The Classification of Finite Generalized Quadrangles Admitting a Group Acting Transitively on Ordered Pentagons

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Abstract

Let $\mathcal{S}$ be a thick generalized quadrangle and let G be a group of automorphisms of $\mathcal{S}$. If G acts transitively on the set of non-degenerate ordered pentagons, then $\mathcal{S}$ is one of the classical generalized quadrangles $W(q)$, $Q(4,q)$, $Q(5,q)$ or $H(3,q^2)$. The possibilities for G in each case are determined. We do not use the classification of the finite simple groups (from which this result also follows).

1 Introduction and Main Result

A finite generalized quadrangle (GQ) of order (s,t), $s,t \in \mathbf{N} \setminus \{0\}$, is an incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint non-empty sets of objects called points and lines respectively, and for which I is a symmetric point-line incidence relation satisfying axioms (GQ1), (GQ2) and (GQ3).

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(GQ1) Each point is incident with $1 + t$ lines and two distinct points are incident with at most one line.

(GQ2) Each line is incident with $1+s$ points and two distinct lines are incident with at most one point.

(GQ3) For every non-incident pair $(x, L) \in \mathcal{P} \times \mathcal{B}$, there exists a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $x \, I \, M \, I \, y \, I \, L$.

For terminology, notation, results, etc., concerning finite GQ, see the monograph [5], hereafter denoted by FGQ. We now introduce some further terminology. A finite GQ of order (s, t) is *thick* if $s, t \geq 2$ (the non-thick GQ are the grids (order $(s, 1)$) and the dual grids (order $(1, t)$)). A *pentagon* in a GQ is a subconfiguration consisting of five distinct points and five distinct lines such that each line (respectively point) is incident with exactly two points (respectively lines). An *ordered pentagon* is a pentagon in which the elements are ordered in such a way that two consecutive elements are incident. A *skeleton* is a subconfiguration $\Omega = (Q; L, p)$ where Q is a quadrilateral (i. e. a subquadrangle of order $(1, 1)$) and L (respectively p) is a line (respectively point) not in Q but incident with a point p_1 (respectively line L_1) of Q, where $p_1 \, I \, L_1$.

It follows easily from 9.8.3 of FGQ that the classical GQ $W(q)$, $Q(4, q)$, $Q(5, q)$ and $H(3, q^2)$ admit an automorphism group G acting transitively on the set of skeletons (the automorphism group of the GQ $H(4, q^2)$ is not transitive on ordered triples of concurrent lines). We will see in 2.1.1 that this is equivalent with G acting transitively on the set of ordered pentagons. The converse is also true. Suppose the GQ $\mathcal{S}$ admits a group G acting transitively on the set of ordered pentagons. Then G is a group with a (B, N)-pair of type B_2 and using the classification of the finite simple groups one can show that $\mathcal{S}$ must be classical (for an explicit proof, see [1]), but in the present case different from $H(4, q^2)$. The aim of this paper is to give a proof of this result without using the classification of the finite simple groups. We will also characterize the groups G. Therefore, we need the following notation.

Let $PGO_5(q)$ be the projective general orthogonal group in 4-dimensional projective space over the field $GF(q)$ and let q be odd. This group has a unique normal subgroup of index 2, namely the simple group $PSO_5(q)$, sometimes denoted by $O_5(q)$. Now let θ be a field automorphism of $GF(q)$ and denote by g_θ the element of $P\Gamma L_5(q)$ corresponding to the semilinear transformation with identity matrix and field automorphism θ. Let g be any element of $PGO_5(q) \setminus PSO_5(q)$. Then we denote by $PGO_5^\theta(q)$ the group generated by $PSO_5(q)$ and $g_\theta g$. Now consider the projective general unitary group $PGU_4(q)$ (still assuming q odd). This group has a unique normal subgroup of index 2 (if $q \equiv 1 \bmod 4$) or 4 (if $q \equiv 3 \bmod 4$), namely the simple group $PSU_4(q)$, also denoted by $U_4(q)$. The corresponding quotient group is cyclic and hence $PGU_4(q)$ has a unique subgroup H of index 2 containing $PSU_4(q)$. Let θ be a field automorphism of $GF(q^2)$ and let $g_\theta \in P\Gamma L_4(q^2)$ be defined similarly as above. Then we denote by $PGU_4^\theta(q)$ the group generated by H and $g_\theta g$, where $g \in PGU_4(q) \setminus H$.

