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THE CLASSIFICATION OF COMPLETE LOCALLY CONFORMALLY FLAT MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

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The main purpose of this note is to give a classification of complete locally conformally flat manifolds of nonnegative Ricci curvature. Such classification for the compact case has been obtained by various authors in the past decade.

1. Introduction. Recall that an n-dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat if it admits a coordinate covering $\{U_\alpha, \phi_\alpha\}$ such that the map $\phi_\alpha\colon (U_\alpha, g_\alpha) \to (S^n, g_0)$ is a conformal map, where g_0 is the standard metric on S^n. It follows from this definition that the Weyl tensor of g vanishes. In particular, the full curvature tensor of g can be recovered from the Ricci tensor of g (an alternating sum). Thus conditions on the Ricci tensor of such manifolds impose very strong restrictions on their metrics. In the first part of this note we confirm this by showing,

THEOREM 1. *If (M^n, g) is a complete locally conformally flat Riemannian manifold with $\mathrm{Ric}(g) \geq 0$, then the universal cover $\widetilde{M}$ of M with the pulled-back metric is either conformally equivalent to S^n, R^n or is isometric to $R \times S^{n-1}$. If M itself is compact, then $\widetilde{M}$ is either conformally equivalent to S^n or isometric to R^n, $R \times S^{n-1}$, where S^n and S^{n-1} are spheres of constant curvature.*

The second part of Theorem 1 was obtained by various authors as consequences of investigating more general classes of manifolds, see the work of Schoen and Yau ([**SY**]) for references. An elementary proof for this case was also given recently by Noronha ([**No**]).

We remark that although the validity of Theorem 1 is not surprising, many similar problems in Riemannian geometry still remain open in the noncompact case, while the compact case has long been solved. The difficulty usually lies in the lack of analytic techniques for noncompact manifolds. The analysis in our case does carry through ([**SY**]) essentially because of the developing map as outlined below.

Our argument for the complete case uses heavily the results of Schoen and Yau. Let us outline the idea here. In [**SY**], Schoen and Yau proved that the developing map for locally conformally flat manifolds with nonnegative scalar curvature is injective, thus exhibiting them as quotients of domains in the sphere by Kleinian groups. Just as in the study of Kleinian groups in the works of Patterson ([**Pa**]) and Sullivan ([**Su**]), Schoen-Yau studied the Hausdorff dimension of the complement of the image of $\widetilde{M}$ under the developing map, and proved that it can be controlled by properties of Green's functions of the conformal Laplacian on $\widetilde{M}$. Our observation is that under the condition of $\mathrm{Ric} \geq 0$, Green's function gives a much stronger control on the Hausdorff dimension than in the case of nonnegative scalar curvature. In fact, we will show that the Hausdorff dimension is zero. Theorem 1 is a consequence of this fact and the splitting theorem of Cheeger-Gromoll ([**CG**]).

In the second part, we study locally conformally flat manifolds under the more general condition of $\mathrm{Ric} \geq -\Lambda^2$, and prove,

THEOREM 2. *If (M^n, g) is a compact locally conformally flat manifold with*

$$\mathrm{Ric} \geq -\Lambda^2, \quad \mathrm{diam}(M) \leq D,$$

then $b_i(M, R) \leq C(n, \Lambda D)$ for any i, where $C(n, \Lambda D)$ is a constant depending only on n and ΛD.

Theorem 2 is a consequence of a general result about elliptic inequalities based on Moser iteration and P. Li's lemma. This line of thought was initiated by P. Li and later developed by Gallot, Besson and Berard, among others (see [**Be**]). Theorem 2 is basically known without being explicitly stated; we find it illuminating to put it here since it gives a parallel to Gromov's famous estimate for Betti numbers for Riemannian manifolds with lower sectional curvature and diameter bounds. And together with a corollary of Theorem 1, it gives strong evidence to the validity of the following conjecture, which was the author's initial motivation for studying locally conformally flat manifolds.

Conjecture. There are only finitely many homotopy (homeomorphism, diffeomorphism) types of locally conformally flat manifolds satisfying

$$\operatorname{Ric}(M) \geq -\Lambda^2, \quad \operatorname{diam}(M) \leq D, \quad \operatorname{vol}(M) \geq V.$$

2. Nonnegative Ricci curvature. As pointed out in the introduction, we will use heavily the results from [**SY**]. Since [**SY**] is a long paper with many results, we will summarize here what is needed for our argument.

