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The Classification of Monopoles for the Classical Groups

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Abstract. By studying a construction of Nahm, we compute the moduli spaces of monopoles with maximal symmetry breaking at infinity for $SU(N)$, $SO(N)$ and $Sp(N)$; these are found to be equivalent to spaces of holomorphic maps from $\mathbb{P}_1$ into flag manifolds.

Introduction

Let P be a principal G-bundle over $\mathbb{R}^3$, G a compact group, ∇ a connection on P with curvature F, φ (the "Higgs field") a section of ad(P), the associated adjoint bundle: (∇, φ) is a *monopole* if it solves the Bogomoln'yi equation, $F = *\nabla\varphi$, and if it satisfies the boundary condition of having finite action, with φ tending toward a finite limit at infinity, with values in a fixed G-orbit in ad(P). Such monopoles, particularly for the group $SU(2)$, have been extensively studied in recent years, from various points of view [JT, Hi, Mu]. One particularly successful construction, due to Nahm [N], describes these monopoles in terms of solutions to some non-linear ordinary differential equations, Nahm's equations. A theorem, whose full proof is due to Hitchin [Hi], shows that for $SU(2)$, there is a natural equivalence between $SU(2)$ monopoles and an appropriate class of solutions to Nahm's equations. Using this, Donaldson was able to give a description of the moduli space of $SU(2)$ monopoles:

Theorem [D1]. *Given an isomorphism $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$, compatible with the usual metrics there is a natural correspondence between a circle bundle $\tilde{M}_k$ defined over the moduli space of $SU(2)$ monopoles of charge k, and the complex manifold R_k of rational maps $f: \mathbb{P}_1 \to \mathbb{P}_1$ of degree k, with $f(\infty) = 0$.*

In terms of the monopole, the extra circle corresponds to the choice of a framing at infinity; see [AHi, Hu].

Recently, in [HuM], a proof was given of the validity of Nahm's construction for all the classical groups, for monopoles with maximal symmetry breaking at infinity. This condition means that if G is the gauge group with maximal torus T,

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the Higgs field at infinity takes values in a fixed adjoint orbit of the form G/T. These monopoles have a discrete classification given by their "topological charge." an element of $\pi_2(G/T)$ defined in terms of their Higgs field at infinity. This charge is essentially an r-tuple $(m_1, \ldots, m_r)$ of integers, where $r = \mathrm{rank}(G)$; it can be shown that, once suitable normalizations are made, $m_i \geqq 0$. For $SU(N)$ the result is then:

Theorem [N, HuM]. *There is a natural correspondence between:*

A) *Elements of a non-empty union $M_m = M(m_1, \ldots, m_{N-1})$ of connected components of the moduli space of $SU(N)$ monopoles of charge $m = (m_1, \ldots, m_{N-1})$ with φ asymptotic to a conjugate of* $\mathrm{diag}(\mu_1, \ldots, \mu_N)$, *$\mu_1 < \cdots < \mu_N$, and*

B) *Conjugacy classes under $U(m_j)$, for each non-zero m_j, of analytic $u(m_j)$-valued functions ${}^iT_j(z)$, $i = 1, 2, 3$, on the intervals (μ_j, μ_{j+1}), such that,*

1) *The ${}^iT_j$ solve Nahm's equations*

$$\frac{d\,{}^iT_j}{dz} + \frac{1}{2} \sum_{k,l=1}^{3} \varepsilon_{ikl} [{}^kT_j, {}^lT_j] = 0.$$

2) *At a boundary point μ_j, setting $t = (z - \mu_j)$, with the convention $m_0 = m_N = 0$,*

i) *if $m_{j-1} < m_j$,*
—there exist finite, non-zero limits $C_j^i = \lim_{t \to 0^-} {}^iT_{j-1}(t)$ and ${}^iT_{j-1}(t)$ is analytic at $t = 0$.

—For $t > 0$, one can conjugate ${}^iT_j(t)$ by a unitary matrix so that one has the expansion near $t = 0$:

$$\begin{array}{cccc} & \longleftarrow (m_{j-1}) \longrightarrow & \longleftarrow (m_j - m_{j-1}) \longrightarrow & \\ {}^iT_j \cong & \left(\begin{array}{c:c} C_j^i + O(t) & O\left(t^{\frac{m_j - m_{j-1} - 1}{2}}\right) \\ \hdashline O\left(t^{\frac{m_j - m_{j-1} - 1}{2}}\right) & \dfrac{r_j^i}{t} + O(1) \end{array}\right) & & \begin{array}{c} m_{j-1} \\ m_j - m_{j-1} \end{array} \end{array}$$

The upper diagonal block is analytic in t; the lower diagonal block is meromorphic in t, and the off-diagonal blocks are of the form $t^{\left(\frac{m_j - m_{j-1} - 1}{2}\right)} \times$ (analytic in t).

ii) *if $m_{j-1} > m_j$,*
one has the same boundary behaviour, but with the roles of (μ_{j-1}, μ_j), (μ_j, μ_{j+1}) reversed.

iii) *if $m_{j-1} = m_j$,*
one has finite analytic limits C_i^+, C_i^-, of the ${}^iT$ from both sides of μ_j; if one sets

$$A^{\pm}(\zeta) = (C_2^{\pm} + iC_3^{\pm}) + (2iC_1^{\pm})\zeta + (C_2^{\pm} - iC_3^{\pm})\zeta^2,$$

one asks that $A^+(\zeta) - A^-(\zeta)$ be of rank at most one for all ζ.

3) *For $m_{j-1} \neq m_j$ the residues r_j^i define an irreducible representation of $su(2)$.*

For the groups $SO(N)$, $Sp(k)$ one embeds $SO(N)$ into $SU(N)$, $Sp(k)$ into $SU(2k)$

in the natural way. One then has the following table.

A G-monopole for $G =$	with G-charges:	embedded in $SU(N)$	As an $SU(N)$-monopole, its Higgs field is asymptotic to diag(μ_i) with:	and is has $SU(N)$ charges m_i, with
$Sp(k)$	$\rho_1,\ldots,\rho_k$	$N = 2k$	$\mu_i = -\mu_{2k+1-i}$ $i = 1,\ldots,k$	$m_i = m_{2k-i} = \rho_i$ $i = 1,\ldots,k.$
$SO(2k)$	$\rho_1,\ldots,\rho_{k-2}$ ρ_+,ρ_-	$N = 2k$	$\mu_i = -\mu_{2k+1-i}$ $i = 1,\ldots,k$	$m_i = m_{2k-i} = \rho_i$ $i = 1,\ldots,k-2$ $m_{k-1} = m_{k+1} = \rho_+ + \rho_-$ $m_k = 2\rho_+.$
$SO(2k+1)$	$\rho_1,\ldots,\rho_k$	$N = 2k+1$	$\mu_i = -\mu_{2k+2-i}$ $i = 1,\ldots,k+1$	$m_i = m_{2k+1-i} = \rho_i$ $i = 1,\ldots,k-1$ $m_k = m_{k+1} = 2\rho_k$

With this in mind, one then has

Theorem [N, HuM]. *There is a natural equivalence between*

A) *Elements of a non-empty union M_ρ of components of the moduli space of G-monopoles of charge ρ, with φ asymptotic under the inclusion of G into $SU(N)$ to a conjugate of* diag(μ_j), $\mu_1 < \cdots < \mu_N$, *and*

B) *Conjugacy classes under $U(m_j)$ for the $m_j \neq 0$, of analytic $u(m_j)$-valued functions ${}^iT_j(z)$ on (μ_j, μ_{j+1}) $i = 1,2,3$, satisfying conditions:*

   1), 2), 3) *as above, and*

   4) *There are matrices c_j with*

$$^iT_{N-j+1}(-z)^T = (c_j)(^iT_j(z))(c_j)^{-1},$$

   c_j, c_{j-1} being compatible in the obvious way at the boundary points, and with

$$c_{N-j+1} = c_j^T \quad \textit{for} \quad Sp, \quad c_{N-j+1} = -c_j^T \quad \textit{for} \quad SO.$$

Remark. M_m is essentially the union of the components of the moduli space containing multi-monopoles looking like the sum of simple monopoles spaced far apart; we shall see later that M_m is in fact connected. It is widely expected that the monopole moduli itself is connected: this is in fact the case for $SU(2)$ and $SU(3)$ [T]. If so, then the theorem above applies to all monopoles.

Once this correspondence with solutions to Nahm's equations is proven, it is natural to try to emulate Donaldson and describe the space M_ρ. This is the purpose of this article; we prove a conjecture of Atiyah and Murray:

Theorem. *Let G be a classical compact group ($SU(n)$, $Sp(n)$, $SO(n)$); let T be a maximal torus of G and g_0 a fixed element of G.*

Given an isomorphism $\mathbb{R}^3 \sim \mathbb{R} \times \mathbb{C}$, compatible with the usual matrics, there is a space $\tilde{M}_\rho$ mapping to M_ρ, which over the open set in M_ρ of irreducible monopoles is a principal T-bundle, and a natural bijective correspondence between $\tilde{M}_\rho$ and the *complex variety R_ρ of rational holomorphic maps*

$$f: \mathbb{P}_1 \to G/T$$

of degree ρ such that $f(\infty) = g_0 T$.

The fibre of $\tilde{M}_\rho$ over M_ρ corresponds to a choice of framing of the monopole at infinity; see Sect. 4.

The strategy of the proof follows that of Donaldson:

—One first divides Nahm's equations into a "real" equation and a "complex" equation, in such a way that the complex equation is invariant under a complex group $\mathscr{G}$ of gauge transformations; one then shows by variational methods that each $\mathscr{G}$-orbit contains an essentially unique solution to the real equation.

—One then classifies the $\mathscr{G}$-orbits in terms of rational maps. Essentially, we will find that the algebraic data we can extract from a $\mathscr{G}$-orbit describes a rational map in terms of "poles and residues."

The result, and its proof, constitute another example of the remarkable link between solutions to the anti-self-duality equations and holomorphic objects. (Both monopoles and Nahm's equations are examples of the anti-self-duality condition on $\mathbb{R}^4$, reduced by translational symmetries.) The correspondence with holomorphic objects functions essentially by forgetting part of the structure of a solution to the anti-self-duality equations. Conversely, one shows that one can recover the solution to the anti-self-duality equations from the holomorphic object, essentially by solving a variational problem. Other examples of this general pattern can be found in [D2, D3, UY, B].

The paper is organized as follows: the first four sections deal with the case of $SU(N)$; the fifth will show how to extend the result to the case of $SO(N)$ and $Sp(N)$. In Sect. 1, following Donaldson, we will show how Nahm's equations divide into two parts, one invariant under a real group of gauge transformations, the other under a large complex group $\mathscr{G}$ of gauge transformations. In Sect. 2, we show how each $\mathscr{G}$-orbit contains an essentially unique solution to the real equations. Section 3 classifies solutions to the complex equations in terms of rational maps. Section 4 will interpret this rational map in terms of the twistor construction of monopoles, generalizing [Hu].

1. Nahm's Equations and Nahm Complexes

We begin, as in [D1], by setting $B(z) = \sum_{i=1}^{3} {}^iT(z)dp_i$. (We drop the subscript index where we are not concerned about which interval (μ_j, μ_{j+1}) we are discussing, or when this is implicitly obvious.) Nahm's equations are then equivalent to the anti-self-duality equations $dB + B \wedge B = -*(dB + B \wedge B)$ on the 4-space $\mathbb{R}^4 = \{(z, p_1, p_2, p_3)\}$: to get Nahm's equations, one has simply gauged the dz-component of B to zero. It is then natural to reinsert this (skew-adjoint) component ${}^0T(z)$ into the equations; setting

$$\nabla_z({}^iT) = \frac{d^iT}{dz} + [{}^0T, {}^iT], \tag{1.1}$$

Nahm's equations become

$$\nabla_z({}^iT) + \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}[{}^jT, {}^kT] = 0, \quad i = 1, 2, 3. \tag{1.2}$$

This equation is invariant under unitary gauge transformations; if $u(z)$ is such a transformation, then

$$\begin{aligned} u({}^iT) &= u\,{}^iTu^{-1}, \quad i = 1, 2, 3 \\ u({}^0T) &= u\,{}^0Tu^{-1} - \frac{du}{dz}u^{-1} \end{aligned} \tag{1.3}$$

maps one solution of (1.2) to another.