In this paper we will show :

MAIN RESULT. *Let $\mathcal{S}$ be a finite thick generalized quadrangle and let G be a group of automorphisms of $\mathcal{S}$. Then G acts transitively on the set of ordered pentagons if and only if $\mathcal{S}$ is one of the classical generalized quadrangles $W(q)$, $Q(4,q)$, $Q(5,q)$ or $H(3,q^2)$, and G contains one of the following groups:*

(i) $PGO_5(q)$, *if* $\mathcal{S} \cong \mathcal{W}(\mathrm{II})$ *or* $\mathcal{S} \cong \mathcal{Q}(\triangle, \mathrm{II})$,

(ii) $PGO_5^\theta(q)$, *if* $\mathcal{S} \cong \mathcal{W}(\mathrm{II})$ *or* $\mathcal{S} \cong \mathcal{Q}(\triangle, \mathrm{II})$ *with q an odd square and θ a field automorphism of $GF(q)$ of order a power of 2,*

(iii) $PGU_4(q)$, *if* $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \mathrm{II})$ *or* $\mathcal{S} \cong \mathcal{H}(\ni, \mathrm{II}^{\in})$,

(iv) $PGU_4^\theta(q)$, *if* $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \mathrm{II})$ *or* $\mathcal{S} \cong \mathcal{H}(\ni, \mathrm{II}^{\in})$ *with q odd and θ a field automorphism of $GF(q^2)$ of order a power of 2.*

The groups will be further discussed in Section 2.4.

Part of the motivation of our proof is to stimulate people to aim for weaker hypotheses (such as the (B,N)-pair hypothesis above), still giving a proof without relying on the classification of the finite simple groups.

2 Proof of the Main Result

In this section, we denote by $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ a finite thick generalized quadrangle of order (s,t) and by G a group of automorphisms of $\mathcal{S}$ acting transitively on the set of ordered pentagons.

2.1 Some General Facts

2.1.1 Skeletons

Let $\Pi = (p_1, L_1, \ldots, p_5, L_5)$, with $p_1 \, I \, L_1 \, I \, p_2 \, I \ldots I \, p_5 \, I \, L_5 \, I \, p_1$, be an ordered pentagon in $\mathcal{S}$. Let p'_i be the point of L_{i+2} collinear with p_i (where indices are taken modulo 5). Consider the skeleton $\Omega = (Q; L_5, p'_4)$, where Q is the quadrilateral $(p_1, L_1, p_2, L_2, p_3, L_3, p'_1, p_1p'_1)$. Then Π completely determines Ω and vice versa. Hence G acts transitively on the set of skeletons.

2.1.2 Property (H)

Fix a point u of $\mathcal{S}$ and let (x, y, z) be a triad of points of $u^\perp$. Suppose $z \in cl(x, y)$. If $z \in \{x, y\}^{\perp\perp}$, then $y \in \{x, z\}^{\perp\perp}$. Now suppose $z \notin \{x, y\}^{\perp\perp}$ and let v respectively v' be a point in $\{x, y\}^\perp \setminus z^\perp$ respectively $\{x, z\}^\perp \setminus y^\perp$. This defines a unique skeleton $(Q; L_0, z')$ respectively (Q', L'_0, y'), where $uz \sim L_0 \, I \, v$ respectively $uy \sim L'_0 \, I \, v'$, where $z \sim z' \, I \, vy$ respectively $y \sim y' \, I \, v'z$, and where $Q = (v, vy, y, yu, u, ux, x, xv)$ respectively $Q' = (v', v'z, z, zu, u, ux, x, xv')$. By the transitivity of G on skeletons, we see that there is an automorphism mapping the ordered quadruple (u, x, y, z) onto the ordered quadruple (u, x, z, y). Indeed, z is collinear with z' and incident with the unique line uy containing u and concurrent with L_0; z' is mapped onto y', and the unique line containing u and concurrent with L'_0 is the line uy. As $y' \sim y \, I \, uy$, the point z is mapped onto y. Hence $y \in cl(x, z)$ and so u has property (H). By transitivity, every point of $\mathcal{S}$ has property (H), and dually, every line has property (H). By 5.6.2 of FGQ $\mathcal{S}$ must satisfy, up to duality, one of the following conditions:

- *(i)* $\mathcal{S}$ is isomorphic to $H(4, q^2)$, $q^2 = s$.
- *(ii)* Every point and every line in $\mathcal{S}$ are regular.
- *(iii)* Every hyperbolic line has exactly 2 points, and dually.
- *(iv)* Every hyperbolic line in $\mathcal{S}$ has exactly 2 points and every line of $\mathcal{S}$ is regular.

It is readily seen that $H(4, q^2)$ does not admit a group of automorphisms acting transitively on the set of skeletons. Also, case (*ii*) implies that $\mathcal{S}$ is isomorphic to $W(2^e)$ for some positive integer e (see 5.2.1 and 3.2.1 of FGQ). So from now on, we may assume that every hyperbolic line has exactly 2 points. Using the same kind of argument to show property (H), one produces a collineation mapping (u, x, y, z) onto (u, x', y', z'), where (x, y, z) and (x', y', z') are two arbitrary triads in $u^\perp$ (note that $z \notin \{x, y\}^{\perp\perp}$ as required for the argument). So all triads in $u^\perp$ have a constant number of centers. By 1.7.1(i) of FGQ, this constant equals $1 + t/s$, so s divides t.

Now we consider the cases (*iii*) and (*iv*) separately.