By the definition of locally conformally flat manifolds and a standard monodromy argument (as in the proof of analytic continuation), it is easy to construct a conformal map $\Phi: \widetilde{M} \to S^n$ which is unique up to conformal transformations of S^n. Φ is called the developing map. It is an easy consequence of the existence of the developing map that any compact simply connected locally conformally flat manifold is conformally equivalent to S^n (originally due to Kuiper ([**Ku1**], [**Ku2**])). In the general case, the significance of the developing map is at least twofold. Firstly, it gives in a natural way a compactification for $\widetilde{M}$ which makes the analysis easier when $\widetilde{M}$ is not compact. Secondly, when Φ is injective, it gives a uniformization for locally conformally flat manifolds, exhibiting them as quotients of domains in the sphere by Kleinian groups. The major result of [**SY**] is to find a class of manifolds for which the developing maps are injective. In order to state the results from [**SY**], we need to consider the conformal Laplacian L_g, which, when acting on a function ϕ, is defined as

$$L_g\phi = \Delta\phi - \frac{n-2}{4(n-1)}R(g)\phi,$$

where $R(g)$ is the scalar curvature of g and Δ is the usual (negative) Laplacian. L_g is conformally invariant in the sense that for any conformal metric $g_* = u^{4/(n-2)}g$, we have

$$L_{g_*}(\phi) = u^{-(n+2)/(n-2)}L_g(u\phi). \tag{1}$$

Letting $\phi = 1$, we get the Yamabe equation:

$$\Delta u - \frac{n-2}{4(n-1)}Ru = -\frac{n-2}{4(n-1)}u^{(n+2)/(n-2)}R(g_*). \tag{2}$$

By the help of the developing map, it is quite standard to show that the conformal Laplacian L_g of $\widetilde{M}$ has a minimal Green's function on $\widetilde{M}$, denoted by G_p, where p is the pole. We will now state the result we need from [**SY**] as the following lemma.

LEMMA 1 ([SY]). *Let (M^n, g) be a complete locally conformally flat Riemannian manifold with nonnegative scalar curvature and $\Phi: \widetilde{M} \to S^n$ the developing map. Then,*

(1) *Φ is injective (Theorem 4.5 in [SY]).*

(2) *$\partial\Phi(\widetilde{M}) \subset S^n$ is of codimension at least two (Propositions 3.3 and 4.4 in [SY]).*

(3) *For any $\varepsilon > 0$, any open set O containing p (Proposition 2.4(iii) in [SY]),*

$$\int_{\widetilde{M}\setminus O} G^{(n+\varepsilon)/(n-2)}\, dv_g < \infty.$$

As pointed out in the introduction, our strategy is to give a good estimate for $\dim(\partial\Phi(\widetilde{M}))$, where dim is the Hausdorff dimension. The idea in [SY] is to consider the quantity

$$d(M) = \inf\left\{ r \,\middle|\, \int_{M\setminus O} G^{2r/(n-2)}\, dv_g < \infty \right\},$$

and proved $\dim(\partial\Phi(\widetilde{M})) \le d(\widetilde{M})$. The starting point of our investigation is that this inequality is not sharp for the following trivial example, and in trying to give a sharp estimate for this example we obtained a proof of Theorem 1.

EXAMPLE. Consider (R^n, ω_0) and (S^n, g_0) where the metrics are the standard metrics. Let $\Psi: (R^n, \omega_0) \to (S^n, g_0)$ be the stereographic projection (which is the developing map for (R^n, ω_0)) defined as

$$\Psi(y_1, \dots, y_n) = \left(\frac{2y_1}{1+|y|^2}, \dots, \frac{2y_n}{1+|y|^2}, \frac{|y|^2-1}{|y|^2+1}\right),$$
$$\Psi^{-1}(x_1, \dots, x_n, \xi) = \left(\frac{x_1}{1-\xi}, \dots, \frac{x_n}{1-\xi}\right).$$

Then,

$$(\Psi^{-1})^*(\omega_0) = \frac{1}{(1-\xi)^2} g_0 = u^{4/(n-2)} g_0,$$
$$\Psi^*(g_0) = \frac{4}{(1+|y|^2)^2}\omega_0,$$

where $u = 1/(1-\xi)^{(n-2)/2}$. The Green's function for (R^n, ω) at 0 is

$$G_0(y) = \frac{1}{(n-2)\omega_{n-1}}|y|^{2-n} = c_n|y|^{2-n}.$$

The Green's function for (S^n, g_0) at S (the south pole) is

$$\begin{aligned} H_S(x_1, \dots, x_n, \xi) &= u(0, \dots, -1)^{-(n+2)/(n-2)}u \\ &\quad \cdot (\Psi^{-1})^*(G_0)(x_1, \dots, x_n, \xi) \\ &= 2^{(n+2)/2}c_n(1+\xi)^{(2-n)/2}. \end{aligned}$$