The isomorphisms $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ in the hypotheses of the theorem allow one to introduce complex coordinates $(z + ip_1), (p_2 + ip_3)$ and to write, in a corresponding fashion

$$\alpha = \tfrac{1}{2}({}^0T + i\,{}^1T), \quad \beta = \tfrac{1}{2}({}^2T + i\,{}^3T). \tag{1.4}$$

Nahm's equations then become:

1. The "complex equation",

$$\frac{d\beta}{dz} + 2[\alpha, \beta] = 0. \tag{1.5}$$

2. The "real equation",

$$F(\alpha, \beta) = \frac{d}{dz}(\alpha + \alpha^*) + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0. \tag{1.6}$$

Let $g(z)$ lie in $Gl(m, \mathbb{C})$: setting

$$g(\alpha) = g\alpha g^{-1} - \frac{1}{2}\frac{dg}{dz}g^{-1}, \quad g(\beta) = g\beta g^{-1}, \tag{1.7}$$

one finds that such a transformation preserves solutions to the complex equation; the real equation, on the other hand, is preserved only if $g(z)$ is unitary.

We now consider boundary behaviour, on an interval (μ_j, μ_{j+1}).

Definition. μ_j is a *superior* (respectively *inferior, neutral*) boundary point of (μ_j, μ_{j+1}) if $m_j > m_{j-1}$ (respectively $<$, $=$).

μ_{j+1} is a *superior* (respectively *inferior, neutral*) boundary point of (μ_j, μ_{j+1}) if $m_j > m_{j+1}$ (respectively $<$, $=$).

At a boundary point μ_j, we will denote by

—k or k_j the absolute value $|m_j - m_{j-1}|$ of the "jump"
—$\bar{m}$ or $\bar{m}_j$ the maximum of (m_j, m_{j-1})
—$\underline{m}$ or $\underline{m}_j$ the minimum of (m_j, m_{j-1})
—$\underline{j}$ the index of the smaller dimension (either $(j-1)$ or j).

At a boundary point μ of (μ_j, μ_{j+1}), setting $t = z - \mu$, the boundary conditions of a solution to Nahm's equations are then, up to a unitary conjugation:

If μ is inferior, α_j, β_j are analytic at $t = 0$, with values $\underline{\alpha}, \underline{\beta}$ at $t = 0$. (1.8a)

If μ is neutral, α_j, β_j are analytic at $t = 0$. (1.8b)

If μ is superior, splitting $\mathbb{C}^{m_j}$ as $\mathbb{C}^m \oplus \mathbb{C}^k$, one has, near $t=0$:

$$\alpha_j = \begin{pmatrix} U & t^{(k-1)/2}V \\ t^{(k-1)/2}W & X \end{pmatrix}, \quad \beta_j = \begin{pmatrix} P & t^{(k-1)/2}Q \\ t^{(k-1)/2}R & S \end{pmatrix}.$$

with: i) U, V, W and P, Q, R analytic at $t=0$. (1.8c)

ii) X, S are meromorphic, with simple poles at $t=0$, and residues x, s, such that, in an appropriate basis:

$$x = \operatorname{diag}\left(\frac{-(k-1)}{4}, \frac{2-(k-1)}{4}, \ldots, \frac{(k-1)}{4}\right),$$

$$s = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & 0 & & \vdots \\ \vdots & 1 & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Furthermore, at μ_j, α, β satisfy the patching conditions:

If $m_j \neq m_{j-1}$ (μ_j not neutral), then the limits $\underline{\alpha}, \underline{\beta}$ of (1.8a) are equal to the $U(0)$, $P(0)$ of (1.8c). (1.9a)

If $m_j = m_{j-1}$ (μ_j neutral), one has limits $\alpha_\pm, \beta_\pm$ from both sides of μ_j; there then exist column vectors u, w in $\mathbb{C}^{m_j}$ with

$$\beta_+ - \beta_- = -\tfrac{1}{2}uw^T,$$

$$(\alpha_+ + \alpha_+^*) - (\alpha_- + \alpha_-^*) = \tfrac{1}{2}(-u\bar{u}^T + \bar{w}w^T). \quad (1.9b)$$

One can then choose for each jumping point μ_j, a v_j,

If $m_j \neq m_{j-1}$, let $v_j \in \mathbb{C}^{\bar{m}}$ be a unit vector in the $-\left(\frac{k-1}{4}\right)$ eigenspace of the residue x of (1.8c). There is an S^1 of possible choices. (1.10a)

If $m_j = m_{j-1}$, let v_j be the couple (u, w) of (1.9b); when $(u, w) \neq (0, 0)$, there is again an S^1 of possible choices; when $(u, w) = (0, 0)$, the solution is continuous at μ_j and corresponds to a $U(N-1)$ monopole embedded in $SU(N)$. (1.10b)

We will now define our complex gauge transformations, taking these boundary conditions into account.

Definition (1.11). Let $\mathscr{G}$ be the set of $(N-1)$-tuples $g = (g_1, \ldots, g_{N-1})$, with g_j a C^1-map,

$$g_j: [\mu_j, \mu_{j+1}] \to Gl(m_j, \mathbb{C})$$

smooth on the interior, such that

1. At a superior boundary point μ, g_j preserves the decomposition $\mathbb{C}^{m_j} = \mathbb{C}^m + \mathbb{C}^k$, and g_j has off diagonal blocks whose derivatives are $O(t^{(k-1)/2})$.
2. g satisfies the following patching condition
   i) at a non-neutral boundary point μ, let $\bar{g}$ denote the limit from the superior

side, and $\underline{g}$ the limit from the inferior side; let h denote the $\underline{m} \times \underline{m}$ upper diagonal block of $\bar{g}$; then $h = \underline{g}$.
ii) at a neutral boundary point, the limits from both sides coincide.

We define $\mathscr{G}_{\mathbb{R}} \subset \mathscr{G}$ to be the set of those g's with g unitary.

Definition (*1.12*). A *Nahm complex* is a triple

$$(\alpha, \beta, v) = ((\alpha_1, \ldots, \alpha_{N-1}), (\beta_1, \ldots, \beta_{N-1}), (v_1, \ldots, v_N))$$

with

$$\alpha_j, \beta_j : (\mu_j, \mu_{j+1}) \to gl(m_j, \mathbb{C})$$

and i) if $m_j \neq m_{j-1}, v_j \in \mathbb{C}^{\bar{m}_j}$
ii) if $m_j = m_{j-1}, v_j = (u_j, w_j) \in \mathbb{C}^{m_j} \times \mathbb{C}^{m_j}$
satisfying the conditions:

1. $(d\beta_j/dz) + 2[\alpha_j, \beta_j] = 0$.
2. α_j, β_j smooth in (μ_j, μ_{j+1}).
3. α, β satisfy the boundary conditions (1.8 a, b, c) up to the action of an element of $\mathscr{G}$.
4. β (but not necessarily α) satisfies the patching condition in (1.9 a, b).
5. For $m_j \neq m_{j-1}$, the v_j satisfy (1.10 a), but are not necessarily unit vectors. For $m_j = m_{j-1}, v_j$ is a couple (u, w) satisfying $(\beta_+ - \beta_-) = -\frac{1}{2} uw^T$, as in (1.9 b).

Definition (*1.13*). Two Nahm complexes (α, β, v) (α', β', v') are equivalent if there exists $g \in \mathscr{G}$ with

$$g(\alpha) = \alpha',$$
$$g(\beta) = \beta',$$
$$g_{\bar{m}_j}(\mu_j)(v_j) = v'_j \quad \forall_j. \qquad \text{(See (1.7))}$$

A *real Nahm complex* is a Nahm complex such that

1. α, β solve the real equations.
2. α, β satisfy the boundary conditions (1.8 a, b, c) up to the action of an element of $\mathscr{G}_{\mathbb{R}}$.
3. $(\alpha + \alpha^*)$ satisfies the patching condition corresponding to the one for α in (1.9 a, b).
4. When $m_j \neq m_{j-1}$, the v_j are unit vectors, and when $m_j = m_{j-1}$ the v_j satisfy (1.10 b).

Proposition (1.14). *There is a natural bijective correspondence between*

1. *Solutions to Nahm's equations, satisfying the conditions of Theorem* 2, *along with v_j satisfying* (*1.10 a, b*), (*with v_j unit vectors for $m_j \neq m_{j-1}$*); *modulo the action of* $\prod_{j=1}^{N-1} U(m_j)$.
2. *Real Nahm complexes, modulo the action of* $\mathscr{G}_{\mathbb{R}}$.

Proof. Given a solution to Nahm's equations, one can, by unitary gauge transformations which are non-constant only on a compact subset of the intervals (μ, μ_{j+1}) arrange for the boundary and patching conditions to be satisfied. One then has a real Nahm complex. Conversely, by going to a flat ($T_0 = 0$) gauge, one obtains a solution to Nahm's equations.

Proposition (1.15). ("*Normal form*").

a) *Away from a boundary point, or at an inferior or neutral boundary point, the α and β of a Nahm complex are locally equivalent* (i.e., *by an element of $\mathcal{G}$*) *to*

$$\alpha(z) = 0, \quad \beta(z) = \beta_0, \quad \textit{a constant.}$$

b) *At a superior boundary point μ, setting $t = z - \mu$, a Nahm complex is locally equivalent to, in the splitting $\mathbb{C}^m = \mathbb{C}^{\underline{m}} \oplus \mathbb{C}^k$:*

$$\alpha(t) = \frac{1}{t}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} -\frac{(k-1)}{4} & & \\ & \ddots & \\ & & \frac{(k-1)}{4} \end{matrix} \end{array}\right]$$

$$\beta(t) = \begin{array}{c} \begin{array}{ccc} \longleftarrow \underline{m} \longrightarrow & \longleftarrow k-1 \longrightarrow & \longleftarrow 1 \longrightarrow \end{array} \\ \left[\begin{array}{c|cccc|c} h & & & 0 & & t^{(k-1)/2}\cdot g \\ \hline t^{(k-1)/2}\cdot f & 0 & \cdots & & & t^{k-1}e_0 \\ \hline & 1/t & & 0 & & t^{k-2}e_1 \\ 0 & & \ddots & & & \vdots \\ & 0 & & \ddots & & te_{k-2} \\ & & & & 1/t & e_{k-1} \end{array}\right] \end{array} \begin{array}{c} \updownarrow \underline{m} \\ \updownarrow 1 \\ \\ \updownarrow k-1 \\ \\ \end{array}$$

$$v = (v_i), \; v_i = \delta_{i, \underline{m}+1}$$

with $e_i \in \mathbb{C}$, $f: \mathbb{C}^{\underline{m}} \to \mathbb{C}$, $g: \mathbb{C} \to \mathbb{C}^{\underline{m}}$, $h: \mathbb{C}^{\underline{m}} \to \mathbb{C}^{\underline{m}}$ linear maps. Furthermore, h is the limit of β from the inferior side.

Proof. The complex equations, modulo gauge, are locally trivial; solving

$$\frac{dw}{dz} = -2\alpha w \tag{1.16}$$

for a basis of vectors w_i, then performing the gauge transformation S with $S^{-1} = (w_1, \ldots, w_{m_j})$ gives the result in a). For b), one considers the equation for the transformed $w' = D \cdot w$, where $D = \mathrm{diag}(1, \ldots, 1, t^{(k-1)/2}, \ldots, t^{(k-1)/2})$. Using the boundary condition (1.8 a), the equation for w' has a regular singular point at $t = 0$. Applying the theory of such o.d.e.'s (see, e.g. [Ha]), and transforming back to w, one sees that, as in [D1],

1. There is a unique $w_1(t)$ satisfying (1.16) with
$$\lim_{t\to 0}\,(t^{-(k-1)/2}w_1(z)-v)=0.$$
2. Setting $w_i(t)=\beta^{i-1}(t)w_1(t)$, then $w_i(t)$ solves (1.16) and
$$\lim_{t\to 0}\,(t^{(i-1)-(k-1)/2}w_i(t)-x^{(i-1)}v)=0.$$
3. There are solutions $u_1(t)\cdots u_m(t)$ to (1.16), whose last k components vanish at $t=0$ to order $(k+1)/2$, and which are linearly independent at $t=0$.