2.2 Case (*iii*)

In this case, also the dual of $\mathcal{S}$ has hyperbolic lines of length 2, hence also $1 + s/t$ is an integer. It follows that $s = t$. As each triad of points has either 0 or 2 centers, each point of $\mathcal{S}$ is antiregular (cf. 1.3.6(iii) of FGQ). Dually, each line of $\mathcal{S}$ is antiregular. Also $s = t$ is odd by 1.5.1(i) of FGQ.

Assume that $\mathcal{S}'$ is a thick subquadrangle of $\mathcal{S}$ of order (s', t'), and that $\mathcal{S}'$ does not admit a proper thick subquadrangle. Let Ω and Ω' be skeletons contained in $\mathcal{S}'$. Further, let θ be an automorphism of $\mathcal{S}$ mapping Ω to Ω'. If $\mathcal{S}'^\theta \neq \mathcal{S}'$, then $\mathcal{S}' \cap \mathcal{S}'^\theta$ is a proper thick subquadrangle of $\mathcal{S}'$, a contradiction. Consequently $\mathcal{S}'^\theta = \mathcal{S}'$, and so the automorphism group G' of $\mathcal{S}'$ acts transitively on the set of skeletons of $\mathcal{S}'$.

In $\mathcal{S}'$ any triad of points has at most two centers, that is, each point of $\mathcal{S}'$ is antiregular. Hence $s' \geq t'$ by 1.3.6(i) of FGQ. Similarly, $t' \geq s'$. Consequently, $s' = t'$ and then by 1.3.6(iii) of FGQ each triad of points respectively lines has 0 or 2 centers. Also, $s' = t'$ is odd. Let us now forget the GQ $\mathcal{S}$; so "$\perp$" means perpendicular in $\mathcal{S}'$, etc.

Since $\mathcal{S}'$ has no proper thick subquadrangles, the identity is the only automorphism of $\mathcal{S}'$ fixing a given skeleton of $\mathcal{S}'$. The group H' of automorphisms of $\mathcal{S}'$ fixing an ordered quadrilateral $Q := (p_1, L_1, p_2, \ldots, L_4)$ and a line L through p_1, $L_4 \neq L \neq L_1$, has even order $s' - 1$. Let σ be an involution of the group H'.

By 1.3.2 of FGQ an affine plane $\pi(p_4, p_1)$ of order s' may be constructed as follows. Points of $\pi(p_4, p_1)$ are the points of $p_4^\perp$ that are not on L_4. Lines are the pointsets $\{p_4, z\}^{\perp\perp} \setminus \{p_4\}$, with $p_4 \sim z \not\sim p_1$, and $\{p_4, u\}^{\perp} \setminus \{p_1\}$, with $p_1 \sim u \not\sim p_4$. Let us denote the pointset of a line M of $\mathcal{S}'$ by M^*. The involution σ induces either the identity or an involution σ' in the plane $\pi(p_4, p_1)$. In the latter case, σ' fixes the point p_3, the line L_3^*, the parallel class of lines defined by L_3^*, the line $\{p_4, p_2\}^{\perp} \setminus \{p_1\} = U$ which contains p_3, the parallel class of lines defined by U (as $L_1^{\sigma} = L_1$), and the parallel class containing the line $\{p_4, v\}^{\perp} \setminus \{p_1\} = V$ with $p_1 \neq v\, I\, L$ (as $L^{\sigma} = L$)

If σ' is the identity, then, by 2.4 of FGQ, σ is the identity, a contradiction; if σ' is a Baer involution, then, again by 2.4 of FGQ, the fixed elements of σ form a subquadrangle of order $(\sqrt{s'}, \sqrt{s'})$ of $\mathcal{S}'$, a contradiction. Hence σ' is a homothety of $\pi(p_4, p_1)$ with center p_3. Consequently σ fixes every line through p_1 and every point of $\{p_1, p_3\}^{\perp}$. As $\mathcal{S}'$ does not contain a proper thick subquadrangle, the fixed elements of σ are p_1, p_3, the points of $\{p_1, p_3\}^{\perp}$, the lines through p_1, and the lines through p_3.

Now let x and y be distinct points of $\mathcal{S}'$ on the line N of $\mathcal{S}'$, with p_1 not incident with N. By the transitivity properties of G' there is an involution γ of $\mathcal{S}'$ fixing p_1 linewise and fixing N. Let $r^{\gamma} \neq r$, with r a point of $\mathcal{S}'$ on N. Further, let δ be an element of G' fixing p_1, N, and for which $r^{\delta} = x$ and $r^{\gamma\delta} = y$. Then $\delta^{-1}\gamma\delta$ fixes p_1 linewise and maps x onto y. Now it is clear that the subgroup of G' fixing p_1 linewise acts transitively on $\mathcal{P}' \setminus \sqrt{}^{\perp}_{\infty}$, with $\mathcal{P}'$ the pointset of $\mathcal{S}'$. Then by 8.2.4 of FGQ the GQ $\mathcal{S}'$ is an EGQ (elation generalized quadrangle) with base point p_1 and elation group $\overline{G}$.