Similarly, for the north pole N,

$$H_N = 2^{(n+2)/2}c_n(1-\xi)^{(2-n)/2}.$$

From the formula for G_0, we see that $d(R^n, \omega_0) = \frac{n}{2}$. This shows that (3) of Lemma 1 is sharp. But obviously $\dim(\partial\Psi(R^n)) = 0$. Thus the inequality $\dim(\partial\Phi(\widetilde{M})) \leq d(\widetilde{M})$ is not sharp when $M = R^n$.

Since the functions in the above examples are explicit, it is not hard to give an analytic proof that $\dim(\partial\Psi(R^n)) = 0$. Because this proof illustrates the idea for the proof of Theorem 1, we will first give a proof in this case.

To this end, as in **[SY]**, we consider the concept of capacity, which is easier to handle analytically than the Hausdorff dimension.

Definition. For a subset $S \subset (M^n, g)$, we define

$$C_p(S) = \inf_{\phi}\left\{\int_M |\nabla_g\phi|^p\,dx \colon \phi \in C_0^\infty,\ \phi|_O = 1\right\},$$

where O is some open set containing S.

The relation between capacity and the Hausdorff dimension is that if $C_p(S) = 0$, then $\dim(S) \leq n - p$ (**[AM]**).

Example (continued). We now give an analytic proof that $\dim(\partial\Psi(R^n)) = 0$. In fact, choose a function $\phi_a\colon R^n \to R$ such that

$$\phi_a(y_1, \dots, y_n) = \begin{cases} 0, & |y| \leq a, \\ 1, & |y| \geq 2a, \end{cases}$$

and $|\nabla_{\omega_0}\phi_a| \leq 2/a$. Note that

$$|\nabla_{\Psi^*(g_0)}\phi_a| = |\nabla_{4\omega_0/(1+|y|^2)^2}\phi_a| = \frac{1+|y|^2}{2}|\nabla_{\omega_0}\phi_a|.$$

Thus,

$$\int_{S^n\setminus N} |\nabla_{g_0}(\Psi^{-1})^*(\phi_a)|^{n-\varepsilon}\, dv_{g_0} = \int_{R^n} |\nabla_{\Psi^*(g_0)}\phi_a|^{n-\varepsilon}\, dv_{\Psi^*(g_0)}$$
$$= \int_{R^n} \left(\frac{1+|y|^2}{2}|\nabla_{\omega_0}\phi_a|\right)^{n-\varepsilon} \left(\frac{2}{1+|y|^2}\right)^n dv_{\omega_0}$$
$$\le \frac{2^n}{a^{n-\varepsilon}} \int_{a\le|y|\le 2a} (1+|y|^2)^{-\varepsilon}\, dv_{\omega_0} \le \frac{2^n}{a^{n-\varepsilon}}(1+a^2)^{-\varepsilon} \int_{a\le|y|\le 2a} dv_{\omega_0}$$
$$\le \frac{2^n}{a^{n-\varepsilon}}(1+a^2)^{-\varepsilon}\cdot\omega_n(2a)^n = \frac{4^n\omega_n a^\varepsilon}{(1+a^2)^\varepsilon} \to 0 \quad \text{as } a\to\infty.$$

Thus $C_{n-\varepsilon}(\partial\Psi(R^n)) = 0$ for any $\varepsilon > 0$. Hence $\dim(\partial\Psi(R^n)) = 0$.

We are now ready to prove the following main lemma.

