

THE CLASSICAL MECHANICS FOR BOSE - FERMI SYSTEMS *)

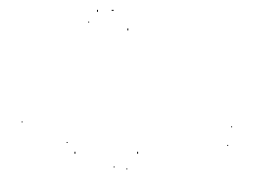
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A B S T R A C T

In this paper we study in a systematic way the classical mechanics of systems described by c number variables and by Grassmann variables. We derive the general form of the non-relativistic action and we study the theory of canonical transformations. For a general action, we show that the Jacobian matrices of the canonical transformations acting on N Grassmann variables form a group $O(N,N)$. This group becomes $O(N)$ for the non-relativistic action, due to the presence of second class constraints. We study some examples which give rise to a correct classical description of the spin. Considering a relativistic extension of one of these models we get a first quantized "substratum" for the superfield theories.

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1. - INTRODUCTION

This paper is devoted to a systematic study of the classical mechanics of systems described by usual c number variables and by Grassmann variables. Such a mechanics has been called pseudomechanics in Ref. 1), and it was shown that the pseudomechanics is the classical limit (in the sense $\hbar \rightarrow 0$) of a general quantum theory with Bose and Fermi operators.

There are various reasons why such a theory is of interest. In particular, the pseudomechanics is interesting by itself because one can develop better intuition about a particular limit of quantum theory, and this can be very useful to construct new models. An example of such a construction is given in Section 6. It is shown there how a first quantized version of supersymmetric field theories²⁾ arises through Lorentz invariant extension of a certain non-relativistic model.

Another important reason is that in this way we get a deeper understanding of the quantization of Fermi trajectories using the "path integral" method with Grassmann variables³⁾.

In some dual theories with Fermi variables⁴⁾, a detailed knowledge of the classical limit is also interesting, because it allows a reinterpretation in terms of Fermionic relativistic strings^{5),6)}.

In field theories in which the interest is mainly in the classical limit, the techniques developed here are certainly useful. We have in mind some early attempts to describe the interaction of the gravitational field with matter fields⁷⁾, in which the use of Grassmann variables at the classical level was strongly suggested.

In this paper we develop the pseudomechanics up to the level of canonical transformations, but certainly it would be possible to extend the theory, as in the classical case, up to the level of adiabatic invariants, perturbation theory and so on. We feel that such developments would be useful to treat non-linear field equations (at the level of classical solutions). We have in mind, in particular, soliton theories with Fermi fields, in which these fields would be treated as Grassmann variables.

Another interesting point is that using the pseudomechanics it is possible to get a very satisfactory description of a spin $\frac{1}{2}$ particle ^{8),9)}.

The study of a general pseudomechanics was performed in Ref. 1) ; here we want to be more definite, so our first objective is to achieve a general formulation of non-relativistic systems. This is done in Section 2 where we derive the general expression for the non-relativistic Lagrangian. The reason to restrict ourselves to non-relativistic systems is that, in general, the relativistic systems are constrained ones, so their description depends on the nature of the constraints. However, our general Lagrangian of Section 2 can be successfully applied to certain relativistic systems, for instance, to the Dirac field.

In Section 3 we study the theory of canonical transformations on a general Lagrangian, and we derive necessary and sufficient conditions that the Jacobian matrix of a given transformation must satisfy in order to be a canonical one. We get also the important result that the group of canonical transformations acting on Grassmann variables only is the pseudo-orthogonal group $O(N,N)$, where N is the number of Grassmann variables.

In Section 4 we analyze our general non-relativistic Lagrangian and we show that it gives rise to second class constraints ¹⁰⁾. Afterwards Dirac brackets are introduced and the properties of contact transformations are reconsidered. We find that the group of canonical transformations acting only on Grassmann variables is the orthogonal group $O(N)$. Finally, we study the problem of quantization of Dirac brackets ; in particular, we show that after quantization the Grassmann variables become the generators of the real Clifford algebra C_N .

Section 5 is devoted to the study of some simple applications. In particular, we consider a non-relativistic particle with internal degrees of freedom described by Grassmann variables. One of these models is the same as investigated in Ref. 8) ; it describes a non-relativistic spin $\frac{1}{2}$ particle. The second example is based on the Grassmann algebra G_4 ⁹⁾. This model provides a correct description of the spin. However, the interest of this last model lies in the fact that it possesses a Lorentz invariant generalization without any enlargement of the algebra G_4 .

The problem of the relativistic extension is studied in Section 6. We postulate Poincaré invariance, and invariance under translations in the space of Grassmann variables. This last property is suggested by the general

structure of the non-relativistic Lagrangian, since it is invariant under such translations. We show that these requirements together lead unavoidably to supersymmetry transformations.

Finally in Section 7, the relativistic supersymmetric model is studied. It is found that the algebra of the Dirac brackets of space-time variables acquires a structure which is in agreement with the results of the general analysis of the relativistic algebra of space-time variables ^{11),12)}.

2. - CLASSICAL LIMIT

The purpose of this section is to give a general expression for the non-relativistic Lagrangian of a pseudoclassical system.

In Ref. 1) it was shown that the limit $\hbar \rightarrow 0$ of a quantum system is a pseudoclassical one ; it follows that all we need is a general formulation of a non-relativistic quantum system. This formulation has been given by Schwinger ¹³⁾. In principle, the Schwinger approach can be used for relativistic systems too, but from the experience with such systems we know that, in general, constraints are present. These constraints modify the theory in a way out of our control, so the relativistic systems must be examined separately in any case.

We will not repeat here the Schwinger approach, but we will recover it by evaluating the propagator for infinitesimal time differences in the coherent state representation. The use of this representation is very convenient in view of the classical limit we want to take.

Considering, for sake of simplicity, one degree of freedom for both Bosonic and Fermionic variables, we define the following coherent states

$$|z\rangle = e^{-\frac{1}{2}z^*z} e^{z a^\dagger} |0\rangle, \quad [a, a^\dagger] = 1, \quad (2.1)$$

$$|\theta\rangle = e^{-\frac{1}{2}\theta^*\theta} e^{\theta b^\dagger} |0\rangle, \quad [b, b^\dagger]_+ = 1, \quad (2.2)$$

where θ and θ^* are Grassmann variables :

$$\theta^2 = \theta^{*2} = \theta\theta^* + \theta^*\theta = 0. \quad (2.3)$$

The propagator in the coherent state representation is :

$$\begin{aligned} \langle z', \theta'; t + \Delta t | z, \theta; t \rangle &= \langle z', \theta'; t | e^{-i \frac{\Delta t}{\hbar} H} | z, \theta; t \rangle = \\ &= \langle z', \theta'; t | z, \theta; t \rangle e^{-i \frac{\Delta t}{\hbar} H(z^*, \theta'; z, \theta)}, \end{aligned}$$

where we have assumed a normal ordered Hamiltonian. For infinitesimal Δt , using (2.1) and (2.2) we get

$$\begin{aligned} \langle z', \theta'; t + \Delta t | z, \theta; t \rangle &\cong \\ &\cong \exp \left\{ i \frac{\Delta t}{\hbar} \left[i \frac{\hbar}{2} (z^* \dot{z} - \dot{z}^* z) + \right. \right. \\ &\left. \left. + i \frac{\hbar}{2} (\theta^* \dot{\theta} - \dot{\theta}^* \theta) - H(z^*, z; \theta^*, \theta) \right] \right\}. \end{aligned} \quad (2.4)$$

The symmetry of this expression among Bose and Fermi variables is a very remarkable one.

The classical limit of the expression (2.4) is a matter of the behaviour of the variables z and θ when $\hbar \rightarrow 0$. We do not know anything about Fermi systems yet ; however, we know that in the classical limit $\hbar(z^* \dot{z} - \dot{z}^* z)$ is a finite quantity. By analogy we will postulate that in the limit $\hbar \rightarrow 0$ the quantity $\hbar(\theta^* \dot{\theta} - \dot{\theta}^* \theta)$ is finite too. We will take care of this fact by defining the following real variables

$$q_B = \sqrt{\frac{\hbar}{2m\omega}} (z + z^*); \quad p_B = -i \sqrt{\frac{\hbar m \omega}{2}} (z - z^*); \quad (2.5a)$$

$$q_F = \sqrt{\frac{\hbar}{2m\omega}} (\theta + \theta^*); \quad p_F = -i \sqrt{\frac{\hbar m \omega}{2}} (\theta - \theta^*). \quad (2.5b)$$

Here m is the mass and ω is an arbitrary constant. Substituting in (2.4), we get

$$\langle z', \theta'; t + \Delta t | z, \theta; t \rangle \cong e^{i \frac{\Delta t}{\hbar} L}, \quad (2.6)$$

where L is

$$L = \frac{1}{2} [P_B \dot{q}_B - q_B \dot{P}_B] + \frac{i}{2} [m\omega q_F \dot{q}_F + \frac{P_F \dot{P}_F}{m\omega}] - H(q_B, P_B; q_F, P_F). \quad (2.7)$$

It follows that the "classical paths" are obtained by minimizing the "classical action"

$$S = \int_{t_i}^{t_f} L dt, \quad (2.8)$$

with L given by (2.7).

The difference between Bose and Fermi kinetic terms in (2.7) is due to the properties of anticommutation of the Grassmann variables. We notice also that in the Bosonic term there is no more reference to the constants m and ω so it is convenient to redefine the Fermi variables :

$$\xi_1 = \sqrt{m\omega} q_F \quad ; \quad \xi_2 = \frac{1}{\sqrt{m\omega}} P_F, \quad (2.9)$$

or in terms of the old variables :

$$\xi_1 = \sqrt{\frac{\hbar}{2}} (\theta + \theta^*) \quad ; \quad \xi_2 = -i \sqrt{\frac{\hbar}{2}} (\theta - \theta^*). \quad (2.10)$$

Neglecting a total time derivative, we get the following final expression for the pseudoclassical action

$$S = \int_{t_i}^{t_f} dt \left[\frac{i}{2} \sum_{\alpha=1,2} \xi_\alpha \dot{\xi}_\alpha - p \dot{q} - H(q, p; \xi_\alpha) \right]. \quad (2.11)$$

This action is in a "Hamiltonian form" with respect to the Bose variables, but in a "Lagrangian form" with respect to the Fermi ones ; that means that the equations of motion for the Grassmann variables are intrinsically of the first order in the time derivative. It is surprising how naturally this result comes out from this approach.

Since we want to analyze the non-relativistic case, we choose $H(q, p; \xi_\alpha)$ of the form

$$H(q, p; \xi_\alpha) = \frac{p^2}{2m} + V(q, \xi_\alpha), \quad (2.12)$$

from which we get

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}.$$

It follows that

$$S = \int_{t_i}^{t_f} dt \left[\frac{i}{2} \sum_{\alpha=1,2} \xi_\alpha \dot{\xi}_\alpha + \frac{1}{2} m \dot{q}^2 - V(q, \xi_\alpha) \right]. \quad (2.13)$$

This expression is easily generalized to an arbitrary number of degrees of freedom. Concerning this point we observe that the case with an odd number of ξ_α is, strictly speaking, not contained in our approach, still it is a very interesting one, as we will see later on. So our general pseudoclassical action is :

$$S = \int_{t_i}^{t_f} dt \left[\frac{i}{2} \sum_{\alpha=1}^N \xi_\alpha \dot{\xi}_\alpha + \frac{1}{2} \sum_{i=1}^n m_i \dot{q}_i^2 - V(q_i, \xi_\alpha) \right]. \quad (2.14)$$

In the case $N = 2M$ we can introduce complex variables

$$\begin{aligned} \eta_{\frac{\alpha+1}{2}} &= \frac{1}{\sqrt{2}} (\xi_\alpha + i \xi_{\alpha+1}), \\ \eta_{\frac{\alpha+1}{2}}^* &= \frac{1}{\sqrt{2}} (\xi_\alpha - i \xi_{\alpha+1}), \quad \alpha = 1, 3, \dots, 2M-1, \end{aligned} \quad (2.15)$$

in terms of which we have instead of (2.14)

$$\begin{aligned} S = \int_{t_i}^{t_f} dt & \left[\frac{i}{2} \sum_{\alpha=1}^M (\eta_\alpha^* \dot{\eta}_\alpha - \dot{\eta}_\alpha^* \eta_\alpha) + \right. \\ & \left. + \frac{1}{2} \sum_{i=1}^n m_i \dot{q}_i^2 - V(q_i, \eta_\alpha) \right]. \end{aligned} \quad (2.16)$$

This last form contains the Dirac field as a particular case.