Let u_1 and u_2 be distinct points of the plane $\pi(p_4, p_1)$. If ζ is the elation in $\overline{G}$ mapping u_1 onto u_2, then ζ induces a translation ζ' in $\pi(p_4, p_1)$ mapping u_1 onto u_2. Consequently $\pi(p_4, p_1)$ is a translation plane. Interchanging the roles of p_1 and p_4, we see that $\pi(p_1, p_4)$ is also a translation plane. It follows that, if ∞ is the point at infinity of $\pi(p_4, p_1)$ defined by the parallel class of lines containing L_3^*, then the projective completion $\overline{\pi(p_4, p_1)}$ of $\pi(p_4, p_1)$ is a dual translation plane with translation point ∞.

Next, let x_1 and x_2 be points of $\pi(p_4, p_1)$ not on L_3^* and not on $\{p_2, p_4\}^{\perp} \setminus \{p_1\} = U$. By the transitivity of G' on the skeletons of $\mathcal{S}'$, there is an automorphism η in G' fixing p_1, p_2, p_3, p_4 mapping $x_1 p_4$ onto $x_2 p_4$ and mapping x_1 onto x_2. Then η induces an automorphism η' of $\pi(p_4, p_1)$ fixing L_3^*, U and mapping x_1 onto x_2. Hence $\pi(p_4, p_1)$ is Desarguesian [4]. Then by 5.2.7 of FGQ, the GQ $\mathcal{S}'$ is isomorphic to $Q(4, s')$. Consequently every line of $\mathcal{S}'$ is regular, a contradiction.

We conclude that case (iii) cannot occur.

2.3 Case (*iv*)

2.3.1 Some general observations

We first fix the notation.

Throughout, Ω will denote the skeleton $((p_1, L_1, p_2, L_2, p_3, L_3, p_4, L_4); L, p)$, where $Q = (p_1, \ldots, L_4)$, $L\,I\,p_1$ and $p\,I\,L_1$. The group G_Ω of automorphisms in G fixing Ω has order $k \in \mathbf{N} \setminus \{0\}$.

The line L_1 is regular in $\mathcal{S}$. For any two lines M and M' meeting L_1 in different points, we call the collection $\{M, M'\}^{\perp\perp}$ of $s+1$ lines the *regulus* through M and M'; it is denoted by MM'. Every line of the regulus MM' meets L_1. The set of all lines through a point p will be denoted by p^*.

Consider an involution σ that fixes p_1, L_1 and p_2. Suppose that σ does not fix all points on L_1. Let x be such a point on L_1 which is not fixed by σ, and let M be a line through x different from L_1. The regulus MM^σ meets p_1^* respectively p_2^* in exactly one line R_1 respectively R_2. Clearly the regulus MM^σ is fixed by σ, and since p_1 and p_2 are fixed by σ, also R_1 and R_2 are fixed. Varying M through x, one sees that the number of fixed reguli MM^σ is t. But this number also equals t_1t_2, where t_i is the number of lines through p_i, different from L_1 and fixed by σ, $i = 1, 2$. If moreover σ fixes a third point u on L_1, then the number t_u of fixed lines through u (different from L_1) satisfies $t_1t_u = t_2t_u = t_1t_2 = t$ implying $t_1 = t_2 = t_u = \sqrt{t}$.

2.3.2 A useful lemma

Let θ ($\neq 1$) be an automorphism in G fixing all points on L_1 and the lines L_2, L_4 and L. Let $r\,I\,L_1$, $p_2 \neq r \neq p_1$, $p_1\,I\,M$, $M \neq L_1$, $r\,I\,N$, $N \neq L_1$. Then the regulus MN is fixed by θ if and only if $M^\theta = M$ and $N^\theta = N$. Let $t_i + 1$ be the number of fixed lines through p_i, $i = 1, 2$, and let $t' + 1$ be the number of fixed lines through r. If w is the number of fixed reguli consisting of $s+1$ lines concurrent with L_1, then clearly $w = t_1t'$. Analogously, $w = t_2t'$ and $w = t_1t_2$. Hence $t_1 = t_2 = t'$. So θ fixes a constant number $t'+1$ of lines through every point of L_1. Let $\mathcal{P}'$ be the union of the points on all these lines and let $\mathcal{B}'$ be the set of lines meeting $\mathcal{P}'$ in at least two points. Clearly, every element of $\mathcal{B}'$ is incident with exactly $s+1$ elements of $\mathcal{P}'$ (by the regularity of L_1). By 2.3.1 of FGQ, $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$, with I' the restriction of I to $\mathcal{P}' \times \mathcal{B}' \cup \mathcal{B}' \times \mathcal{P}'$, is a subquadrangle of order (s, t'). If $t' < t$, then by 5.3.5 of FGQ and the transitivity properties of G, the GQ $\mathcal{S}$ is the classical GQ $Q(5, s)$. Now let $t = t'$. As $\theta \neq 1$, we have $p_3^\theta \neq p_3$ by 2.4.1 of FGQ. For every point x on L_2 with $p_2 \neq x \neq p_3$, the conjugate θ^η of θ, where η is a collineation fixing L_1, L_2, L_4, p_3 and mapping p_3^θ to x, fixes every line concurrent with L_1 and maps p_3 to x. Therefore $\mathcal{S}$ is half Moufang and hence Moufang by [7]. Since all lines are regular, we have $\mathcal{S} \cong \mathcal{Q}(\triangle, \int)$ or $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \int)$ by 3.3.1 of FGQ.

