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THE CLASSIFICATION PROBLEM FOR TORSION-FREE ABELIAN GROUPS OF FINITE RANK

SIMON THOMAS

1. Introduction

In 1937, Baer [5] introduced the notion of the type of an element in a torsion-free abelian group and showed that this notion provided a complete invariant for the classification problem for torsion-free abelian groups of rank 1. Since then, despite the efforts of such mathematicians as Kurosh [23] and Malcev [25], no satisfactory system of complete invariants has been found for the torsion-free abelian groups of finite rank $n \geq 2$. So it is natural to ask whether the classification problem is genuinely more difficult for the groups of rank $n \geq 2$. Of course, if we wish to show that the classification problem for the groups of rank $n \geq 2$ is intractible, it is not enough merely to prove that there are 2^ω such groups up to isomorphism. For there are 2^ω pairwise nonisomorphic groups of rank 1, and we have already pointed out that Baer has given a satisfactory classification for this class of groups. In this paper, following Friedman-Stanley [11] and Hjorth-Kechris [15], we shall use the more sensitive notions of descriptive set theory to measure the complexity of the classification problem for the groups of rank $n \geq 2$.

Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. Thus the collection of torsion-free abelian groups of rank $1 \leq r \leq n$ can be naturally identified with the set $S(\mathbb{Q}^n)$ of all nontrivial additive subgroups of $\mathbb{Q}^n$. Notice that $S(\mathbb{Q}^n)$ is a Borel subset of the Polish space $\mathcal{P}(\mathbb{Q}^n)$ of all subsets of $\mathbb{Q}^n$, and hence $S(\mathbb{Q}^n)$ can be regarded as a *standard Borel space*; i.e. a Polish space equipped with its associated σ-algebra of Borel subsets. (Here we are identifying $\mathcal{P}(\mathbb{Q}^n)$ with the space $2^{\mathbb{Q}^n}$ of all functions $h : \mathbb{Q}^n \to \{0, 1\}$ equipped with the product topology.) Furthermore, the natural action of $GL_n(\mathbb{Q})$ on the vector space $\mathbb{Q}^n$ induces a corresponding Borel action on $S(\mathbb{Q}^n)$; and it is easily checked that if $A, B \in S(\mathbb{Q}^n)$, then $A \cong B$ iff there exists an element $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) = B$. It follows that the isomorphism relation on $S(\mathbb{Q}^n)$ is a countable Borel equivalence relation. (If X is a standard Borel space, then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. The Borel equivalence relation E is said to be *countable* iff every E-equivalence class is countable.)

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Notation 1.1. Throughout this paper, the isomorphism relation on $S(\mathbb{Q}^n)$ will be denoted by $\cong_n$.

The notion of Borel reducibility will enable us to compare the complexity of the isomorphism relations for $n \geq 1$. If E, F are Borel equivalence relations on the standard Borel spaces X, Y, respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel map $f : X \to Y$ such that xEy iff $f(x)Ff(y)$. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$. Most of the Borel equivalence relations that we shall consider in this paper arise from group actions as follows. Let G be a *lcsc* group; i.e. a locally compact second countable group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g.x$ of G on X. The corresponding G-orbit equivalence relation on X, which we shall denote by E_G^X, is a Borel equivalence relation. In fact, by Kechris [18], E_G^X is Borel bireducible with a countable Borel equivalence relation. Conversely, by Feldman-Moore [10], if E is an arbitrary countable Borel equivalence relation on the standard Borel space X, then there exists a countable group G and a Borel action of G on X such that $E = E_G^X$.

It is clear that $(\cong_n) \leq_B (\cong_{n+1})$ for all $n \geq 1$; and our earlier question on the difficulty of the classification problem for the torsion-free abelian groups of rank $n \geq 2$ can be rephrased as the question of whether $(\cong_1) <_B (\cong_n)$ when $n \geq 2$. Before we discuss Hjorth's solution [14] of this problem and state the main theorems of this paper, it will be helpful to first give a brief account of some of the general theory of countable Borel equivalence relations. (A detailed development of the theory can be found in Jackson-Kechris-Louveau [16].)

The least complex countable Borel equivalence relations are those which are *smooth*; i.e. those countable Borel equivalence relations E on a standard Borel space X for which there exists a Borel function $f : X \to Y$ into a standard Borel space Y such that xEy iff $f(x) = f(y)$. Next in complexity come those countable Borel equivalence relations E such that E is Borel bireducible with the *Vitali equivalence relation* E_0 defined on $2^{\mathbb{N}}$ by xE_0y iff $x(n) = y(n)$ for almost all n. More precisely, by Harrington-Kechris-Louveau [13], if E is a countable Borel equivalence relation, then E is nonsmooth iff $E_0 \leq_B E$. Furthermore, by Dougherty-Jackson-Kechris [9], if E is a countable Borel equivalence relation on a standard Borel space X, then the following three properties are equivalent:

(1) $E \leq_B E_0$.
(2) E is *hyperfinite*; i.e. there exists an increasing sequence

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

of finite Borel equivalence relations on X such that $E = \bigcup_{n \in \mathbb{N}} F_n$. (Here an equivalence relation F is said to be *finite* iff every F-equivalence class is finite.)
(3) There exists a Borel action of $\mathbb{Z}$ on X such that $E = E_{\mathbb{Z}}^X$.