Setting $\tilde{S}\cdot{}^{-1}(t)=(u_1(t),\dots,u_m(t),w_1(t),\dots,w_k(t))$, $\tilde{S}$ transforms one to a gauge with $\alpha=0$, and β a constant matrix of the form

$$\beta=\left[\begin{array}{c|ccccc|c} h & & & 0 & & & g \\ \hline & 0 & & & & 0 & e_0 \\ & 1 & \ddots & & & \vdots & \vdots \\ F & & \ddots & \ddots & & \vdots & \vdots \\ & & & \ddots & \ddots & 0 & e_{k-2} \\ & & & & & 1 & e_{k-1} \end{array}\right]$$

However $\tilde{S}$ is not regular at $t=0$; if one sets $S^{-1}=\tilde{S}^{-1}\cdot\mathrm{diag}(1\cdots 1, t^{-(k-1)/2}, t^{(-k+3)/2},\dots,t^{(k-1)/2})$, then S is regular at $t=0$, and, in fact, lies in $\mathscr{G}$. Under the action of S, α, β become of the form stated in the proposition; as S lies in $\mathscr{G}$, one sees that all but the top row of F is zero.

2. Solving the Real Equation

We now want to show that each $\mathscr{G}$-equivalence class of Nahm complexes contains a solution to the real equation, unique up to the action on $\mathscr{G}_{\mathbb{R}}$; gauging ${}^0T$ to zero, this, by Proposition (1.14) is the same as showing that each equivalence class of Nahm complexes yields a solution to Nahm's equations, satisfying the conditions of Theorem 2; this solution is unique up to the action of the $U(m_j)$.

We prove this in two steps: we first show that we can obtain on each interval (μ_j,μ_{j+1}) solutions of the desired form, satisfying the boundary conditions. We then show that the solutions on the different intervals can be patched together uniquely, to give us our global solution. The first step follows Donaldson [D1, Sect. 2] closely, so many details will be omitted.

A) *Solution on an Interval.* Fixing a solution (α,β) to the complex equations over an interval $[c,d]$, the real equation $F(g(\alpha,\beta))=0$ for g a (sufficiently smooth) map
$$g\colon [c,d]\to Gl(m,\mathbb{C})$$
has a nice variational description in terms of a functional $\mathscr{L}$ on the space of such

maps. $F=0$ is the Euler-Lagrange equation for

$$\mathscr{L}(g)=\int_c^d |g(\alpha)+g(\alpha)^*|^2+2|g(\beta)|^2\,dz. \tag{2.1}$$

One notes that $\mathscr{L}$ is invariant under unitary gauge transformations. It is convenient to factor this out. Let $\mathscr{H}=Gl(m,\mathbb{C})/U(m)$ be the space of positive hermitian matrices; one defines

$$h=h(g)=g^*g\colon (c,d)\to\mathscr{H}. \tag{2.2}$$

Now suppose that (α,β) is such that it can be gauged to a constant $(0,\beta_0)$ on $[c,d]$. One has the existence and unicity property:

Proposition (2.3) [D1, pp. 395–397]. *For any h_+,h_- in $\mathscr{H}$, there is a continuous $g\colon[c,d]\to Gl(m,\mathbb{C})$, with $h(g)=h_-,h_+$ at c, d respectively such that $(\tilde{\alpha},\tilde{\beta})=g(0,\beta_0)$ satisfies the real equation $F(\tilde{\alpha},\tilde{\beta})=0$ in (c,d).*

If g_1,g_2 are any two such g's with $h_1(z)=g_1^*g_1, h_2(z)=g_2^*g_2$ taking on equal values at the endpoints c,d, then $h_1(z)=h_2(z)$ throughout (c,d); the solution is thus unique up to a unitary gauge transformation.

The main tool for proving uniqueness is a "convexity lemma" for the eigenvalues of h: for $h\in\mathscr{H}$ with eigenvalues λ_i, set $\Phi(h)=\log\max(\lambda_i)_{i=1}^m\in\mathbb{R}$.

Lemma (2.4) [D1, p. 396]. *If $(\tilde{\alpha},\tilde{\beta})=g(\alpha,\beta)$ over $[c,d]$, then*

$$\frac{d^2}{dz^2}\Phi(h)\geqq -2(|F(\alpha,\beta)|+|F(\tilde{\alpha},\tilde{\beta})|),$$

$$\frac{d^2}{dz^2}\Phi(h^{-1})\geqq -2(|F(\alpha,\beta)|+|F(\tilde{\alpha},\tilde{\beta})|)$$

in the weak sense.

To extend (2.3) to an interval where (α,β) have poles, one must first put (α,β) into a "nice" form.

Lemma (2.5) [D1, p. 398]. *Let (α,β,v) be a Nahm complex. Then there is an equivalent "nice" Nahm complex (α',β',v') with:*

i) *$F(\alpha',\beta')$ bounded,*
ii) *$|\alpha'-\alpha'^*|$ bounded,*
iii) *$|v_j|=1$, when $m_j\neq m_{j-1}$.*
iv) *The residues of the matrices ${}^iT$ corresponding to α',β' at superior boundary points are conjugate under a $U(k)$ transformation to the standard irreducible representation τ_i of $SU(2)$ on $\mathbb{C}^k\subset\mathbb{C}^{\bar{m}}$.*

With this, one has

Theorem (2.6) [D1, pp. 399–402]. *Let (α,β,v) be a Nahm complex, nice in the sense of (2.5). Then there is a $g\colon[\mu_j,\mu_{j+1}]\to Gl(m_j,\mathbb{C})$ continuous, smooth on the interior, with $h(\mu_j)=h(\mu_{j+1})=1$, and (dg/dz) bounded in (μ_j,μ_{j+1}) such that*

1. *$g(\alpha,\beta,v)$ solves the real equation, with $\alpha=\alpha^*$ over (μ_j,μ_{j+1}),* i.e., *defines a solution to Nahm's equations.*

2. *This solution is bounded, continuous at an inferior or neutral boundary point; at a superior boundary point μ there is a unitary matrix v with*

$$|(v^iTv^{-1}-(\tau_i/(z-\mu)))|$$

bounded.

Any two such g's give solutions which are conjugate by a unitary matrix.

There remain two properties of the solutions ${}^iT(z)$ on (μ_j, μ_{j+1}) to be proven. One is that the ${}^iT$ have the correct analytic and meromorphic behaviour; the other is that they have the correct block decomposition at a superior boundary point.

In the $SU(2)$ case treated in [D1], to show that the solutions are meromorphic, one appeals to the theorem of Hitchin [Hi], which says that the ${}^iT$ of (2.6) yield an $SU(2)$ monopole; this in turn gives one enough regularity to assert that the ${}^iT$ have the appropriate analyticity. In the $SU(N)$ case, one cannot appeal to the similar theorem in [HuM]; the reason is that the construction of the monopole from the ${}^iT$ in this case assumes the analytic behaviour of the ${}^iT$. One must therefore proceed differently.

We will exploit the fact that Nahm's equations can be expressed in terms of flows on the Jacobian of a curve embedded in $T\mathbb{P}_1$. We briefly recall the pertinent details of this construction; more can be found in [HuM], [Hi]. Let ζ be a standard coordinate of $\mathbb{P}_1$, and let $\eta\to\eta(d/d\zeta)$ be the associated fiber coordinate in $T\mathbb{P}_1$. Now write

$$\begin{aligned}A(z,\zeta)&=A_0(z)+A_1(z)\zeta+A_2(z)\zeta^2\overset{\text{def}}{=}({}^2T+i\,{}^3T)(z)+2i\,{}^1T(z)\zeta+({}^2T-i\,{}^3T)(z)\zeta^2\\&=2\beta(z)+4i\alpha(z)\zeta-2\beta^*(z)\zeta^2.\end{aligned}$$

Nahm's equations can be rewritten as:

$$\frac{dA}{dz}+\left[\frac{A_1}{2}+A_2\zeta, A\right]=0. \tag{2.7}$$

Let $\mathcal{O}(k)$ denote the lift to $T\mathbb{P}_1$ of the standard line bundles $\mathcal{O}(k)$ on $\mathbb{P}_1$; one also defines a line bundle L^z, for $z\in\mathbb{C}$ by the transition function $\exp(z\eta/\zeta)$ from $\{\zeta\neq\infty\}$ to $\{\zeta\neq 0\}$. Now define the sheaf X_z over $T\mathbb{P}_1$:

$$0\to\mathcal{O}(-2)^{\oplus m}\xrightarrow{(\eta\,\mathrm{Id}-A(z,\zeta))}\mathcal{O}^{\oplus m}\to X_z\to 0. \tag{2.8}$$

X_z is supported on the (compact) curve

$$S=\{(\eta,\zeta)\,|\det(\eta\,\mathrm{Id}-A(z,\zeta))=0\}. \tag{2.9}$$

Because Nahm's equations are in Lax form, S is an invariant of the flow. One can show that if A solves Nahm's equations, $X_z=X_0\otimes L^z$. Also, one can prove that

$$H^0(S,X_z)\cong H^0(S\cap(\zeta=\zeta_0),X_z)\quad\text{for all}\quad\zeta_0. \tag{2.10}$$

To obtain $A(z,\zeta)$ from a flow $X_0\otimes L^z$, one

1. forms the (rank m) vector bundle V over $\mathbb{C}$ whose fiber at a generic z will be $H^0(S,X_z)$ (more properly, V will be a direct image sheaf under a projection $S\times\mathbb{C}\to\mathbb{C}$).

2. One then defines a natural geometric endomorphism $\tilde{A}(z,\zeta)$ of V_z, for the z verifying (2.10), by the diagram:

$$\begin{array}{ccc} H^0(S,X_z) & \xrightarrow{\sim} & H^0(S\cap(\zeta=\zeta_0),X_z) \\ \tilde{A}(z,\zeta_0)\Big\downarrow & & \Big\downarrow \begin{array}{l}\text{multiplication}\\ \text{by the fiber}\\ \text{coordinate } \eta\end{array} \\ H^0(S,X_z) & \xrightarrow{\sim} & H^0(S\cap(\zeta=\zeta_0),X_z) \end{array} \tag{2.11}$$

Note that as the sections on the left are supported on discrete points distinct from $\eta=\infty$, the map is well defined.

3. One trivializes V by an appropriate connection; $\tilde{A}(z,\zeta)$ then becomes a matrix $A(z,\zeta)$, and solves Nahm's equations. One such connection could just be the connection ∇_0 defined by evaluation at $\zeta=0$ (in an appropriate trivialization). In a ∇_0 constant basis, by (2.11), one has that $\tilde{A}(z,0)=2\beta(z)$ is constant in z. This, however, is incorrect; the connection appropriate to solving Nahm's equations is $\nabla=\nabla_0+(A_1(z)/2)dz$.

It is clear from the above that the solution one obtains is analytic, whenever (2.10) holds, and so wherever $A(z,\zeta)$ is finite. There remains the problem of a superior boundary point. Note that steps 1 and 2 are essentially algebraic; $\tilde{A}(z,\zeta)$ is meromophic in any geometrically defined basis of V. In particular, with respect to the ∇_0-flat basis, the matrix of $\tilde{A}$ is meromorphic. We will denote this matrix by $B(z,\zeta)$. Any essential singularity of the solution $A(z,\zeta)$ of Nahm's equations is then due to the passage from a ∇_0-flat basis to a ∇-flat basis. Let $f(z)$ be the matrix linking the two; one has:

$$A(z,\zeta)=f(z)B(z,\zeta)f(z)^{-1}, \tag{2.12}$$

$$A_1(z)=-2\frac{df}{dz}f^{-1}, \tag{2.13}$$

$$B_1(z)=-2f^{-1}\frac{df}{dz}. \tag{2.14}$$

Now suppose one is at a superior boundary point of (μ_j,μ_{j+1}); for definiteness, take this to be μ_j. One can choose the matrix $g(\mu_j)$ of Theorem (2.6) to preserve the decomposition $\mathbb{C}^{m_j}=\mathbb{C}^{m_{j-1}}\oplus\mathbb{C}^k$ at μ_j. Furthermore, referring to Proposition (1.15) and the fact that $\beta=\text{constant}$ in the ∇_0-gauge, to Lemma (2.5) and to Theorem (2.6), one sees that near μ_j

$$f=g\cdot k\cdot\operatorname{diag}(t^{\lambda_i}) \tag{2.15}$$

with $t=(z-\mu_j)$, $(\lambda_1,\ldots,\lambda_{m_j})=(0\ldots0,(k-1)/2,(k-3)/2,\ldots,-(k-1)/2)$, $k\in\mathcal{G}$, and dg/dt bounded along the real axis at μ_j. Differentiating, one obtains:

$$(gk)^{-1}\frac{d(gk)}{dz}=\operatorname{diag}(t^{\lambda_i})\left(-\frac{B_1}{2}-\operatorname{diag}\left(\frac{\lambda_i}{t}\right)\right)\operatorname{diag}(t^{-\lambda_i}).$$

As B_1 is meromorphic, one sees that $M=(gk)^{-1}(d(gk)/dz)$ has the block decomposition near μ_j

$$M = \begin{pmatrix} P(t) & t^{\rho/2}Q(t) \\ t^{\rho/2}R(t) & S(t) \end{pmatrix} \tag{2.16}$$

with P, Q, R, S meromorphic, and $\rho = 0, 1$ with $\rho, (k-1)$ having the same parity. As $d(gk)/dz$ is bounded, P, Q, R, S are in fact analytic at $t = 0$. Solving for (gk), one sees that (gk) has a similar decomposition, with Q, R vanishing at $t = 0$. In turn, the ${}^iT$'s of (2.6) also have a similar decomposition, with S meromorphic, P, Q, R analytic, and for $k > 1$, $t^{\rho/2}Q, t^{\rho/2}R$ vanishing at $t = 0$.