Lemma 2. *Let (M^n, g) be a locally conformally flat manifold with nonnegative Ricci curvature. Let $\Phi\colon (\widetilde{M}, g) \to (S^n, g_0)$ be the developing map. Then,*

$$\dim(\partial\Phi(\widetilde{M})) = 0.$$

Proof. By Lemma 1, Φ is injective, thus we can view $\widetilde{M}$ as a subset of S^n, and there is a function $u\colon \widetilde{M} \to R^+$ such that $\Phi^* g_0 = u^{-4/(n-2)} g$. Without loss of generality, we assume $\Phi(p) = N$. By equation (1), we calculate,

$$\begin{aligned} L_g(u^{-1}\cdot\Phi^*(H_N)) &= L_{(\Phi^{-1})^* g}((\Phi^{-1})^*(u^{-1})\cdot H_N) \\ &= L_{[(\Phi^{-1})^*(u)]^{4/(n-2)} g_0}((\Phi^{-1})^*(u^{-1})\cdot H_N) \\ &= [(\Phi^{-1})^*(u)]^{-(n+2)/(n-2)} L_{g_0}(H_N) \\ &= u(p)^{-(n+2)/(n-2)}\delta_N \end{aligned}$$

thus $L_g(u(p)^{(n+2)/(n-2)}\cdot u^{-1}\cdot\Phi^*(H_N)) = \delta_p$. Using (2) of Lemma 1 and the minimality of G_p, it is standard to conclude that $G_p = u(p)^{(n+2)/(n-2)}\cdot u^{-1}\cdot\Phi^*(H_N)$ (see [SY], p. 55). Therefore, the integrability condition in (3) in Lemma 1 is equivalent to

$$\int_{\widetilde{M}\setminus O} u^{-(n+\varepsilon)/(n-2)}\, dv_g < \infty. \tag{3}$$

(Note that H_N is bounded in $S^n\setminus O$.) Now for any $a > 0$, we choose, as in the example, a function ϕ_a on $\widetilde{M}$, such that

$$\phi_a(x) = \begin{cases} 0, & d_g(x, p) \le a, \\ 1, & d_g(x, p) \ge 2a, \end{cases}$$

and $|\nabla_g \phi_a| \le 2/a$. Then

$$\begin{aligned}
\int_{\Phi(\widetilde{M})} |\nabla_{g_0}(\Phi^{-1})^*(\phi_a)|^{n-\varepsilon}\, dv_{g_0} &= \int_{\widetilde{M}} |\nabla_{\Phi^*(g_0)}\phi_a|^{n-\varepsilon}\, dv_{\Phi^*(g_0)} \\
&= \int_{\widetilde{M}} |(\nabla_g \phi_a) u^{2/(n-2)}|^{n-\varepsilon} u^{-2n/(n-2)}\, dv_g \\
&\le \left(\frac{2}{a}\right)^{n-\varepsilon} \int_{a\le d(x,p)\le 2a} u^{-2\varepsilon/(n-2)}\, dv_g \\
&\le \left(\frac{2}{a}\right)^{n-\varepsilon} \left(\int_{a\le d(x,p)\le 2a} u^{-(n+\varepsilon)/(n-2)}\, dv_g\right)^{2\varepsilon/(n+\varepsilon)} \\
&\quad \cdot \left(\int_{a\le d(x,p)\le 2a} dv_g\right)^{(n-\varepsilon)/(n+\varepsilon)} \\
&\le \left(\frac{2}{a}\right)^{n-\varepsilon} \left(\int_{\widetilde{M}\setminus O} u^{-(n+\varepsilon)/(n-2)}\, dv_g\right)^{2\varepsilon/(n+\varepsilon)} \\
&\quad \cdot (\mathrm{vol}_{\widetilde{M}}(B_p(2a)))^{(n-\varepsilon)/(n+\varepsilon)} \\
&\le C \cdot \frac{1}{a^{n-\varepsilon}} [\omega_n (2a)^n]^{(n-\varepsilon)/(n+\varepsilon)} \\
&= C a^{-\varepsilon(n-\varepsilon)/(n+\varepsilon)} \to 0 \quad (\text{as } a \to +\infty),
\end{aligned}$$

where in the last inequality we have used (3) and the Bishop volume comparison theorem. Thus $C_{n-\varepsilon}(\partial\Phi(\widetilde{M})) = 0$ for any $\varepsilon > 0$. Hence $\dim(\partial\Phi(\widetilde{M})) = 0$. □

Proof of Theorem 1. Since any manifold with more than one end contains a line, it follows from the Cheeger-Gromoll splitting theorem that a manifold of nonnegative Ricci curvature has at most two ends. Consider the developing map $\Phi\colon \widetilde{M} \to S^n$. Each end of $\widetilde{M}$ gives a connected component of $\partial\Phi(\widetilde{M})$; therefore, $\partial\Phi(\widetilde{M})$ has at most two connected components. By Lemma 2, $\partial\Phi(\widetilde{M})$ consists of at most two points. We therefore have the following three cases.

