in X^{*+}$ and for disjoint subsets E, F of I we have $x^*_{E\cup F} = x^*_E + x^*_F,\ \|x^*_{E\cup F}\| = \|x^*_E\| + \|x^*_F\|$. So, we can define the measure $\nu_{x^*} \in \mathfrak{M}(I)^+$ as follows: $\forall E \subseteq I, \quad \nu_{x^*}(E) = \|x^*_E\|$. Note that this map $X^{*+} \ni x^* \to \nu_{x^*} \in \mathfrak{M}(I)^+$ is linear, monotone (i.e. $x^* \geq y^* \geq 0$ implies $\nu_{x^*} \geq \nu_{y^*}$) and $\|\nu_{x^*}\| = \|x^*\|$ (see Lemmas 2 and 3 of [1]).

We claim that $\nu_{x^*} \in M^+$. Clearly, $\nu_{x^*}(\{i\}) = 0,\ \forall i \in I$, whence, if $a(\varphi) > 0$, we get $\nu_{x^*} \in M^+$. Assume that $a(\varphi) = 0$ and pick $f \in \ell_\varphi(I)^+$ such that $I_\varphi(f) \leq 1$ and $\|x^*_E\| = x^*(Q(f_E)),\ \forall E \subseteq I$ (see Lemma 2 of [1]). Define $G_1 = \{i \in I : |f_i| \geq 1\},\ G_k = \{i \in I : \frac{1}{k} \leq |f_i| < \frac{1}{k-1}\},\ k \geq 2$, and observe that $|G_k| < \infty,\ k \geq 1$, because we suppose that $a(\varphi) = 0$. We have:
(a) $\nu_{x^*}(I \setminus \bigcup_{k\geq 1} G_k) = \|x^*_{I\setminus \cup_{k\geq 1} G_k}\| = x^*(Q(f_{I\setminus\cup_{k\geq 1}G_k})) = x^*(0) = 0$.
(b) $\sum_{k\geq 1} \varphi(\frac{1}{k}) \cdot |G_k| \leq I_\varphi(f) < \infty$.
(c) Let $E \subseteq I$ be such that $\nu_{x^*}(E) > 0$. Then:

$$0 < \nu_{x^*}(E) = \|x^*_E\| = x^*(Q(f_E)) = x^*_E(Q(f_E)) \leq \|Q(f_E)\| \cdot \|x^*_E\|,$$

whence we get $1 \leq \|Q(f_E)\|$, i.e., $d(f_E, h_\varphi(\mathcal{S})) \geq 1$. Hence, $\forall \lambda > 1,\ \forall g \in h_\varphi(\mathcal{S})$, we have $I_\varphi(\lambda(f_E - g)) = \infty$. Pick $n \in \mathbb{N}$ and choose $k_o \in \mathbb{N}$ such that, $\forall k > k_o,\ (1+\frac{1}{n})\frac{1}{k} \geq (1+\frac{1}{2n})\frac{1}{k-1}$. Then, since $f_{E\cap(\cup_{i=1}^{k_o} G_i)} \in \mathcal{S}$, we have:

$$\sum_{k\geq 1} \varphi([1+\tfrac{1}{n}]\tfrac{1}{k}) \cdot |G_k \cap E| \geq \sum_{k>k_o} \varphi([1+\tfrac{1}{2n}]\tfrac{1}{k-1}) \cdot |G_k \cap E|$$
$$\geq I_\varphi([1+\tfrac{1}{2n}][f_E - f_{E\cap(\cup_{i=1}^{k_o} G_i)}]) = \infty,$$

and this completes the proof of the claim.

If $\nu \in \mathfrak{M}(I)^+$, define $x^*_\nu : X^+ \to \mathbb{R}$ as follows:

$$\forall h \in \ell_\varphi(I)^+,\ x^*_\nu(Q(h)) = \inf \sum_{k=1}^{n} \delta(h_{E_k}) \cdot \nu(E_k),$$

where the infimum is taken over all finite pairwise disjoint partitions $\{E_k\}_{k=1}^n$ of I. By Lemmas 4, 5 and 6 of [1] and defining

$$\forall h \in \ell_\varphi(I),\ x^*_\nu(Q(h)) = x^*_\nu(Q(h^+)) - x^*_\nu(Q(h^-)),$$

we have that $x^*_\nu \in X^{*+}$ and $\|x^*_\nu\| \leq \|\nu\| = \nu(I)$. In addition, if $\nu \in M^+$ and $x^* \in X^{*+}$ (see [1, Theorems 2 and 3]), then: (i) $\|(x^*_\nu)_E\| = \nu(E),\ \forall E \subseteq I$; (ii) $x^*_{\nu_{x^*}} = x^*,\ \nu_{x^*_\nu} = \nu$. Hence the positive cones M^+ and X^{*+} are order isomorphic and isometric. If $\nu \in \mathfrak{M}(I)$ and $x^* \in X^*$, define $\nu_{x^*} = \nu_{x^{*+}} - \nu_{x^{*-}},\ x^*_\nu = x^*_{\nu^+} - x^*_{\nu^-}$. With this extension we obtain an order isomorphism and isometry between X^* and M. The projection $P : \mathfrak{M}(I) \to M$ is defined as $P(\nu) = \nu_{x^*_\nu},\ \forall \nu \in \mathfrak{M}(I)$. $\square$

