The Classification of Finite Metahamiltonian p-Groups *

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Abstract

A finite non-abelian group G is called metahamiltonian if every subgroup of G is either abelian or normal in G. If G is non-nilpotent, then the structure of G has been determined. If G is nilpotent, then the structure of G is determined by the structure of its Sylow subgroups. However, the classification of finite metahamiltonian p-groups is an unsolved problem. In this paper, finite metahamiltonian p-groups are completely classified up to isomorphism.

Keywords minimal non-abelian groups, Hamiltonian groups, metahamiltonian groups, $\mathcal{A}_2$-groups *2000 Mathematics subject classification:* 20D15.

1 Introduction

To determine a finite group by using its subgroup structure is an important theme in the group theory. Let G be a finite non-abelian p-group. If every proper subgroup of G is abelian then G is called *minimal non-abelian*, which was classified by Redei [19]. If every subgroup of G is normal in G then G is called *Hamiltonian*, which was classified by Dedekind [9]. The classifications of minimal non-abelian p-groups and Hamiltonian groups are two classical results in the theory of finite p-groups.

As a generalization of minimal non-abelian group, many authors investigate finite p-groups with many abelian subgroups. Among these works, the classification of $\mathcal{A}_2$-groups is the most important one. A finite non-abelian p-group G is called an $\mathcal{A}_2$-*group* if G is not minimal non-abelian and all of its subgroups of index p are either abelian or minimal non-abelian. Many scholars studied and classified $\mathcal{A}_2$-groups, see [6, 7, 10, 11, 20, 26]. Resent years, several important classes of p-groups which contain $\mathcal{A}_2$-group are determined. For example, Xu et al. [21] classified finite p-groups all of whose non-abelian proper subgroups are generated by two elements. An et al. [1, 2, 16, 17, 18] classified finite p-groups with a minimal non-abelian subgroup of index p. Zhang et al. [27] classified finite p-groups all of its subgroups of index p^3 are abelian.

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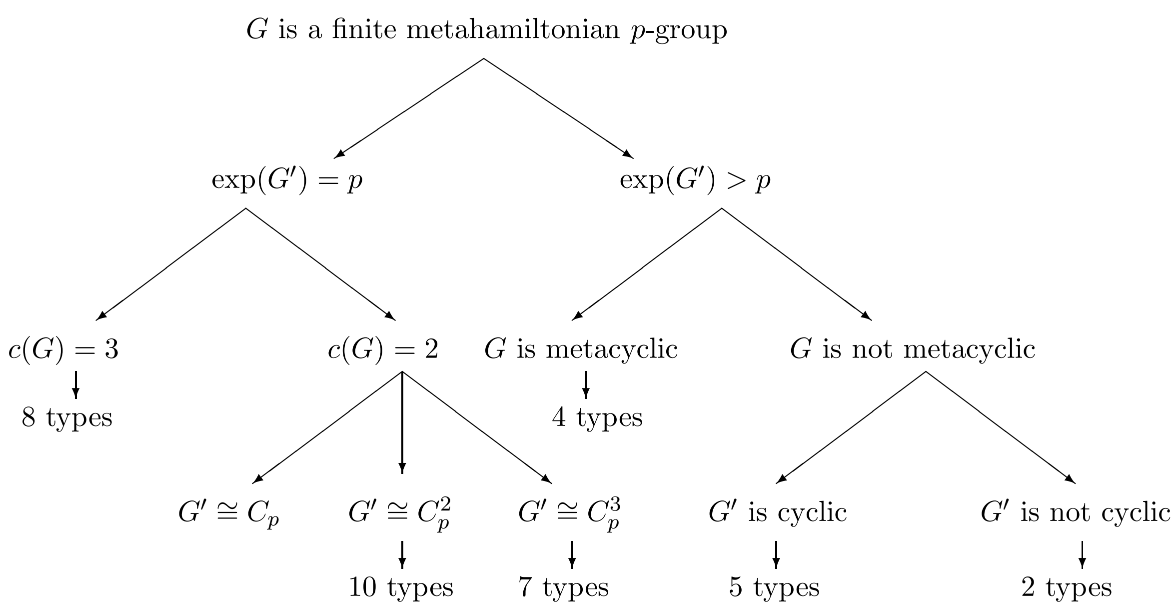
As a generalization of Hamilton groups, many authors investigate finite p-groups with many normal subgroups. For example, Passman [15] classified finite p-groups all of whose non-normal subgroups are cyclic. Zhang et al. [23, 24, 25] classified finite p-groups all of whose non-normal subgroups have orders $\leq p^3$.

A non-abelian group G is called *metahamiltonian* if every proper subgroup of G is either abelian or normal in G. Obviously, $\mathcal{A}_2$-groups are metahamiltonian. Groups in [15, 23] are also metahamiltonian. Thus the class of metahamiltonian p-groups is much larger than that of minimal non-abelian p-groups and Hamilton p-groups. The classification of metahamiltonian p-groups is an old problem. The present paper is devoted to the classification.

By the way, Nagrebeckii [13] determined the structure of finite non-nilpotent metahamiltonian groups. Obviously, a nilpotent group is metahamiltonian if and only if all its Sylow subgroups are metahamiltonian. Hence finite metahamiltonian groups are completely determined.

This paper is divided into four sections. Section 2 is a preliminary. In section 3, we classify finite metahamiltonian p-groups whose derived group is of exponent p, and the case of exponent $> p$ is dealt with in section 4.

The sketch of the classification of metahamiltonian p-groups is as follows.



2 Preliminaries

Let G be a finite p-group. For a positive integer t, G is said to be an *$\mathcal{A}_t$-group* if the greatest index of non-abelian subgroups is p^{t-1}. So $\mathcal{A}_1$-groups are just the minimal non-abelian p-groups.

Let G be a finite p-group. We define

$$\Lambda_m(G) = \{a \in G \mid a^{p^m} = 1\}, \qquad V_m(G) = \{a^{p^m} \mid a \in G\},$$
$$\Omega_m(G) = \langle \Lambda_m(G) \rangle = \langle a \in G \mid a^{p^m} = 1 \rangle, \text{ and } \mho_m(G) = \langle V_m(G) \rangle = \langle a^{p^m} \mid a \in G \rangle.$$

G is called *p-abelian* if $(ab)^p = a^p b^p$ for all $a, b \in G$. We use $c(G)$ and $d(G)$ to denote the nilpotency class and minimal number of generators, respectively.

We use C_n and C_n^m to denote the cyclic group and the direct product of m cyclic groups of order n, respectively. We use $M_p(m,n)$ to denote groups

$$\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle, \text{ where } m \geq 2,$$

and use $M_p(m,n,1)$ to denote groups

$$\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle,$$

where $m + n \geq 3$ for $p = 2$ and $m \geq n$. We can give a presentation of minimal non-abelian p-groups as follows:

Theorem 2.1. (See [19]) *Let G be a minimal non-abelian p-group. Then G is Q_8, $M_p(m,n)$, or $M_p(m,n,1)$.*

A finite p-group G is called *metacyclic* if it has a cyclic normal subgroup N such that G/N is also cyclic.

In 1973 King [12] classified metacyclic p-groups. In 1988 Newman and Xu (see [14, 22]) found new presentations for these groups. Theorem 2.2 is quoted from [22].

Theorem 2.2. (1) *Any metacyclic p-group G, p odd, has the following presentation:*

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ a^b = a^{1+p^r} \rangle$$

where $r,\ s,\ t,\ u$ are non-negative integers with $r \geq 1$ and $u \leq r$. Different values of the parameters $r,\ s,\ t$ and u with the above conditions give non-isomorphic metacyclic p-groups. It is denoted to $< r, s, t, u >_p$ in this paper.

(2) *Let G be a metacyclic 2-group. Then G has one of the following three kinds of presentations:*

(I) *G has a cyclic maximal subgroup. Hence G is dihedral, semi-dihedral, generalized quaternion, or an ordinary metacyclic group presented by*

$$G = \langle a, b \mid a^{2^n} = 1, b^2 = 1, a^b = a^{1+2^{n-1}} \rangle.$$

(II) *Ordinary metacyclic 2-groups:*

$$G = \langle a, b \mid a^{2^{r+s+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s}}, a^b = a^{1+2^r} \rangle,$$

where r, s, t, u *are non-negative integers with* $r \geq 2$ *and* $u \leq r$. *It is denoted to be* $< r, s, t, u >_2$ *in this paper.*

(III) *Exceptional metacyclic 2-groups:*

$$G = \langle a, b \mid a^{2^{r+s+v+t'+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, a^b = a^{-1+2^{r+v}} \rangle,$$

where r, s, v, t, t', u *are non-negative integers with* $r \geq 2$, $t' \leq r$, $u \leq 1$, $tt' = sv = tv = 0$, *and if* $t' \geq r - 1$ *then* $u = 0$. *Groups of different types or of the same type but with different values of parameters are not isomorphic to each other. It is denoted to be* $< r, s, v, t, t', u >_2$ *in this paper.*

Lemma 2.3. (See [8]) *Suppose that* G *is a finite* p*-group. Then* G *is metacyclic if and only if* $G/\Phi(G')G_3$ *is metacyclic.*

Lemma 2.4. (See [5, Lemma J(i)]) *Let* G *be a metacyclic* p*-group. Then* G *is an* $\mathcal{A}_n$*-group if and only if* $|G'| = p^n$.

In [4], the properties of metahamiltonian p-groups are given as follows:

Theorem 2.5. *Let* G *be a metahamiltonian* p*-group. Then* $c(G) \leq 3$. *In particular,* G *is metabelian.*

Theorem 2.6. *Let* G *be a finite* p*-group.* G *is metahamiltonian if and only if* G' *is contained in every non-abelian subgroup of* G.

Theorem 2.7. *Suppose that* G *is a finite metahamilton* p*-group. If* $d(G) = 2$ *and* $\exp(G') > p$, *then* G *is metacyclic.*

Theorem 2.8. *Suppose that* G *is a finite metahamiltonian* p*-group having an elementary abelian derived group. If* $c(G) = 3$, *then* G *is an* $\mathcal{A}_2$*-group.*

Corollary 2.9. *Suppose that* G *is a finite metahamiltonian* p*-group having an elementary abelian derived group. If* $c(G) = 3$, *then* $d(G) = 2$ *and* p *is odd.*

3 Finite metahamiltonian p-groups whose derived group is of exponent p

In this section, we determine finite metahamiltonian p-groups whose derived group is of exponent p. In order to avoid tedious calculations, we provide a proof which relies on some results obtained in other papers. These papers are [2, 3, 17, 26].