(1) If $\partial\Phi(\widetilde{M})$ is empty, then $\widetilde{M}$ is conformally equivalent to S^n.

(2) If $\partial\Phi(\widetilde{M})$ has only one point, then $\widetilde{M}$ is conformally equivalent to R^n.

(3) If $\partial\Phi(\widetilde{M})$ has two points, by composing Φ with a conformal transformation of S^n, we can assume $\partial\Phi(\widetilde{M}) = \{S, N\}$. Writing the metric of S^n in polar coordinates, we have $g = u(t, x)(dt^2 + \sin^2 t\, d\sigma)$ where $d\sigma$ is the standard metric on S^{n-1}. On the other hand, by the splitting theorem, $\widetilde{M}$ is isometric to $R \times N$ with N closed and

simply connected, hence conformally equivalent to S^{n-1}. Therefore, the metric g can be written as $g = dr^2 + f^2(x)\,d\sigma$. It follows that the function u is independent of x. By a change of the parameter t, we conclude that $\widetilde{M}$ is isometric to $R \times S^{n-1}$ with S^{n-1} of constant curvature.

In the case when M is compact, we only need to show that in case (2) $\widetilde{M}$ is actually isometric to R^n. In fact, from (2), there is a positive function u on M with $\omega_0 = u^{4/(n-2)}g$. The Yamabe equation (2) implies

$$\Delta u - \frac{n-2}{4(n-1)} Ru = 0.$$

Thus u satisfies the maximal principle. Since M is compact, u is a constant. This shows $\widetilde{M}$ is isometric to R^n. □

COROLLARY. *If (M^n, g) is an open locally conformally flat manifold with*

$$\mathrm{Ric} \geq 0, \quad \mathrm{vol}(B_p(r)) \geq cr^n$$

for some point $p \in M$ and some constant $c > 0$, where $B_p(r)$ is the geodesic ball of radius r around p, then M^n is conformally equivalent to R^n.

Proof. It is well known that $\pi_1(M)$ is finite and $\widetilde{M}$ has only one end; thus $\widetilde{M}$ is conformally equivalent to R^n. This implies that $\pi_1(M)$ is torsion free, hence trivial. Therefore, M is conformally equivalent to R^n. □

REMARK. This corollary says that the local model in the sense of M. Anderson (**[An]**) for the class in the conjecture in §1 is conformally equivalent to R^n. This gives evidence that the conjecture is correct.

We end this section with a family of examples of conformally flat metrics on R^n with nonnegative Ricci curvature and various volume growth.

EXAMPLE. Let (R^n, ω_0) be the standard flat metric on R^n. Consider $g = (r^2+1)^{-2\alpha}\omega_0$, a globally conformally flat metric. It follows easily from a direct computation that

$$\mathrm{Ric}_{ii} = \frac{4\alpha(n-2)(1-\alpha)(r^2 - x_i^2)}{(r^2+1)^2} + \frac{4(n-1)\alpha}{(r^2+1)^2}.$$

Thus when $0 \leq \alpha \leq 1$, we have $\mathrm{Ric} \geq 0$. It's also easy to see,

(a) $\frac{1}{2} < \alpha \leq 1$: $\mathrm{Ric} \geq 0$, noncomplete;

(b) $\alpha = \frac{1}{2}$: Ric ≥ 0, complete, $\mathrm{vol}(B(r)) = c_n r$;
(c) $0 \leq \alpha < \frac{1}{2}$: Ric ≥ 0, complete, $\mathrm{vol}(B(r)) = c_n r^n$.

3. Estimating Betti numbers. As pointed out in §1, Theorem 2 is a consequence of a general result stated in **[Be]** and the following well-known Weizenböck formula. Since the proof is simple, for completeness, we will give a detailed proof of Theorem 2 here. The first part of the proof is the standard Moser iteration. The second part is what is known as Peter Li's lemma.

LEMMA 3 (**[G1]**). *Let (M^n, g) be a compact locally conformally flat Riemannian manifold and ϕ a harmonic p-form. Then*

$$\Delta|\phi|^2 = 2|\nabla\phi|^2 + \frac{2p(n-2p)}{n-2} R_{ij}\phi^{ii_2\cdots i_p}\phi^{j}_{i_2\cdots i_p} + \frac{2p(p-1)\cdot p!}{(n-1)(n-2)} R|\phi|^2,$$

where R_{ij} is the Ricci tensor of g.