Now we distinguish between several numerical cases.

2.3.3 k and s both odd

The subgroup H of G fixing Q and L has order $(s-1)k$ and hence contains some involution σ. If σ would fix a third point u on L_1, then σ would be inside G_Ω (taking without loss of generality $u = p$), contradicting the hypothesis k odd. Hence σ has no fixed points, different from p_1 and p_2, on L_1. From Section 2.3.1 follows immediately that $t_1 t_2 = t$, where $t_i + 1$ is the number of fixed lines (for σ) through p_i, $i = 1, 2$.

Suppose $t_i > 1$ for $i = 1, 2$. Let $M_i \neq L_1$ be any line fixed by σ and incident with p_i and let x_i be the point of M_i collinear with p_{i+2}, $i = 1, 2$. Clearly x_i is also fixed by σ and considering the point x_2' of M_2 collinear with x_1, one sees that σ fixes a skeleton unless $x_2' = x_2$. Since k is odd, σ cannot fix a skeleton and hence there arises a dual $(t_1+1) \times (t_2+1)$-grid having as points the points p_1, p_2, the t_1 points x_1 and the t_2 points x_2 (obtained by letting vary M_1 and M_2). So there are three points with at least $t_1 + 1$ centers and three points with at least $t_2 + 1$ centers. By the observation in 2.1.2, $t_i \leq t/s$ and hence $t = t_1 t_2 \leq t^2/s^2$, implying $s^2 \leq t$, so $t = s^2$ (cf. 1.2.3 of FGQ) and $t_1 = t_2 = t/s = s$. So we have $|\{p_1, p_3, x_2\}^{\perp\perp}| \geq 1 + s$, hence the left hand side is actually equal to $1 + s$. By the transitivity on triads in $x_1^\perp$, we have that x_1 is 3-regular. By transitivity every point of $\mathcal{S}$ is 3-regular, and consequently, by 5.3.2(i) of FGQ, $\mathcal{S}$ is the classical GQ $Q(5, s)$.

Next we consider the case $t_2 = 1$. Then of course $t_1 = t$ and so σ fixes all lines through p_1. Consider two arbitrary points y, z on L_2 with $z \neq p_2 \neq y$

and $y \neq p_3$. The subgroup H^* of G fixing p_1, p_2, y, L_4 and L has also order $(s-1)k$ and acts transitively on the points of L_2 different from p_2 and y. Let $\theta \in H^*$ map y^σ (which is different from y) to z. Then the map $\theta^{-1}\sigma\theta$ maps y to z and fixes all lines through p_1. Varying the point p_2 on L_1 and varying afterwards the line L_1 through p_1, we see that there exists a group K of automorphisms of $\mathcal{S}$ fixing all lines through p_1 and acting transitively on $\mathcal{P} \setminus \sqrt{\infty}^{\perp}$. By 8.2.4(iv) of FGQ, there is a unique subgroup $E \trianglelefteq K$ of order s^2t acting regularly on $\mathcal{P} \setminus \sqrt{\infty}^{\perp}$; this normal subgroup is the set of all automorphisms of $\mathcal{S}$ fixing all lines through p_1 and having no fixed point in $\mathcal{P} \setminus \sqrt{\infty}^{\perp}$. Let E_{L_2} be the subgroup of E, of order s, fixing L_2.

Now consider the subgroup H_1 of G which fixes L_4, p_1, L_1, p_2, L_2 and L. This group acts transitively on the points of L_1 different from p_1 and p_2. Clearly $E_{L_2} \leq H_1$. If $\eta \in E_{L_2} \setminus \{1\}$ and $\delta \in H_1$, then $\delta^{-1}\eta\delta$ fixes p_1 linewise, fixes L_2, and has no fixed point collinear with p_1. Hence $\delta^{-1}\eta\delta \in E_{L_2}$, and so $E_{L_2} \trianglelefteq H_1$. Hence E_{L_2} acts on the points of L_1 different from p_1 and p_2 in orbits of equal length d. So d has to divide both $s-1$ and s, hence $d = 1$. It follows that every element of E_{L_2} fixes all points on L_1. By the regularity of L_1 it is immediately clear that every element of E_{L_2} fixes each line of $\{L_4, L_2\}^{\perp\perp}$. Now let M be a line meeting L_1, with p_1 not incident with M, p_2 not incident with M, and $M \notin \{L_4, L_2\}^{\perp\perp}$. Let N be the line through p_1 which belongs to $\{M, L_2\}^{\perp\perp}$. Each element of E_{L_2} fixes $\{N, L_2\}^{\perp\perp}$ and the common point of M and L_1. Hence each element of E_{L_2} fixes M. It easily follows that each element of E_{L_2} fixes each line concurrent with L_1. By transitivity of G on the lines of $\mathcal{S}$, $\mathcal{S}$ is half Moufang and hence Moufang by [7]. So, by [3], $\mathcal{S}$ is classical (cf. 9.1 of FGQ). Since all lines of $\mathcal{S}$ are regular, we have $\mathcal{S} \cong \mathcal{Q}(\triangle, \int)$ or $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \int)$.