The equations of motion which follow from (2.14) are ¹⁴⁾

$$m_i \ddot{q}_i = - \frac{\partial V}{\partial q_i}, \quad (2.17)$$

$$\dot{\xi}_\alpha = -i \frac{\partial V}{\partial \xi_\alpha}. \quad (2.18)$$

We see that $V(q_i, \xi_\alpha)$ is the analogue of a potential, so we define as a free motion the motion with $V=0$. It follows that for a free motion the ξ_α are constants of motion; however, the equations of motion are invariant under translations $\xi_\alpha \rightarrow \xi_\alpha + \epsilon_\alpha$ and the variation of the free Lagrangian is just a total time derivative, viz.

$$\delta L^{free} = \frac{i}{2} \sum_{\alpha=1}^N \epsilon_\alpha \dot{\xi}_\alpha, \quad (2.19)$$

or, in terms of the variables η_α , $\eta_\alpha \rightarrow \eta_\alpha + \epsilon_\alpha$,

$$\delta L^{free} = \frac{i}{2} \sum_{\alpha=1}^N (\epsilon_\alpha^* \dot{\eta}_\alpha - \dot{\eta}_\alpha^* \epsilon_\alpha). \quad (2.20)$$

Finally the dimensions of the variables ξ_α (or those of the η_α 's) are

$$[\xi_\alpha] = [\text{action}]^{1/2} = m^{1/2} \ell t^{-1/2} \quad (2.21)$$

These considerations lead one to suspect that the variables ξ_α are the appropriate ones to describe spin. We will come back to this point later on.

3. - CANONICAL TRANSFORMATIONS

We now leave the non-relativistic case studied in the previous section, and we study the theory of the canonical transformations on general Bose-Fermi systems defined by an action of the form ¹⁾

$$S = \int_{t_i}^{t_f} dt L(q_i, \dot{q}_i; \theta_\alpha, \dot{\theta}_\alpha), \quad (3.1)$$

$i = 1, \dots, n; \alpha = 1, \dots, N,$

where the θ_α are Grassmann variables.

We recall the main results obtained in Ref. 1) ; the Hamiltonian is

$$H = \dot{q}_i p^i + \dot{\theta}_\alpha \pi^\alpha - L, \quad (3.2)$$

and the Hamilton equations read :

$$\begin{aligned} \dot{p}^i &= - \frac{\partial H}{\partial q_i} & ; & & \dot{q}_i &= \frac{\partial H}{\partial p^i} ; \\ \dot{\pi}^\alpha &= - \frac{\partial H}{\partial \theta^\alpha} & ; & & \dot{\theta}_\alpha &= - \frac{\partial H}{\partial \pi^\alpha} ; \end{aligned} \quad (3.3)$$

where

$$p^i = \frac{\partial L}{\partial \dot{q}_i} & ; & \pi^\alpha = \frac{\partial L}{\partial \dot{\theta}_\alpha}. \quad (3.4)$$

Using the notation E_i and O_i for dynamical variables which are even and odd elements of the Grassmann algebra, respectively, the Poisson brackets are

$$\begin{aligned} \{E_1, E_2\} &= \left(\frac{\partial E_1}{\partial q_i} \frac{\partial E_2}{\partial p^i} - \frac{\partial E_2}{\partial q_i} \frac{\partial E_1}{\partial p^i} \right) + \\ &+ \left(\frac{\partial E_1}{\partial \theta_\alpha} \frac{\partial E_2}{\partial \pi^\alpha} - \frac{\partial E_2}{\partial \theta_\alpha} \frac{\partial E_1}{\partial \pi^\alpha} \right), \end{aligned} \quad (3.5)$$

$$\begin{aligned} \{O, E\} &= - \{E, O\} = \left(\frac{\partial O}{\partial q_i} \frac{\partial E}{\partial p^i} - \frac{\partial E}{\partial q_i} \frac{\partial O}{\partial p^i} \right) - \\ &- \left(\frac{\partial O}{\partial \theta_\alpha} \frac{\partial E}{\partial \pi^\alpha} + \frac{\partial E}{\partial \theta_\alpha} \frac{\partial O}{\partial \pi^\alpha} \right), \end{aligned} \quad (3.6)$$

$$\{O_1, O_2\} = \left(\frac{\partial O_1}{\partial q_i} \frac{\partial O_2}{\partial p^i} + \frac{\partial O_2}{\partial q_i} \frac{\partial O_1}{\partial p^i} \right) - \left(\frac{\partial O_1}{\partial \Theta_\alpha} \frac{\partial O_2}{\partial \Pi^\alpha} + \frac{\partial O_2}{\partial \Theta_\alpha} \frac{\partial O_1}{\partial \Pi^\alpha} \right). \quad (3.7)$$

The properties of these Poisson brackets were discussed in Ref. 1) where it was shown that their algebra is a graded Lie algebra.

Now we will consider an arbitrary contact transformation, which will be defined by the condition that the new and the old Lagrangians differ by a total time derivative ¹⁶⁾ :

$$\bar{L}(\bar{q}_i, \dot{\bar{q}}_i; \bar{\Theta}_\alpha, \dot{\bar{\Theta}}_\alpha) = L(q_i, \dot{q}_i; \Theta_\alpha, \dot{\Theta}_\alpha) - \frac{d}{dt} \phi(q_i, \bar{q}_i; \Theta_\alpha, \bar{\Theta}_\alpha). \quad (3.8)$$

By differentiation we get the following relations :

$$\bar{p}^i = - \frac{\partial \phi}{\partial q_i} ; \quad p^i = \frac{\partial \phi}{\partial q_i} ; \quad (3.9)$$

$$\bar{\Pi}^\alpha = - \frac{\partial \phi}{\partial \Theta_\alpha} ; \quad \Pi^\alpha = \frac{\partial \phi}{\partial \Theta_\alpha} ; \quad (3.10)$$

and moreover using for \bar{H} the expression (3.2) in terms of the new variables we find

$$\bar{H} = H + \frac{\partial \phi}{\partial t}. \quad (3.11)$$

These equations are conveniently combined by introducing the differential form,

$$d\phi(q_i, \bar{q}_i; \Theta_\alpha, \bar{\Theta}_\alpha) = dq_i p^i - d\bar{q}_i \bar{p}^i + d\Theta_\alpha \Pi^\alpha - d\bar{\Theta}_\alpha \bar{\Pi}^\alpha - (H - \bar{H}) dt. \quad (3.12)$$

Up to this moment there is no difference among Bose and Fermi variables. Let us consider, however, an infinitesimal contact transformation (ICT)

$$\begin{aligned} q_i &= \bar{q}_i + \delta q_i & ; & & p^i &= \bar{p}^i + \delta p^i ; \\ \theta_\alpha &= \bar{\theta}_\alpha + \delta \theta_\alpha & ; & & \pi^\alpha &= \bar{\pi}^\alpha + \delta \pi^\alpha . \end{aligned} \quad (3.13)$$

A general ICT can mix Bose and Fermi variables, however, we will restrict our ICT to ones which maintain the odd or even character of any variable ; this requires that $\delta \theta_\alpha$ and $\delta \pi^\alpha$ are odd variables, and that δq_i and δp^i are even ones. On substituting the expressions (3.13) into (3.12) and defining

$$F = \delta q_i p^i + \delta \theta_\alpha \pi^\alpha - \phi , \quad (3.14)$$

we have

$$\delta q_i = \frac{\partial F}{\partial p^i} ; \quad \delta p^i = - \frac{\partial F}{\partial q_i} ; \quad (3.15)$$

$$\delta \theta_\alpha = - \frac{\partial F}{\partial \pi^\alpha} ; \quad \delta \pi^\alpha = - \frac{\partial F}{\partial \theta_\alpha} ; \quad (3.16)$$

$$H = \bar{H} + \frac{\partial F}{\partial t} . \quad (3.17)$$

The function $F(q_i, p^i; \theta_\alpha, \pi^\alpha; t)$ is called the generator of the ICT. We notice the difference of signs in the equations for Fermi variables with respect to Bose ones.

Our restriction on the ICT's can be now formulated by requiring that F is an even element of the Grassmann algebra.

Writing the variation of a general dynamical variable A under an ICT, we get :

$$\delta A = \left(\frac{\partial A}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial F}{\partial q_i} \frac{\partial A}{\partial p_i} \right) - \left(\frac{\partial F}{\partial \pi^\alpha} \frac{\partial A}{\partial \theta_\alpha} + \frac{\partial F}{\partial \theta_\alpha} \frac{\partial A}{\partial \pi^\alpha} \right).$$

Using Eqs. (3.5) and (3.6) and the fact that F is even, we get

$$\delta A = \{A, F\}. \quad (3.18)$$

Now we want to characterize the canonical transformations from a group theoretical point of view. To this end we need to find the conditions which the Jacobian matrix of a contact transformation must satisfy. It is convenient to introduce the uniformizing variables,

$$z_i : z_1 = q_1, z_2 = p^1, z_3 = q_2, z_4 = p^2, \dots, \\ z_{2n-1} = q_n, z_{2n} = p^n.$$

$$z_\alpha : z_1 = \theta_1, z_2 = \pi^1, z_3 = \theta_2, z_4 = \pi^2, \dots, \\ z_{2N-1} = \theta_N, z_{2N} = \pi^N, \quad (3.19)$$

and the matrices

$$A_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & & & & \ddots \end{bmatrix}; i, j = 1, \dots, 2n; \quad (3.20)$$

$$B_{\alpha\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & & & & \ddots \end{bmatrix}; \alpha, \beta = 1, \dots, 2N; \quad (3.21)$$

with the properties

$$A^T = A^{-1} = -A, \quad (3.22)$$

$$B^T = B^{-1} = B. \quad (3.23)$$

With these notations the equations (3.15) and (3.16) for an ICT can be written

$$\delta z_i = A_{ij} \frac{\partial F}{\partial z_j}, \quad (3.24)$$

$$\delta \bar{z}_\alpha = B_{\alpha\beta} \frac{\partial F}{\partial \bar{z}_\beta}, \quad (3.25)$$

and analogously for the canonical equations

$$\dot{z}_i = A_{ij} \frac{\partial H}{\partial z_j}, \quad (3.26)$$

$$\dot{\bar{z}}_\alpha = B_{\alpha\beta} \frac{\partial H}{\partial \bar{z}_\beta}. \quad (3.27)$$

These equations can be obtained from the following Lagrangian in phase space

$$L = -\frac{1}{2} \dot{z} A^{-1} z - \frac{1}{2} \dot{\bar{z}} B^{-1} \bar{z} - H, \quad (3.28)$$

which, using the properties of A and B, can be rewritten

$$L = \frac{1}{2} \dot{z} A z - \frac{1}{2} \dot{\bar{z}} B \bar{z} - H.$$

We notice that this Lagrangian differs from the ordinary one only in a total time derivative.

When we perform a canonical transformation it is useful to introduce the following notations for the various Jacobian submatrices

$$\begin{aligned} M_{ij} &= \frac{\partial \bar{z}_j}{\partial z_i} ; & F_{\alpha j} &= \frac{\partial \bar{z}_j}{\partial \bar{z}_\alpha} ; \\ G_{i\beta} &= \frac{\partial \bar{z}_\beta}{\partial z_i} ; & N_{\alpha\beta} &= \frac{\partial \bar{z}_\alpha}{\partial \bar{z}_\beta}. \end{aligned} \quad (3.29)$$

Now it is convenient to introduce matrices in the complete space of the z_i and ζ_α variables ; in particular the matrices we are interested in are

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad (3.30)$$

and

$$\mathcal{K} = \begin{pmatrix} M & G \\ F & N \end{pmatrix}. \quad (3.31)$$

The matrix \mathcal{K} is of the kind considered in Ref. 17) because (M,N) are even and (G,F) are odd.