Theorem 3.1. *Suppose that* G *is a finite metahamiltonian* p*-group with* $\exp(G') = p$. *Then* G *is one of the following non-isomorphic groups:*

(A) *groups with* $|G'| = p$.

(B) $c(G) = 3$. *In this case,* p *is odd,* $d(G) = 2$ *and* $G \in \mathcal{A}_2$.

(B1) $\langle a_1, b \mid a_1^p = a_2^p = a_3^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_i, a_j] = 1\rangle$, *where* $p \geq 5$ *for* $m = 1$, $p \geq 3$ *and* $1 \leq i, j \leq 3$;

(B2) $\langle a_1, b \mid a_1^p = a_2^p = b^{p^{m+1}} = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_1, a_2] = 1\rangle$, *where* $p \geq 3$;

(B3) $\langle a_1, b \mid a_1^{p^2} = a_2^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_1, a_2] = 1\rangle$, *where* $p \geq 3$ *and* $\nu = 1$ *or a fixed quadratic non-residue modulo* p;

(B4) $\langle a_1, a_2, b \mid a_1^9 = a_2^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [a_2, a_1] = 1\rangle$.

(B5) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p\rangle$, *where* $p \geq 5$, ν *is a fixed square non-residue modulo* p;

(B6) $\langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^{-p}b^{-lp}, [c, b] = a^{-p}\rangle$, *where* $p \geq 5$, $4l = \rho^{2r+1} - 1$, $r = 1, 2, \ldots, \frac{1}{2}(p-1)$, ρ *is the smallest positive integer which is a primitive root modulo* p;

(B7) $\langle a, b \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^3\rangle$;

(B8) $\langle a, b \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^{-3}\rangle$.

(C) $c(G) = 2$ *and* $G' \cong C_p^2$.

(C1) $K \times A$, *where* $K = \langle a_1, a_2, b \mid a_1^4 = a_2^4 = 1, b^2 = a_1^2, [a_1, a_2] = 1, [a_1, b] = a_2^2, [a_2, b] = a_1^2\rangle$ *and* A *is abelian such that* $\exp(A) \leq 2$;

(C2) $K \times A$, *where* $K = \langle a_1, a_2, b, d \mid a_1^4 = a_2^4 = 1, b^2 = a_1^2, d^2 = a_2^2, [a_1, a_2] = 1, [a_1, b] = a_2^2, [a_2, b] = a_1^2, [a_1, d] = a_1^2, [a_2, d] = a_1^2 a_2^2, [b, d] = 1\rangle$ *and* A *is abelian such that* $\exp(A) \leq 2$.

(C3) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3}} = 1, [a_1, a_2] = a_1^{p^{m_1}}, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = 1\rangle$, $m_1 > 1$ *for* $p = 2$, $m_1 \geq m_2 \geq m_3$, *and* A *is abelian such that* $\exp(A) \leq p^{m_2}$;

(C4) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3}} = 1, [a_1, a_2] = 1, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = a_1^{\nu p^{m_1}}\rangle$, $p > 2$, ν *is a fixed square non-residue modulo* p, $m_1 - 1 = m_2 \geq m_3$ *or* $m_1 = m_2 \geq m_3$, *and* A *is abelian such that* $\exp(A) \leq p^{m_2}$;

(C5) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3}} = 1, [a_1, a_2] = 1, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = a_1^{kp^{m_1}} a_2^{-p^{m_2}}\rangle$, $1 + 4k \notin (F_p)^2$ *for* $p > 2$, $k = 1$ *for* $p = 2$, $m_1 = m_2 \geq m_3$ *and* A *is abelian such that* $\exp(A) \leq p^{m_2}$;

(C6) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3}} = 1, [a_1, a_2] = 1, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = a_1^{p^{m_1}}\rangle$, $m_1 - 1 = m_2 \geq m_3$ *and* A *is abelian such that* $\exp(A) \leq p^{m_2}$;

(C7) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_1, a_2] = a_3^{p^{m_3}}, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = 1\rangle$, $m_1 \geq m_2 = m_3 + 1$ *and* A *is abelian such that* $\exp(A) \leq p^{m_3}$;

(C8) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_1, a_2] = a_3^{p^{m_3}}, [a_1, a_3] = a_2^{\nu p^{m_2}}, [a_2, a_3] = 1\rangle$, $p > 2$, ν *is a fixed square non-residue modulo* p, $m_1 \geq m_2 = m_3 + 1$ *or* $m_1 > m_2 = m_3$ *and* A *is abelian such that* $\exp(A) \leq p^{m_3}$;

(C9) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_1, a_2] = a_3^{p^{m_3}}, [a_1, a_3] = a_2^{kp^{m_2}} a_3^{-p^{m_3}}, [a_2, a_3] = 1\rangle$, $1 + 4k \notin (F_p)^2$ *for* $p > 2$, $k = 1$ *for* $p = 2$, $m_1 > m_2 = m_3$ *and* A *is abelian such that* $\exp(A) \leq p^{m_3}$;

(C10) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2}} = a_3^{p^{m_3+1}} = 1, [a_1, a_2] = a_3^{p^{m_3}}, [a_1, a_3] = a_1^{p^{m_1}}, [a_2, a_3] = 1\rangle$, $m_1 \geq m_2 = m_3 + 1$ *and* A *is abelian such that* $\exp(A) \leq p^{m_3}$.

(D) $c(G) = 2$ *and* $G' \cong C_p^3$.

(D1) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_2, a_3] = a_1^{p^{m_1}}, [a_1, a_3] = a_2^{\eta p^{m_2}}, [a_1, a_2] = a_3^{p^{m_3}}, [a_3^p, a_1] = [a_3^p, a_2] = 1\rangle$, *where* p *is odd,* $m_1 = m_2 + 1 = m_3 + 1$ *and* η *is a fixed square non-residue modulo* p, *and* A *is abelian with* $\exp(A) \leq p^{m_3}$;

(D2) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_2, a_3] = a_1^{p^{m_1}}, [a_1, a_3] = a_2^{lp^{m_2}} a_3^{-p^{m_2}}, [a_1, a_2] = a_3^{p^{m_3}}, [a_3^p, a_1] = [a_3^p, a_2] = 1\rangle$, *where* p *is odd,* $m_1 = m_2 + 1 = m_3 + 1$ *and* $1 + 4l \notin (F_p)^2$, *and* A *is abelian with* $\exp(A) \leq p^{m_3}$;

(D3) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{2^{m_1+1}} = a_2^{2^{m_2+1}} = a_3^{2^{m_3+1}} = 1, [a_2, a_3] = a_1^{2^{m_1}}, [a_3, a_1] = a_2^{2^{m_2}}, [a_1, a_2] = a_2^{2^{m_2}} a_3^{2^{m_3}}, [a_3^2, a_1] = [a_3^2, a_2] = 1\rangle$, *where* $m_1 = m_2 + 1 = m_3 + 1$, *and* A *is abelian with* $\exp(A) \leq 2^{m_3}$;

(D4) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_2, a_3] = a_1^{p^{m_1}}, [a_1, a_3] = a_2^{\eta p^{m_2}}, [a_1, a_2] = a_3^{p^{m_3}}, [a_3^p, a_1] = [a_3^p, a_2] = 1\rangle$, *where* p *is odd,* $m_1 = m_2 = m_3 + 1$ *and* η *is a fixed square non-residue modulo* p, *and* A *is abelian with* $\exp(A) \leq p^{m_3}$;

(D5) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3+1}} = 1, [a_2, a_3] = a_1^{p^{m_1}}, [a_1, a_3] = a_1^{p^{m_1}} a_2^{lp^{m_2}}, [a_1, a_2] = a_3^{p^{m_3}}, [a_3^p, a_1] = [a_3^p, a_2] = 1\rangle$, *where* p *is odd,* $m_1 = m_2 = m_3 + 1$ *and* $1 + 4l \notin (F_p)^2$, *and* A *is abelian with* $\exp(A) \leq p^{m_3}$;

(D6) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{2^{m_1+1}} = a_2^{2^{m_2+1}} = a_3^{2^{m_3+1}} = 1, [a_2, a_3] = a_1^{2^{m_1}} a_2^{2^{m_2}}, [a_3, a_1] = a_2^{2^{m_2}}, [a_1, a_2] = a_3^{2^{m_3}}, [a_3^2, a_1] = [a_3^2, a_2] = 1\rangle$, *where* $m_1 = m_2 = m_3 + 1$, *and* A *is abelian with* $\exp(A) \leq 2^{m_3}$;

(D7) $K\times A$, *where* $K=\langle a,b,c \mid a^4=b^4=c^4=1,[b,c]=a^2b^2,[c,a]=b^2c^2,[a,b]=c^2,[c^2,a]=[c^2,b]=1\rangle$, *and* A *is abelian with* $\exp(A)\le 2$.

Proof By Theorem 2.5, $c(G)\le 3$. If $c(G)=3$, then, by Theorem 2.8, $G\in\mathcal{A}_2$. Checking groups listed in [4, Lemma 2.4], we get groups (B1)–(B8). In the following, we may assume that $c(G)=2$. Let N be a minimal non-abelian subgroup of G. By Theorem 2.6, $G'\le N$. Since $G'\le Z(G)$, $G'\le \Omega_1(Z(N))=\Omega_1(\Phi(N))$. It follows from Theorem 2.1 that $G'\le C_p^3$. If $G'\cong C_p$, then G is of Type (A) in the theorem. If $G'\cong C_p^2$, then, by the following Lemma 3.2, G is a group of Type (C1)–(C10) in the theorem. For the case of $G'\cong C_p^3$, Lemma 3.3 gives groups of Type (D1)–(D5) in the theorem. Finally, it is omitted to check that such groups are non-isomorphic metahamiltonian p-groups. □

Lemma 3.2. *Suppose that G is a metahamilton p-group. If $G'\cong C_p^2$ and $c(G)=2$, then G is a group of Type* (C1)–(C10) *as defined in Theorem* 3.1.

Proof Let the type of G/G' be $(p^{m_1},p^{m_2},\dots,p^{m_r})$, where $m_1\ge m_2\ge\cdots\ge m_r$. Let

$$G/G'=\langle a_1G'\rangle\times\langle a_2G'\rangle\times\cdots\times\langle a_rG'\rangle,\text{ where } o(a_iG')=p^{m_i},\ i=1,2,\dots,r.$$

Then $G=\langle a_1,a_2,\dots,a_r\rangle$.