This completes the case k and s both odd.

2.3.4 k odd and s even

We consider here the subgroup H_1 of G fixing $L_4, p_1, L_1, p_2, L_2, L$ and p. This group has order sk and hence it contains an involution σ. If σ would fix a point not incident with L_1, then it would fix a skeleton and this contradicts the fact that k is odd. So the only fixed points for σ are on L_1. Let x be any point on L_1 and let y be any point collinear with x but not on L_1. If

$y^\sigma \sim y$, then clearly x is fixed by σ. If $y \not\sim y^\sigma = y'$, then $\{y, y'\}^\perp$ contains a fixed point (indeed, it contains $t+1$ elements and since s divides t, t is even). Hence there is a point collinear with y which is fixed by σ. This must clearly be x and so we showed that σ fixes every point on L_1. By Lemma 2.3.2, $\mathcal{S} \cong \mathcal{Q}(\triangle, \int)$ or $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \int)$.

2.3.5 k and s both even

From these assumptions follows the existence of an involution σ fixing the skeleton Ω. This involution has as fixed elements the points and lines of a thick proper subquadrangle $\mathcal{S}'$ of order (s', t'). If $s = s'$, then by the transitivity properties of G, it follows from 5.3.5 of FGQ that $\mathcal{S}$ is the classical GQ $Q(5, s)$. Hence suppose $s \neq s'$. By Section 2.3.1, $t' = \sqrt{t}$.

Let R be a line of $\mathcal{S}'$ and let z be a point of R not in $\mathcal{S}'$. Then any line through z, different from R, has no point in common with $\mathcal{S}'$; so there exists a line M external to $\mathcal{S}'$. The line $M' = M^\sigma$ does not meet M and the $s+1$ lines of $\{M, M'\}^{\perp\perp}$ are permuted by σ. Since $s+1$ is odd, $\{M, M'\}^{\perp\perp}$ contains at least one fixed line N, which contains in turn exactly $s'+1$ fixed points. A line containing any of these $s'+1$ points and concurrent with both M and M' is fixed by σ. Hence there are exactly $s'+1$ lines of $\mathcal{S}'$ concurrent with a given external line. By Proposition 2.2.1 of FGQ, we have $1 + s' = 1 + t'$. But again by 2.2.1 of FGQ, we also have $s \geq s't' = t$. Hence $s = t$ and $\mathcal{S}$ is the classical GQ $Q(4, q)$ (by the regularity of every line).

2.3.6 k even and s odd

We subdivide this case in two subcases: t odd and t even.

Subcase t even. Here we consider the subgroup H_2 of G fixing L_4, p_1, L_1, p_2, L and p. This group has order stk and hence contains a non-trivial Sylow 2-subgroup S. Suppose 2^n respectively 2^m divides t respectively k and 2^{n+1} respectively 2^{m+1} does not. Then $|S| = 2^{n+m}$. Let σ be an involution in the center of S and suppose that σ fixes a second line through p_2, say L_2. Since s is odd σ fixes a second point on L_4 respectively L_2. If σ would fix all points on L_1, then its fixed elements would be the points and lines of a thick proper subquadrangle of order (s, t'). Then, by the transitivity properties of G, it again follows from 5.3.5 of FGQ, that $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \int)$. Hence we assume now that σ does not fix all points of L_1. By Section 2.3.1, σ fixes $\sqrt{t}$ lines different from L_1 through p_2. The group S acts on these lines and the stabilizer S_M in S of any of these lines M has at most order 2^m, since S_M is a subgroup of a conjugate of the group G_Ω of order k. So the length of the orbit of M under S has at least order $|S|/2^m = 2^n$. Since M was arbitrary, this means that $\sqrt{t}$ must be divisible by 2^n, hence 2^{2n} divides t, a contradiction. Hence σ does not fix any line through p_2 besides L_1. But then, by 2.3.1, σ fixes every point on L_1. On L_4 there is a second fixed point p_4. If σ does not fix all points of L_4, then, by 2.3.1, p_4 is on at least two fixed lines, a contradiction as every fixed line contains p_1. Hence σ fixes every point of L_4, and similarly it fixes every point of L. Now consider the group T generated by all conjugates of σ by elements of the subgroup H_3 of G fixing p_1, L_1, p_2. Every element of T fixes each point of L_1. Let M be a line different from L_1 containing p_1, and let M', $\overline{M}'$ be distinct lines different from L_1 containing p_2. Then there is a $\delta \in H_3$ such that $L_4^\delta = M$, $N^\delta = \overline{M}'$, $N^{\sigma\delta} = M'$, where N is any line through p_2 different from L_1. Then $\delta^{-1}\sigma\delta \in T$ fixes each point of M and maps M' onto $\overline{M}'$. Now let x be a point collinear with p_2 and not incident with L_1. Denote by O_x the orbit of x under T. Clearly, we have $|O_x| \geq t$. If $|O_x| = t$, then by the transitivity property just mentioned, all points of O_x are collinear to one single point of M, $M \neq L_1$ and M through p_1. But M is arbitrary, so we obtain a dual $(t+1) \times (t+1)$-grid. Consequently we have a regular pair of points and by 1.3.6(i) of FGQ, $s \geq t$. But all lines are regular, so $t \geq s$ implying $s = t$, which means that $\mathcal{S}$ is the classical GQ $Q(4, s)$. Hence we may assume $|O_x| > t$. This means that there exists a collineation γ in T preserving L_2 and not inducing the identity map on the set of points of L_2. By interchanging roles of p_1 and p_2, we see that there exists an element $\xi \in G$ fixing all points of L_1, all points of L_2 and mapping L_4^γ onto L_4. Then $\gamma\xi = \zeta$ fixes L_2, L_4, all points of L_1, but not every point of L_2. Let $x^\zeta \neq x$, $x \, I \, L_2$. Choose on L_2 distinct points y, z, not on L_1. By the transitivity properties of G, there is an element $\delta \in H_3$ which fixes L_4, maps x onto y, and x^ζ onto z. Then $\delta^{-1}\zeta\delta$ fixes all points of L_1, fixes L_4 and L_2, and maps y onto z. Consequently the group T_0 generated by all conjugates of ζ by elements of H_3 fixes all points of L_1, fixes L_4 and L_2, and acts transitively on the s points of L_2 not on L_1. Hence s divides $|T_0|$. Assume that no element of $T_0 \setminus \{1\}$ fixes a third line through p_1. Then T_0 acts faithfully and semi-regularly on the lines through p_1, different from L_1 and L_4. Hence $|T_0|$ divides $t-1$, so s divides $t-1$. As s divides t, we have a contradiction.