Any variation of the new variables \bar{z} and $\bar{\zeta}$ can be expressed in terms of \mathcal{K} because we have

$$\delta \bar{z}_j = \delta z_i \frac{\partial \bar{z}_j}{\partial z_i} + \delta \zeta_\alpha \frac{\partial \bar{z}_j}{\partial \zeta_\alpha} + \frac{\partial \bar{z}_j}{\partial t} \delta t, \quad (3.32)$$

$$\delta \bar{\zeta}_\rho = \delta \zeta_\alpha \frac{\partial \bar{\zeta}_\rho}{\partial \zeta_\alpha} + \delta z_i \frac{\partial \bar{\zeta}_\rho}{\partial z_i} + \frac{\partial \bar{\zeta}_\rho}{\partial t} \delta t, \quad (3.33)$$

from which by introducing the row vector :

$$\bar{\mathcal{Z}} = (z_i, \zeta_\alpha), \quad (3.34)$$

we get

$$\delta \bar{\mathcal{Z}} = \delta \mathcal{Z} \mathcal{K} + \frac{\partial \bar{\mathcal{Z}}}{\partial t} \delta t. \quad (3.35)$$

If we try to write this equation in the "transposed" version, we have to change the sign of the second term on the right-hand side of Eq. (3.32). This suggests to introduce an operation on matrices of type \mathcal{K} :

$$\mathcal{N} = \left(\begin{array}{c|c} A_{ij} & B_{ip} \\ \hline C_{\alpha j} & D_{\alpha p} \end{array} \right) \equiv \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right), \quad (3.36)$$

with (A,D) even and (B,C) odd variables ; the operation is

$$\mathcal{N}' = \left(\begin{array}{c|c} A^T & C^T \\ \hline -B^T & D^T \end{array} \right), \quad (3.37)$$

and it has the following properties :

$$(\mathcal{N}_1 \mathcal{N}_2)' = \mathcal{N}_2' \mathcal{N}_1', \quad (3.38)$$

and

$$\mathcal{N}'''' = \mathcal{N}; \quad (3.39)$$

so we have defined a sort of involution. We notice that in order to obtain Eq. (3.38) we have used the following property of matrices with odd elements

$$(C_1 C_2)^T = -C_2^T C_1^T. \quad (3.40)$$

The "''" operation can be defined in an analogous way on the vectors as well :

$$\mathcal{Z}' = \begin{pmatrix} z_i \\ -z_\alpha \end{pmatrix}, \quad \mathcal{Z}'' = (z_i, -z_\alpha), \quad \mathcal{Z}''' = \begin{pmatrix} z_i \\ z_\alpha \end{pmatrix}. \quad (3.41)$$

So, for instance, we can take the "''" of Eq. (3.35) :

$$\delta \bar{\mathcal{Z}}' = \mathcal{N}' \delta \mathcal{Z}' + \frac{\partial \bar{\mathcal{Z}}'}{\partial t} \delta t. \quad (3.42)$$

Taking the derivative of a dynamical variable we can get another vector :

$$\frac{\partial A}{\partial \mathcal{Z}} = \begin{pmatrix} \frac{\partial A}{\partial z_i} \\ \frac{\partial A}{\partial z_\alpha} \end{pmatrix}, \quad (3.43)$$

whose properties of transformation are :

$$\frac{\partial A}{\partial \bar{z}} = \mathcal{K} \frac{\partial \bar{A}}{\partial z}. \quad (3.44)$$

If we want to take the "" of this relation we have to distinguish between the cases of A being even or odd ; we have, respectively :

$$A\text{-even} : \left(\frac{\partial A}{\partial \bar{z}} \right)' = \left(\frac{\partial \bar{A}}{\partial z} \right)' \mathcal{K}', \quad (3.45)$$

$$A\text{-odd} : \left(\frac{\partial A}{\partial \bar{z}} \right)' = \left(\frac{\partial \bar{A}}{\partial z} \right)' \mathcal{K}'''. \quad (3.46)$$

Now we are in the position to determine the conditions \mathcal{K} must satisfy in order to describe a canonical transformation.

Let us start with the Lagrangian (3.28) which can be rewritten in the form :

$$L = \frac{1}{2} \dot{z} \mathcal{K} z' - H. \quad (3.47)$$

Writing again the condition (3.8) for a canonical transformation we get :

$$d\phi = \frac{1}{2} \left[\dot{z} \mathcal{K} z' - \dot{\bar{z}} \mathcal{K} \bar{z}' \right] dt - (H - \bar{H}) dt,$$

and using (3.35) :

$$d\phi = \frac{1}{2} dz \left[\mathcal{K} z' - \mathcal{K} \mathcal{K} \bar{z}' \right] - (H - \bar{H} + \frac{1}{2} \frac{\partial \mathcal{K}}{\partial t} \mathcal{K} z) dt.$$

On imposing the condition that $d\phi$ is a perfect differential we find

$$\mathcal{K} \mathcal{K} \mathcal{K}' = \mathcal{K}. \quad (3.48)$$

If the transformation is an invertible one, it follows :

$$\mathcal{K}^{-1} = \mathcal{K} \mathcal{K}' \mathcal{K}^{-1}.$$

Hence, on using

$$\mathcal{K}' = \mathcal{K}^{-1}, \quad (3.49)$$

the identity

$$\mathcal{K}''' \mathcal{K} \mathcal{K} = \mathcal{K} \quad (3.50)$$

follows.

In this formalism we can rewrite the Poisson brackets (3.5) in the following way :

$$\begin{aligned} \{E_1, E_2\} &= \left(\frac{\partial E_1}{\partial \mathcal{K}} \right)''' \mathcal{K} \left(\frac{\partial E_2}{\partial \mathcal{K}} \right), \\ \{0, E\} &= \left(\frac{\partial 0}{\partial \mathcal{K}} \right)' \mathcal{K} \left(\frac{\partial E}{\partial \mathcal{K}} \right), \\ \{E, 0\} &= \left(\frac{\partial E}{\partial \mathcal{K}} \right)''' \mathcal{K} \left(\frac{\partial 0}{\partial \mathcal{K}} \right), \\ \{0_1, 0_2\} &= \left(\frac{\partial 0_1}{\partial \mathcal{K}} \right)' \mathcal{K} \left(\frac{\partial 0_2}{\partial \mathcal{K}} \right). \end{aligned} \quad (3.51)$$

Using (3.50), (3.44), (3.45) and (3.46), it is easy to show that the Poisson brackets are invariant under a canonical transformation.

Now we want to show the converse part, that is, if \mathcal{K} defines a canonical transformation it follows that condition (3.48) is identically satisfied. Let us consider an infinitesimal contact transformation. Using (3.29), (3.24) and (3.25) we have

$$\mathcal{K} \approx 1 + \mathcal{G} \mathcal{K}, \quad (3.52)$$

where

$$g = \begin{pmatrix} m & g \\ f & n \end{pmatrix}, \quad (3.53)$$

with

$$\begin{aligned} m_{ij} &= \frac{\partial^2 F}{\partial z_i \partial z_j} ; & f_{\alpha i} &= \frac{\partial^2 F}{\partial z_\alpha \partial z_i} ; \\ g_{i\beta} &= - \frac{\partial^2 F}{\partial z_i \partial z_\beta} ; & n_{\alpha\beta} &= - \frac{\partial^2 F}{\partial z_\alpha \partial z_\beta} . \end{aligned} \quad (3.54)$$

These submatrices have the following properties :

$$m^T = m ; \quad f^T = -g ; \quad n^T = -n . \quad (3.55)$$

For any matrix of the form (3.52), condition (3.48) becomes

$$g \cdot \mathcal{L} + g' \cdot \mathcal{L}' = 0, \quad (3.56)$$

and we can easily verify that, due to relations (3.55), the condition (3.56) is satisfied.

Finally, we can state the following

THEOREM : Given an infinitesimal contact transformation

$$\delta \bar{z} = \bar{z} - \bar{z}' = \frac{\partial F}{\partial \bar{z}} \cdot \mathcal{L}', \quad (3.57)$$

the Jacobian matrix \mathcal{K} satisfies identically the condition

$$\mathcal{K} \cdot \mathcal{L} \cdot \mathcal{K}' = \mathcal{L}. \quad (3.58)$$

Conversely, given a Jacobian matrix which satisfies (3.58), the corresponding transformation does not change the equations of motion and the Poisson brackets.

Let us now discuss this result. If the odd and even variables are not mixed by the transformation, that is when $G = F = 0$, condition (3.58) becomes

$$MAM^T = A \quad ; \quad NBNT^T = B. \quad (3.59)$$

The first of these two conditions is nothing but the usual one, and it amounts to say that the canonical transformations on the even variables form a symplectic group. What we can learn from the second condition is that the canonical transformations on the odd variables form a group which is the invariance group of the matrix B. However, the matrix B can be diagonalized into the form :

$$B_0 = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}, \quad (3.60)$$

so it follows that this group of invariance is isomorphic to $O(N,N)$, the pseudo-orthogonal group in $(N+N)$ dimensions. The fact that B can be diagonalized using the orthogonal matrix

$$O = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & & \\ 1 & 1 & 0 & 0 & \dots & \\ 0 & 0 & 1 & -1 & & \\ 0 & 0 & 1 & 1 & & \\ & \vdots & & & \ddots & \end{bmatrix}, \quad (3.61)$$

means that we can define new variables

$$\zeta = O\gamma, \quad (3.62)$$

in terms of which the equations of motion are :

$$\dot{\zeta}_{2\alpha-1} = \frac{\partial H}{\partial \zeta_{2\alpha-1}}, \quad (3.63)$$

$$\dot{\zeta}_{2\alpha} = -\frac{\partial H}{\partial \zeta_{2\alpha}}, \quad \alpha = 1, \dots, N, \quad (3.64)$$

and this in turn requires that for transformations which involve only $\zeta_{2\alpha-1}$, or $\zeta_{2\alpha}$, the group of the canonical transformation is $O(N)$.

4. - CANONICAL FORMALISM FOR THE NON-RELATIVISTIC CASE,
AND QUANTIZATION

Now let us come back to the general non-relativistic action

(2.14) :

$$S = \int_{t_i}^{t_f} dt \left[\frac{i}{2} \sum_{\alpha=1}^N \xi_{\alpha} \dot{\xi}_{\alpha} + \frac{1}{2} \sum_{i=1}^n m_i \dot{q}_i^2 - V(q_i, \xi_{\alpha}) \right], \quad (4.1)$$

from the definition (3.4) we get

$$\pi^{\alpha} = - \frac{i}{2} \xi_{\alpha}. \quad (4.2)$$

These equations are constraint equations, and putting :

$$\chi_{\alpha} = \pi^{\alpha} + \frac{i}{2} \xi_{\alpha}, \quad (4.3)$$

we get from our Poisson brackets (3.7) :

$$\{\chi_{\alpha}, \chi_{\beta}\} = -i \delta_{\alpha\beta}. \quad (4.4)$$

So the χ_{α} 's are second class constraints ¹⁰⁾.

We observe also that the Hamiltonian which arises from (4.1) is :

$$H = \frac{1}{2} \sum_{i=1}^n m_i \dot{q}_i^2 + V(q_i, \xi_{\alpha}), \quad (4.5)$$

and we see that there is no contribution from the ξ_{α} 's to the kinetic energy, as expected. [The kinetic term in (4.1) is homogeneous of first degree in the time derivative.]

The situation which we are faced with is analogous to the classical one, so also in this case we will try to define Dirac brackets.

Let us consider as in Ref. 1) the following grading ¹⁸⁾ of the vector space V of the dynamical variables

$$V = T_0 \oplus T_1, \quad (4.6)$$

where the even variables belong to T_0 and the odd ones to T_1 , and let us denote a homogeneous element by v_h , $h=0,1$. It follows that the vector space V is an anticommutative graded algebra with respect to the natural multiplication

$$v_h v_k = (-1)^{hk} v_k v_h. \quad (4.7)$$

We recall here that the indices $(0,1)$ are effectively the elements of the group with two elements Z_2 .

Let us now suppose, in general, to have constraints χ_i^h where h is the grading index ; then we define the matrix :

$$C_{ij}^{hk} = \{ \chi_i^h, \chi_j^k \}, \quad (4.8)$$

with the property

$$C_{ij}^{hk} = - (-1)^{hk} C_{ji}^{kh}. \quad (4.9)$$

We do not know any general proof of the fact that C_{ij}^{hk} admits an inverse, however, we strongly suspect that to be the case. In particular, if the even and odd constraints are not mixed, i.e., if the C matrix is of the form

$$C = \left[\begin{array}{c|c} C_{ij}^{00} & 0 \\ \hline 0 & C_{\alpha\beta}^{11} \end{array} \right], \quad (4.10)$$

it is easy to repeat Dirac's proof ¹⁰⁾ of the existence of the inverse for any of the two submatrices. We observe also that the submatrix $C_{\alpha\beta}^{11}$ is a symmetric one, so the number of second class constraints of the odd type does not need to be even.