If $m_1=1$, then G/G' is elementary abelian. By Theorem 2.6, $G'\le\langle x,y\rangle$ for every non-commutative pair $x,y\in G$ and hence $\langle x,y\rangle$ is minimal non-abelian with order p^4. Such groups were classified in [3]. By checking the results in [3], we get the groups (C3)–(C5) where $m_1=m_2=m_3=1$ and (C1)–(C2). In the following, we may assume that $m_1>1$.

Let i be the minimal integer such that $a_i\notin Z(G)$. That is, there exists $j>i$ such that $[a_i,a_j]\ne 1$. If $i\ne 1$, then $a_1\in Z(G)$. Replacing a_1 with a_1a_j, we get $a_1\notin Z(G)$. If $i=1$, then we also have $a_1\notin Z(G)$.

Let j be the minimal integer such that $[a_1,a_j]\ne 1$. If $j\ne 2$, then $[a_1,a_2]=1$. Replacing a_2 with a_2a_j, we get $[a_1,a_2]\ne 1$. If $j=2$, then we also have $[a_1,a_2]\ne 1$.

Let k be the minimal integer such that $[a_k,a_l]\notin\langle[a_1,a_2]\rangle$. If $k>2$, then, for all integer s, we have

$$[a_1,a_s]\in\langle[a_1,a_2]\rangle \text{ and } [a_2,a_s]\in\langle[a_1,a_2]\rangle.$$

(1) If $[a_1,a_l]=1$, then, replacing a_2 with a_2a_l, we have $[a_2,a_k]\notin\langle[a_1,a_2]\rangle$. (2) If $[a_1,a_l]=[a_1,a_2]^\alpha$ where $(\alpha,p)=1$, then, letting $[a_1,a_k]=[a_1,a_2]^\beta$ and replacing a_2 with $a_2a_ka_l^{\alpha^{-1}\beta}$, we have $[a_2,a_l]\notin\langle[a_1,a_2]\rangle$. Hence we may assume that $k\le 2$.

Let l be the minimal integer such that $[a_k,a_l]\notin\langle[a_1,a_2]\rangle$. If $l\ne 3$, then $[a_1,a_3]\in\langle[a_1,a_2]\rangle$ and $[a_2,a_3]\in\langle[a_1,a_2]\rangle$. Replacing a_3 with a_3a_l, we have $[a_k,a_3]\notin\langle[a_1,a_2]\rangle$. Hence we may assume that $l=3$.

Let $K = \langle a_1, a_2, a_3\rangle$. Then $|K'| = |G'| = p^2$. Such groups K were determined in [2]. By checking [2, Table 4], K is one of the groups (C3)–(C10) in Theorem 3.1. If $r = 3$, then $G = K$. In the following we may assume that $r \geq 4$.

Case 1: K is one of the groups of Type (C3)–(C6) in Theorem 3.1.

In this case, $G' = \langle a_1^{p^{m_1}}, a_2^{p^{m_2}}\rangle$ and $[a_1, a_3] = a_2^{p^{m_2}}$. Assume that $a_4^{p^{m_4}} = a_1^{\alpha p^{m_1}} a_2^{\beta p^{m_2}}$. Replacing a_4 with $a_4 a_1^{-\alpha p^{m_1 - m_4}}$, we have $a_4^{p^{m_4}} = a_2^{\beta p^{m_2}}$ since $m_1 > 1$.

If $p > 2$ or $m_2 > 1$, then, replacing a_4 with $a_4 a_2^{-\beta p^{m_2 - m_4}}$, we have $a_4^{p^{m_4}} = 1$. If $p = 2$ and $m_2 = 1$, then we claim that there exists an $x \in \{a_4, a_4 a_2\}$ such that $x^2 \in \langle a_1^{2^{m_1}}\rangle$. Otherwise, $a_4^2 = a_2^2$. Since $[a_4, a_2] = (a_4 a_2)^2 \notin \langle a_1^{2^{m_1}}\rangle$, $\langle a_4, a_2\rangle$ is not abelian. It follows from Theorem 2.6 that $a_1^{2^{m_1}} \in \langle a_4, a_2\rangle$. Hence $[a_4, a_2] = a_1^{2^{m_1}} a_2^2$. Thus $\langle a_4 a_2, a_2 a_1^{2^{m_1 - 1}}\rangle$ is neither abelian nor normal in G, a contradiction. Replacing a_4 with x or $x a_1^{2^{m_1 - 1}}$, we have $a_4^2 = 1$.

Hence we may assume that $a_4^{p^{m_4}} = 1$. We claim that $[a_1, a_4] \in \langle a_2^{p^{m_2}}\rangle$. Otherwise, we may assume that $[a_1, a_4] = a_1^{\gamma p^{m_1}} a_2^{\alpha p^{m_2}}$ where $(\gamma, p) = 1$. By calculation, $\langle a_1, a_4 a_3^{-\alpha}\rangle$ is neither abelian nor normal in G, a contradiction. Hence $[a_1, a_4] \in \langle a_2^{p^{m_2}}\rangle$.

Let $L = \langle a_1, a_2, a_4\rangle$. If $[a_1, a_4] \neq 1$, then, by suitable replacement, we may assume that $[a_1, a_4] = a_2^{p^{m_2}}$. In this case, we claim that $L' = G'$. If not, then $L' = \langle a_2^{p^{m_2}}\rangle$. Since $G' \not\leq \langle a_2, a_4\rangle$, $[a_2, a_4] = 1$ by Theorem 2.6. Since $K' = G'$, $K' = \langle a_2^{p^{m_2}}, [a_2, a_3]\rangle$. Hence we may assume that $[a_2, a_3] = a_1^{sp^{m_1}} a_2^{tp^{m_2}}$ where $(s, p) = 1$. If $(t, p) = 1$, then $\langle a_1^{sp^{m_1 - m_2}} a_2^t, a_3 a_4^{-1}\rangle$ is neither abelian nor normal in G, a contradiction. If $t = 0$ and $m_1 > m_2$, then $\langle a_1 a_2, a_3 a_4^{-1}\rangle$ is neither abelian nor normal in G, a contradiction. If $t = 0$ and $m_1 = m_2$, then $\langle a_1 a_2, a_3 a_4^{s-1}\rangle$ is neither abelian nor normal in G, also a contradiction.

By a similar argument as above, for $4 \leq i \leq r$, we may assume that $a_i^{p^{m_i}} = 1$ and $[a_1, a_i] = 1$ or $a_2^{p^{m_2}}$. Moreover, we have:

$$(*)\quad \text{If } [a_1, a_i] = a_2^{p^{m_2}}, \text{ then } L' = G' \text{ where } L = \langle a_1, a_2, a_i\rangle.$$

For $3 \leq i < j \leq r$, $[a_i, a_j] = 1$ by Theorem 2.6.

Let j be the maximal integer such that $[a_1, a_j] = a_2^{p^{m_2}}$. Then $[a_1, a_k] = 1$ for $j < k \leq r$. For $3 \leq k < j$, if $[a_1, a_k] = a_2^{p^{m_2}}$, then $[a_1, a_k a_j^{-1}] = 1$. Replacing a_k with $a_k a_j^{-1}$ if necessary, we get $[a_1, a_k] = 1$.

Let $J = \langle a_1, a_2, a_j\rangle$. Then J is one of the groups of Type (C3)–(C6) in Theorem 3.1 since $J' = \langle a_1^{p^{m_1}}, a_2^{p^{m_2}}\rangle$. We claim that $[a_2, a_k] = 1$ for $3 \leq k \leq r$ and $k \neq j$. If not, then we will reduce contradictions on two subcases respectively.

Subcase 1: J is the group of Type (C3) in Theorem 3.1.

In this subcase, $[a_2, a_j] = 1$. We may assume that $[a_2, a_k] = a_2^{\gamma p^{m_2}} a_1^{\beta p^{m_1}}$ where $(\beta, p) = 1$. If $(\gamma, p) = 1$, then $\langle a_1^{\beta p^{m_1 - m_2}} a_2^{\gamma}, a_k\rangle$ is neither abelian nor normal in G, a contradiction. If $\alpha = 0$ and $m_1 > m_2$, then $\langle a_1 a_2, a_k\rangle$ is neither abelian nor normal in G, a contradiction. If $\alpha = 0$ and $m_1 = m_2$, then $\langle a_1 a_2, a_k a_j^{\beta}\rangle$ is neither abelian nor normal in G, also a contradiction.

Subcase 2: J is one of the groups of Type (C4)–(C7) in Theorem 3.1.

In this subcase, $[a_1, a_2] = 1$ and $G' = \langle [a_2, a_j], a_2^{p^{m_2}} \rangle$. Hence we may assume that $[a_2, a_k] = a_2^{\gamma p^{m_2}} [a_2, a_j]^\beta$ where $(\beta, p) = 1$. Let $x = a_k^{-\beta^{-1}} a_j$. Then $[a_1, x] = a_2^{p^{m_2}}$ and $[a_2, x] = a_2^{-\beta^{-1}\gamma p^{m_2}}$. If $(\gamma, p) = 1$, then $\langle a_2, a_k a_j^{-\beta} \rangle$ is neither abelian nor normal in G, a contradiction. If $\alpha = 0$, then $\langle a_1, a_2, x\rangle' = \langle a_2^{p^{m_2}} \rangle$. This contradicts (*).

In this case, $G = J \times A$ where $A = \langle a_3 \rangle \times \cdots \times \langle a_{j-1} \rangle \times \langle a_{j+1} \rangle \times \cdots \times \langle a_r \rangle$. Hence we get the groups (C3)–(C7) in Theorem 3.1.

Case 2: K is one of the groups of Type (C7)–(C10) in Theorem 3.1.

In this case, $G' = \langle a_s^{p^{m_s}}, a_3^{p^{m_3}} \rangle$ where $s = 1$ or 2, $[a_1, a_2] = a_3^{p^{m_3}}$ and $[a_2, a_3] = 1$. Assume that $a_4^{p^{m_4}} = a_s^{\alpha p^{m_s}} a_3^{\beta p^{m_3}}$.