Proof of Theorem 2. Let us assume that Ric $\geq -\Lambda^2$ and $\mathrm{diam}(M) = D$. Then $R \geq -n(n-1)\Lambda^2$. It follows from Lemma 3 that

$$\Delta|\phi|^2 \geq 2|\nabla\phi|^2 - c(n, p)\Lambda^2|\phi|^2,$$

where $c(n, p)$ is a constant depending only on n and p. In what follows constants will always be denoted in this way, while their values may change. From the definition of the Laplacian, we have

$$\Delta|\phi|^2 = 2|\nabla|\phi||^2 + 2|\phi|\Delta|\phi|.$$

Thus,

$$|\phi|\Delta|\phi| \geq |\nabla\phi|^2 - |\nabla|\phi||^2 - c(n, p)\Lambda^2|\phi|^2.$$

By the Schwarz inequality, it is easy to see $|\nabla\phi|^2 \geq |\nabla|\phi||^2$; therefore,

$$-\Delta|\phi| \leq c(n, p)\Lambda^2|\phi|.$$

Multiply both sides by $|\phi|^{2k-1}$ for $k > 1/2$, and integrate by parts,

$$\int \nabla(|\phi|^{2k-1}) \cdot \nabla|\phi| \leq c(n, p)\Lambda^2 \int |\phi|^{2k},$$

that is,

$$\|\nabla|\phi|^k\|_2 \leq \frac{k\Lambda c(n, p)}{\sqrt{2k-1}} \| \, |\phi|^k\|_2.$$

Recall the Sobolev inequality for a Riemannian manifold says (**[Be]**),

$$\|f\|_{2n/(n-2)} \leq \frac{1}{V^{1/n}}(c(n, D\Lambda) \cdot D\|\nabla f\|_2 + \|f\|_2)$$

for any $f \in W^{1,2}$, where $V = \mathrm{vol}(M)$. Using the Sobolev inequality, we continue the previous inequality,

$$\|\phi\|_{2nk/(n-2)}^2 \le \left(1 + c(n, p, D\Lambda)\frac{k}{\sqrt{2k-1}}\right)^{2/k} V^{-2/nk}\|\phi\|_{2k}^2.$$

Let $k = (\frac{n}{n-2})^i$, and multiply all inequalities with $i = 0, 1, \dots$, we deduce,

$$\|\phi\|_\infty^2 \le \prod_{i=0}^{\infty}\left(1 + c(n, p, D\Lambda)\frac{\tau^i}{\sqrt{2\tau^i - 1}}\right)^{2/\tau^i} V^{-1}\|\phi\|_2^2,$$

where we have denoted $\tau = \frac{n}{n-2}$. It is easy to see that the product in the above inequality converges.

Let H^p be the space of harmonic p-forms with the L^2 inner product. By the Hodge theory, $\dim(H^p) = b_p$. Let $\psi_1, \dots, \psi_{b_p}$ be an orthonormal basis for H^p. Consider the following function on M,

$$f(x) = \sum_{i=1}^{b_p} \frac{|\psi_i(x)|^2}{\int_M |\psi_i|^2}.$$

Note f is independent of the choice of orthonormal basis. Let $f(x_0) = \max f$. Define a map $H^p \xrightarrow{s} \Lambda^p(T_{x_0}^* M)$ by $s(\psi) = \psi(x_0)$. Then $H^p = \mathrm{Ker}\, s \oplus (\mathrm{Ker}\, s)^\perp$. Let $\{\phi_i\}$ be an orthonormal basis adapted to this decomposition, there are at most $\dim(\mathrm{Ker}\, s)^\perp$ of the ϕ_i's with $\phi_i(x_0) \neq 0$. Thus,

$$f(x_0) \le \dim(\mathrm{Ker}\, s)^\perp \cdot \max_i \frac{\sup|\phi_i|^2}{\int_M |\phi_i|^2} \le \binom{n}{p} \cdot \sup_\phi \frac{\|\phi\|_\infty^2}{\|\phi\|_2^2}.$$

Therefore,

$$\begin{aligned} b_p(M, R) &= \int_M f(x)\, dv_g \le f(x_0) \cdot V \le \binom{n}{p} \sup_\phi \frac{\|\phi\|_\infty^2}{\|\phi\|_2^2} \cdot V \\ &\le \binom{n}{p} \prod_{i=0}^{\infty}\left(1 + c(n, p, D\Lambda)\frac{\tau^i}{\sqrt{2\tau^i - 1}}\right)^{2/\tau^i} \\ &\le C(n, D\Lambda). \end{aligned}$$

□

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