All the cases we will be interested in are such that C^{-1} exists ¹⁹⁾. In that case, Dirac brackets can be defined as follows

$$\{ v_a, v_b \}^* = \{ v_a, v_b \} - \{ v_a, \chi_i^e \} (C^{-1})_{ij}^{em} \{ \chi_j^m, v_b \}. \quad (4.11)$$

Let us now study the properties of such a bracket. First of all, as a consequence of the relation

$$(C^{-1})_{ij}^{\ell m} \in T_{\ell+m}, \quad (4.12)$$

it follows that

$$\{V_h, V_k\}^* \in T_{h+k}. \quad (4.13)$$

Furthermore, using the property

$$(C^{-1})_{ij}^{mn} = -(-1)^{(m^2+mn+n^2)} (C^{-1})_{ji}^{nm}, \quad (4.14)$$

and Eq. (4.7) we can show

$$\begin{aligned} \{V_h, V_k + W_k\}^* &= \{V_h, V_k\}^* + \{V_h, W_k\}^*, \\ \{V_h, V_k\}^* &= -(-1)^{hk} \{V_k, V_h\}^*, \\ \{V_e, V_m V_n\}^* &= (-1)^{me} V_m \{V_e, V_n\}^* + \{V_e, V_m\}^* V_n, \\ \{V_e V_m, V_n\}^* &= V_e \{V_m, V_n\}^* + (-1)^{mn} \{V_e, V_n\}^* V_m, \\ (-1)^{en} \{V_e, \{V_m, V_n\}^*\}^* &+ (-1)^{me} \{V_m, \{V_n, V_e\}^*\}^* \\ &+ (-1)^{nm} \{V_n, \{V_e, V_m\}^*\}^* = 0. \end{aligned} \quad (4.15)$$

These properties are the same which are satisfied by the original Poisson brackets, moreover :

$$\{\chi_i^e, A\}^* = 0, \quad (4.16)$$

where A is an arbitrary dynamical variable.

Using the same argument as in Ref. 1), it follows that we can quantize the theory assuming for the quantum operators the quantization rule

$$\psi_n \psi_e - (-1)^{n\ell} \psi_e \psi_n = i\hbar \{ \psi_n, \psi_e \}^* . \quad (4.17)$$

In the case described by the action (4.1), we have :

$$C_{\alpha\beta} = -i \delta_{\alpha\beta} ,$$

from which

$$(C^{-1})_{\alpha\beta} = i \delta_{\alpha\beta} . \quad (4.18)$$

It follows

$$\{ \xi_\alpha, \xi_\beta \}^* = -i \delta_{\alpha\beta} , \quad (4.19)$$

where we have used the canonical Poisson brackets for the ξ_α

$$\begin{aligned} \{ \xi_\alpha, \xi_\beta \} &= \{ \pi^\alpha, \pi^\beta \} = 0, \\ \{ \xi_\alpha, \pi^\beta \} &= -\delta_{\alpha\beta} . \end{aligned} \quad (4.20)$$

The equations of motion for ξ_α are :

$$\dot{\xi}_\alpha = \{ \xi_\alpha, H \}^* = -i \frac{\partial H}{\partial \xi_\alpha} = -i \frac{\partial V}{\partial \xi_\alpha} , \quad (4.21)$$

which are the correct ones [see Eq. (2.18)], and where we have used the obvious property

$$\{ \xi_\alpha, f(\xi) \}^* = -i \frac{\partial f}{\partial \xi_\alpha} . \quad (4.22)$$

Now the dynamical variables depend only on the ξ_α , so it is easy to obtain the following expressions for the Dirac brackets of any pair of variables

$$\begin{aligned} \{ E_1, E_2 \}^* &= \left(\frac{\partial E_1}{\partial q_i} \frac{\partial E_2}{\partial p_i} - \frac{\partial E_2}{\partial q_i} \frac{\partial E_1}{\partial p_i} \right) + \\ &+ i \frac{\partial E_1}{\partial \xi_\alpha} \frac{\partial E_2}{\partial \xi_\alpha} ; \end{aligned} \quad (4.23)$$

$$\{0, \epsilon\}^* = - \{\epsilon, 0\}^* = \left(\frac{\partial 0}{\partial q_i} \frac{\partial \epsilon}{\partial p_i} - \frac{\partial \epsilon}{\partial q_i} \frac{\partial 0}{\partial p_i} \right) - i \frac{\partial 0}{\partial \xi_\alpha} \frac{\partial \epsilon}{\partial \xi_\alpha}, \quad (4.24)$$

$$\{0_1, 0_2\}^* = \left(\frac{\partial 0_1}{\partial q_i} \frac{\partial 0_2}{\partial p_i} + \frac{\partial 0_2}{\partial q_i} \frac{\partial 0_1}{\partial p_i} \right) - i \frac{\partial 0_1}{\partial \xi_\alpha} \frac{\partial 0_2}{\partial \xi_\alpha}. \quad (4.25)$$

The action (4.1) can be rewritten in the phase space, omitting a total time derivative, in the following way :

$$S = \int_{t_i}^{t_f} dt \left[- \frac{i}{2} \sum_{\alpha=1}^N \dot{\xi}_\alpha \xi_\alpha + \frac{1}{2} \sum_{i=1}^n \dot{z}_i A_{ij} z_j - H \right], \quad (4.26)$$

where the z_i are defined in (3.19), the matrix A is defined in (3.20) and H is given in (4.5). We see that this expression is formally the same as expression (3.28) with $B_{\alpha\beta} = -i\delta_{\alpha\beta}$, and this is a symmetric matrix like that of the previous section. We emphasize this point because the symmetry is the only property that we have really used in all the derivations. It follows that the Dirac brackets can be rewritten in the form used in the previous section

$$\begin{aligned} \{\epsilon_1, \epsilon_2\}^* &= \left(\frac{\partial \epsilon_1}{\partial \mathcal{Z}} \right)' \mathcal{L} \left(\frac{\partial \epsilon_2}{\partial \mathcal{Z}} \right), \\ \{\epsilon, 0\}^* &= - \{0, \epsilon\}^* = \left(\frac{\partial \epsilon}{\partial \mathcal{Z}} \right)' \mathcal{L} \left(\frac{\partial 0}{\partial \mathcal{Z}} \right), \\ \{0_1, 0_2\}^* &= \left(\frac{\partial 0_1}{\partial \mathcal{Z}} \right)' \mathcal{L} \left(\frac{\partial 0_2}{\partial \mathcal{Z}} \right), \end{aligned} \quad (4.27)$$

where

$$\mathcal{L} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad (4.28)$$

and

$$B_{\alpha\beta} = -i \delta_{\alpha\beta};$$

the variable Z is defined by

$$\mathcal{Z} = (z_i, \xi_\alpha). \quad (4.30)$$

We notice that everything is formally as in the general case, but now the variables ξ_α are not the co-ordinates in the phase space, and the Poisson brackets are replaced by the Dirac ones ; these two differences are obviously related. In particular the number of the ξ_α is arbitrary and not only even as it would be in the previous section.

From these considerations it follows that we can show also in this case the theorem of the previous section, the only difference being the different structure of the matrix B . So we will have again that the canonical transformations are characterized by the condition

$$\mathcal{N} \mathcal{B} \mathcal{N}' = \mathcal{B}, \quad (4.31)$$

with \mathcal{B} determined by Eqs. (4.28) and (4.29).

We observe that, for canonical transformations acting on the variables ξ_α only, that is $\xi_\alpha \rightarrow f_\alpha(\xi_\alpha)$ we must have :

$$N B N^T = B. \quad (4.32)$$

Looking at the definition of B it follows that

$$N N^T = 1. \quad (4.33)$$

So we get the important result that the group of the canonical transformations on the variables ξ_α is the orthogonal group in N dimensions, that is $O(N)$.

To conclude this section we will make some remarks about the quantization. As we said we have to quantize the Dirac brackets so our fundamental quantization rules will be

$$[q_i, p^j] = i\hbar \delta_i^j, \quad (4.34)$$

$$[\xi_\alpha, \xi_\beta]_+ = \hbar \delta_{\alpha\beta}. \quad (4.35)$$

We see that the generators of the Grassmann algebra G_N become, after quantization, the generators of the Clifford algebra C_N .

From this point of view it is very simple to understand the previous result about the group of canonical transformations. In fact, in the quantized theory the group of canonical transformations is the group of the automorphisms of the graded Lie algebra (4.17). However, the automorphism group of the subalgebra C_N [Eq. (4.35)] is precisely $O(N)$ as is it well known from the theory of Clifford algebras²⁰.

5. - SOME ELEMENTARY EXAMPLES

In this section we will discuss some simple cases of the general non-relativistic action introduced in Section 2 and discussed in Section 4. We are interested in describing a massive particle with some degrees of freedom, which will be related to the Grassmann variables. In particular, we want to describe particles with spin, that means that we need to specify the behaviour of the chosen Grassmann algebra G_N , under a group $SU(2)$ [or $O(3)$] which will be identified with the spin group. However, we require that the over-all rotation group of transformations on the variables q_i and ξ_α is a symmetry group. So, in particular, these transformations are canonical ones and we want to realize them in a linear way.

We know from the previous section that if we have linear canonical transformations on the variables ξ_α only, then these transformations belong to a group $O(N)$ under which the ξ_α transform as the components of a N vector. It follows, in the general case of an algebra G_N , that we have to identify the spin group as a subgroup of the canonical group $O(N)$.

The first interesting case is clearly G_3 ; however, G_4 is another interesting possibility due to the isomorphy of $O(4)$ to $SU(2) \times SU(2)$, so we can identify the spin group with one of these $SU(2)$.

Grassmann algebras G_N , of higher order ($N > 4$) can be treated in an analogous fashion therefore we concentrate on the two prototypes, G_3 and G_4 , respectively. However, as a warming up exercise, let us start with the study of G_1 and G_2 in which, of course, the ξ_α will transform as scalars under the rotation group.

G₁

The most general Lagrangian is

$$L = \frac{i}{2} \xi \dot{\xi} + \frac{1}{2} m \vec{v}^2 - V(\vec{q}), \quad (5.1)$$

due to the fact that the only even form in ξ is $\xi^2 = 0$. This is a very trivial case because Fermi and Bose variables are decoupled, and furthermore ξ is a constant of motion :

$$\dot{\xi} = 0. \quad (5.2)$$

G₂

The most general Lagrangian is

$$L = \frac{i}{2} (\xi_1 \dot{\xi}_1 + \xi_2 \dot{\xi}_2) + \frac{1}{2} m \vec{v}^2 - V_1(\vec{q}) - \frac{i}{2} (\xi_1 \xi_2 - \xi_2 \xi_1) V_2(\vec{q}), \quad (5.3)$$

or introducing complex co-ordinates [see Eq. (2.15)]

$$\eta = \frac{1}{\sqrt{2}} (\xi_1 + i \xi_2), \quad \eta^* = \frac{1}{\sqrt{2}} (\xi_1 - i \xi_2), \quad (5.4)$$

it follows

$$L = \frac{i}{2} (\eta^* \dot{\eta} - \dot{\eta}^* \eta) + \frac{1}{2} m \vec{v}^2 - V_1(\vec{q}) - \eta^* \eta V_2(\vec{q}). \quad (5.5)$$

The equation of motion for η is

$$\dot{\eta} = -i V_2 \eta. \quad (5.6)$$

If we consider the case $V_2(\vec{q}) = \omega$ with a constant ω we get :

$$\eta(t) = e^{-i\omega(t-t_i)} \eta_i, \quad (5.7)$$

where

$$\eta_i = \eta(t_i). \quad (5.8)$$

So $\eta(t)$ performs a periodic motion with period $T = 2\pi/\omega$. From the Lagrangian (5.5), we get the Hamiltonian

$$H = \frac{1}{2} m \vec{v}^2 + V_1 + \omega \eta^* \eta. \quad (5.9)$$

In particular we have that the part of the Hamiltonian which contains the Grassmann variables is a constant of motion by itself :

$$E_\eta = \omega \eta^*(t) \eta(t) = \omega \eta_i^* \eta_i. \quad (5.10)$$

It is very amusing to calculate the "phase integral" for the Grassmann variables :

$$J = \oint (d\eta \Pi_\eta + d\eta^* \Pi_{\eta^*}), \quad (5.11)$$

where the integral is extended to one period of motion, and

$$\Pi_\eta = -\frac{i}{2} \eta^*, \quad \Pi_{\eta^*} = -\frac{i}{2} \eta. \quad (5.12)$$

It follows :

$$J = \frac{2\pi}{\omega} E_\eta. \quad (5.13)$$

At this point one is strongly tempted to follow the quantization procedure of Bohr and Sommerfeld and to say that

$$J = n h, \quad (5.14)$$

where h is the Planck constant and n is an integer. This would give

$$E_\eta = n \hbar \omega. \quad (5.15)$$

However, it is easy to realize that in this case the Bohr-Sommerfeld procedure is at least ambiguous ; indeed as we will see in a moment, n can assume only two values $n=0,1$. It follows that if we do not take care of the ground state energy properly, we get wrong values for the eigenvalues of the energy.