If $p > 2$ or $m_3 > 1$, then, replacing a_4 with $a_4 a_s^{-\alpha p^{m_s - m_4}} a_3^{-\beta p^{m_3 - m_4}}$, we have $a_4^{p^{m_4}} = 1$. If $p = 2$, $m_3 = 1$ and $m_s > 1$, then, we claim that there exists an $x \in \{a_4, a_4 a_3\}$ such that $x^2 \in \langle a_s^{2^{m_s}} \rangle$. Otherwise, $a_4^2 = a_s^{\alpha 2^{m_s}} a_3^2$. Replacing a_4 with $a_4 a_s^{-\alpha p^{m_s - m_4}}$, we have $a_4^2 = a_3^2$. Since $[a_4, a_3] = (a_4 a_3)^2 \notin \langle a_s^{2^{m_s}} \rangle$, $\langle a_4, a_3 \rangle$ is non-abelian. It follows from Theorem 2.6 that $G' \le \langle a_4, a_3 \rangle$. Hence $[a_4, a_3] = a_s^{2^{m_s}} a_3^2$. By calculation, $\langle a_4 a_3, a_3 a_s^{2^{m_s - 1}} \rangle$ is neither abelian nor normal in G, a contradiction. Replacing a_4 with x or $x a_s^{2^{m_s - 1}}$, we have $a_4^2 = 1$. If $p = 2$ and $m_s = m_3 = 1$, then $s = 2$ since $m_1 > 1$. Hence K is a group of Type (C9). In this case, $[a_1, a_3] = a_2^2 a_3^2$. we claim that there exists an involution in $\{a_4, a_4 a_2, a_4 a_3, a_4 a_2 a_3\}$. Otherwise, since $a_4^2 \neq 1$, we have

$$a_4^2 = a_2^2,\ a_3^2 \text{ or } a_2^2 a_3^2.$$

If $a_4^2 = a_3^2$, then, by replacing a_2, a_3 with $a_3, a_2 a_3$ respectively, it is reduced to $a_4^2 = a_2^2$. If $a_4^2 = a_2^2 a_3^2$, then, by replacing a_2, a_3 with $a_2 a_3, a_2$ respectively, it is also reduced to $a_4^2 = a_2^2$. Hence we may assume that $a_4^2 = a_2^2$. Since $(a_4 a_2)^2 = [a_4, a_2] \neq 1$, $L = \langle a_4, a_2 \rangle$ is not abelian. It follows from Theorem 2.6 that $G' \le L$. Hence we may assume that $[a_4, a_2] = a_3^2 a_2^{2\alpha}$. If $[a_4, a_2] = a_3^2 a_2^2$, then $\langle a_1 a_4, a_2 \rangle$ is neither abelian nor normal in G, a contradiction. If $[a_4, a_2] = a_3^2$, then $(a_4 a_2)^2 = a_3^2$. Since $(a_4 a_2 a_3)^2 \neq 1$, $[a_4 a_2, a_3] = [a_4, a_3] = (a_4 a_2 a_3)^2 \neq 1$. Since $M = \langle a_4 a_2, a_3 \rangle$ is not abelian, $G' \le \langle a_4 a_2, a_3 \rangle$ by Theorem 2.6. Hence we may assume that $[a_4, a_3] = [a_4 a_2, a_3] = a_2^2 a_3^{2\alpha}$. Since $(a_4 a_3)^2 \neq 1$, $[a_4, a_3] \neq a_2^2 a_3^3$. Hence $[a_4, a_3] = a_2^2$. In this case, $\langle a_1 a_4 a_2, a_3 \rangle$ is neither abelian nor normal in G, a contradiction.

By the above argument, we may assume that $a_4^{p^{m_4}} = 1$. Let $\{s, t\} = \{1, 2\}$. Since $G' \not\le \langle a_t, a_4 \rangle$, $[a_t, a_4] = 1$ by Theorem 2.6. By the definition relations of (C7)–(C10), $m_t > m_3$. It follows from Theorem 2.6 that $[a_t a_3, a_4] = 1$ since $G' \not\le \langle a_t a_3, a_4 \rangle$. Hence $[a_3, a_4] = 1$. We claim that $[a_s, a_4] \in \langle a_3^{p^{m_3}} \rangle$. Otherwise, we may assume that $[a_s, a_4] = a_s^{\alpha p^{m_s}} a_3^{\beta p^{m_3}}$ where $(\alpha, p) = 1$. By calculation, $\langle a_s, a_t^\beta a_4^{s-t} \rangle$ is neither abelian or normal in G, a contradiction. Hence we may assume that $[a_s, a_4] = a_3^{\beta p^{m_3}}$.

We claim that $[a_s, a_4] = 1$. If not, then, $(\beta, p) = 1$ and we may assume that $[a_s, a_4] = a_3^{p^{m_3}}$ by suitable replacement. We will reduce contradictions on three subcases respectively.

Subcase 1: $s = 2, t = 1$ and $m_2 > m_3$.

In this subcase, K is one of the groups of Type (C7)–(C8). By the definition relations of Type (C7)–(C8), $[a_1, a_3] = a_2^{\eta p^{m_2}}$ where $\eta = 1$ or ν. By calculation, $\langle a_1 a_4, a_2 a_3 \rangle$ is neither abelian or normal in G, a contradiction.

Subcase 2: $s = 2, t = 1$ and $m_2 = m_3$.

In this subcase, K is one of the groups of Type (C8)–(C9). If K is one of the groups of Type (C8), then $[a_1, a_3] = a_2^{\nu p^{m_2}}$. By calculation, $\langle a_1 a_4^{1-\nu}, a_2 a_3 \rangle$ is neither abelian or normal in G, a contradiction. If K is one of the groups of Type (C9), then $[a_1, a_3] = a_2^{kp^{m_2}} a_3^{-p^{m_3}}$ where $(k, p) = 1$. By calculation, $\langle a_1 a_4, a_2^k a_3^{-1} \rangle$ is neither abelian or normal in G, a contradiction.

subcase 3: $s = 1, t = 2$.

In this subcase, K is a group of Type (C10). By the definition relations of Type (C10), $[a_1, a_3] = a_1^{p^{m_3}}$. By calculation, $\langle a_1, a_2 a_3 a_4^{-1} \rangle$ is neither abelian or normal in G, also a contradiction.

Hence $[a_s, a_4] = 1$. By a similar argument, for $4 \leq i \leq r$, we may assume that $a_i^{p^{m_i}} = 1$. Moreover, $[a_1, a_i] = [a_2, a_i] = [a_3, a_i] = 1$. For $4 \leq i < j \leq r$, $[a_i, a_j] = 1$ by Theorem 2.6. In this case, $G = K \times A$ where $A = \langle a_4 \rangle \times \langle a_5 \rangle \times \cdots \times \langle a_r \rangle$. Hence we get the groups of Type (C7)–(C10) in Theorem 3.1. □

Lemma 3.3. *Suppose that G is a metahamilton p-group. If $G' \cong C_p^3$ and $c(G) = 2$, then G is a group of Type* (D1)–(D7) *as defined in Theorem* 3.1.

Proof Let the type of G/G' be $(p^{m_1}, p^{m_2}, \ldots, p^{m_r})$, where $m_1 \geq m_2 \geq \cdots \geq m_r$, $G/G' = \langle a_1 G' \rangle \times \langle a_2 G' \rangle \times \cdots \times \langle a_r G' \rangle$, where $o(a_i G') = p^{m_i}$, $i = 1, 2, \ldots, r$. Then $G = \langle a_1, a_2, \ldots, a_r \rangle$. If $[a_i, a_j] \neq 1$, then $G' = \langle a_i^{p^{m_i}}, a_j^{p^{m_j}}, [a_i, a_j] \rangle$ by Theorem 2.6. Hence we have:

$$\textbf{(*)} \quad \text{If } a_i^{p^{m_i}} = 1, \text{ then } a_i \in Z(G).$$

Let i be the minimal integer such that $a_i^{p^{m_i}} \neq 1$. If $i \neq 1$, then

$$a_1^{p^{m_1}} = \cdots = a_{i-1}^{p^{m_{i-1}}} = 1$$

and hence $a_1, \ldots a_{i-1} \in Z(G)$ by (*). We claim that $m_i = m_1$. If not, then $(a_1 a_j)^{p^{m_1}} = 1$ for $j \geq i$. It follows that $a_1 a_j \in Z(G)$ by (*) and hence $a_j \in Z(G)$ for $j \geq i$. This contradicts $|G'| = p^3$. Hence we may assume that $a_1^{p^{m_1}} \neq 1$.

Let j be the minimal integer such that $a_j^{p^{m_j}} \notin \langle a_1^{p^{m_1}} \rangle$. If $j \neq 2$, then we may assume that $a_k^{p^{m_k}} = a_1^{\alpha_k p^{m_1}}$ for $2 \leq k \leq j-1$. By Theorem 2.6, $[a_k, a_1] = 1$. Replacing a_k with $a_k a_1^{-\alpha_k p^{m_1 - m_k}}$, we get $a_k^{p^{m_k}} = 1$. By (*), $a_k \in Z(G)$ for $2 \leq k \leq j-1$. We claim that $m_j = m_2$. If not, then $(a_2 a_k)^{p^{m_2}} = 1$ for $k \geq j$. It follows that $a_2 a_k \in Z(G)$ by (*) and hence $a_k \in Z(G)$ for $k \geq j$. This contradicts $|G'| = p^3$. Hence we may assume that $a_2^{p^{m_2}} \notin \langle a_1^{p^{m_1}} \rangle$.

Let k be the minimal integer such that $a_k^{p^{m_k}} \notin \langle a_1^{p^{m_1}}, a_2^{p^{m_2}} \rangle$. If $k \neq 3$, then we may assume that $a_w^{p^{m_w}} = a_1^{\alpha_w p^{m_1}} a_2^{\beta_w p^{m_2}}$ for $3 \leq w \leq k-1$. We claim that $m_k = m_3$. If not, then $m_3 > m_k$. Without loss of generality, we may assume that $m_{k-1} > m_k$. Replacing a_w with $a_w a_1^{-\alpha_w p^{m_1 - m_w}} a_2^{\beta_w p^{m_2 - m_w}}$, we get $a_w^{p^{m_w}} = 1$. By (*), $a_w \in Z(G)$ for $3 \leq w \leq k-1$. For $w \geq k$, since $(a_3 a_w)^{p^{m_3}} = 1$, $a_3 a_w \in Z(G)$ by (*). It follows that $a_w \in Z(G)$ for $w \geq k$. This contradicts $|G'| = p^3$. Hence we may assume that $a_3^{p^{m_3}} \notin \langle a_1^{p^{m_1}}, a_2^{p^{p^{m_3}}} \rangle$.

If $r = 3$, then, by [17, Theorem 8.1], G is a group of Type (D1)–(D7) in Theorem 3.1. In the following we may assume that $r \geq 4$.