To complete our discussion of the solutions on the interval, one must show that the off-diagonal blocks vanish to the correct order at $t = 0$, for a superior boundary point with $k > 1$. One notes that, setting the off diagonal blocks in the Nahm complex to zero, and applying (2.6), one obtains a different solution ${}^iT'$ which is block diagonal, i.e., a sum of a $gl(k)$ and a $gl(\underline{m})$ solution. We view ${}^iT$ as a perturbation of ${}^iT'$, and set

$$\Delta^iT = {}^iT - {}^iT' = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \tag{2.17}$$

From what we know from the above, a_i, b_i, c_i, d_i are Frobenius series $t^\gamma \times$ (analytic in t), with $\gamma \geqq 0$ for d_i, $\gamma > 0$ for b_i, c_i, a_i. One has the equation for Δ^iT:

$$\frac{d(\Delta^iT)}{dt} + \tfrac{1}{2}\Sigma\,\varepsilon_{ijk}([\Delta^jT, {}^kT'] + [{}^jT, \Delta^kT] + [\Delta^jT, \Delta^kT]) = 0. \tag{2.18}$$

This equation has a linear term with a simple pole at $t = 0$, and a smooth quadratic term; as we know a priori that the solutions are t^γ (analytic in t) $\gamma \geqq 0$, we find that, as in the linear case, $-\gamma$ must be an eigenvalue of the residue of the linear term. As

$$\operatorname{res}({}^iT') = \begin{pmatrix} 0 & 0 \\ 0 & \tau_i \end{pmatrix}$$

with $\tau_i (k \times k)$ matrices defining an irreducible representation of $su(2)$ one finds, for the terms b_i, that

$$\frac{d}{dz}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 0 & -\tau_3 & \tau_2 \\ \tau_3 & 0 & -\tau_1 \\ -\tau_2 & \tau_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \text{lower order} = 0,$$

and a similar equation for the c_i. This polar term has a representation theoretic interpretation; each column of the matrix of b's can be thought of as an element of the representation $S^2 \otimes S^{k-1}$ of $su(2)$; in these terms, the action of the matrix of τ's on the b's is given by $\Sigma e_i \otimes e_i$, where the e_i's form the standard basis of $su(2)$. In terms of Casimir operators:

$$\Sigma\, e_i \otimes e_i = \tfrac{1}{2}(C(S^2 \otimes S^{k-1}) - C(S^2) \otimes 1 - 1 \otimes C(S^{k-1})).$$

Decomposing $S^2 \otimes S^{k-1}$ into irreducibles as $S^{k+1} \oplus S^{k-1} \oplus S^{k-3}$, and using $C(S^k) = -k(k+2)/4$, one gets as possibilities for γ: $(k-1)/2$ on S^{k+1}, -1 on S^{k-1}

$-(k+1)/2$, on S^{k-3}. The only positive γ is $(k-1)/2$, and so b_i vanishes to order $(k-1)/2$. The proof for the c_i is the same.

Remark: Once one knows the boundary behaviour of the ${}^iT$, it is easy to show, using (2.13), that if $g(\mu_j)$ respects the decomposition $\mathbb{C}^{m_j} = \mathbb{C}^{\underline{m}} \oplus \mathbb{C}^k$ at a superior boundary point, then g satisfies at μ_j the constraint for lying in $\mathscr{G}$; in fact, the g of (2.15) has analytic diagonal blocks, and off-diagonal blocks of the form $t^{(k+1)/2}\cdot$analytic.

Given a Nahm complex, then, we have produced a solution to Nahm's equations over (μ_j, μ_{j+1}) with the correct boundary behaviour. Alternately, we have, in the orbit of $\mathscr{G}$, a solution to the real equations.

There arises the question of how many different solutions one can produce in this way. Note that one obtains gauge equivalent solutions from g, g' if $g'g^{-1}(\mu_j)$, $g'g^{-1}(\mu_{j+1})$ lie in $U(m_j)$. At a neutral or inferior boundary point μ one's degree of freedom is thus $g^*g(\mu) \in \mathscr{H}(m_j) = Gl(m_j, \mathbb{C})/U(m_j)$. At a superior boundary point, one has "used up" part of this freedom to produce the "nice" Nahm complex of (2.5); g lives in $Gl(\underline{m}) \times U(k)$ and so the degrees of freedom, modulo unitary transformations, is $\mathscr{H}(\underline{m})$. Summing this up:

Theorem (2.18). *Let $r_j = \min(m_j, m_{j-1})$. Let (α, β, v) be a Nahm complex. The solutions to the real equation, over the interval (μ_j, μ_{j+1}) lying in the $\mathscr{G}$-orbit of (α, β, v) are, modulo the action of $\mathscr{G}_{\mathbb{R}}$, parametrized by*

$$\mathscr{H}(r_j) \times \mathscr{H}(r_{j+1})$$

with the convention $\mathscr{H}(0) = \{\text{point}\}$.

B) Patching Together the Intervals. We now show inductively in $(j-i)$ that for a given Nahm complex the set m_{ij} of solutions on an interval (μ_i, μ_j), $i<j$ satisfying the boundary conditions at all boundary points and the patching conditions at the "interior" boundary points μ_k, $i<k<j$, modulo $\mathscr{G}_{\mathbb{R}}$, is parametrized by $\mathscr{H}(r_i) \times \mathscr{H}(r_j)$. As $\mathscr{H}(r_1) \times \mathscr{H}(r_N)$ is a singleton, this will prove existence and unicity.

Consider then (μ_i, μ_j), (μ_j, μ_k), $i<j<k$. We would like to show that fixing $h_i \in \mathscr{H}(r_i)$, $h_k \in \mathscr{H}(r_k)$ there is a unique $h_j \in \mathscr{H}(r_j)$ such that the solutions on $(\mu_i, \mu_j), (\mu_j, \mu_k)$, corresponding to $(h_i, h_j), (h_j, h_k)$, respectively satisfy the patching condition at μ_j.

Without loss of generality, let us take $h_i = h_k = 1$. Let us fix a "base" solution (α, β) for both intervals corresponding to $h_j = 1$. For each h_j, there is a $g \in \mathscr{G}$ such that $h = g^*g$ satisfies $h(\mu_i) = 1, h(\mu_j) = h_j, h(\mu_k) = 1$, and the transform $(\tilde{\alpha}, \tilde{\beta}) = g(\alpha, \beta)$ also provides a solution over (μ_i, μ_j) and (μ_j, μ_k). The key ingredient is:

Lemma (2.19). *Let h be as above, and define $\varphi = \Phi(h)$, $\psi = \Phi(h^{-1})$ as in (2.4). Let $\Delta\varphi', \Delta\psi'$, denote the jumps in φ', ψ' at μ_j. (Here $'$ denotes d/dz.) Then, for some constant K,*

$$\Delta\varphi' \leqq -\varphi(\mu_j)\left(\frac{1}{(\mu_k - \mu_j)} + \frac{1}{(\mu_j - \mu_i)}\right) + K,$$

$$\Delta\psi' \leqq -\psi(\mu_j)\left(\frac{1}{(\mu_k - \mu_j)} + \frac{1}{(\mu_j - \mu_i)}\right) + K.$$

Proof. One begins by noting that, as h is analytic near any μ_l, one sided limits $\varphi'_{\pm}(\mu_l)$ of φ' exist at μ_l. Also, at any interior boundary point μ_l of (μ_i, μ_j), from the equation

$$\tfrac{1}{2}g' = g\alpha - \tilde{\alpha}g,$$

and the fact that both $\alpha, \tilde{\alpha}$ satisfy the patching conditions, one gets a similar patching for g'; this continuity of g' gives us

$$\varphi'_{-}(\mu_l) \leqq \varphi'_{+}(\mu_l) + k_1.$$

From the fact (2.4) that $\varphi'' \geqq 0$ weakly, one easily obtains, by Rolle's theorem, that on each interval (μ_l, μ_{l+1}), $i \leqq l < l+1 \leqq j$,

$$\varphi'_{+}(\mu_l)(\mu_{l+1} - \mu_l) \leqq \varphi(\mu_{l+1}) - \varphi(\mu_l) \leqq \varphi'_{-}(\mu_{l+1})(\mu_{l+1} - \mu_l),$$

and so

$$\varphi(\mu_{l+1}) - \varphi(\mu_l) \leqq \varphi'_{-}(\mu_j)(\mu_{l+1} - \mu_l) + k_2,$$

and so after summing, $(\varphi(\mu_i) = \varphi(\mu_k) = 0)$,

$$\varphi(\mu_j) \leqq \varphi'_{-}(\mu_j)(\mu_j - \mu_i) + k_3.$$

Similarly,

$$(\mu_k - \mu_i)\varphi'_{+}(\mu_j) \leqq -\varphi(\mu_j) + k_4.$$

which is what we need. The proof for ψ is similar.

Proposition (2.20). *Let μ_j be a non-neutral boundary point. Fixing h_i, h_k, there is a unique $h_j \in \mathscr{H}(r_j)$ such that the corresponding solutions on (μ_i, μ_j) and (μ_j, μ_k) satisfy the patching conditions at μ_j.*

Proof. We compute the norm $\operatorname{tr}((\Delta(\tilde{\alpha} + \tilde{\alpha}^*))^2)$ of the jump $\Delta(\tilde{\alpha} + \tilde{\alpha}^*)$ at μ_j as a function of $\Delta\alpha, \Delta\alpha^*$, and h, and find:

$$\operatorname{tr}(\Delta(\tilde{\alpha} + \tilde{\alpha}^*)^2) = \operatorname{tr}[h(\Delta\alpha)h^{-1} + \Delta\alpha^* - \tfrac{1}{2}(\Delta h')h^{-1}]^2. \tag{2.21}$$

where $h, \Delta h'$ are evaluated at μ_j.

Let us suppose that $h(\mu_j)$ has distinct eigenvalues, and diagonalize h by unitary gauge transformations on each side of μ_j. This modifies $\Delta\alpha$ by an essentially irrelevant unitary factor. Setting $h = \operatorname{diag}(e^{t_i})$ and expanding (2.21), one obtains

$$\sum_{ij}(\Delta\alpha_{ij}\Delta\alpha_{ji} + \Delta\bar{\alpha}_{ji}\Delta\bar{\alpha}_{ij}) + \frac{1}{4}\sum_i(\Delta t_i')^2 - \sum_i(\Delta\alpha_{ii} + \Delta\bar{\alpha}_{ii})\Delta t_i' + \sum_{i,j} e^{t_i - t_j}\Delta\alpha_{ij}\Delta\bar{\alpha}_{ij},$$

where $t_i = t_i(\mu_j)$, $\Delta t_i' = \Delta t_i'(\mu_j)$. This is of the form (quadratic polynomial in $\Delta t_i'$) + positive. As $\Delta\varphi' = \Delta(\max t_i)'$, $\Delta\psi' = \Delta(\min t_i)'$, one obtains as the quadratic term in the polynomial is positive, a bound

$$\operatorname{tr}(\Delta(\tilde{\alpha} + \tilde{\alpha}^*)^2) \geqq c((\Delta\varphi')^2 + (\Delta\psi')^2) + K$$

for some positive c, which extends to the case of non-distinct eigenvalues. As h_j tends to infinity in $\mathscr{H}(r_j)$, the same must be true of either φ^2 or ψ^2; by Lemma (2.19), one of $\Delta\varphi'$, $\Delta\psi'$ also tends to infinity.