Let us analyze the quantum theory. As we know [see Eq. (4.35)] the quantum theory is obtained by putting

$$[\xi_i, \xi_j]_+ = \hbar \delta_{ij}. \quad (5.16)$$

By introducing :

$$b = \frac{1}{\sqrt{2\hbar}} (\xi_1 + i \xi_2),$$

$$b^\dagger = \frac{1}{\sqrt{2\hbar}} (\xi_1 - i \xi_2),$$

we get

$$[b, b^\dagger]_+ = 1 ; \quad [b, b]_+ = [b^\dagger, b^\dagger]_+ = 0. \quad (5.17)$$

So we have

$$\eta \rightarrow \sqrt{\hbar} b \quad ; \quad \eta^* \rightarrow \sqrt{\hbar} b^\dagger, \quad (5.18)$$

and

$$E_\eta \rightarrow \hbar \omega b^\dagger b. \quad (5.19)$$

We see that Eq. (5.15) is formally correct if we take $n=0,1$, because these are the eigenvalues of the operator $b^\dagger b$. However, if we assume the Lagrangian (5.3) as a quantum Lagrangian, the Hamiltonian would be

$$\hbar \omega b^\dagger b - \frac{1}{2} \hbar \omega. \quad (5.20)$$

It follows that the ordering plays a crucial rôle, contrarily to the Bose case in which the ordering has no effect for big quantum numbers. The point is that for Fermi systems we have no way to get big quantum numbers apart from possibly going to some statistical system.

We can see from this simple example the reasons why the classical limit of Fermi systems is in general only a formal one ; in fact the classical limit requires to take the limit of big quantum numbers together with the limit $\hbar \rightarrow 0$.

G₃

The most general Lagrangian is

$$L = \frac{i}{2} \dot{\vec{\xi}} \cdot \dot{\vec{\xi}} + \frac{1}{2} m \vec{V}^2 - V_1(\vec{q}) - \xi_i \xi_j V_{ij}(\vec{q}), \quad (5.21)$$

where, as we anticipated, we take $\vec{\xi}$ as a vector under the rotation group. It follows that the antisymmetric tensor V_{ij} must be of the form

$$V_{ij} = \frac{i}{2} \epsilon_{ijk} V_k(\vec{q}), \quad (5.22)$$

where $V_k(\vec{q})$ transforms as a pseudovector.

From the Appendix we have that the generators of the $O(3)$ group are

$$S_{ij} = -\frac{i}{2} [\xi_i, \xi_j], \quad (5.23)$$

so it is natural to introduce the pseudovector

$$\vec{S} = -\frac{i}{2} \vec{\xi} \wedge \vec{\xi}, \quad (5.24)$$

in terms of which we can rewrite the Lagrangian (5.21) in the form

$$L = \frac{i}{2} \dot{\vec{\xi}} \cdot \dot{\vec{\xi}} + \frac{1}{2} m \vec{V}^2 - V_1(\vec{q}) + \vec{S} \cdot \vec{V}(\vec{q}). \quad (5.25)$$

The Dirac brackets of \vec{S} are

$$\{S_i, S_j\}^* = \epsilon_{ijk} S_k.$$

This Lagrangian has been introduced in Ref. 8). We see that this Lagrangian offers a correct pseudoclassical description of the spin. Let us exhibit the equations of motion for $\vec{\xi}$ and \vec{S} . We have :

$$\dot{\vec{\xi}} = \vec{\xi} \wedge \vec{V}, \quad (5.26)$$

$$\dot{\vec{S}} = \vec{S} \wedge \vec{V}. \quad (5.27)$$

Thus, $\vec{\xi}$ and \vec{S} obey the same equations of motion ; this must be so, because after quantization we do not expect any difference between $\vec{\xi}$ and \vec{S} . Let us check this point ; the quantization rule is :

$$[\xi_i, \xi_j]_+ = \hbar \delta_{ij}. \quad (5.28)$$

So it follows (see the Appendix) that

$$\vec{\xi} = \sqrt{\frac{\hbar}{2}} \vec{\sigma}, \quad (5.29)$$

where $\vec{\sigma}$ are the Pauli matrices. From (5.24) and the properties of the Pauli matrices we have, of course

$$\vec{S} = \frac{1}{2} \hbar \vec{\sigma}. \quad (5.30)$$

This is an interesting point : due to the fact that the Clifford algebra C_3 admits a non-linear representation, it follows that the Lagrangian (5.21), which was even in the ξ_i variables, becomes odd after quantization.

As a slight extension we can consider the potential \vec{V} as depending on the velocity ; in this case we can describe a spin orbit coupling :

$$L = \frac{i}{2} \dot{\vec{\xi}} \cdot \dot{\vec{\xi}} + \frac{1}{2} m \vec{v}^2 + \mu (\vec{q} \wedge \vec{v}) \cdot \vec{S}. \quad (5.31)$$

The equations of motion are

$$\dot{\vec{\xi}} = \mu \vec{\xi} \wedge (\vec{q} \wedge \vec{v}), \quad (5.32)$$

$$\dot{\vec{p}} = \mu \vec{v} \wedge \vec{S}. \quad (5.33)$$

Defining the mechanical angular momentum

$$\vec{M} = m \vec{q} \wedge \vec{v}, \quad (5.34)$$

we can calculate the time derivatives of the orbital momentum $\vec{q} \wedge \vec{p}$ and of the spin \vec{S} . We obtain :

$$\dot{\vec{L}} = \frac{d}{dt} (\vec{q} \wedge \vec{p}) = \frac{M}{m} \vec{M} \wedge \vec{S}, \quad (5.35)$$

$$\dot{\vec{S}} = \frac{M}{m} \vec{S} \wedge \vec{M}. \quad (5.36)$$

We see that also in this case the equations for $\dot{\vec{L}}$ and for $\dot{\vec{S}}$ are the same, and moreover the total angular momentum $\vec{L} + \vec{S}$ is a constant of motion.

Now we want to describe a model with a G_4 algebra as illustrated at the beginning of this section.

G_4

The Lagrangian will be of the form ⁹⁾

$$L = \frac{i}{2} \sum_{\alpha=1}^4 \xi_{\alpha} \dot{\xi}_{\alpha} + \frac{1}{2} m \vec{v}^2 - V(\vec{q}, \xi_{\alpha}). \quad (5.37)$$

The generators of the group $O(4)$ are

$$S_{\alpha\beta} = -\frac{i}{2} [\xi_{\alpha}, \xi_{\beta}], \quad (5.38)$$

for which we have the Dirac brackets

$$\{S_{\alpha\beta}, S_{\gamma\delta}\}^* = -(\delta_{\alpha\delta} S_{\beta\gamma} + \delta_{\beta\gamma} S_{\alpha\delta} - \delta_{\alpha\gamma} S_{\beta\delta} - \delta_{\beta\delta} S_{\alpha\gamma}). \quad (5.39)$$

The group $O(4)$ is decomposed into the direct product $SU(2) \times SU(2)$ by means of the following substitutions :

$$S_i = \frac{1}{2} \left(S_{i4} + \frac{1}{2} \epsilon_{ijk} S_{jk} \right), \quad (5.40)$$

$$N_i = \frac{1}{2} \left(-S_{i4} + \frac{1}{2} \epsilon_{ijk} S_{jk} \right), \quad (5.41)$$

$i, j, k = 1, 2, 3.$

These combinations satisfy the algebra

$$\{S_i, S_j\}^* = \epsilon_{ijk} S_k, \quad (5.42)$$

$$\{N_i, N_j\}^* = \epsilon_{ijk} N_k, \quad (5.43)$$

$$\{S_i, N_j\}^* = 0. \quad (5.44)$$

Now let us choose S_i as the generators of the physical spin group. This implies that we have to identify the behaviour of our dynamical variables with respect to S_i . The variables ξ_α transform as components of a vector under $O(4)$, that is ξ_α belongs to the representation $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ of $SU(2) \otimes SU(2)$. It follows that it is possible to construct from ξ_α spinors under S_i . The decomposition of the vector is made by introducing the matrix

$$\vec{\sigma} \cdot \vec{\xi} + i \xi_4 = \begin{pmatrix} \xi_3 + i \xi_4 & \xi_1 - i \xi_2 \\ \xi_1 + i \xi_2 & -\xi_3 + i \xi_4 \end{pmatrix}, \quad (5.45)$$

and its conjugate

$$\vec{\sigma} \cdot \vec{\xi} - i \xi_4 = \begin{pmatrix} \xi_3 - i \xi_4 & \xi_1 - i \xi_2 \\ \xi_1 + i \xi_2 & -\xi_3 - i \xi_4 \end{pmatrix}. \quad (5.46)$$

The behaviour of these quantities under S_i is easily deduced. One finds

$$\{S_i, \vec{\sigma} \cdot \vec{\xi} + i \xi_4\}^* = -\frac{i}{2} (\vec{\sigma} \cdot \vec{\xi} + i \xi_4) \sigma_i, \quad (5.47)$$

$$\{S_i, \vec{\sigma} \cdot \vec{\xi} - i \xi_4\}^* = +\frac{i}{2} \sigma_i (\vec{\sigma} \cdot \vec{\xi} - i \xi_4). \quad (5.48)$$

From these relations it follows that it is convenient to introduce the spinors :

$$\eta_\alpha = \frac{1}{\sqrt{2}} (\vec{\sigma} \cdot \vec{\xi} - i \xi_4)_{\alpha 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_3 - i \xi_4 \\ \xi_1 + i \xi_2 \end{pmatrix}, \quad (5.49)$$

$$\eta_\alpha^* = \frac{1}{\sqrt{2}} (\vec{\sigma} \cdot \vec{\xi} + i \xi_4)_{1\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_3 + i \xi_4 \\ \xi_1 - i \xi_2 \end{pmatrix}, \quad (5.50)$$

$$\eta^\alpha = \epsilon^{\alpha\beta} \eta_\beta = \frac{1}{\sqrt{2}} (\vec{\sigma} \cdot \vec{\xi} + i \xi_4)_{2\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i \xi_2 \\ -\xi_3 + i \xi_4 \end{pmatrix}, \quad (5.51)$$

$$\eta^{\alpha*} = \epsilon^{\alpha\beta} \eta_\beta^* = \frac{1}{\sqrt{2}} (\vec{\sigma} \cdot \vec{\xi} - i \xi_4)_{\alpha 2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 - i \xi_2 \\ -\xi_3 - i \xi_4 \end{pmatrix}, \quad (5.52)$$

where

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Such spinors have the following behaviour under S_i :

$$\{S_i, \eta_\alpha\}^* = \frac{i}{2} (\sigma_i)_{\alpha\beta} \eta_\beta, \quad (5.53)$$

$$\{S_i, \eta^\alpha\}^* = -\frac{i}{2} \eta^\beta (\sigma_i)_{\beta\alpha}, \quad (5.54)$$

$$\{S_i, \eta_\alpha^*\}^* = -\frac{i}{2} \eta_\beta^* (\sigma_i)_{\beta\alpha}, \quad (5.55)$$

$$\{S_i, \eta^{\alpha*}\}^* = \frac{i}{2} (\sigma_i)_{\alpha\beta} \eta^{\beta*}. \quad (5.56)$$

It follows that η_α and $\eta^{\alpha*}$ are $SU(2)$ spinors, and that η_α^* and η^α transform according to the contravariant representation.