We claim that there are suitable a_1, a_2, a_3 such that the following condition:

(**) For all $x \in G'$, there exists $b \in \langle a_1, a_2, a_3 \rangle$ such that $x = b^{p^{m_3}}$.

If (**) holds, then for $i > 3$, there exists $b_i \in \langle a_1, a_2, a_3 \rangle$ such that $a_i^{p^{m_i}} = b_i^{p^{m_3}}$. By Theorem 2.6, $[a_i, b_i] = 1$. Replacing a_i with $a_i b_i^{-p^{m_3 - m_i}}$, we get $a_i^{p^{m_i}} = 1$. By (*), $a_i \in Z(G)$. Hence we get the groups (D1)–(D7) in Theorem 3.1.

In the following, we prove that we may choose suitable a_1, a_2, a_3 satisfying the condition (**). If $p > 2$ or $m_2 > 1$, then (**) holds. Hence, we only need to deal with the case where $p = 2$ and $m_2 = 1$.

Case 1. $m_1 > 1$.

If $[a_2, a_3] \neq 1$, then we may assume that $[a_2, a_3] = a_2^{2i} a_3^{2j} a_1^{2^{m_1}}$ by Theorem 2.6. If $[a_2, a_3] = a_2^2 a_3^{2j} a_1^{2^{m_1}}$, then $\langle a_2 a_1^{2^{m_1 - 1}}, a_3 \rangle$ is neither abelian nor normal in G, a contradiction. If $[a_2, a_3] = a_2^3 a_1^{2^{m_1}} = (a_3 a_1^{2^{m_1-1}})^2$, then $\langle a_3 a_1^{2^{m_1-1}}, a_3 \rangle$ is neither abelian nor normal in G, a contradiction. Hence $[a_2, a_3] = a_1^{2^{m_1}}$. In this case, it is easy to check that $G' = V_1(\langle a_1, a_2, a_3 \rangle)$. Hence (**) holds.

Case 2. $m_1 = 1$.

By an argument similar to the beginning of the proof of Theorem 3.1, we may choose suitable a_1, a_2, a_3 such that the commutative group of $K = \langle a_1, a_2, a_3 \rangle$ is of order at least 4.

If there are two elements in $\{1, a_1, a_2, a_3, a_1 a_2, a_1 a_3, a_2 a_3, a_1 a_2 a_3\}$ such that the squares are equal to each other, then, by Theorem 2.6, they are commutative. It follows that there is an involution in $\{a_1, a_2, a_3, a_1 a_2, a_1 a_3, a_2 a_3, a_1 a_2 a_3\}$. By (*), this involution is in the center of K, which contradicts $|K'| \geq 4$. Hence

$$G' = V_1(K) = \{1, a_1^2, a_2^2, a_3^2, (a_1 a_2)^2, (a_1 a_3)^2, (a_2 a_3)^2, (a_1 a_2 a_3)^2\}.$$

That is, (**) holds. □

4 Finite metahamiltonian p-groups whose derived group is of exponent $> p$

Theorem 4.1. *Suppose that G is a finite metahamiltonian p-group with $\exp(G') > p$. Then G is isomorphic to one of the following non-isomorphic groups:*

(E) *G is metacyclic.*

(E1) $\langle a, b \mid a^{p^{r+s+u}} = 1,\ b^{p^{r+s+t}} = a^{p^{r+s}},\ a^b = a^{1+p^r}\rangle$, *where* $r \geq 1$, $u \leq r$, $r+1 \geq s+u \geq 2$, *and if* $p = 2$ *then* $r \geq 2$;

(E2) $\langle a, b \mid a^{2^3} = b^{2^m} = 1, a^b = a^{-1}\rangle$, *where* $m \geq 1$;

(E3) $\langle a, b \mid a^{2^3} = 1, b^{2^m} = a^4, a^b = a^{-1}\rangle$, *where* $m \geq 1$;

(E4) $\langle a, b \mid a^{2^3} = b^{2^m} = 1, a^b = a^3\rangle$, *where* $m \geq 1$.

(F) *G is not metacyclic and G' is cyclic and* $|G'| \geq p^2$.

(F1) $K \times A$, *where* $K = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s}} = 1, a^b = a^{1+p^r}\rangle$, $u \leq r$, $r+1 > s+u \geq 2$, *and* $A \neq 1$ *is abelian such that* $\exp(A) \leq p^{(r+1)-(s+u)}$;

(F2) $K \times A$, *where* $K = \langle a, b \mid a^{p^{r+t+u}} = 1, b^{p^r} = 1, a^b = a^{1+p^{r+t}}\rangle$, $t \geq 1$, $r \geq u \geq 2$, *and* $A \neq 1$ *is abelian such that* $\exp(A) \leq p^{t+(r+1)-u}$;

(F3) $K \times A$, *where* $K = \langle a, b \mid a^{p^{r+s}} = 1, b^{p^{r+s+t}} = 1, a^b = a^{1+p^r}\rangle$, $t \geq 1$, $r+1 > s \geq 2$, *and* $A \neq 1$ *is abelian such that* $\exp(A) \leq p^{(r+1)-s}$;

(F4) $K \times A$, *where* $K = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r}\rangle$, $stu \neq 0$, $r+1 > s+u \geq 2$, *and* $A \neq 1$ *is abelian such that* $\exp(A) \leq p^{(r+1)-(s+u)}$;

(F5) $(K \rtimes B) \times A$, *where* $K = \langle a, b \mid a^{p^{r+t+u}} = 1, b^{p^r} = 1, a^b = a^{1+p^{r+t}}\rangle$, $B = \langle b_1\rangle \times \langle b_2\rangle \times \cdots \times \langle b_f\rangle$ *such that* $o(b_i) = p^{r_i}$, $[a, b_i] = a^{p^{r+t_i}}$, $[b, b_i] = 1$, $\max\{t, u-2\} < t_1 < t_2 < \cdots < t_f < t+u$, $r+t > r_1+t_1 > r_2+t_2 > \cdots > r_f+t_f \geq t+u \geq t+2$, *and* A *is abelian such that* $\exp(A) \leq p^{t+(r+1)-u}$.

(G) *the type of G' is* (p^α, p) *where* $\alpha \geq 2$.

(G1) $\langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1+m_2}} = a_2^{p^{m_2+1}} = a_3^p = 1,\ [a_1, a_2] = a_1^{p^{m_1}},\ [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = 1\rangle$, *where* $p > 2$ *and* $m_1 > m_2 \geq 1$;

(G2) $K \times A$, *where* $K = \langle a_1, a_2, a_3 \mid a_1^{p^{m_1+1+k}} = a_2^{p^{m_2+1}} = a_3^{p^{m_3}} = 1, [a_1, a_2] = a_1^{p^{m_1}}, [a_1, a_3] = a_2^{p^{m_2}}, [a_2, a_3] = 1\rangle$, $m_1 \geq m_2 \geq m_3$, $1 \leq k \leq \min\{m_1 - m_3, m_2 - m_3 + 1, m_2 - 1\}$ *and* A *is abelian such that* $\exp(A) \leq p^{m_2-k}$.

Proof If G is metacyclic, then, by Lemma 4.2, G is a group of Type (E1)–(E4) in the theorem. In the following, we may assume that G is not metacyclic. If G' is cyclic, then, by Lemma 4.5, G is a group of Type (F1)–(F5) in the theorem. If G' is not cyclic, then, by Lemma 4.6, G is a group of Type (G1)–(G2) in theorem. Finally, it is omitted to check that such groups are non-isomorphic metahamiltonian p-groups. □

Lemma 4.2. *Suppose that G is a metacyclic p-group and $|G'| \geq p^2$. If G is metahamiltonian, then G is a group of Type* (E1)–(E4) *as defined in Theorem* 4.1.

Proof Case 1: $p > 2$ or G is an ordinary metacyclic 2-group. That is,

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r} \rangle,$$

where $r \geq 1$, $u \leq r$, and if $p = 2$ then $r \geq 2$.

Since $|G'| \geq p^2$, we have $s + u \geq 2$. We only need to prove that $r + 1 \geq s + u$. Otherwise, $r + 1 < s + u$. By calculation,

$$[a^{p^{r+1}}, b] = a^{-p^{r+1}}(a^{p^{r+1}})^b = a^{p^{2r+1}} \neq 1.$$

Hence $\langle a^{p^{r+1}}, b \rangle$ is neither abelian nor normal in G, a contradiction. Thus $r+1 \geq s+u$ and G is a group of Type (E1) in Theorem 4.1.

Case 2: $p = 2$ and G is not an ordinary metacyclic 2-group.

Let $o(a) = 2^n$ and $H = \langle a^{2^{n-2}}, b \rangle$. Since $H' = \langle a^{2^{n-1}} \rangle$, H is not abelian. It follows that $H \trianglelefteq G$. By Theorem 2.6, $a^2 \in H$. Hence $n \leq 3$ and $|G'| = 4$. By Lemma 2.4, $G \in \mathcal{A}_2$. By [4, Lemma 2.4], we get groups of (E2)–(E4) in Theorem 4.1. $\square$

We need the following two lemmas on number theory. Proofs are omitted.

Lemma 4.3. *Suppose that $U = U(p^n)$ is the multiplicative group containing of all the invertible elements of $\mathbb{Z}/p^n\mathbb{Z}$, where p is an odd prime and n is a positive integer. That is,*

$$U = \{x \in \mathbb{Z}/p^n\mathbb{Z} \mid (x, p) = 1\}.$$

Let $S(U) \in \mathrm{Syl}_p(U)$. Then

$$S(U) = \{x \in U \mid x \equiv 1 \pmod{p}\},$$

and $S(U)$ is cyclic with order p^{n-1}. $S_i(U)$ where $0 \leq i < n$, the unique subgroup of $S(U)$ of order p^i, is

$$S_i(U) = \{x \in U \mid x \equiv 1 \pmod{p^{n-i}}\}.$$

Lemma 4.4. *Suppose that $U = U(2^n)$ is the multiplicative group containing of all invertible elements of $\mathbb{Z}/2^n\mathbb{Z}$, where $n \geq 2$ is a positive integer. Then*

$$\begin{aligned} U &= \langle -1 \rangle \times \langle 1 + 2^2 \rangle (\cong C_2 \times C_{2^{n-2}}) \\ &= \{\varepsilon + i2^m \mid \varepsilon = \pm 1, 2 \leq m \leq n, 1 \leq i \leq 2^{n-m} \text{ and } i \text{ is odd}\} \end{aligned}$$

For $m < n$, the order of $\varepsilon + i2^m$ is 2^{n-m} and $\langle \varepsilon + i2^m \rangle = \langle \varepsilon + j2^m \rangle$ for all odd j.