In short, the map $\rho: \mathscr{H}(r_j) \to \mathbb{R}^+$, sending h_j to $\operatorname{tr}(\Delta(\tilde{\alpha} + \tilde{\alpha}^*)^2)$, is proper, and so it has a minimum.

There are two possibilities for a critical point $(\tilde{\alpha}, \tilde{\beta})$ of ρ: either $\Delta(\tilde{\alpha}+\tilde{\alpha}^*)=0$, or the differential of $\Delta(\tilde{\alpha}+\tilde{\alpha}^*)$ is singular. We now exclude the latter. We choose a unitary gauge so that $\tilde{\alpha}=\tilde{\alpha}^*$: performing an infinitesimal gauge change $(1+sx)$ preserving the real equations with x self adjoint, $x(\mu_i)=x(\mu_k)=0$, we obtain

$$\frac{d}{ds}(\Delta(\tilde{\alpha}+\tilde{\alpha}^*)) = -2(\Delta x' + \Delta x'^*).$$

By the infinitesimal version of (2.19), the norm of $\Delta x'$ is bounded below by that of $x(\mu_j)$ and so the differential is nowhere singular.

The only critical points of ρ are the zeros of $\Delta(\tilde{\alpha}+\tilde{\alpha}^*)$; as the differential of $\Delta(\tilde{\alpha}+\tilde{\alpha}^*)$ is nowhere zero, the zeroes are isolated; a min-max argument shows that it is unique.

Proposition (2.21). *Let μ_j be a neutral boundary point. Then, as above, fixing h_i, h_k there is a unique h_j such that the corresponding solutions on (μ_i, μ_j), (μ_j, μ_k) satisfy the patching conditions (1.9b) at μ_j.*

Proof. Again, let α, β denote our "base" solution; by the proof of the preceding proposition, we can suppose $\Delta\alpha=0$. Let u, w be the vectors such that $\Delta\beta = -\frac{1}{2}uw^T$. Let g be a gauge transformation corresponding to h_j: if $(\tilde{\alpha}, \tilde{\beta}) = g(\alpha, \beta)$, we want, at μ_j

$$\Delta(\tilde{\alpha}+\tilde{\alpha}^*) = \tfrac{1}{2}(-guu^*g^* + g^{-1*}\bar{w}w^Tg^{-1}),$$

and so, at μ_j, we want

$$M \stackrel{\text{def}}{=} \Delta(g')g^{-1} + g^{-1*}\Delta(g')^* - guu^*g^* + g^{-1*}\bar{w}w^Tg^{-1} = 0. \tag{2.22}$$

Computing $\operatorname{tr}(M^2) = \operatorname{tr}(g^*M^2g^{*-1})$, one finds:

$$\operatorname{tr}(\Delta(h')h^{-1} - huu^* + \bar{w}w^Th^{-1})^2. \tag{2.23}$$

Again, by a unitary change of gauge, it is sufficient to consider $h = \operatorname{diag}(e^{t_i})$. The diagonal in g^*Mg^{*-1} contributes to $\operatorname{tr}(M^2)$ terms of the form:

$$\rho_i^2 = (\Delta(t_i') - e^{t_i}u_i\bar{u}_i + w_i\bar{w}_ie^{-t_i})^2$$

and the off-diagonal, terms of the form:

$$\sigma_{ij} = e^{t_i+t_j}u_i\bar{u}_iu_j\bar{u}_j + e^{-t_i-t_j}w_i\bar{w}_iw_j\bar{w}_j - u_i\bar{u}_jw_j\bar{w}_i - u_j\bar{u}_iw_i\bar{w}_j.$$

The σ_{ij}'s are of the form (positive-constant): as for the ρ_i, ordering the eigenvalues by $t_1 \leqq \cdots \leqq t_m$, one has that either $\rho_1 \to +\infty$, or $\rho_m \to -\infty$ uniformly as $h_j \to \infty$, by Lemma (2.19), and so the map $f: \mathscr{H}(m_j) \to \mathbb{R}^+$ defined by $f(h_j) = \operatorname{tr}(M^2)$ is again proper, and f has a minimum.

Again, we show that the differential of $M(h_j)$ is nowhere singular. Let $\tilde{\alpha} = \tilde{\alpha}^*$, and make, as above, an infinitesimal gauge transformation $g = 1 + sx$, x self adjoint, preserving the real equation; we have

$$\frac{d}{ds}(M) = 2\Delta(x') - xuu^* - uu^*x - xww^* - ww^*x.$$

This will be nonzero if x is nonzero, by the infinitesimal version of the argument given above; if one conjugates so that $x = \operatorname{diag}(t_i)$ then one has for the diagonal terms of dM/ds:

$$\frac{d}{ds}(M)_{ii} = 2\Delta(t_i') - 2t_i u_i \bar{u}_i - 2t_i w_i \bar{w}_i$$

If $t_1 \leqq \cdots \leqq t_m$, then, at μ_j, $\Delta(t_m') < -Kt_m$, $\Delta(t_1') > -Kt_1$ for some positive K, by the infinitesimal version of (2.19), and so $|(d/ds)(M)| > c|x|$, for some positive c.

As in the preceding proposition, there is then a unique h_j for which $M(h_j) = 0$.

Combining (2.20) (2.21), one has by induction that there is on (μ_1, μ_N), a set of solutions parametrized by $\mathscr{H}(r_1) \times \mathscr{H}(r_N)$, i.e., a point.

Theorem (2.22). *Each $\mathscr{G}$-equivalence class of Nahm complexes contains a unique $\mathscr{G}_{\mathbb{R}}$-equivalence class of real Nahm complexes.*

Remark. It is perhaps appropriate here to insert a remark about the continuity of this procedure, i.e., if $(\alpha, \beta, v)(t)$ is a continuous family (in some appropriate sense) of Nahm complexes, is the procedure which assigns to $(\alpha, \beta, v)(t)$ the corresponding real complex $(\alpha', \beta', v')(t)$ continuous? The answer is yes, and the reason is to be found in Lemma (2.4). Suppose that $(\alpha, \beta, v)(0) = (\alpha', \beta', v')(0)$; then $F((\alpha, \beta)(t))$ is small for small t, and using lemma (2.4), one finds that fixing $h(\mu_1) = h(\mu_N) = 1$ the h corresponding to the gauge transformation sending $(\alpha, \beta, v)(t)$ to $(\alpha', \beta', v')(t)$ is then also small.

3. Nahm Complexes and Rational Maps

We will start by giving a sheaf theoretic formulation of a based rational map $F: \mathbb{P}_1 \to SU(N)/T$ suited to our purposes. As is usual, in what follows, we denote by the same symbol a vector bundle and its sheaf of sections. Let E be the trivial rank n bundle over $\mathbb{P}_1$, with a fixed global basis $\{e_1, \ldots, e_N\}$. One can define a standard flag of subbundles:

$$E_0^+ = \{0\}, \quad E_1^+ = \langle e_1 \rangle, \quad E_2^+ = \langle e_1, e_2 \rangle, \ldots, E_N^+ = E.$$

Let $\bar{E}_i^+$ denote the "anti-standard" flag:

$$\bar{E}_i^+ = \langle e_N, e_{N-1}, \ldots, e_{N-i+1} \rangle.$$

A map $F: \mathbb{P}_1 \to SU(N)/T$ can then be thought of as a flag $E_1^- \subset E_2^- \subset \cdots \subset E_{N-1}^- \subset E$ of subbundles of E. We "base" the map F by asking that E_i^- coincide with $\bar{E}_i^+$ at ∞, i.e., $F(\infty) = \bar{E}_i^+$.

If the map F is of degree $m = (m_1, \ldots, m_{N-1})$, one has that E_i^-/E_{i-1}^- is the line bundle $\mathcal{O}(k_{N-i+1})$, where $k_i = (m_i - m_{i-1})$ (see, e.g., [M]) ($m_0 = m_N = 0$); on the other hand, $E_i^+/E_{i-1}^+ \cong \mathcal{O}$. Now consider the sum $E_i^+ + E_{N-i}^-$; except over a finite set of points, $E_i^+ + E_{N-i}^- = E$, and so the sheaf

$$Q_i \stackrel{\text{def}}{=} E/(E_i^+ + E_{N-i}^-)$$

is supported on a finite set of points. Similarly, $E_{i-1}^+ + E_{N-i}^-$ is of dimension $(N-1)$, anywhere where either $E_i^+ + E_{N-i}^-$ or $E_{i-1}^+ + E_{N-i+1}^-$ is of dimension N, i.e., away from the intersection of the supports of Q_i and Q_{i-1}. In other words, the sheaf

$$P_i \overset{\text{def}}{=} E/(E_{i-1}^+ + E_{N-i}^-)$$

is a line bundle away from $\operatorname{supp}(Q_i) \cap \operatorname{supp}(Q_{i-1})$. Furthermore, one has exact sequences

$$0 \to \mathcal{O} \to P_i \xrightarrow{\pi_i} Q_i \to 0, \tag{3.1}$$

$$0 \to \mathcal{O}(k_i) \to P_i \xrightarrow{\rho_i} Q_{i-1} \to 0, \tag{3.2}$$

for $i = 1, \ldots, N$. Note that $Q_0 = Q_N = 0$. One has an exact sequence (see [HuM, Proposition 1.12])

$$0 \to E \to \begin{array}{c} P_1 \\ \oplus \\ P_2 \\ \oplus \\ \vdots \\ P_{N-1} \\ \oplus \\ P_N \end{array} \longrightarrow \begin{array}{c} Q_1 \\ \oplus \\ Q_2 \\ \vdots \\ \oplus \\ Q_{N-1} \end{array} \to 0 \tag{3.3}$$

where the map between the last two terms is $(p_1 \cdots p_N) \mapsto (\pi_1(p_1) - \rho_2(p_2), \pi_2(p_2) - \rho_3(p_3), \ldots, \pi_{N-1}(p_{N-1}) - \rho_N(p_N))$. Embedding E in the center term, E_i^+ is composed of sections of the form $(s_1 \cdots s_i, 0 \cdots 0)$ and E_i^- of sections of the form $(0, \ldots, 0, t_{N-i+1}, \ldots, t_N)$.

As Q_i is supported on points, $h^1(\mathbb{P}_1, Q_i) = 0$, and so, as the space of rational maps of fixed degree is connected, $h^0(\mathbb{P}_1, Q_i) = h^0 - h^1$ is a constant. For the generic rational map $[G]$, $\operatorname{supp}(Q_i)$ is m_i points, over each of which Q_i is the skyscraper sheaf $\mathbb{C}$.

Therefore,

$$h^0(\mathbb{P}_1, Q_i) = m_i. \tag{3.4}$$

(One also has that generically, $P_i \cong \mathcal{O}(m_i)$.)

Proposition (3.5). *There is a natural equivalence between*

—Based rational maps

$$F: \mathbb{P}_1 \to SU(N)/T,$$

—Equivalence classes under automorphisms of pairs (S, e), where S is a sequence of $\mathcal{O}$-modules of the form (3.3) with:

—Q_i supported over a finite set of points which doesn't include ∞,

—$h^0(\mathbb{P}_1, Q_i) = m_i$,

—P_i, Q fitting into exact sequences (3.1), (3.2),

and $e = \{e_1, \ldots, e_n\}$ is a basis of E at ∞ with $e_i \in P_i$.