The $SU(2)$ invariant form is

$$\sum_{\alpha=1,2} \eta_\alpha^* \eta_\alpha = \sum_{\alpha=1,2} \eta^{\alpha*} \eta^\alpha,$$

and we have

$$\frac{i}{2} \sum_{\alpha=1}^4 \xi_\alpha \dot{\xi}_\alpha = \frac{i}{2} \sum_{\alpha=1,2} (\eta_\alpha^* \dot{\eta}_\alpha - \dot{\eta}_\alpha^* \eta_\alpha).$$

Hence, our Lagrangian (5.37) can be rewritten in the form

$$L = \frac{i}{2} \sum_{\alpha=1,2} (\eta_{\alpha}^* \dot{\eta}_{\alpha} - \dot{\eta}_{\alpha}^* \eta_{\alpha}) + \frac{1}{2} m \vec{v}^2 - V(\vec{q}, \eta_{\alpha}). \quad (5.57)$$

In terms of the η_{α} the spin S_i can be written :

$$S_i = \frac{1}{2} \eta^* \sigma_i \eta, \quad (5.58)$$

and we notice that the Dirac brackets for the η_{α} are :

$$\begin{aligned} \{\eta_{\alpha}, \eta_{\beta}\}^* &= \{\eta_{\alpha}^*, \eta_{\beta}^*\}^* = 0, \\ \{\eta_{\alpha}, \eta_{\beta}^*\}^* &= -i \delta_{\alpha\beta}. \end{aligned} \quad (5.59)$$

In this model we can also introduce an interaction term analogous to that considered in G_3 :

$$L = \frac{i}{2} \sum_{\alpha=1,2} (\eta_{\alpha}^* \dot{\eta}_{\alpha} - \dot{\eta}_{\alpha}^* \eta_{\alpha}) + \frac{1}{2} m \vec{v}^2 + \vec{S} \cdot \vec{V}(\vec{q}), \quad (5.60)$$

with \vec{S} given by (5.58). The equations of motion are :

$$\dot{\eta} = \frac{i}{2} (\vec{V} \cdot \vec{\sigma}) \eta, \quad (5.61)$$

from which we get the equations of motion for the spin vector \vec{S} :

$$\dot{\vec{S}} = \vec{S} \wedge \vec{V}. \quad (5.62)$$

We see that the equations of motion for \vec{S} are the same as in G_3 . An analogous conclusion can be reached for a spin-orbit coupling. However, we must observe that in the case of G_4 we can in principle admit more complicated interactions, like $(\eta^* \eta)^2$, which are not allowed in the G_3 model.

Moreover, the G_3 and G_4 models are different after quantization. In fact, as we saw the model G_3 describes a single spin $\frac{1}{2}$ particle. For the G_4 model, it follows from (5.59) that on quantizing

$$[\eta_{\alpha}^*, \eta_{\beta}]_+ = \hbar \delta_{\alpha\beta}. \quad (5.63)$$

That is we have two Fermi oscillators ; but this means that we can construct the following states starting from the ground state :

$$|0, \vec{q}\rangle, \quad \eta_{\alpha}^{*} |0, \vec{q}\rangle, \quad \eta_{\alpha}^{*} \eta_{\beta}^{*} |0, \vec{q}\rangle. \quad (5.64)$$

In other words we have a scalar state (the ground state), a spin $\frac{1}{2}$ state, and another scalar state, because

$$S_i (\eta_{\alpha}^{*} \eta_{\beta}^{*} |0, \vec{q}\rangle) = 0. \quad (5.65)$$

We notice also

$$S_i (\eta_{\alpha}^{*} |0, \vec{q}\rangle) = \eta_{\beta}^{*} |0, \vec{q}\rangle \frac{1}{2} \hbar (\sigma_i)_{\beta\alpha}. \quad (5.66)$$

6. - RELATIVIZATION OF THE G_4 MODEL AND SUPERSYMMETRIES

In this section we discuss the relativization of the G_4 model ⁹⁾, and we show the connection with supersymmetry transformations ^{2), 4)}.

As we have already observed, the free non-relativistic pseudo-classical Lagrangian is invariant under translations both in the q_{\pm} and in the ξ_{α} . Let us consider, in particular, the free Lagrangian for the G_4 algebra [see Eq. (5.57)]

$$L = \frac{i}{2} \sum_{\alpha=1,2} (\eta_{\alpha}^{*} \dot{\eta}_{\alpha} - \dot{\eta}_{\alpha}^{*} \eta_{\alpha}) + \frac{1}{2} m \vec{V}^2. \quad (6.1)$$

If we suppose that the invariance of the Lagrangian under translation in η_{α} is of the same importance as the invariance under ordinary translations, we are led to search for a relativistic generalization of the Lagrangian (6.1) which satisfies the following requirements :

- i) Poincaré invariance ;
- ii) invariance under translations in the η_{α} 's space.

Such a generalization could be made for the G_3 model too ; however, it is easy to realize that this would mean an enlargement of the original Grassmann algebra. On the contrary, as we shall see, in the G_4 model, such an extension is possible without any enlargement.

We describe the time evolution of the relativistic system, using an invariant parameter τ (which we choose having the dimensions of a length) such that the Lagrangian is invariant under reparametrization ; that means that the Lagrangian must be homogeneous of the first degree in the derivatives with respect to τ .

Another obvious requirement is that the relativistic Lagrangian goes in the limit $c \rightarrow \infty$ in the Lagrangian (6.1).

Let us start by noticing that given a two-spinor η_α , it is possible to construct from it the following four vector :

$$V_\mu = \frac{i}{2} \left(\eta^* \sigma_\mu \frac{d\eta}{d\tau} - \frac{d\eta^*}{d\tau} \sigma_\mu \eta \right), \quad (6.2)$$

in which

$$(\sigma_\mu)_{\alpha\beta} \equiv (1, -\vec{\sigma}). \quad (6.3)$$

V_μ is a proper four-vector if we define the parity operation on spinors as

$$\eta^\alpha \rightarrow -i (\tilde{\sigma}_0)^{\alpha\dot{\beta}} \eta_{\dot{\beta}}^*, \quad (6.4)$$

where

$$(\tilde{\sigma}_\mu)^{\alpha\dot{\beta}} \equiv (1, \vec{\sigma}). \quad (6.5)$$

We observe also that if we choose the parameter τ equal to ct , it follows that V_0 is proportional to the kinetic part for the η_α variables in the Lagrangian (6.1). We see that the relativistic Lagrangian L_R must be a scalar function of \dot{x}_μ and V_μ (if we want to get the non-relativistic limit). However, due to the fact that L_R must be homogeneous of the first degree in the derivatives with respect to τ , it results that

it cannot possibly be a linear function of V_μ . But the non-relativistic Lagrangian was invariant under translations in the η_α due to its linearity in V_μ ; so the variation of V_μ under translations must be compensated from an analogous variation of another four-vector, which can be \dot{x}_μ only. We see that we are forced to require that under translations in the η_α \dot{x}_μ transforms as well.

It follows that the Lagrangian L_R must be a scalar function of a linear combination of \dot{x}_μ and V_μ which we choose as

$$\dot{x}_\mu - \frac{1}{mc} V_\mu. \quad (6.6)$$

In writing this expression we have required that the only dimensional constant is the mass m , and we have absorbed a dimensionless factor in the scale used for x_μ .

The vector (6.6) is invariant under the translation

$$\eta_\alpha \rightarrow \eta_\alpha + \epsilon_\alpha, \quad (6.7)$$

if we require that at the same time x_μ transforms as

$$x_\mu \rightarrow x_\mu + \frac{i}{2mc} (\epsilon^* \sigma_\mu \eta - \eta^* \sigma_\mu \epsilon). \quad (6.8)$$

We see that (6.7) and (6.8) are the usual supersymmetry transformations.

Now we can satisfy all the required properties choosing L_R of the form

$$L_R = -mc \sqrt{\left(\dot{x}_\mu - \frac{1}{mc} V_\mu\right)^2}. \quad (6.9)$$

The non-relativistic limit is easily obtained by putting $\tau = ct$ and performing the limit $c \rightarrow \infty$; the result is

$$L_R \approx -mc^2 + \frac{i}{2} (\eta^* \dot{\eta} - \dot{\eta}^* \eta) + \frac{1}{2} m \vec{v}^2 + O(1/c). \quad (6.10)$$

which is the correct limit.

It is interesting to observe that in the present context the existence of supersymmetry transformations arises as a purely relativistic effect as it is explicitly shown by Eq. (6.8). In particular, this equation implies that the transformation is of the order $1/c$ in the spatial components of the position four-vector, and of the order $1/c^2$ in the time.

7. - STUDY OF THE RELATIVISTIC G_4 MODEL

In this section we study the relativistic model introduced in the previous section. The reasons for a detailed study are due to the relation of this model with supersymmetries and to the fact that it describes particles with spin.

Introducing the variables

$$\theta^\alpha = \frac{1}{\sqrt{2m}} \eta^\alpha, \quad (7.1)$$

and using natural units $c = \hbar = 1$, the Lagrangian (6.9) can be rewritten as

$$L_R = -m \left[(\dot{x}_\mu - i\theta^* \sigma_\mu \dot{\theta} + i\dot{\theta}^* \sigma_\mu \theta)^2 \right]^{1/2}. \quad (7.2)$$

The momentum conjugate to x_μ is

$$P_\mu = - \frac{\partial L_R}{\partial \dot{x}^\mu} = - \frac{m^2}{L_R} \left[\dot{x}_\mu - i\theta^* \sigma_\mu \dot{\theta} + i\dot{\theta}^* \sigma_\mu \theta \right]. \quad (7.3)$$

Using this expression we can get easily the other quantities of interest

$$\Pi_\alpha = \frac{\partial L_R}{\partial \dot{\theta}^\alpha} = -i\theta^\beta (P^\mu \sigma_\mu)_{\beta\alpha}, \quad (7.4)$$

$$\Pi_{\dot{\alpha}} = \frac{\partial L_R}{\partial \dot{\theta}^{\dot{\alpha}}} = -i(P^\mu \sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta, \quad (7.5)$$

$$\frac{\partial L_R}{\partial \theta^\alpha} = i\dot{\theta}^\beta (P^\mu \sigma_\mu)_{\beta\alpha}, \quad (7.6)$$

$$\frac{\partial L_R}{\partial \theta^{\dot{\alpha}}} = i (P^\mu \sigma_\mu)_{\dot{\alpha}\beta} \dot{\theta}^\beta. \quad (7.7)$$

In these expressions we have used the convention

$$\theta^{\dot{\alpha}} = (\theta^\alpha)^*, \quad (7.8)$$

and we notice the following property

$$(\pi_\alpha)^* = -\pi_{\dot{\alpha}}. \quad (7.9)$$

The equations of motion are

$$\dot{P}_\mu = 0, \quad (7.10)$$

and

$$\dot{\pi}_\alpha = i \dot{\theta}^\beta (P^\mu \sigma_\mu)_{\beta\alpha}. \quad (7.11)$$

From (7.11), using (7.4) it follows ²¹⁾

$$\dot{\theta}^\alpha = 0, \quad (7.12)$$

as we expected from the invariance of L_R under θ^α translations.

We notice that (7.4) and (7.5) are constraint equations, in fact they express π_α ($\pi_{\dot{\alpha}}$) as a function of P_μ and $\theta^{\dot{\alpha}}$ (θ^α). These constraints are the relativistic generalizations of the constraints (4.2) as it can be easily verified performing the non-relativistic limit.