Lemma 4.5. *Suppose that G is a metahamilton p-group and G is not metacyclic. If $|G'| \geq p^2$ and G' is cyclic, then G is a group of Type* (F1)–(F5) *in Theorem* 4.1.

Proof By Theorem 2.7, $d(G) > 2$. Let $G' = \langle c \rangle$, the type of G/G' be $(p^{m_1}, p^{m_2}, \ldots, p^{m_w})$ where $m_1 \geq m_2 \geq \cdots \geq m_w$. Let

$$G/G' = \langle a_1G' \rangle \times \langle a_2G' \rangle \times \cdots \times \langle a_wG' \rangle \text{ where } o(a_iG') = p^{m_i},\ i = 1, 2, \ldots, w.$$

Then $G = \langle a_1, a_2, \dots, a_w \rangle$.

Let i be the minimal integer such that $a_i \notin C_G(G/\mho_1(G'))$. Then there exists $j > i$ such that $G' = \langle [a_i, a_j] \rangle$. If $i \neq 1$, then $a_1 \in C_G(G/\mho_1(G'))$. Replacing a_1 with $a_1 a_j$, we have $G' = \langle [a_1, a_i] \rangle$.

Let j be the minimal integer such that $G' = \langle [a_1, a_j] \rangle$. If $j \neq 2$, then $[a_1, a_2] \in \mho_1(G')$. Replacing a_2 with $a_2 a_j$, we have $G' = \langle [a_1, a_2] \rangle$.

Let $K = \langle a_1, a_2 \rangle$. By Theorem 2.7, K is metacyclic. Hence K is one of the groups in Theorem 4.2. That is, K is one of the groups (E1)–(E4) in Theorem 4.1.

Step 1: We claim that K is one of the groups of Type (E1) in Theorem 4.1.

If not, then we may assume that $K = \langle a, b \rangle$ satisfying the relations of Type (E2)–(E4) in Theorem 4.1. That is,

$$a^{2^3} = 1, b^{2^m} \in \mho_1(K') = \langle a^4 \rangle \text{ and } [a, b] \equiv a^2 \pmod{\mho_1(K')}.$$

Obviously, $G' = K' = \langle a^2 \rangle$ and $m_3 = m_4 = \cdots = m_w = 1$.

Case 1: $a_3^2 \in \mho_1(K')$ and $[a_3, b] \in \mho_1(K')$.

If $[a_3, b] = a^4$, then $\langle a_3, b \rangle$ is neither abelian nor normal in G, a contradiction. If $[a_3, b] = 1$, then $\langle a_3 a^2, b \rangle$ is neither abelian nor normal in G, a contradiction.

Case 2: $a_3^2 \in \mho_1(K')$ and $[a_3, b] \equiv a^2 \pmod{\mho_1(K')}$.

If $[a_3, a] \equiv a^2 \pmod{\mho_1(K')}$, then $(a_3 a)^2 \in \mho_1(K')$ and $[a_3 a, b] \in \mho_1(K')$. Replacing a_3 with $a_3 a$, it is reduced to Case 1. Hence we may assume that $[a_3, a] \in \mho_1(K')$. Since $[a_3, a^2] = [a_3, a]^2 = 1$, $[a_3, G'] = 1$. By calculation, $1 = [a_3^2, b] = [a_3, b]^2 [a_3, b, a_3] = [a_3, b]^2$. Hence $[a_3, b] \in \mho_1(K')$, a contradiction.

Case 3: $a_3^2 \equiv a^2 \pmod{\mho_1(K')}$.

If $[a_3, a] \in \mho_1(K')$, then, replacing a_3 with $a_3 a$, it is reduced to Case 1 or Case 2. Hence we may assume that $[a_3, a] \equiv a^2 \pmod{\mho_1(K')}$. Since $a_3^2 \equiv a^2 \pmod{\mho_1(K')}$, $[a_3^2, b] = [a^2, b] = a^4$. It follows that $[a_3, b] \equiv a^2 \pmod{\mho_1(K')}$. Since $(a_3 a)^2 \equiv a^2 \pmod{\mho_1(K')}$, similar reason as above gives that $[a_3 a, b] \equiv a^2 \pmod{\mho_1(K')}$. Hence $[a, b] \in \mho_1(K')$, a contradiction.

Step 2: By suitable replacement, we may assume $a_i^{p^{m_i}} = 1$, where $3 \le i \le w$. Moreover, $[a_i, a_j] = 1$ for all $3 \le i, j \le w$.

By Step 1, $K \cong \langle r, s, t, u \rangle_p$ where $r \geq 1$, $u \leq r$, $r + 1 \geq s + u$, and if $p = 2$ then $r \geq 2$. Assume that

$$K = \langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r} \rangle.$$

Let $L = \langle a, a_i \rangle$ and $x_i \in L$ such that $L = \langle a, x_i \rangle$ and $\langle x_i \rangle \cap \langle a \rangle$ has minimal order. We claim that $x_i^{p^{m_i}} = 1$. Otherwise, we may assume that

$$\langle x_i \rangle \cap \langle a \rangle = \langle a^{p^\alpha} \rangle \text{ and } \langle [x_i, a] \rangle = \langle a^{p^\beta} \rangle \text{ where } \alpha \geq r \text{ and } \beta \geq r.$$

Then there exist integers y and z such that $(yz,p)=1$, $x_i^{p^{m_i}}=a^{yp^{\alpha}}$ and $[x_i,a]=a^{zp^{\beta}}$. By calculation,

$$\begin{aligned}(x_i a^{-yp^{\alpha-m_i}})^{p^{m_i}} &= x_i^{p^{m_i}}[x_i,a^{yp^{\alpha-m_i}}]^{\binom{p^{m_i}}{2}}[x_i,a^{yp^{\alpha-m_i}},x_i]^{\binom{p^{m_i}}{3}}a^{-yp^{\alpha}}\\ &= a^{yzp^{\alpha+\beta-m_i}\binom{p^{m_i}}{2}}[a^{yzp^{\alpha+\beta-m_i}\binom{p^{m_i}}{3}},x_i]\end{aligned}$$

Noting that $\beta\geq r\geq 2$ for $p=2$, we have $(x_i a^{-yp^{\alpha-m_i}})^{p^{m_i}}\in\langle a^{p^{\alpha+1}}\rangle$, which is contrary to the choice of x_i. Replacing a_i with x_i, we have $a_i^{p^{m_i}}=1$ where $3\leq i\leq w$.

For $3\leq i,j\leq w$, we claim that $[a_i,a_j]=1$. Otherwise, Theorem 2.6 gives that $G'\leq\langle a_i,a_j\rangle$. It is easy to see that $\langle a_i,a_j\rangle$ is not metacyclic. This contradicts Theorem 2.7.

Step 3: K is one of the following groups:

(A) $\langle a,b\mid a^{p^{r+s+u}}=1, b^{p^{r+s}}=1, a^b=a^{1+p^r}\rangle$, where $r\geq 2$ for $p=2$ and $r+1\geq s+u\geq 2$;

(B) $\langle a,b\mid a^{p^{r+t+u}}=1, b^{p^{r}}=1, a^b=a^{1+p^{r+t}}\rangle$, where $t\geq 1$ and $r\geq u\geq 2$;

(C) $\langle a,b\mid a^{p^{r+s}}=1, b^{p^{r+s+t}}=1, a^b=a^{1+p^r}\rangle$, where $r\geq 2$ for $p=2$, $t\geq 1$ and $r+1\geq s\geq 2$;

(D) $\langle a,b\mid a^{p^{r+s+u}}=1, b^{p^{r+s+t}}=a^{p^{r+s}}, a^b=a^{1+p^r}\rangle$, where $r\geq 2$, $stu\neq 0$ and $r+1\geq s+u\geq 2$.

Assume that $K=\langle a,b\mid a^{p^{r+s+u}}=1, b^{p^{r+s+t}}=a^{p^{r+s}}, a^b=a^{1+p^r}\rangle$. If $t=0$, then we have $(ba^{-1})^{p^{r+s}}=1$ for $p>2$ and $(ba^{2^u-2^{r-1}-1})^{2^{r+s}}=1$ for $p=2$. Replacing b with ba^{-1} or $ba^{2^u-2^{r-1}-1}$ respectively, we get a group of Type (A). In the following we may assume that $t\geq 1$. If $s=0$, then $(a^{-1}b^{p^t})^{p^r}=1$. Replacing a and b with b and $a^{-1}b^{p^t}$, respectively, we get a group of Type (B). If $u=0$, then we get a group of Type (C). If $su\neq 0$, then we get a group of Type (D).

Step 4: Determine G in which K is a direct factor. That is, $G=K\times A$. Since $K'=G'$, A is abelian.

Case 1: K is a group of Type (A) in Step 3.

Let $d\in A$ and $o(d)=p^e$. By calculation,

$$[a^{p^{s+u-1}}d,b]=a^{p^{r+s+u-1}}\neq 1.$$

It follows that

$$a^{p^r}\in\langle(a^{p^{s+u-1}}d)^{p^e}\rangle=\langle a^{p^{e+s+u-1}}\rangle.$$

Hence $e+s+u-1\leq r$. By the arbitrariness of d, we get $\exp(A)\leq p^{(r+1)-(s+u)}$. Since G is not metacyclic, $A\neq 1$. It follows that $r+1>s+u$. Hence we get a group of Type (F1) in Theorem 4.1.

Case 2: K is a group of Type (B) in Step 3.

Let $d \in A$ and $o(d) = p^e$. By calculation,

$$[a^{p^{u-1}}d, b] = a^{p^{r+t+u-1}} \neq 1.$$

It follows that

$$a^{p^{r+t}} \in \langle (a^{p^{u-1}}d)^{p^e} \rangle = \langle a^{p^{e+u-1}} \rangle.$$

Hence $e + u - 1 \leq r + t$. By the arbitrariness of d, we get $\exp(A) \leq p^{t+(r+1)-u}$. Hence G is a group of Type (F2) in Theorem 4.1.

Case 3: K is a group of Type (C) or (D) in Step 3.