Hence there is an element $\theta \in T_0$ fixing all points of L_1, fixing L_2 and L_4, and fixing a third line, say L, containing p_1. By Lemma 2.3.2, $\mathcal{S} \cong \mathcal{Q}(\triangle, \int)$ or $\mathcal{S} \cong \mathcal{Q}(\bigtriangledown, \int)$, so s and t have the same parity, a contradiction.

Subcase t odd. Here we consider the subgroup H_4 of G fixing Q and p, which has order $(t-1)k$. We consider a Sylow 2-subgroup S in H_4. Suppose 2^n respectively 2^m divides $t-1$ respectively k and 2^{n+1} respectively 2^{m+1} does not. Then $|S| = 2^{n+m}$. Let σ be an involution in the center of S. If σ does not fix every point on L_1, then it fixes exactly $\sqrt{t}-1$ lines different from L_1 and L_4 through p_1. Let M be one of these lines. The group S acts on these $\sqrt{t}-1$ fixed lines and similarly as in the previous case, we obtain that the size of the orbit of M is divisible by 2^n. Hence 2^n divides $\sqrt{t}-1$, so 2^{n+1} divides $(\sqrt{t}-1)(\sqrt{t}+1) = t-1$, a contradiction. Hence σ fixes all points on L_1. Assume there is no element of G fixing all points of L_1, the lines L_2, L_4, and at least three lines containing p_1 or p_2. Let T' be the group generated by the congugates of σ by the elements of the subgroup H_5 of G fixing L_4, p_1, L_1, p_2 and L_2. Then the elements of T' fix all points of L_1, fix L_2 and L_4. Suppose first that σ does not fix all points of L_2. Then T' acts transitively on the set A of points of L_2 different from p_2. By an argument similar to the one in the case t even, we obtain a collineation of $\mathcal{S}$ fixing all points of L_1, the line L_2 and at least three lines containing p_1. Hence σ fixes all points of L_2. Then the fixed elements of σ are all elements of the $(s+1)\times(s+1)$-grid containing the lines L_2 and L_4. Let N be a line through p_2 different from L_1 and L_2. Then $N^\sigma \neq N$. Let H_3 be the subgroup of G fixing p_1, L_1, p_2, let M be a line different from L_1 containing p_1, and let M', $\overline{M}'$ be distinct lines different from L_1 containing p_2. Then there is a $\delta \in H_3$ such that $L_4^\delta = M$, $N^\delta = \overline{M}'$, $N^{\sigma\delta} = M'$. So $\delta^{-1}\sigma\delta$ fixes each point of L_1, fixes each point of M and maps M' onto $\overline{M}'$. Let T'' be the subgroup of G generated by the conjugates of σ by the elements of the group H_3. Then the elements of T'' fix all points of L_1. Using an argument analogous to the one in the case t even (replacing T by T''), we find that necessarily $\mathcal{S} \cong \mathcal{Q}(\triangle, \int)$. So we may assume that $\mathcal{S}$ admits a collineation fixing all points of L_1, the lines L_2, L_4, and at least three lines containing p_1 or p_2. Then by Lemma 2.3.2, the GQ $\mathcal{S}$ is isomorphic to either $Q(4,s)$ or $Q(5,s)$.