Introducing the canonical Poisson brackets (see Section 3)

$$\{x_\mu, P_\nu\} = -g_{\mu\nu}, \quad (7.13)$$

$$\{\theta^\alpha, \pi_\beta\} = -\delta_\beta^\alpha, \quad (7.14)$$

$$\{\theta^{\dot{\alpha}}, \pi_{\dot{\beta}}\} = -\delta_{\dot{\beta}}^{\dot{\alpha}}, \quad (7.15)$$

We can easily verify that, as expected from the non-relativistic limit, the constraints (7.4) and (7.5) are second class ones ; in fact, by putting

$$\begin{aligned} D_\alpha &= \Pi_\alpha + i \theta^\beta (P \cdot \sigma)_{\beta\alpha}, \\ D_{\dot{\alpha}} &= \Pi_{\dot{\alpha}} + i (P \cdot \sigma)_{\dot{\alpha}\beta} \theta^\beta, \end{aligned} \quad (7.16)$$

we get :

$$\begin{aligned} \{D_\alpha, D_\beta\} &= \{D_{\dot{\alpha}}, D_{\dot{\beta}}\} = 0, \\ \{D_\alpha, D_{\dot{\beta}}\} &= -2i (P \cdot \sigma)_{\beta\alpha}. \end{aligned} \quad (7.17)$$

From Eq. (3.14) for the generator of an ICT

$$F = -\delta X_\mu P^\mu + \delta \theta^\alpha \Pi_\alpha + \delta \theta^{\dot{\alpha}} \Pi_{\dot{\alpha}} - \phi, \quad (7.18)$$

we get the expression for the generator of the supersymmetry transformation

$$\begin{aligned} F &= -i \epsilon^{\dot{\alpha}} (P \cdot \sigma)_{\dot{\alpha}\beta} \theta^\beta + i \theta^{\dot{\alpha}} (P \cdot \sigma)_{\dot{\alpha}\beta} \epsilon^\beta + \\ &+ \epsilon^\alpha \Pi_\alpha + \epsilon^{\dot{\alpha}} \Pi_{\dot{\alpha}} = \\ &= \epsilon^\alpha \left[\Pi_\alpha - i \theta^\beta (P \cdot \sigma)_{\beta\alpha} \right] + \\ &+ \epsilon^{\dot{\alpha}} \left[\Pi_{\dot{\alpha}} - i (P \cdot \sigma)_{\dot{\alpha}\beta} \theta^\beta \right], \end{aligned} \quad (7.19)$$

where we have used [see Eq. (6.8)]:

$$\delta X_\mu = i \epsilon^{\dot{\alpha}} (\sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta - i \theta^{\dot{\alpha}} (\sigma_\mu)_{\dot{\alpha}\beta} \epsilon^\beta. \quad (7.20)$$

It is convenient to define the following odd generators

$$\begin{aligned} G_\alpha &= \Pi_\alpha - i \theta^\beta (P \cdot \sigma)_{\beta\alpha}, \\ G_{\dot{\alpha}} &= \Pi_{\dot{\alpha}} - i (P \cdot \sigma)_{\dot{\alpha}\beta} \theta^\beta, \end{aligned} \quad (7.21)$$

with Poisson brackets

$$\{G_\alpha, G_\beta\} = \{G_{\dot{\alpha}}, G_{\dot{\beta}}\} = 0, \quad (7.22)$$

$$\{G_\alpha, G_{\dot{\beta}}\} = 2i(P \cdot \sigma)_{\dot{\beta}\alpha}. \quad (7.23)$$

Moreover, we have :

$$\{G_\alpha, D_\beta\} = \{G_\alpha, D_{\dot{\beta}}\} = 0, \quad (7.24)$$

$$\{G_{\dot{\alpha}}, D_\beta\} = \{G_{\dot{\alpha}}, D_{\dot{\beta}}\} = 0; \quad (7.25)$$

these equations mean that the D_α are invariant under supersymmetry transformations ; furthermore, looking at their expressions we can recognize the so-called supergauge covariant derivatives ²⁾.

From the homogeneity of L_R in the derivatives with respect to τ , the constraint

$$P^2 = m^2, \quad (7.26)$$

follows, as expected. This constraint is a first class one.

Before taking into account the presence of second class constraints, it is worth looking at the Lorentz group generators. The properties of transformation that we require are :

$$\delta \theta^\alpha = \frac{1}{4} \omega^{\mu\nu} (\sigma_{\mu\nu} \theta)^\alpha, \quad (7.27)$$

$$\delta \theta^{\dot{\alpha}} = \frac{1}{4} \omega^{\mu\nu} (\tilde{\sigma}_{\mu\nu} \theta)^{\dot{\alpha}}, \quad (7.28)$$

$$\delta x^\mu = \omega^{\mu\nu} x_\nu, \quad (7.29)$$

where

$$(\sigma_{\mu\nu})^{\alpha}_{\beta} = \frac{1}{2} \left[(\tilde{\sigma}_{\mu})^{\alpha\dot{\gamma}} (\sigma_{\nu})_{\dot{\gamma}\beta} - (\tilde{\sigma}_{\nu})^{\alpha\dot{\gamma}} (\sigma_{\mu})_{\dot{\gamma}\beta} \right], \quad (7.30)$$

$$\begin{aligned} (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} &= \frac{1}{2} \left[(\sigma_{\nu})_{\dot{\beta}\gamma} (\tilde{\sigma}_{\mu})^{\gamma\dot{\alpha}} - (\sigma_{\mu})_{\dot{\beta}\gamma} (\tilde{\sigma}_{\nu})^{\gamma\dot{\alpha}} \right] = \\ &= \left[(\sigma_{\mu\nu})^{\alpha}_{\beta} \right]^*. \end{aligned} \quad (7.31)$$

Using again the expression (7.18) for the generator of an ICT

we find

$$\begin{aligned} F &= \frac{1}{2} \omega^{\mu\nu} \left[-P_{\mu} x_{\nu} + P_{\nu} x_{\mu} - \right. \\ &\left. - \frac{1}{2} \left(\Pi_{\alpha} (\sigma_{\mu\nu})^{\alpha}_{\beta} \Theta^{\beta} - \Theta^{\dot{\alpha}} (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \Pi_{\dot{\beta}} \right) \right]. \end{aligned} \quad (7.32)$$

So we can define the quantity

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad (7.33)$$

where

$$L_{\mu\nu} = -P_{\mu} x_{\nu} + P_{\nu} x_{\mu}, \quad (7.34)$$

and

$$S_{\mu\nu} = -\frac{1}{2} \left[\Pi \sigma_{\mu\nu} \Theta + \Theta^* \tilde{\sigma}_{\mu\nu} \Pi^* \right] \quad (7.35)$$

are the orbital and the intrinsic angular momentum, respectively.

The Poisson brackets for $L_{\mu\nu}$ and $S_{\mu\nu}$ are :

$$\begin{aligned} \{L_{k\lambda}, L_{\mu\nu}\} &= g_{\lambda\mu} L_{k\nu} - g_{k\mu} L_{\lambda\nu} + \\ &+ g_{k\nu} L_{\lambda\mu} - g_{\lambda\nu} L_{k\mu}, \end{aligned} \quad (7.36)$$

$$\begin{aligned} \{S_{k\lambda}, S_{\mu\nu}\} &= g_{\lambda\mu} S_{k\nu} - g_{k\mu} S_{\lambda\nu} + \\ &+ g_{k\nu} S_{\lambda\mu} - g_{\lambda\nu} S_{k\mu}, \end{aligned} \quad (7.37)$$

$$\{S_{k\lambda}, L_{\mu\nu}\} = 0, \quad (7.38)$$

from which we get the correct Poisson brackets for the total angular momentum $M_{\mu\nu}$.

We notice that with respect to the Poisson brackets, θ^α and $\dot{\theta}^\alpha$ are spinors under the generators $S_{\mu\nu}$, because we have

$$\{\theta^\alpha, L_{\mu\nu}\} = \{\dot{\theta}^\alpha, L_{\mu\nu}\} = 0. \quad (7.39)$$

The behaviour of D_α and G_α is different, and they are spinors only under the full generator $M_{\mu\nu}$; in fact we have :

$$\begin{aligned} \{D_\alpha, L_{\mu\nu}\} &= -i \theta^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}, \\ \{D_{\dot{\alpha}}, L_{\mu\nu}\} &= -i (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta, \\ \{G_\alpha, L_{\mu\nu}\} &= i \theta^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}, \\ \{G_{\dot{\alpha}}, L_{\mu\nu}\} &= i (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta, \end{aligned} \quad (7.40)$$

and

$$\begin{aligned} \{D_\alpha, S_{\mu\nu}\} &= -\frac{1}{2} D_\beta (\sigma_{\mu\nu})^\beta{}_\alpha + i \theta^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}, \\ \{D_{\dot{\alpha}}, S_{\mu\nu}\} &= -\frac{1}{2} (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}\beta} D^\beta + i (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta, \\ \{G_\alpha, S_{\mu\nu}\} &= -\frac{1}{2} G_\beta (\sigma_{\mu\nu})^\beta{}_\alpha - i \theta^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}, \\ \{G_{\dot{\alpha}}, S_{\mu\nu}\} &= -\frac{1}{2} (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}\beta} G^\beta - i (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\dot{\alpha}\beta} \theta^\beta, \end{aligned} \quad (7.41)$$

from which the correct rules of transformation under $M_{\mu\nu}$ follow.

After having explored the properties of the Poisson brackets, we have to take into account the constraints that are present in the theory.

The first class constraint (7.26) can be treated in the usual way ¹⁰⁾, in the sense that the Hamiltonian will be the canonical one (which is zero due to homogeneity) plus a linear combination of the first class constraints

$$H = \lambda (P^2 - m^2). \quad (7.42)$$

In order to deal with the second class constraints, we need the introduction of the Dirac brackets according to the general theory discussed in Section 4.

In the present case the matrix C_{ij}^{hk} consists only of the part $C_{\alpha\beta}^{11}$. More precisely, we have

$$\begin{aligned} C^{11} &= \begin{bmatrix} 0 & \{D_\alpha, D_{\dot{\beta}}\} \\ \{D_{\dot{\alpha}}, D_\beta\} & 0 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & -2i (P \cdot \sigma)_{\dot{\beta}\alpha} \\ -2i (P \cdot \sigma)_{\dot{\alpha}\beta} & 0 \end{bmatrix}, \end{aligned} \quad (7.43)$$

from which we get easily the expression for the inverse matrix

$$[(C^{11})^{-1}]^{\alpha\dot{\beta}} = \frac{i}{2P^2} (P \cdot \tilde{\sigma})^{\alpha\dot{\beta}}, \quad (7.44)$$

$$[(C^{11})^{-1}]^{\dot{\alpha}\beta} = \frac{i}{2P^2} (P \cdot \tilde{\sigma})^{\beta\dot{\alpha}}. \quad (7.45)$$

It follows that the Dirac bracket for two general dynamical variables is

$$\begin{aligned} \{\xi, \eta\}^* &= \{\xi, \eta\} - \frac{i}{2} \{\xi, D_\alpha\} \frac{(P \cdot \tilde{\sigma})^{\alpha\dot{\beta}}}{P^2} \{D_{\dot{\beta}}, \eta\} - \\ &- \frac{i}{2} \{\xi, D_{\dot{\alpha}}\} \frac{(P \cdot \tilde{\sigma})^{\dot{\alpha}\beta}}{P^2} \{D_\beta, \eta\}. \end{aligned} \quad (7.46)$$

We can immediately see a very important difference between the relativistic and the non-relativistic case. In fact, in the non-relativistic case the constraints D_α were functions only of the Grassman type variables. So, the presence of the constraints did not change the structure of the Poisson brackets among the usual Lagrangian variables. However, in the relativistic case, the constraints D_α involve the momentum P_μ conjugate to x_μ . That means that the presence of the constraints modifies the algebra of the space-time variables.

This a very important point but it is not unexpected. We know indeed that, for relativistic systems with spin, the algebra of the space-time variables is not a trivial extension of the non-relativistic algebra ; but there are deep modifications like the lack of commutativity for the components of the covariant position four vector ^{11),12)}. So let us see in detail what the Dirac brackets are, for the various interesting quantities. We get

$$\{\theta^\alpha, \theta^\beta\}^* = -\frac{i}{2} \frac{(P \cdot \tilde{\sigma})^{\alpha\beta}}{P^2}, \quad (7.47)$$

and

$$\{\theta^\alpha, \pi_\beta\}^* = -\frac{1}{2} \delta_\beta^\alpha, \quad (7.48)$$

and analogous relations for the complex conjugate quantities.

Due to the fact that the constraints are invariant under supersymmetry transformations, it follows that the Dirac bracket of a supersymmetry generator with any other dynamical variable is the same as the Poisson bracket,

$$\{G_\alpha, \eta\}^* = \{G_\alpha, \eta\}. \quad (7.49)$$

This equation is very important because it means that the constraints do not destroy the supersymmetry properties of the model.

Now let us go to the space-time variables. Due to the fact that under Dirac brackets the constraints are strongly zero, it is convenient to re-express the spin tensor using the constraint equations :

$$S_{\mu\nu} = \frac{i}{2} \left[\theta^{\dot{\alpha}} (P \cdot \sigma)_{\dot{\alpha}\beta} (\tilde{\sigma}_{\mu\nu})^{\beta\gamma} \theta^{\gamma} - \theta^{\dot{\alpha}} (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}\beta} (P \cdot \sigma)^{\beta\gamma} \theta^{\gamma} \right]. \quad (7.50)$$

This expression can be simplified using the following identity :

$$\sigma_{\mu} \tilde{\sigma}_{\nu} \sigma_{\rho} = \sigma_{\mu} g_{\rho\nu} - \sigma_{\nu} g_{\mu\rho} + \sigma_{\rho} g_{\mu\nu} + i \epsilon_{\mu\rho\nu\lambda} \sigma^{\lambda}. \quad (7.51)$$

The result is :

$$S_{\mu\nu} = - \epsilon_{\mu\nu\rho\lambda} P^{\rho} \Sigma^{\lambda}, \quad (7.52)$$

where

$$\Sigma^{\mu} = \theta^{\dot{\alpha}} (\sigma^{\mu})_{\dot{\alpha}\beta} \theta^{\beta}. \quad (7.53)$$

We notice also that $S_{\mu\nu}$ is a transverse tensor

$$P^{\mu} S_{\mu\nu} = 0. \quad (7.54)$$

The Dirac brackets between x_{μ} and x_{ν} become :

$$\{x_{\mu}, x_{\nu}\}^* = \frac{S_{\mu\nu}}{p^2}, \quad (7.55)$$

according to the general analysis of the position four-vector ¹¹⁾. However, x_{μ} is still the variable conjugate to P_{μ} , because the constraints are translationally invariant. Hence,

$$\{P_{\mu}, \eta\}^* = \{P_{\mu}, \eta\}, \quad (7.56)$$

where η is any dynamical variable.