Proof. We have shown how, from a map F, one can obtain a pair (S, e); we now show how to invert this procedure. We first prove that E is locally free, and trivial. Define sections s_i of E of the form $(s_{i1}, \ldots, s_{i,i-1}, 1, 0, \ldots, 0)$, where 1 represents the section of $\mathcal{O} = \ker(\pi_i: P_i \to Q_i)$ which is equal to e_i at ∞. Such sections exist, by the surjectivity of $P_i \to Q_i, P_i \to Q_{i-1}$; starting with the section of $\mathcal{O}$, one "zigzags up" the sequence (3.3). Let $r = (r_1, \ldots, r_N)$ be any other section, local or global of E. The last term, r_N, is a section of $\mathcal{O} = P_N$, and so there is a function t_N such that $r - t_N s_N$ is of the form $(r'_1, \ldots, r'_{N-1}, 0)$, r'_{N-1} is now a section of $\ker(\pi_{N-1}) \cong \mathcal{O}$, and so there is a function t_{N-1} with $r - t_N s_N - t_{N-1} s_{N-1}$ of the form $(r''_1, \ldots, r''_{N-2}, 0, 0)$. Iterating this procedure, one has $r = \sum t_i s_i$; if r is a global section, the t_i's are constants; E is a trivial bundle, with preferred basis s_i.

One then defines the subsheaf E_i^- as the subsheaf of E of sections of the form $(0, \ldots, 0, u_{N-i+1}, \ldots, u_N)$. Near ∞, and in fact away from the support of the Q_i, E_i^- is obviously locally free. On the other hand, away from ∞ one can choose trivializations of $\mathcal{O}(k_i) = \ker \pi_i$; with respect to these trivializations, define local sections w_i of E of the form $(0, \ldots, 0, 1, w_{i,N-i+2}, \ldots, w_{i,N})$, using the same zigzag procedure as above, but moving downward. $(w_1, \ldots, w_i)$ then forms a basis away from ∞ for E_i^-, which is then locally free; this is proven by the same procedure as above.

There remains to show that the E_i^-'s embed in E as sub-bundles. This is equivalent to showing that E/E_i^- is locally free. However, this quotient is given by

$$0 \to E/E_i^- \to \bigoplus_{i=1}^{N-i} P_i \to \bigoplus_{i=1}^{N-i-1} Q_i \to 0,$$

and so one uses the same techniques as above to show that E/E_i^- is indeed locally free.

One can use this to prove

Theorem (3.6). *There is a natural bijective correspondence between*

—based rational maps $\mathbb{P}_1 \to SU(N)/T$

and

—$\mathcal{G}$-equivalence classes of Nahm complexes (α, β, v).

Proof. A) We begin by giving the map associated to a Nahm complex. Following (1.15), one remarks that on the interior of each interval (μ_j, μ_{j+1}), the only invariant is the conjugacy class of β_j. An equivalent datum is the sheaf Q_j defined by the sequence

$$0 \to \mathcal{O}(-1)^{\oplus m_j} \xrightarrow{(\eta 1 - \beta_j(u))} \mathcal{O}^{\oplus m_j} \to Q_j \to 0. \tag{3.7}$$

Here η is the standard coordinate on $\mathbb{P}_1$, and u is a point in (μ_j, μ_{j+1}). Q_j is invariant under conjugation of β_j; it is supported over the eigenvalues of β_j, and its isomorphism class at an eigenvalue determines the Jordan form of β_j. For example, for a double eigenvalue a, the diagonal Jordan form corresponds to $(\mathcal{O}/\langle \eta - a \rangle)^{\oplus 2}$, while the non-diagonal Jordan form corresponds to $\mathcal{O}/\langle (\eta - a)^2 \rangle$.

In any case, from the sequence (3.7), one sees that

$$H^0(\mathbb{P}_1, Q_j) \cong H^0(\mathbb{P}_1, \mathcal{O}^{\oplus m_j}) \cong \mathbb{C}^{m_j}.$$

The definition of the P_j is obtained from the behavior of the Nahm complex near a boundary point μ_j. We distinguish three cases:

1. $\underline{m_j > m_{j-1}}$:

The normal form proposition (1.15) tells us that β_{j-1} can be taken to be a constant near μ_j and that β_j is conjugate to a matrix with block form:

$$\left[\begin{array}{c|cccc}
 & & & & g_1 \\
\beta_{j-1} & & 0 & & \vdots \\
 & & & & g_{m_{j-1}} \\
\hline
f_1, \ldots\ldots, f_{m_{j-1}} & 0 & & & e_0 \\
 & 1 & \ddots & & \vdots \\
0 & & \ddots & 0 & \vdots \\
 & 0 & & 1 & e_{k-1}
\end{array}\right] \tag{3.8}$$

with v_j the $(m_{j-1}+1)$'th basis vector. This form is determined up to the action of $Gl(m_{j-1}, \mathbb{C})$, where $s(\beta_{j-1}, f, g) = (s\beta_{j-1}s^{-1}, fs^{-1}, sg)$. One computes that:

$$\det(\eta\mathbb{1} - \beta_j) = \det(\eta\mathbb{1} - \beta_{j-1})(\eta^k - e_{k-1}\eta^{k-1} - \cdots - e_0) - f(\eta\mathbb{1} - \beta_{j-1})_{\mathrm{adj}} \cdot g, \tag{3.9}$$

where adj denotes the classical adjoint (matrix of cofactors). The e_i's are thus determined by the spectrum of β_j and by β_{j-1}, f, and g. Let π denote the projection onto the first m_{j-1} coordinates. We define P_j and the maps $P_j \to Q_j$, $P_j \to Q_{j-1}$ by the commuting diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O}(-1) \arrow[r, "(\eta\mathbb{1} - \beta_{j-1})"] & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O} \arrow[r] & Q_{j-1} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) \arrow[u, "\pi"] \arrow[r, "\binom{\eta\mathbb{1} - \beta_j}{-s_j}"] \arrow[d, equal] & (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} \arrow[u, "{(\pi, -g)}"'] \arrow[r] \arrow[d] & P_j \arrow[u] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) \arrow[r, "(\eta\mathbb{1} - \beta_j)"] & \mathbb{C}^{m_j} \otimes \mathcal{O} \arrow[r] & Q_j \arrow[r] & 0
\end{tikzcd} \tag{3.10}$$

Here $s_j: \mathbb{C}^{m_j} \to \mathbb{C}$ is defined by $s_j = 0$ on $\mathbb{C}^{m_{j-1}} \subset \mathbb{C}^{m_j}$, and $s_j(\beta_j^n(v_j)) = \delta_{n,k-1}$, $(k = m_j - m_{j-1})$; in the basis used above, $s_j = (0, \ldots, 0, 1)$.

This definition is $Gl(m_{j-1}, \mathbb{C})$-invariant. From (3.10) one easily sees that P_j maps surjectively to Q_j, with kernel $\mathcal{O}$, proving (3.1); as for the map $P_j \to Q_{j-1}$, it is also surjective. To compute its kernel, let $i: \mathbb{C}^{m_{j-1}} \to \mathbb{C}^{m_j}$ be the injection into the first m_{j-1} coordinates, and let $i_v: \mathbb{C} \to \mathbb{C}^{m_j}$ map 1 to v_j. Then one has the diagram, with exact rows, defining a sheaf R_j and maps $R_j \to Q_{j-1}$, $R_j \to P_j$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O}(-1) & \xrightarrow{(\eta\mathbb{1}-\beta_{j-1})} & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O} & \longrightarrow & Q_{j-1} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j-1}\\ -f\end{smallmatrix}\right)} & (\mathbb{C}^{m_{j-1}} \oplus \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & R_j & \longrightarrow & 0 \\
 & & \downarrow\iota & & \downarrow\left(\begin{smallmatrix}i, & i_v\\ 0, & 0\end{smallmatrix}\right) & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j}\\ -s_j\end{smallmatrix}\right)} & (\mathbb{C}^{m_j} \otimes \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & P_j & \longrightarrow & 0
\end{array}
\qquad (3.11)
$$

R_j injects into P_j. Let $t(\eta)\colon \mathbb{C}^{m_j} \to \mathbb{C}$ be defined by $t(\eta)(a_1, \ldots, a_{m_{j-1}}, b_1 \cdots b_k) = \eta^{k-1} b_k + \cdots + \eta b_2 + b_1$, and let p be the polynomial $(\eta^k - e_{k-1}\eta^{k-1} \cdots - e_0)$. Then one has a commuting diagram, away from $\eta = \infty$:

$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j}\\ -s_j\end{smallmatrix}\right)} & (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & P_j & \longrightarrow & 0 \\
 & & \downarrow\pi & & \downarrow\left(\begin{smallmatrix}\pi, & -g\\ t, & p\end{smallmatrix}\right) & & \downarrow & & \\
0 & \to & \mathbb{C}^{m_{j-1}} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j-1}\\ -f\end{smallmatrix}\right)} & (\mathbb{C}^{m_{j-1}} \oplus \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & R_j & \longrightarrow & 0
\end{array}
\qquad (3.12)
$$

which defines away from $\eta = \infty$ a left inverse for the map $R_j \to P_j$ of (3.11). The map $P_j \to R_j$ has a pole of order k at infinity, and so $R_j \cong P_j(-k)$. One now notes that the map $R_j \to Q_{j-1}$ of (3.11) is surjective, with kernel $\mathcal{O}$; however this map factors through P_j, showing that P_j satisfies (3.2).

The basis of P_j one chooses at ∞ is that defined by the section $(0, 1)$ of $(\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O}$.

2. $m_{j-1} > m_j$:

One proceeds as above, with the following modifications: one changes all the j indices to $(j-1)$, and all the $(j-1)$ indices to j; one then replaces P_{j-1} by R_j, and R_{j-1} by P_j.

3. $m_{j-1} = m_j$:

One can gauge the Nahm complex to constant matrices β_j, β_{j-1}, with $\beta_j = \beta_{j-1} + uw^T$. One then defines P_j, and the maps by the commuting diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{(\eta\mathbb{1}-\beta_{j-1})} & \mathbb{C}^{m_j} \otimes \mathcal{O} & \longrightarrow & Q_{j-1} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j-1}\\ w^T\end{smallmatrix}\right)} & (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & P_j & \longrightarrow & 0 \\
 & & \| & & \downarrow\left(\begin{smallmatrix}1 & -u\\ 0 & 1\end{smallmatrix}\right) & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{\left(\begin{smallmatrix}\eta\mathbb{1}-\beta_{j}\\ w^T\end{smallmatrix}\right)} & (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} & \longrightarrow & P_j & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) & \xrightarrow{(\eta\mathbb{1}-\beta_{j})} & \mathbb{C}^{m_j} \otimes \mathcal{O} & \longrightarrow & Q_j & \longrightarrow & 0
\end{array}
\qquad (3.13)
$$

It is trivial to check that (3.1) and (3.2) are satisfied. Again, the trivialization one chooses at infinity is that given by the vector $(0, 1) \in \mathbb{C}^{m_j} \oplus \mathbb{C}$.

B) Suppose we are given the sheaf P_j, Q_j corresponding to a rational map. We want to invert the procedure outlined above, and obtain a Nahm complex. The first step is to obtain the matrices β_j up to conjugacy. To do this, one notes that as Q_j has support not intersecting $\eta=\infty$, there is a well defined multiplication map $\times\eta: Q_j \to Q_j$. We define β_j to be the induced endomorphism of $H^0(\mathbb{P}_1, Q_j)$.

From the sheaves P_j, and the maps $P_j \to Q_j$, $P_j \to Q_{j-1}$, we will now obtain the boundary data of a Nahm complex. To begin, as P_j is a line bundle near ∞, one has a natural identification of $P_j(-l)$, $l>0$, as sections of P_j vanishing to order l at ∞; for $l<0$, as sections with poles of order $-l$.

There are therefore sequences

$$\begin{gathered} 0 \to \mathcal{O}(-l) \to P_j(-l) \to Q_j \to 0, \\ 0 \to \mathcal{O}(k_j - l) \to P_j(-l) \to Q_{j-1} \to 0. \end{gathered} \tag{3.14}$$

Also, as we have a fixed coordinate η, the trivializations of P_j at ∞ induce trivializations of $P_j(-l)$.

We define, using (3.14),

$$H^0(\mathbb{P}_1, P_j(-1)) \cong H^0(\mathbb{P}_1, Q_j) \overset{\text{def}}{\cong} W_j = \mathbb{C}^{m_j},$$

$$H^0(\mathbb{P}_1, P_j(-k_j-1)) \cong H^0(\mathbb{P}_1, Q_{j-1}) \overset{\text{def}}{\cong} V_j = \mathbb{C}^{m_{j-1}}.$$

We distinguish three cases:

1. $m_j > m_{j-1}$:

One has a natural inclusion $i: V_j \to W_j$, as sections vanishing to order $(-k_j-1)$. There is also a natural projection $\pi: W_j \to V_j$ defined by $H^0(\mathbb{P}_1, P_j(-1)) \to H^0(\mathbb{P}_1, Q_{j-1})$; $\pi \circ i = \mathrm{Id}$. We can decompose W_j as $i(V_j) \oplus \ker(\pi)$.