Let $d \in A$ and $o(d) = p^e$. By calculation,

$$[a^{p^{s+u-1}}d, b] = a^{p^{r+s+u-1}} \neq 1.$$

It follows that

$$a^{p^r} \in \langle (a^{p^{s+u-1}}d)^{p^e} \rangle = \langle a^{p^{e+s+u-1}} \rangle.$$

Hence $e+s+u-1 \leq r$. By that arbitrariness of d, we get $\exp(A) \leq p^{(r+1)-(s+u)}$. Since G is not metacyclic, $A \neq 1$. It follows that $r + 1 > s + u$. Hence we get a group of Type (F3) or (F4) in Theorem 4.1.

Step 5: Determine G in which K is not a direct factor.

Let $G = H \times A$, where $K < H$ and A is as large as possible for K. Since $K' = G'$, A is abelian. By Step 2, we may assume that $H = K \rtimes B$ where $B = \langle b_1 \rangle \times \langle b_2 \rangle \times \cdots \times \langle b_f \rangle$ such that $o(b_i) = p^{r_i}$, $o(bG') \geq r_1 \geq r_2 \geq \cdots \geq r_f$.

We claim that K is neither a group of Type (C) nor (D) in Step 3. Otherwise, by calculation, $\langle ab^{-p^t} \rangle \cap \langle a \rangle = 1$. Since $G' \not\leq \langle ab^{-p^t}, b_i \rangle$, Theorem 2.6 gives that $[ab^{-p^t}, b_i] = 1$. Similar reason gives that $[b, b_i] = 1$. Hence $H = K \times B$, which is contrary to the choice of H.

If K is a group of Type (A) in Step 3, then we claim that $s = 0$. Otherwise, by calculation, $\langle ab \rangle \cap \langle a \rangle \leq \langle a^{p^{r+1}} \rangle$. Since $G' \not\leq \langle ab, b_i \rangle$, Theorem 2.6 gives that $[ab, b_i] = 1$. Similar reason gives that $[b, b_i] = 1$. Hence $H = K \times B$, which is contrary to the choice of H.

By the above argument, we may assume that

$$K = \langle a, b \mid a^{p^{r+t+u}} = 1, b^{p^r} = 1, a^b = a^{1+p^{r+t}} \rangle,$$

where $t \geq 0$ and $r \geq u \geq 2$. Since $G' \not\leq \langle b, b_i \rangle$, Theorem 2.6 gives that $[b, b_i] = 1$.

Let j be the minimal positive integer such that $[a, b_i]$ has maximal order. We may assume that $j = 1$, replacing b_1 with $b_1 b_j$ when it is necessary. Similarly, we may assume that $\langle [a, b_1] \rangle \geq \langle [a, b_2] \rangle \geq \cdots \geq \langle [a, b_f] \rangle$.

Assume that $[a,b_i]=a^{\gamma_i p^{r+t_i}}$ where $(\gamma_i,p)=1$. Then $t\le t_1\le t_2\le\cdots\le t_f$. Note that $a^b=a^{1+\gamma_i p^{r+t_i}}$. By Lemma 4.3 and 4.4, there exists positive integer w such that

$$(1+\gamma_i p^{r+t_i})^j\equiv 1+p^{r+t_i} \pmod{p^{r+t+u}}.$$

Replacing b_i with b_i^w, we have $[a,b_i]=a^{p^{r+t_i}}$.

Case 1: $t_1>t$.

If $t_2=t_1$, then $b_1b_2^{-1}$ is a direct factor of H, a contradiction. So $t_1<t_2$. Similarly, we have

$$t<t_1<t_2<\cdots<t_f.$$

If $(b_1b^{-p^{t_1-t}})^{p^{r_1}}=1$, then $b_1b^{-p^{t_1-t}}$ is a direct factor of H, a contradiction. Hence $(b_1b^{-p^{t_1-t}})^{p^{r_1}}\ne 1$. It follows that $b^{p^{r_1+t_1-t}}\ne 1$. Hence $r_1+t_1-t<r$. Thus

$$r-r_1>t_1-t>0.$$

Similarly, we have

$$r_i+t_i>r_{i+1}+t_{i+1}.$$

By Lemma 4.3 and 4.4, in the multiplicative group consisting of all invertible elements of $\mathbb{Z}/p^{r+t+u}\mathbb{Z}$, the order of $1+p^{r+t_f}$ is p^{t+u-t_f}. Since $[a,b_f^{p^{r_f}}]=1$, we have

$$a^{b_f^{p^{r_f}}}=a^{(1+p^{r+t_f})^{p^{r_f}}}=a.$$

It follows that $r_f\ge t+u-t_f$. Thus

$$t_f+r_f\ge t+u.$$

By calculation,

$$\langle ba^{p^{t-t_1+u-1}}\rangle\cap\langle a\rangle=\langle (ba^{p^{t-t_1+u-1}})^{p^r}\rangle=\langle a^{p^{r+t-t_1+u-1}}\rangle.$$

Let $N=\langle ba^{p^{t-t_1+u-1}},b_1\rangle$. Since $[ba^{p^{t-t_1+u-1}},b_1]=a^{p^{r+t+u-1}}\ne 1$, N is not abelian. By Theorem 2.6, $G'\le N$. It follows that $r+t-t_1+u-1\le r+t$. Thus

$$t_1\ge u-1.$$

Finally, by an argument similar to Step 4, we have $\exp(A)\le p^{t+(r+1)-u}$. Hence we get a group of Type (F5) in Theorem 4.1. In this case, $\exp(A)\le p^r$.

Case 2: $t_1=t$.

Suppose that h is the maximal positive integer such that $t_h=t$. Let $r'=r_h$, $t'=t+(r-r_h)$ and $\tilde K=\langle a,b_h\rangle$. Then

$$\tilde K=\langle a,b_h \mid a^{p^{r'+t'+u}}=1, b_h^{p^{r'}}=1, a^{b_h}=a^{1+p^{r'+t'}}\rangle.$$

If $h<f$, then we let $f'=f-h$. For $1\le i\le f'$, let

$$b'_i = b_{h+i},\ t'_i = t_{h+i},\ \tilde{B} = \langle b'_1\rangle \times \dots \langle b'_{f'}\rangle,\ \tilde{H} = \tilde{K} \rtimes \tilde{B}, \text{ and}$$
$$\tilde{A} = A \times \langle bb_h^{-1}\rangle \times \langle b_1 b_h^{-1}\rangle \dots \langle b_{h-1} b_h^{-1}\rangle.$$

Then $G = \tilde{H} \times \tilde{A}$, where $\tilde{A}$ is as large as possible. Notice that $t'_1 > t'$. By a similar argument to Case 1, we get a group of Type (F5) in Theorem 4.1.

If $h = f$, then we also have $G = \tilde{H} \times \tilde{A}$. The difference in this case from the case $h < f$ is $\tilde{H} = \tilde{K}$. By an argument similar to Step 4, we have $\exp(A) \le p^{t'+(r'+1)-u}$. Hence we get a group of Type (F2) in Theorem 4.1. □

Lemma 4.6. *Suppose that G is a finite metahamilton p-group. If $\exp(G') > p$ and G' is not cyclic, then G is a group of Type (G1)–(G2) in Theorem 4.1.*

Proof Let $H \le G$ such that $d(H) = 2$ and $\exp(H') > p$. By Theorem 2.7, H is metacyclic. By Theorem 2.6, $G' < H$ and hence G' is metacyclic.

Let $N = \mho_1(G')$ and $\bar{G} = G/N$. Then $\bar{G}' \cong C_p^2$. By Theorem 2.7, $d(G) > 2$ and hence $d(\bar{G}) > 2$. By Corollary 2.9, $c(\bar{G}) = 2$. Hence $\bar{G}$ is a group in Theorem 3.2. That is, $\bar{G}$ is a group of Type (C1)–(C10) in Theorem 3.1.

Suppose that $\bar{G}$ is a group of Type (C1) in Theorem 3.1. That is, $\bar{G} = \bar{K} \times \bar{A}$, where

$$\bar{K} = \langle \bar{a}_1, \bar{a}_2, \bar{b} \mid \bar{a}_1^4 = \bar{a}_2^4 = 1, \bar{b}^2 = \bar{a}_1^2, [\bar{a}_1, \bar{a}_2] = 1, [\bar{a}_1, \bar{b}] = \bar{a}_2^2, [\bar{a}_2, \bar{b}] = \bar{a}_1^2\rangle$$

and $\bar{A}$ is abelian such that $\exp(\bar{A}) \le 2$. Then

$$G' = \langle [a_1, b], [a_2, b], \mho_1(G')\rangle = \langle a_1^2, a_2^2\rangle \text{ and } \mho_1(G') = \langle a_1^4, a_2^4\rangle.$$

Let M be a maximal subgroup of $\mho_1(G')$ such that $M \trianglelefteq G$. Then we may assume that

$$M = \langle e, \mho_2(G')\rangle,\ [a_1, a_2] \equiv e^i \pmod M,$$
$$b^2 \equiv a_1^2 e^j \pmod M \text{ and } [a_1, b] \equiv a_2^2 e^k \pmod M.$$

It follows from $[a_1, a_2] \equiv e^i \pmod M$ that $[a_1^2, a_2] \equiv [a_1, a_2^2] \equiv 1 \pmod M$. It follows from $b^2 \equiv a_1^2 e^j \pmod M$ that $[a_1^2, b] \equiv [a_1, b^2] \equiv 1 \pmod M$. On the other hand, it follows from $[a_1, b] \equiv a_2^2 e^k \pmod M$ that $[a_1^2, b] \equiv [a_1, b]^2 [a_1, b, a_1] \equiv a_2^4 \pmod M$. It follows that $a_2^4 \in M$ and hence $M = \langle a_1^8, a_2^4\rangle$.

Let $L = \langle a_1 M, bM\rangle$. Since $\exp(L') = 2$, Theorem 2.8 gives that $c(L) = 2$. It follows that $[a_2^2, b] \equiv 1 \pmod M$. On the other hand, $[a_2^2, b] \equiv [a_2, b]^2 [a_2, b, a_2] \equiv a_1^4 \pmod M$. It follows that $a_1^4 \in M$. Hence $M = \mho_1(G)$, a contradiction.

Similar reasoning gives that $\bar{G}$ is not a group of Type (C2) in Theorem 3.1.