Another interesting Dirac bracket is that between Σ_μ and Σ_ν :

$$\{\Sigma_\mu, \Sigma_\nu\}^* = \frac{S_{\mu\nu}}{p^2}. \quad (7.57)$$

This expression suggests to introduce the complex quantities ¹²⁾

$$z_\mu = x_\mu + i \Sigma_\mu, \quad (7.58)$$

$$z_\mu^* = x_\mu - i \Sigma_\mu, \quad (7.59)$$

for which, using the equations

$$\{\Sigma_\mu, x_\nu\}^* = -\frac{1}{p^2} [P_\mu \Sigma_\nu + P_\nu \Sigma_\mu - g_{\mu\nu} (P \cdot \Sigma)], \quad (7.60)$$

we get the Dirac brackets :

$$\{z_\mu, z_\nu\}^* = 0, \quad (7.61)$$

$$\{z_\mu^*, z_\nu^*\}^* = 0. \quad (7.62)$$

z_μ (z_μ^*) is an interesting quantity because it is invariant under supersymmetry transformations generated by G_α (G_α) ; in fact, we have :

$$\{G_\alpha, z_\mu\}^* = 0, \quad (7.63)$$

$$\{G_\alpha, z_\mu\}^* = -2i (\sigma_\mu)_{\alpha\beta} \theta^\beta. \quad (7.64)$$

Another interesting point is that the separation of $M_{\mu\nu}$ in spin and orbital part loses its meaning. In fact, the Dirac brackets of the spin part with the space-time variables no longer vanish. We have :

$$\{x_\lambda, S_{\mu\nu}\}^* = \frac{1}{p^2} (P_\mu S_{\nu\lambda} - P_\nu S_{\mu\lambda}), \quad (7.65)$$

$$\{P_\lambda, S_{\mu\nu}\}^* = 0. \quad (7.66)$$

Also the Grassmann-type variables have now a different behaviour under $S_{\mu\nu}$,

$$\begin{aligned} \{\theta^\alpha, S_{\mu\nu}\}^* &= \frac{1}{2} (\sigma_{\mu\nu})_{\alpha\beta} \theta^\beta - \\ &- \frac{1}{2} \frac{(P \cdot \tilde{\sigma})_{\alpha\beta}}{p^2} (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\delta} \theta^\delta \end{aligned} \quad (7.67)$$

$$\begin{aligned} \{G_\alpha, S_{\mu\nu}\}^* &= -\frac{1}{2} G_\beta (\sigma_{\mu\nu})^{\beta\alpha} - \\ &\quad - i \dot{\Theta}^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}. \end{aligned} \quad (7.68)$$

However, all the quantities transform correctly under the Poincaré group ; because we get compensations from the "orbital part" of $M_{\mu\nu}$,

$$\begin{aligned} \{X_\lambda, L_{\mu\nu}\}^* &= (g_{\lambda\mu} X_\nu - g_{\lambda\nu} X_\mu) - \\ &\quad - \frac{1}{p^2} (P_\mu S_{\lambda\nu} - P_\nu S_{\lambda\mu}) \end{aligned} \quad (7.69)$$

$$\{\Theta^\alpha, L_{\mu\nu}\}^* = \frac{1}{2} \frac{(P \cdot \tilde{\sigma})^{\alpha\beta}}{p^2} (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\gamma} \Theta^\gamma, \quad (7.70)$$

$$\{G_\alpha, L_{\mu\nu}\}^* = i \dot{\Theta}^\beta (P_\mu \sigma_\nu - P_\nu \sigma_\mu)_{\beta\alpha}. \quad (7.71)$$

We can check the validity of the Lorentz algebra using the following Dirac brackets :

$$\{S_{k\lambda}, S_{\mu\nu}\}^* = (g_{\lambda\mu} - \frac{P_\lambda P_\mu}{p^2}) S_{k\nu} + \dots, \quad (7.72)$$

$$\{L_{k\lambda}, L_{\mu\nu}\}^* = g_{\lambda\mu} L_{k\nu} + \frac{P_\lambda P_\mu}{p^2} S_{k\nu} + \dots, \quad (7.73)$$

$$\{L_{k\lambda}, S_{\mu\nu}\}^* = \frac{P_\lambda P_\mu}{p^2} S_{k\nu} + \dots. \quad (7.74)$$

Equation (7.72) is easily understood, recalling from Eq. (7.54) that $S_{\mu\nu}$ is a transverse tensor.

We can also check that $z_\mu (z_\mu^*)$ is a four-vector, since

$$\{\Sigma_\lambda, M_{\mu\nu}\}^* = (g_{\lambda\mu} \Sigma_\nu - g_{\lambda\nu} \Sigma_\mu). \quad (7.75)$$

It follows from all these Dirac brackets that the theory is a covariant one.

Finally, let us make some remarks about the quantization problem. One possibility is to quantize the Dirac brackets (see Section 4), and there is no formal difficulty in doing so, as it can be easily checked. However, due to the fact that $[x_\mu, x_\nu] \neq 0$, we have no space-time description of our theory (at most we have a momentum space description). From this point of view, it seems more convenient to proceed by imposing the second class constraints as constraints on the states ; for instance, one may require

$$D_\alpha |\Psi\rangle = 0, \quad (7.76)$$

$$\langle \Psi | D_\alpha = 0, \quad (7.77)$$

and quantize with Poisson brackets.

From this point of view the states $|\psi\rangle$ are like chiral superfields, due to the fact that in a representation in which x_μ and θ^α are diagonal, D_α is exactly the chiral projection of the supergauge covariant derivative ²⁾. Furthermore, the physical states $|\psi\rangle$ must satisfy the first class constraint

$$(p^2 - m^2) |\psi\rangle = 0; \quad (7.78)$$

from which it follows that the components of $|\psi\rangle$ are degenerate in mass.

We see from these considerations that this model provides a first quantized "substratum" of the superfield theories, as the theory of the relativistic material point ²²⁾ does for the Klein-Gordon field.

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APPENDIX

In this Appendix we review briefly the properties of a Clifford algebra ²⁰⁾. This algebra is defined by the equations

$$\xi_i \xi_j + \xi_j \xi_i = \delta_{ij}, \quad i, j = 1, \dots, N, \quad (\text{A.1})$$

and it is denoted by C_N .

Any product of ξ_i can be brought into the form

$$(\xi_1)^{\epsilon_1} (\xi_2)^{\epsilon_2} \dots (\xi_N)^{\epsilon_N},$$

where $\epsilon_i = 0, 1$. It follows that there are 2^N independent products, and consequently the dimension of the Clifford algebra C_N is 2^N .

The algebra C_N is closely connected with the Lie algebra of the group $O(N)$. Indeed the quantities

$$S_{ij} = -\frac{i}{2} [\xi_i, \xi_j], \quad (\text{A.2})$$

satisfy the algebra

$$[S_{ij}, S_{hk}] = -i (\delta_{ik} S_{jh} + \delta_{jh} S_{ik} - \delta_{ih} S_{jk} - \delta_{jk} S_{ih}), \quad (\text{A.3})$$

which is the Lie algebra of $O(N)$.

Furthermore, this algebra can be immediately extended to the Lie algebra of $O(N+1)$; in fact, by putting

$$S_{0i} = -S_{i0} = \frac{\xi_i}{i}, \quad (\text{A.4})$$

it follows :

$$[S_{0i}, S_{0j}] = i S_{ij}, \quad (\text{A.5})$$

and

$$[S_{ij}, S_{0k}] = -i (S_{jo} \delta_{ik} - S_{io} \delta_{jk}). \quad (\text{A.6})$$

This equation shows that the ξ_i transform as the components of a N vector under $O(N)$. Defining :

$$S_{AB} = -S_{BA} = \begin{cases} S_{0i}, & A=0, B=i, \\ S_{ij}, & A=i, B=j, \end{cases} \quad (\text{A.7})$$

we get the Lie algebra of $O(N+1)$:

$$[S_{AB}, S_{CD}] = -i (\delta_{AD} S_{BC} + \delta_{BC} S_{AD} - \delta_{AC} S_{BD} - \delta_{BD} S_{AC}) \quad (\text{A.8})$$

Now let us study the irreducible representations of the Clifford algebras. In the case $N = 2n$, we can use the Jordan-Wigner trick, and construct the following matrices

$$\xi_i = \frac{1}{\sqrt{2}} (\sigma_3)_1 \otimes (\sigma_3)_2 \otimes \dots \otimes (\sigma_3)_{i-1} \otimes (\sigma_1)_i \otimes (1)_{i+1} \otimes \dots \otimes (1)_n, \quad (\text{A.9})$$

$$\eta_i = -\frac{1}{\sqrt{2}} (\sigma_3)_1 \otimes (\sigma_3)_2 \otimes \dots \otimes (\sigma_3)_{i-1} \otimes (\sigma_2)_i \otimes (1)_{i+1} \otimes \dots \otimes (1)_n, \\ i = 1, 2, \dots, n.$$

It is immediately seen that the correspondence

$$\begin{aligned} \xi_{2i-1} &\rightarrow \xi_i, \\ \xi_{2i} &\rightarrow \eta_i, \end{aligned} \quad (\text{A.10})$$

gives a representation of C_{2n} . It is also possible to show that the representation (A.10) on the space of the $2^n \times 2^n$ matrices is a faithful one.

In the case $N = 2n+1$, we can start from the matrices (A.9) and construct the matrix

$$\begin{aligned} \tau &= \frac{(2i)^n}{\sqrt{2}} \xi_1 \eta_1 \xi_2 \eta_2 \dots \xi_n \eta_n = \\ &= \frac{1}{\sqrt{2}} (\sigma_3)_1 \otimes (\sigma_3)_2 \otimes \dots \otimes (\sigma_3)_n, \end{aligned} \quad (\text{A.11})$$

which has the properties :

$$\begin{aligned} \tau^2 &= 1/2, \\ \tau \xi_i + \xi_i \tau &= \tau \eta_i + \eta_i \tau = 0. \end{aligned} \quad (\text{A.12})$$

Therefore, the correspondence (A.10) and

$$\xi_{2n+1} \rightarrow \tau, \quad (\text{A.13})$$

give a representation of C_{2n+1} in terms of the $2^n \times 2^n$ matrices. Due to Eq. (A.11) this representation is not a faithful one, i.e., it is non-linear since τ , can be expressed through ξ_i and η_i .

However, there exists a second representation in the space of the $2^n \times 2^n$ matrices. It is given by the correspondence

$$\begin{aligned} \xi_{2i-1} &\rightarrow -\xi_i, \\ \xi_{2i} &\rightarrow -\eta_i, \\ \xi_{2n+1} &\rightarrow -\tau. \end{aligned} \quad (\text{A.14})$$

Since the image of

$$\frac{(2i)^n}{12} \xi_1 \dots \xi_{2n+1}$$

is $+\frac{1}{2}$ for the representations (A.10), (A.13), and $-\frac{1}{2}$ for (A.14), the two representations are inequivalent.

It is possible to show ²⁰⁾ that the Clifford algebra C_{2n+1} is isomorphic to the direct sum of two spaces of $2^n \times 2^n$ matrices. Consequently, a faithful representation of C_{2n+1} is obtained by the correspondence

$$\begin{aligned} \xi_{2i-1} &\rightarrow \begin{pmatrix} \xi_i & | & 0 \\ \hline 0 & | & -\xi_i \end{pmatrix}, \\ \xi_{2i} &\rightarrow \begin{pmatrix} \eta_i & | & 0 \\ \hline 0 & | & -\eta_i \end{pmatrix}, \\ \xi_{2n+1} &\rightarrow \begin{pmatrix} \tau & | & 0 \\ \hline 0 & | & -\tau \end{pmatrix} \end{aligned} \quad (\text{A.15})$$

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