From the sequence $0 \to \mathcal{O} \to P_j(-k_j) \to Q_{j-1} \to 0$, one deduces that there is a unique section v_j of P_j, vanishing to order k_j at ∞, which lies in $\ker(\pi)$ and which coincides with the basis of $P_j(-k)$ at ∞. One then has a decomposition of W_j as:

$$W_j \cong V_j \oplus \langle v_j \rangle \oplus \langle \eta v_j \rangle \oplus \cdots \oplus \langle \eta^{k_j-1} v_j \rangle. \tag{3.15}$$

Now consider the map $\times\eta: H^0(\mathbb{P}_1, Q_j) \to H^0(\mathbb{P}_1, Q_j) \cong W_j$. W_j also decomposes as $H^0(\mathbb{P}_1, P_j(-2)) \oplus \langle \eta^{k-1} v_j \rangle$. On the first summand, multiplication by η is just that induced by $\times\eta: P_j(-2) \to P_j(-1)$; on the other hand, for $\langle \eta^{k-1} v_j \rangle$, one must really evaluate in Q_j, multiply by η, then take the corresponding section. In a basis given by the decomposition (3.15), multiplication by η has the form:

$$\left[\begin{array}{ccc|cccc} & & & & & & g_1 \\ & A & & & 0 & & \vdots \\ & & & & & & g_{m_{j-1}} \\ \hline f_1 & \cdots & f_{m_{j-1}} & 0 & & & e_0 \\ & & & 1 & \ddots & & \vdots \\ & 0 & & & \ddots & 0 & \vdots \\ & & & 0 & & 1 & e_{k-1} \end{array}\right] \tag{3.16}$$

Furthermore, by comparing the operations

$$\times\eta: H^0(\mathbb{P}_1, Q_{j-1}) \to H^0(\mathbb{P}_1, Q_{j-1})$$

and

$$\times\eta: H^0(\mathbb{P}_1, P_j(-k_j-1)) \to H^0(\mathbb{P}_1, P_j(-k_j)),$$

one can see that the block A in the matrix above is precisely β_{j-1}.

2. $m_j < m_{j-1}$:
One proceeds as above, inverting the roles of V_j and W_j.

3. $m_j = m_{j-1}$:
One has the sequences

$$0 \to \mathcal{O} \to P_j \to Q_j \to 0, \tag{3.17}$$

$$0 \to \mathcal{O} \to P_j \to Q_{j-1} \to 0. \tag{3.18}$$

Let ρ be the section of $\mathcal{O}$ in (3.17), σ be the section of $\mathcal{O}$ in (3.18), both taking as value at ∞ our basis element of P_j. One has:

$$H^0(\mathbb{P}_1, P_j(-1)) \cong H^0(\mathbb{P}_1, Q_j) \cong H^0(\mathbb{P}_1, Q_{j-1}), \tag{3.19}$$

$$H^0(\mathbb{P}_1, P_j) \cong H^0(\mathbb{P}_1, Q_{j-1}) \oplus \langle\sigma\rangle \tag{3.20}$$

$$\cong H^0(\mathbb{P}_1, Q_j) \oplus \langle\rho\rangle. \tag{3.21}$$

Consider the map induced on sections by $\times\eta: P_j(-1) \to P_j$; using the isomorphisms (3.19), (3.20), this is of the form, for $q \in H^0(\mathbb{P}_1, P_j(-1))$.

$$\eta q = \beta_{j-1} q + \sigma(w^T \cdot q)$$

for some $w^T: H^0(\mathbb{P}_1, P_j(-1)) \to \mathbb{C}$; if, instead, one uses (3.19), (3.21), one gets

$$\eta q = \beta_j \cdot q + \rho(\tilde{w}^T \cdot q).$$

As $\rho = \sigma$ at ∞, $\sigma - \rho = u \in H^0(\mathbb{P}_1, P_j(-1))$, and so

$$\beta_{j-1} \cdot q + u \cdot (w^T \cdot q) + \rho(w^T \cdot q) = \beta_j \cdot q + \rho(\tilde{w}^T \cdot q),$$

which forces $w = \tilde{w}$, and $\beta_j = \beta_{j-1} + uw^T$. The matrices therefore have the correct rank one jump, and one puts $v_j = (u, w)$.

We have thus obtained

—a conjugacy class $[\beta_j]$ for each interval (μ_j, μ_{j+1});

—the correct data for a Nahm complex in normal form near each boundary point μ_j:

—the "extra data" v_j, for each μ_j.

It is then straightforward to piece together the various "chunks" of Nahm complex defined near each μ_j via gauge transformations which are non-constant only on compact subsets of the (μ_j, μ_{j+1})'s, giving one a Nahm complex.

It is also easy to verify that the constructions of parts A) and B) of the proof are inverses of one another.

4. Twistors and Monopoles

We have shown that equivalence classes of pairs (solutions to Nahm's equations, extra data v) correspond naturally to based rational maps. This section addresses itself to two remaining problems. One is that of the meaning of the rational map in terms of the monopole; the other is the related problem of interpreting the extra data v. These questions have already been answered in the case of $SU(2)$-monopoles [Hu], [AHi], and, as the answer here is essentially the same, we will be brief.

We first recall from [M] the twistor transform for $SU(N)$ monopoles. Let (∇, φ) be an $SU(N)$ monopole over $\mathbb{R}^3$, with (∇, φ) acting on a rank N bundle V. One defines a rank N bundle E over $T\mathbb{P}_1 = \{\text{oriented lines in } \mathbb{R}^3\}$ by $E_l = \{s \in \Gamma(l, v) | (\nabla_u - i\varphi)s = 0\}$, where u is a positive unit vector field along the line l. Let r denote the radial coordinate in $\mathbb{R}^3$.

The boundary conditions of (∇, φ) imply that one can define a full flag E_i^+ of subbundles of E by

$$(E_i^+)_l = \{s \in E_l | s \text{ bounded by } r^k e^{-\mu_{N-i+1} r} \text{ for some } k, \text{ as } r \to \infty\}.$$

Similarly, one defines E_i^- by considering decay in the negative direction. Setting $P_i = E/(E_{i-1}^+ + E_{N-i}^-)$, $Q_i = E/(E_i^+ + E_{N-i}^-)$, one obtains a sequence of the form (3.3), but over $T\mathbb{P}_1$.

Now one restricts this sequence to a fiber $\mathbb{C}_0$ of the projection $T\mathbb{P}_1 \to \mathbb{P}_1$; this is tantamount to considering only the lines in a fixed direction in $\mathbb{R}^3$, and is what corresponds to the splitting $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ of the main theorem. Over $\mathbb{C}_0$, Q_i is supported over a finite set of points, and $h^0(\mathbb{C}_0, Q_i) = m_i$. Furthermore, if one refers to the construction of the solution to Nahm's equations of [HuM], one sees that it is an exact parallel of the construction of the Nahm complex from the rational map in Sect. 3. In other words, E and the two flags of subbundles E_i^+, E_i^-, restricted to $\mathbb{C}_0 \subset T\mathbb{P}_1$, are exactly the restriction to $\mathbb{C}_0 = \mathbb{P}_1 \backslash \{\infty\}$ of the flags involved in the rational map.

Referring to Sect. 3, (compare [AHi]), the meaning of the extra information v associated to a solution to Nahm's equations is clear; in essence, v determines a (unitary) trivialization e_i of E at ∞ in $\mathbb{P}_1$, such that e_i, at infinity, lies in $E_i^+ \cap E_{N-i+1}^-$. Interpreting in terms of scattering data as in [AHi, Sect. 3], one has the following picture. One fixes a direction in $\mathbb{R}^3$, let us say the positive z-axis. The lines in this direction then correspond naturally to $(x + iy) \in \mathbb{C}$. Integrating $(\nabla_z + i\varphi)$, one obtains over $\mathbb{C}$ a bundle E containing two flags E_i^+, E_i^-, which are defined by decay behaviour at $+\infty$, $-\infty$ respectively. The v_i's correspond to a choice of unitary trivialization of the bundle E over $\mathbb{C}$ such that E_i^+ is the standard flag, and such that E_i^- tends to the anti-standard flag as $|x + iy| \to \infty$; such a trivialization is defined from a trivialization of the bundle V over $\mathbb{R}^3$, which is asymptotically flat as $z \to +\infty$. We refer to such a trivialization as a v-framing; for a given (∇, φ), there is a torus T^N of such framings. Given such a framing, a rational map is defined by the flag E_i^-, as in [AHi].

The moduli space $\tilde{M}_m$ we have studied is thus that of pairs (monopoles, v-framing). Its fiber over a point (∇, φ) of the monopole moduli space is then the torus T^N, quotiented by the subgroup of T^N which stabilizes (∇, φ). When the monopole is irreducible, this stabilizer is simply the S^1 of unitary multiples of the identity; the fiber is then the torus T^{N-1} of $SU(N)$.

5. The Cases of $SO(N)$ and $Sp(K)$

We treat the case of $SO(N)$ and $Sp(K)$ monopoles by considering them as $SU(N)$ monopoles with extra structure. (For Sp, we set $N=2K$.) One then just retraces the main steps of the proof for the $SU(N)$ case, checking at each step what extra structure must be added.

A) Nahm's Equations and Nahm Complexes. One has [HuM] that for both SO, Sp, the intervals on which one solves Nahm's equations are symmetric about the origin:

$$(\mu_{N-j-1}, \mu_{N-j}) = (-\mu_{j+1}, -\mu_j).$$

Let the solutions ${}^iT_j$ to Nahm's equations act on vector spaces V_j; one has the extra structure of a nondegenerate pairing

$$c_j: V_j \to V^*_{N-j}$$

with respect to which ${}^iT_j(z)$, ${}^iT_{N-j}(-z)$ are adjoint, i.e.,

$$c^i_j T_j(z) c_j^{-1} = ({}^iT_{N-j}(-z))^T. \tag{5.1}$$

The c_j are compatible with the boundary gluing between the intervals, in the following sense:

—at a non-neutral boundary point μ_j one has a projection and an injection:

$$\pi_j \cdot \mathbb{C}^{\bar{m}} \to \mathbb{C}^{\underline{m}}; \quad i_j: \mathbb{C}^{\underline{m}} \to \mathbb{C}^{\bar{m}}.$$

The compatibility condition is then:

$$\begin{aligned} c_{j-1}\pi_j &= i^T_{N-j+1} c_j, &&\text{for} \quad m_j > m_{j-1}, \\ c_j \pi_j &= i^T_{N-j+1} c_{j-1}, &&\text{for} \quad m_j < m_{j-1} \end{aligned} \tag{5.2}$$

—at a neutral boundary point, one asks that $c_j = c_{j-1}$.

For $SO(N)$, one has the symmetry condition:

$$c_j = -c^T_{N-j}, \tag{5.3}$$

and for $Sp(K)$:

$$c_j = c^T_{N-j}. \tag{5.4}$$

Turning now to Nahm complexes, one asks that

$$\begin{aligned} c_j \alpha_j c_j^{-1} &= \alpha^T_{N-j}, \\ c_j \beta_j c_j^{-1} &= \beta^T_{N-j} \end{aligned} \tag{5.5}$$

for the v_j, one imposes the condition that, at a non-neutral boundary point μ_j, $j \leqq (N+1)/2$:

$$\begin{aligned} c(i_j(\mathbb{C}^{\underline{m}}))(v_{N-j+1}) &= 0, \\ c(\beta^l v_j)(v_{N-j+1}) &= -\delta_{l,\bar{m}-\underline{m}-1} \quad \text{for} \quad l = 0, \ldots, \bar{m} - \underline{m} - 1 \end{aligned} \tag{5.6}$$

and, at a neutral boundary point μ_j, $j \leqq (N+1)/2$, if $v_j = (u, w)$, $v_{N-j+1} = (u', w')$,

$$c_j(u) = -w'. \tag{5.7}$$

One constrains the gauge transformations by

$$g(-z)^{-1T}=cg(z)c^{-1}; \tag{5.8}$$

these are the transformations keeping c_j constant.