Suppose that $\bar{G}$ is a group of Type (C4) in Theorem 3.1. That is, $\bar{G} = \bar{K} \times \bar{A}$, where

$$\bar{K} = \langle \bar{a}_1, \bar{a}_2, \bar{a}_3 \mid \bar{a}_1^{p^{m_1+1}} = \bar{a}_2^{p^{m_2+1}} = \bar{a}_3^{p^{m_3}} = 1, [\bar{a}_1, \bar{a}_2] = 1, [\bar{a}_1, \bar{a}_3] = \bar{a}_2^{p^{m_2}},$$
$$[\bar{a}_2, \bar{a}_3] = \bar{a}_1^{\nu p^{m_1}}\rangle,\ p > 2,\ \nu \text{ is a fixed square non-residue modulo } p,$$
$$m_1 - 1 = m_2 \ge m_3 \text{ or } m_1 = m_2 \ge m_3, \text{ and } \bar{A} \text{ is abelian such that } \exp(\bar{A}) \le p^{m_2}.$$

Then $G' = \langle [a_1,a_3],[a_2,a_3],\mho_1(G')\rangle = \langle [a_1,a_3],[a_2,a_3]\rangle = \langle a_1^{p^{m_1}}, a_2^{p^{m_2}}\rangle$. Since $\langle \bar{a}_1,\bar{a}_3\rangle$ and $\langle \bar{a}_2,\bar{a}_3\rangle$ are not metacyclic, $\langle a_1,a_2\rangle$ and $\langle a_1,a_3\rangle$ are not metacyclic. By Theorem 2.7, $[a_1,a_2]^p = 1$ and $[a_1,a_3]^p = 1$. Moreover, $\exp(G') = p$, a contradiction.

Similar reasoning gives that $\bar{G}$ is not a group of Type (C5)–(C10) in Theorem 3.1.

By the above argument, $\bar{G}$ is a group of Type (C3) in Theorem 3.1. That is, $\bar{G} = \bar{K}\times\bar{A}$, where

$$\bar{K} = \langle \bar{a}_1,\bar{a}_2,\bar{a}_3 \mid \bar{a}_1^{p^{m_1+1}} = \bar{a}_2^{p^{m_2+1}} = \bar{a}_3^{p^{m_3}} = 1, [\bar{a}_1,\bar{a}_2] = \bar{a}_1^{p^{m_1}}, [\bar{a}_1,\bar{a}_3] = \bar{a}_2^{p^{m_2}}, [\bar{a}_2,\bar{a}_3] = 1\rangle,\ m_1 > 1 \text{ for } p = 2,$$
$$m_1 \geq m_2 \geq m_3 \text{ and } \bar{A} \text{ is abelian such that } \exp(\bar{A}) \leq p^{m_2}.$$

Then $G' = \langle a_1^{p^{m_1}}, a_2^{p^{m_2}}\rangle$.

Since $G' \not\leq \langle a_2,a_3\rangle$, $[a_2,a_3] = 1$. Since $\langle \bar{a}_1,\bar{a}_3\rangle$ is not metacyclic, $\langle a_1,a_3\rangle$ is not metacyclic. By Theorem 2.7, $[a_1,a_3]^p = 1$. Let $[a_1,a_3] = a_2^{p^{m_2}}d$ where $d \in \mho_1(G')$. Then $a_2^{p^{m_2+1}}d^p = 1$. It follows that $a_2^{p^{m_2+1}} \in \mho_2(G')$. Hence

$$o(a_1) > p^{m_1+1},\ N = \mho_1(G') = \langle a_1^{p^{m_1+1}}, a_2^{p^{m_2+1}}\rangle = \langle a_1^{p^{m_1+1}}\rangle \text{ and } a_2^{p^{m_2+1}} \in \langle a_1^{p^{m_1+2}}\rangle.$$

Since $G' \not\leq \langle a_2, a_3a_1^{p^{m_1}}\rangle$, $[a_2, a_3a_1^{p^{m_1}}] = 1$ and hence $[a_1^{p^{m_1}}, a_2] = a_1^{p^{2m_1}} = 1$. Assume that the order of a_1 is p^{m_1+1+k} where $k \geq 1$. Then $m_1 > k$.

Let $\bar{A} = \langle \bar{a}_4\rangle \times \langle \bar{a}_5\rangle \times \cdots \times \langle \bar{a}_f\rangle$ and the type of $\bar{A}$ be $(p^{m_4}, p^{m_5}, \dots, p^{m_f})$. For $4 \leq i \leq f$ and $1 \leq j \leq f$, since $G' \not\leq \langle a_i, a_j\rangle$, $[a_i,a_j] = 1$ and hence $a_i \in Z(G)$. Assume that $a_i^{p^{m_i}} = a_1^{sp^{m_1+1}}$. Then $(a_ia_1^{-sp^{m_1+1-m_i}})^{p^{m_i}} = 1$. Let $b_i = a_ia_1^{-sp^{m_1+1-m_i}}$, $A = \langle b_4\rangle \times \langle b_5\rangle \times \cdots \times \langle b_f\rangle$ and $K = \langle a_1,a_2,a_3\rangle$. Then $G = K \times A$.

Assume that $[a_1,a_2] = a_1^{p^{m_1}}a_1^{up^{m_1+1}}$. Then $a_1^{a_2} = a_1^{1+(1+up)p^{m_1}}$. By Lemma 4.3 and Lemma 4.4, there exists a positive integer w such that $(1+(1+up)p^{m_1})^j = 1 + p^{m_1}$. Replacing a_2 and a_3 with a_2^w and a_3^w respectively, we have $[a_1,a_2] = a^{p^{m_1}}$.

By Lemma 4.3 and Lemma 4.4, in the multiplicative group consisting of all invertible elements of $\mathbb{Z}/p^{m_1+1+k}\mathbb{Z}$, the order of $1+p^{m_1}$ is p^{k+1}. Since $a_1^{(1+p^{m_1})^{p^{m_2+1}}} = a_1^{a_2^{p^{m_2+1}}} = a_1$, we have $k+1 \leq m_2+1$. Hence $k \leq m_2$.

Case 1: $k = m_2$.

In this case, $m_1 > m_2$ and $[a_1, a_2^{p^{m_2}}] \neq 1$. It follows that $c(\langle a_1,a_3\rangle) > 2$, Corollary 2.9 gives that $p > 2$ and $\langle a_1,a_3\rangle \in \mathcal{A}_2$. If $m_3 > 1$, then $\langle a_1, a_2^{p^{m_2}}a_3^p\rangle$ is neither abelian nor normal in G, a contradiction. Hence we have $m_3 = 1$. If $A \neq 1$, then, letting $1 \neq e \in A$, $\langle a_1, a_2^{p^{m_2}}e\rangle$ is neither abelian nor normal in G, a contradiction. Hence we have $A = 1$. Assume that $a_3^p = a_1^{vp^{m_1+1}}$. Replacing a_3 with $a_3a_1^{-vp^{m_1}}$, we have $a_3^p = 1$.

Assume that $[a_1,a_3] = a_2^{p^{m_2}}a_1^{wp^{m_1+1}}$. Then $a_2^{p^{m_2+1}}a_1^{wp^{m_1+2}} = 1$. Since

$$(a_2a_1^{wp^{m_1-m_2+1}})^{p^{m_2+1}} = 1,$$

we may assume that

$$(a_2a_1^{wp^{m_1-m_2+1}})^{p^{m_2}} = a_2^{p^{m_2}}a_1^{wp^{m_1+1}}a_1^{xp^{m_1+m_2}}.$$

Replacing a_2 with $a_2a_1^{wp^{m_1-m_2+1}}a_1^{-xp^{m_1}}$, we have $a_2^{p^{m_2+1}}=1$ and $[a_1,a_3]=a_2^{p^{m_2}}$. Hence G is a group of Type (G1) in Theorem 4.1.

Case 2: $k<m_2$.

In this case $[a_1,a_2^{p^{m_2}}]=1$. Since $[a_1,a_3,a_1]=1$, $[a_1^p,a_3]=[a_1,a_3]^p=1$. Since $G'\not\leq\langle a_2,a_3a_1^{p^{m_1-m_3+1}}\rangle$, $[a_2,a_3a_1^{p^{m_1-m_3+1}}]=1$. It follows that

$$1=[a_1^{p^{m_1-m_3+1}},a_2]=a_1^{p^{2m_1-m_3+1}}.$$

Hence $2m_1-m_3+1\geq m_1+1+k$. That is, $m_1-m_3\geq k$. Since $G'\not\leq\langle a_1,a_2^{p^{m_2-m_3+2}}a_3^p\rangle$, $[a_1,a_2^{p^{m_2-m_3+2}}a_3^p]=1$. It follows that

$$a_1^{a_2^{p^{m_2-m_3+2}}}=a_1^{(1+p^{m_1})^{p^{m_2-m_3+2}}}=a_1.$$

By Lemma 4.3 and Lemma 4.4, in the multiplicative group consisting of all invertible elements of $\mathbb{Z}/p^{m_1+1+k}\mathbb{Z}$, the order of $1+p^{m_1}$ is p^{k+1}. Hence we have $m_2-m_3+2\geq k+1$. That is, $k\leq m_2-m_3+1$.

Let $b\in A$ and the order of b be p^e. Since $G'\not\leq\langle a_1,a_2^{p^{m_2-e+1}}b\rangle$, $[a_1,a_2^{p^{m_2-e+1}}b]=1$. It follows that $a_1^{a_2^{p^{m_2-e+1}}}=a_1^{(1+p^{m_1})^{p^{m_2-e+1}}}=a_1$. By Lemma 4.3 and Lemma 4.4, in the multiplicative group consisting of all invertible elements of $\mathbb{Z}/p^{m_1+1+k}\mathbb{Z}$, the order of $1+p^{m_1}$ is p^{k+1}. Hence we have $m_2-e+1\geq k+1$. That is, $e\leq m_2-k$. By the arbitrariness of b, $\exp(A)\leq p^{m_2-k}$.

Assume that $a_3^p=a_1^{vp^{m_1+1}}$. Replacing a_3 with $a_3a_1^{-vp^{m_1}}$, we have $a_3^p=1$.

Assume that $[a_1,a_3]=a_2^{p^{m_2}}a_1^{wp^{m_1+1}}$. Then $a_2^{p^{m_2+1}}a_1^{wp^{m_1+2}}=1$. Replacing a_2 with $a_2a_1^{wp^{m_1-m_2+1}}$, we have $a_2^{p^{m_2+1}}=1$ and $[a_1,a_3]=a_2^{p^{m_2}}$. Hence G is a group of Type (G2) in Theorem 4.1. □

Summarizing, we have the following

Main Theorem. Suppose that G is a finite metahamiltonian p-group. If $\exp(G')=p$, then G is one of the groups listed in Theorem 3.1. If $\exp(G')>p$, then G is one of the groups listed in Theorem 4.1.

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