Proposition 7.2. *Let I be an infinite set, φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$, $\{x_n^*\}_{n\geq 1}$ a sequence in $(\ell_\varphi(I)/h_\varphi(\mathcal{S}))^*$ and $\epsilon > 0$. Then there exists $f \in \ell_\varphi(I)^+$ such that $I_\varphi(f) \leq \epsilon$ and:*
(1) $\nu_{x_n^*}(E) = x_n^*(Q(f_E)),\ \forall n \geq 1,\ \forall E \subseteq I;$
(2) $\nu_{x_n^*}(g) = x_n^*(Q(gf)),\ \forall n \geq 1,\ \forall g \in \ell_\infty(I).$

Proof. (A) If $x^* \in (\ell_\varphi(I)/h_\varphi(\mathcal{S}))^*$, by Lemma 2 of [1], there exists $f \in \ell_\varphi(I)^+$ such that $I_\varphi(f) \leq \epsilon$ and $\nu_{x^{*+}}(E) = x^{*+}(Q(f_E)),\ \nu_{x^{*-}}(E) = x^{*-}(Q(f_E)),\ \forall E \subseteq I$. Hence:

$$\forall E \subseteq I,\ \nu_{x^*}(E) = \nu_{x^{*+}}(E) - \nu_{x^{*-}}(E) = x^{*+}(Q(f_E)) - x^{*-}(Q(f_E)) = x^*(Q(f_E)).$$

So, considering ν_{x^*} as a member of $C(\beta I)^*$, we get that $\nu_{x^*}(g) = x^*(Q(gf)),\ \forall g \in \ell_\infty(I)$.

(B) For each x_n^* take $f_n \in \ell_\varphi(I)^+$ satisfying (A) and such that $I_\varphi(f_n) \leq \epsilon/2^n$. Let $f = \sup_{n\geq 1} f_n$. Then we have $I_\varphi(f) \leq \epsilon$ (see Lemma 1 of [1]) and (1), (2) are fulfilled, $\forall n \geq 1$. □

A Banach space is said to be a *Grothendieck space* (see [4]) if for each sequence $\{x_n^*\}_{n\geq 0}$ in X^* such that $x_n^* \to x_0^*$ in the w*-topology, we have that $x_n^* \to x_0^*$ in the w-topology of X^*.

Proposition 7.3. *Let I be an infinite set and φ an Orlicz function such that $\ell_\varphi(I) \neq h_\varphi(\mathcal{S})$. Then $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is a Grothendieck space.*

Proof. Let $\{x_n^*\}_{n\geq 0}$ be a sequence in $(\ell_\varphi(I)/h_\varphi(\mathcal{S}))^*$ such that $x_n^* \to x_0^*$ in the w*-topology. By Proposition 7.2 there exists $f \in \ell_\varphi(I)^+$ such that, $\forall g \in \ell_\infty(I),\ \forall n \geq 0,\ \nu_{x_n^*}(g) = x_n^*(Q(gf))$. Since $Q(gf) \in \ell_\varphi(I)/h_\varphi(\mathcal{S})$, we have

$$\lim_{n\to\infty} x_n^*(Q(gf)) = x_0^*(Q(gf)).$$

Hence $\nu_{x_n^*} \to \nu_{x_0^*}$ in the w*-topology as members of $C(\beta I)^*$. Since $C(\beta I)$ is Grothendieck, we get $\nu_{x_n^*} \to \nu_{x_0^*}$ in the w-topology of $C(\beta I)^*$. Therefore $x_n^* \to x_0^*$ in the w-topology, because $(\ell_\varphi(I)/h_\varphi(\mathcal{S}))^*$ is a subspace of $C(\beta I)^*$. □

Remarks. Since $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ has the Dunford-Pettis property (M-spaces have the Dunford-Pettis property because they are L_1-preduals) and is a Grothendieck space, we obtain that $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ has no infinite dimensional complemented subspaces Y with B_{Y^*} w*-sequentially compact. Also from Proposition 7.3 we get again that $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ cannot be renormed in order to be smooth, because a Grothendieck smooth space is reflexive ([4, p. 215]) and $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ is not, containing a copy of $C(\beta\mathbb{N} \setminus \mathbb{N})$.

Question. Is $\ell_\varphi(I)/h_\varphi(\mathcal{S})$ primary? Recall that Drewnowski and Roberts proved, under CH, that ℓ_∞/c_0 is primary (see [5]).

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