2.4 The possible groups

Let $\mathcal{S}$ be one of the generalized quadrangles $W(q), Q(4,q), Q(5,q)$ or $H(3,q^2)$ and let G be a group of automorphisms acting transitively on the set of ordered pentagons. By duality, we may restrict ourselves to $Q(4,q)$ and $H(3,q^2)$. First let $\mathcal{S} \neq \mathcal{Q}(\triangle, \in)$. By a theorem of Seitz [6], G has to contain the simple group $O_5(q)$ ($q \neq 2$), respectively $U_4(q)$. If q is even, then $O_5(q)$ respectively $U_4(q)$ coincides with $PGO_5(q)$ respectively $PGU(4,q)$ and each of these groups contains all generalized homologies of the corresponding GQ; hence these groups have the desired transitivity properties. If $\mathcal{S} = \mathcal{Q}(\triangle, \in)$, then $O_5(2)$ is the full automorphism group of $\mathcal{S}$ and $|O_5(2)|$ is the number of ordered pentagons in $\mathcal{S}$. As $O_5(2)$ acts semiregularly on the set of ordered pentagons, and consequently also regularly, it follows that G must coincide with $O_5(2)$ in this case. Suppose now that q is odd.

First let $\mathcal{S} \cong \mathcal{Q}(\triangle, \text{II})$. The full group of automorphisms of $Q(4,q)$ is $PGO_5(q).h = P\Gamma O_5(q)$, where $q = p^h$ with p a prime number. If h is odd, then the group $O_5(q).h$ does not act transitively on the ordered pentagons since its order is not divisible by the number of ordered pentagons. This number of ordered pentagons equals $2|O_5(q)|$, so $O_5(q)$ must have even index in G. Since the outer automorphism group of $O_5(q)$ has a unique involution (this is the diagonal automorphism, see [2]), every extension of $O_5(q)$ in $P\Gamma O_5(q)$ in which $O_5(q)$ has even index must contain $PGO_5(q)$. Suppose now that h is even. Consider a skeleton $(Q; L, x)$ in $\mathcal{S}$. The stabilizer in $O_5(q)$ of Q acts transitively on the points not in Q and incident with the line of Q containing x; but the stabilizer in $O_5(q)$ of Q and x acts on the set V of lines not in Q but incident with the point of Q on L as the permutation group on $GF(q) \setminus \{0\}$ consisting of the elements $\gamma_a : x \mapsto a^2 x$, $a, x \in GF(q) \setminus \{0\}$. If G does not contain $PGO_5(q)$, then it contains an element involving a field automorphism θ, $\theta \neq 1$. So G contains the group $PGO_5^\theta(q)$ generated by $O_5(q)$ and $g_\theta g$, where $g \in PGO_5(q)$ and where g_θ is the element of $P\Gamma L_5(q)$ corresponding to the semilinear transformation with identity matrix and field automorphism θ. Clearly $PGO_5^\theta(q)$ contains an element that fixes Q and x, and acts on V as the permutation $\gamma : x \mapsto bx^\theta$, with b a non-square in $GF(q)$, that is, we may assume that $g \in PGO_5(q) \setminus O_5(q)$. Also, θ has even order n (otherwise $(g_\theta g)^n \in PGO_5(q) \setminus O_5(q)$ and then G contains $PGO_5(q)$). Conversely, each such $PGO_5^\theta(q)$ acts transitively on the set of skeletons of $\mathcal{S}$. Let $n = l.2^e$ with l odd and put $\theta' = \theta^l$. Then θ' has order 2^e, $PGO_5^{\theta'}(q) \leq PGO_5^{\theta}(q)$, and $PGO_5^{\theta'}(q)$ also acts transitively on the set of skeletons, and hence on the set of ordered pentagons. This shows (i) and (ii) of the main result.

Now suppose $\mathcal{S} \cong \mathcal{H}_{\ni}(\text{II}^{\in})$, q odd. First suppose that $q \equiv 3 \bmod 4$. Then the Sylow 2-subgroup P of the outer automorphism group of $U_4(q)$ is a semi-direct product of a cyclic group C_4 (of diagonal automorphisms) with a group of order 2 (the unique involutory field automorphism). Now the number of ordered pentagons equals $4|U_4(q)|$ and so G must contain $U_4(q)$ as a subgroup of index divisible by 4. Hence $G/U_4(q)$ contains a group isomorphic to a subgroup of P of order at least 4. But every such subgroup clearly contains the unique involutory diagonal automorphism (the unique involution of C_4). So G contains the group H (with the notation of the introduction). This is trivially true if $q \equiv 1 \bmod 4$. But now, a similar reasoning as in the case $\mathcal{S} \cong \mathcal{Q}(\triangle, \text{II})$, q odd and h even, of the preceding paragraph shows that only cases (iii) and (iv) of the main result are possible.

This completes the proof of our main result.

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