The proof that for each equivalence class of Nahm complexes one has an essentially unique solution proceeds exactly as before; the unique solution must then be invariant under c. In fact, this case is already considered by Donaldson, who treats $SU(2)$ as $Sp(1)$.

B) *Nahm Complexes and Rational Maps.* One wants an equivalence between Nahm complexes and rational maps into flag manifolds for $SO(N)$ and $Sp(K)$. The flag manifolds one must consider are those of full flags $E_1 \subset E_2 \subset \cdots \subset E_N = \mathbb{C}^N$ satisfying $E_{N-i}=E_i^{\perp}$ with respect to some standard quadratic form. We call these flags *isotropic–coisotropic*. For $SO(N)$, the quadratic form is taken to be

$$\langle (a_i),(b_i)\rangle = \sum_{i=1}^{N} a_i b_{N-i+1}, \tag{5.9}$$

and, for $Sp(K)$

$$\langle (a_i),(b_i)\rangle = \sum_{i=1}^{K} a_i b_{N-i+1} - b_i a_{N-i+1}. \tag{5.10}$$

We will show that, given a c-invariant Nahm complex (α, β, v), the rational map one obtains by the procedure of Sect. 3 is indeed into the submanifold of isotropic-coisotropic flags. To do this, we proceed in several steps, using the sequence $0 \to E \to \oplus P_i \to \oplus Q_i \to 0$ of (3.3).

i) To begin, one notes that our pairing c_j is invariantly a pairing $H^0(\mathbb{P}_1, Q_j)\otimes H^0(\mathbb{P}_1, Q_{N-j}) \to \mathbb{C}$, with respect to which the adjoint of β_j is β_{N-j}. From this, we will define a pairing

$$H^0(\mathbb{P}_1, P_j)\otimes H^0(\mathbb{P}_1, P_{N-j+1}) \to H^0(\mathbb{P}_1, \mathcal{O}(m_j+m_{j-1})),$$

where $\mathcal{O}(m_j+m_{j-1})$ is to be thought of as the sheaf of functions with poles over the support of Q_j and Q_{j-1}.

ii) We then show that this pairing descends to one over $P_j \otimes P_{N-j-1}$.

iii) One then considers the induced pairing on local sections of $E \subset (\oplus P_i)$; one shows that for sections of E, the poles cancel, and one is left with a holomorphic pairing. One also checks that the induced pairing on global sections of E is the standard one. The fact that, in $(\oplus P_i)$, P_j is only paired with P_{N-j+1} then implies that the flags E_i^+, E_i^- defined in Sect. 3 are isotropic-coisotropic.

Let $G=SO(N)$ or $Sp(K)$ be our group. We define

$$\begin{aligned} \tau_j &= -1 \quad \text{if} \quad j > N/2, \quad \text{and} \quad G=SO(N) \\ &= 1 \text{ otherwise,} \\ \sigma_j &= -1 \quad \text{if} \quad j > K = N/2, \quad \text{and} \quad G=Sp(K) \\ &= 1 \text{ otherwise.} \end{aligned} \tag{5.11}$$

Let $\{\,,\}_j$ denote the pairing between sections of Q_j and sections of Q_{N-j} defined by c_j. We define the pairing of sections of P_j with sections of P_{N-j-1} by distinguishing three separate cases:

i) $m_j > m_{j-1}$

In this case, P_j is defined by (3.10):

$$0 \to \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) \xrightarrow{\binom{\eta\mathbb{1}-\beta_j}{-s_j}} (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} \to P_j \to 0 \tag{5.12}$$

and, as $m_{N-j} = m_j > m_{j-1} = m_{N-j+1}$, P_{N-j+1} is defined by (3.11):

$$0 \to \mathbb{C}^{m_{j-1}} \otimes \mathcal{O}(-1) \xrightarrow{\binom{\eta\mathbb{1}-\beta_{N-j+1}}{-\beta_{N-j+1}}} (\mathbb{C}^{m_{j-1}} \oplus \mathbb{C}) \otimes \mathcal{O} \to P_{N-j+1} \to 0. \tag{5.13}$$

Let $\pi = \pi_j : \mathbb{C}^{m_j} \to \mathbb{C}^{m_{j-1}}$ be the projection, and $i = i_{N-j+1} : \mathbb{C}^{m_{j-1}} \to \mathbb{C}^{m_j}$ be the corresponding injection; set $v = v_{N-j+1}$. Referring to (3.8), we define a pairing between $(a,x) \in \mathbb{C}^{m_j} \oplus \mathbb{C} \cong H^0(\mathbb{P}_1, P_j)$ and $(b,y) \in \mathbb{C}^{m_{j-1}} \oplus \mathbb{C} \cong H^0(\mathbb{P}_1, P_{N-j+1})$ by:

$$\begin{aligned}((a,x),(b,y))_j = &\{(\eta\mathbb{1} - \beta_{j-1})^{-1}(\pi(a) - g_j x), b\}_{j-1} \\ &- \{(\eta\mathbb{1} - \beta_j)^{-1}a, i(b) + i_v(y)\}_j + \tau_j x \cdot y.\end{aligned} \tag{5.14}$$

Here η is the standard coordinate on $\mathbb{P}_1$. Now, if $(a,x) = ((\eta\mathbb{1} - \beta_j)u, -s_j \cdot u))$ one finds that

$$((a,x),(b,y))_j = \{\pi(u), b\}_{j-1} - \{u, i(b) + yv_{N-j+1}\}_j - \tau_j s_j(u) \cdot y. \tag{5.15}$$

Referring to (3.10) and (5.6), $s_j = -\tau_j\{\cdot, v_{N-j+1}\}_j$; also by the compatibility condition (5.2) π is the adjoint of i. The r.h.s. of (5.15) therefore vanishes. Similarly, (5.14) also vanishes when $(b,y) = ((\eta\mathbb{1} - \beta_{N-j+1})w, -f_{N-j+1}w)$. Referring to (5.12), (5.13), this means that the product $(,)_j$ descends to:

$$(,)_j : P_i \otimes P_{N-j+1} \to \mathcal{O}(m_j + m_{j-1}).$$

ii) $m_j < m_{j-1}$:

One proceeds in exactly symmetrical fashion; defining the pairing by:

$$\begin{aligned}((a,x),(b,y))_j = &\{(\eta\mathbb{1} - \beta_{j-1})^{-1}(i(a) + i_v(x)), b\}_{j-1} \\ &- \{(\eta\mathbb{1} - \beta_j)^{-1}a, \pi(b) - g_{N-j+1}y\}_j - \tau_j x \cdot y\end{aligned}$$

iii) $m_{j-1} = m_j$:

In this case, one has that P_j, P_{N-j+1} are defined by (3.13),

$$0 \to \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) \xrightarrow{\binom{\eta\mathbb{1}-\beta_{j-1}}{w^T}} (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} \to P_j \to 0,$$

$$0 \to \mathbb{C}^{m_j} \otimes \mathcal{O}(-1) \xrightarrow{\binom{\eta\mathbb{1}-\beta_{N-j}}{w'^T}} (\mathbb{C}^{m_j} \oplus \mathbb{C}) \otimes \mathcal{O} \to P_{N-j+1} \to 0, \tag{5.16}$$

where $\beta_j = \beta_{j-1} + uw^T$, $\beta_{N-j+1} = \beta_{N-j} + u'w'^T$. As above, one defines a pairing between sections of P_j and P_{N-j+1}, by

$$((a,x),(b,y))_j = \{(\eta\mathbb{1} - \beta_{j-1})^{-1}a, b - u'y\}_{j-1} - \{(\eta\mathbb{1} - \beta_j)^{-1}(a - ux), b\}_j + \sigma_j xy, \tag{5.17}$$

again, one checks, using (5.7), that this descends to $P_j \otimes P_{N-j+1}$.

Summing one obtains a bilinear form, with poles, on $\oplus P_i$ defined by

$$((a_i),(b_i)) = \sum_{i=1}^{N} (a_i, b_{N-i+1})_i.$$

One therefore has a pairing on local sections of E. Referring, however, to the sequence (3.3) which defines E, and to the definitions of the forms $(\cdot,\cdot)_j$, one easily sees that the poles cancel, giving a pairing $E \otimes E \to \mathcal{O}$. As remarked above, as the pairing on E is a sum of pairings on $P_i \otimes P_{N-i+1}$, one automatically has that E_i^+, E_i^- are isotropic-coisotropic. To evaluate the pairing on sections of E, it suffices as E is trivial, to evaluate at $\eta = \infty$. Referring to (5.14), (5.17), one sees that only the $\pm\tau_j xy$, $\sigma_j xy$ terms contribute at infinity and, with respect to the standard basis of $H^0(\mathbb{P}_1, E)$, the pairing is indeed the standard one. Also note that if $\{,\}_j = \pm\{,\}_{N-j}$, then $(,)_j = \mp(,)_{N-j+1}$; from (5.3), (5.4), the pairing on E is indeed symmetric for $SO(N)$, antisymmetric for $Sp(N)$.

The last thing to be checked is that one can invert the above procedure. One wants to obtain from an isotropic-coisotropic rational map, a c-invariant Nahm complex, i.e., a non-degenerate pairing

$$\{,\}_j : H^0(\mathbb{P}_1, Q_j) \otimes H^0(\mathbb{P}_1, Q_{N-j}) \to \mathbb{C},$$

such that (5.2) to (5.7) are satisfied.

To do this, we note that away from the support of the Q_i, $E \cong (\oplus P_i)$, and so pairing on E defines a pairing $(,)$ on sections of $(\oplus P_i)$; this extends over all of $\mathbb{P}_1$ to a meromorphic pairing. As E_i^+, E_i^- are isotropic-coisotropic, one finds that

$$(P_j, P_k) \neq 0 \quad \text{only if} \quad k = N - i + 1. \tag{5.18}$$

Let q be a section of Q_j, q' a section of Q_{N-j}. Then, over $\mathbb{P}_1 \backslash \{\infty\}$, there is a section $p = (p_1, \ldots, p_j, 0 \ldots 0)$ in $\oplus P_i$ mapping to $(0 \ldots 0, q, 0 \ldots 0)$ in $\oplus Q_i$; similarly, one has a section $p' = (0, \ldots, 0, p'_{N-j+1}, \ldots, p'_N)$ of $\otimes P_i$ mapping to $(0, \ldots, 0, q', 0 \ldots 0)$. Define

$$\{q, q'\}_j = - \sum_{\text{poles} \neq \infty} \operatorname{res}(p, p'). \tag{5.19}$$

To see that this is well defined, note that p and p' are determined up to addition of sections of E. Suppose that p is a section of E; then, referring to (3.3), $p' = (0, \ldots, 0, p_{N-j+1}, \ldots, p_N)$ can be "completed" to $\tilde{p}' = (p'_1, \ldots, p'_{N-j}, p'_{N-j+1}, \ldots, p'_N)$ a section of E, and, by (5.18), $(p, p') = (p, \tilde{p}')$; but this last expression stays finite as $p, \tilde{p}'$ are sections of E. Similarly, if p' is a section of E, one completes p to $\tilde{p}$, a section of E, and $(p, p') = (\tilde{p}, p') = $ finite.

In a similar vein, one shows that the pairing is non-degenerate. Suppose that $\{q, q'\}_j = 0$ for all q'. This is equivalent to saying that (p, p') stays finite, for all sections $p' = (0, \ldots, 0, p'_{N-j+1}, \ldots, p'_N)$ mapping to $(0, \ldots, 0, Q_{N-j}, 0 \ldots 0)$; "completing," this is the same as $(p, \check{p}')$ finite, for all $\tilde{p}'$ sections of E. However, p already represents a meromorphic section of E; if $(p, \tilde{p}')$ stays finite for all $\tilde{p}'$, then p is holomorphic, and its image $q \in Q_j$ is then zero.

That β_j, β_{N-j} are adjoints of one another follows from the fact that they are defined as the maps induced on global sections by the multiplication map $\times \eta$ of Q_j, Q_{N-j}. For α_{N-j}, α_j, one uses the fact that the pairing respects the trivializations of the bundles over $(\mu_j, \mu_{j+1}), (\mu_{N-j-1}, \mu_{N-j})$ with respect to which the α's are zero. The proof of the rest of relations (5.2) to (5.7) follows from the definition, and the fact that this construction inverts that given above is also straightforward.

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