It turns out that there is also a most complex countable Borel equivalence relation E_∞, which is *universal* in the sense that $F \leq_B E_\infty$ for every countable Borel equivalence relation F, and that $E_0 <_B E_\infty$. (Clearly this universality property uniquely determines E_∞ up to Borel bireducibility.) E_∞ has a number of natural realisations in many areas of mathematics, including algebra, topology and recursion theory. (See Jackson-Kechris-Louveau [16].) For example, E_∞ is Borel bireducible to both the isomorphism relation for finitely generated groups [31] and the isomorphism relation for fields of finite transcendence degree [32].

For many years, it was an open problem whether, up to Borel bireducibility, there existed infinitely many countable Borel equivalence relations E such that $E_0 <_B E <_B E_\infty$. This problem was finally resolved by Adams-Kechris [3], who used Zimmer's superrigidity theory [34] to show that there are actually 2^ω such relations E up to Borel bireducibility. Of these, the only ones which have been extensively studied are the treeable relations, which were originally introduced by Adams [1]. The countable Borel equivalence relation E on the standard Borel space X is said to be *treeable* iff there exists a Borel relation $R \subseteq X \times X$ such that:

(a) $\langle X; R \rangle$ is an acyclic graph; and
(b) the connected components of $\langle X; R \rangle$ are precisely the E-equivalence classes.

For example, whenever a countable free group F has a free Borel action on a standard Borel space X, then the corresponding orbit equivalence relation E_F^X is treeable. (Here F is said to act *freely* on X iff $g.x \neq x$ for all $1 \neq g \in F$ and $x \in X$.) The class of treeable relations is not nearly so well understood as that of the hyperfinite relations. It is known that there exists a universal treeable countable Borel equivalence relation $E_{T\infty}$; but it remains open whether there exists a (necessarily treeable) countable Borel equivalence relation E such that

$$E_0 <_B E <_B E_{T\infty},$$

or even whether $E_{T\infty} \leq_B E$ for every non-hyperfinite countable Borel equivalence relation E. (A fuller account of the notion of treeability can be found in Hjorth-Kechris [15] and Jackson-Kechris-Louveau [16].) We shall return to a further consideration of these questions at the end of Section 5.

Returning to our discussion of the complexity of the isomorphism relation $\cong_n$ on $S(\mathbb{Q}^n)$, it is easily checked that Baer's classification of the rank 1 groups implies that $(\cong_1) \sim_B E_0$. (For example, see [31] or [21].) In [15], Hjorth-Kechris conjectured that $(\cong_n) \sim_B E_\infty$ for all $n \geq 2$; in other words, the classification problem for the torsion-free abelian groups of rank 2 is already as complex as that for arbitrary finitely generated groups. In [14], Hjorth provided some evidence for this conjecture by proving that $E_0 <_B (\cong_n)$ for all $n \geq 2$. (For $n \geq 3$, Hjorth actually proved the stronger result that $\cong_n$ is not treeable. More recently, Kechris [21] has shown that $\cong_2$ is also not treeable.) Later Adams-Kechris [3] used Zimmer's superrigidity theorem for cocycles [34, Theorem 5.2.5] to prove that

$$(\cong_1^*) <_B (\cong_2^*) <_B \cdots <_B (\cong_n^*) <_B \cdots$$

where $(\cong_n^*)$ is the restriction of the isomorphism relation to the class of *rigid* torsion-free abelian groups $A \in S(\mathbb{Q}^n)$. Here an abelian group A is said to be rigid if its only automorphisms are the obvious ones: $a \mapsto a$ and $a \mapsto -a$. In particular, none of the relations $\cong_n^*$ is a universal countable Borel equivalence relation. It was not clear whether the Adams-Kechris result should be regarded as evidence for or against the Hjorth-Kechris conjecture, since very little was known concerning the relationship between $\cong_n^*$ and $\cong_m$ for n, $m \geq 1$. Of course, it is clear that $(\cong_n^*) \leq_B (\cong_n)$ for all $n \geq 1$; and it is easily seen that $(\cong_1^*) \sim_B E_0$ and so $(\cong_1^*) \sim_B (\cong_1)$. But, apart from these easy observations, essentially nothing else was known. Recently, Thomas [29] disproved the Hjorth-Kechris conjecture by showing that $(\cong_3^*) \not\leq_B (\cong_2)$. However, Thomas [29] left open the possibility that there might exist an integer $n > 2$ such that $(\cong_n) \sim_B E_\infty$. The main result in this paper shows that none of the relations $\cong_n$ is universal.

Theorem 1.2. $(\cong^*_{n+1}) \not\leq_B (\cong_n)$ *for all* $n \geq 1$.

As an immediate consequence, we also see that the classification problem for $S(\mathbb{Q}^{n+1})$ is strictly more complex than that for $S(\mathbb{Q}^n)$.

Theorem 1.3. $(\cong_n) <_B (\cong_{n+1})$ *for all* $n \geq 1$.

Theorem 1.2 is an easy consequence of Theorem 1.5. But before we can state Theorem 1.5, we need to recall some notions from ergodic theory and group theory. Let G be a lcsc group and let X be a standard Borel G-space. Throughout this paper, a probability measure on X will always mean a Borel probability measure; i.e. a measure which is defined on the collection of Borel subsets of X. The probability measure μ on X is said to be *nonatomic* if $\mu(\{x\}) = 0$ for all $x \in X$; and μ is said to be *G-invariant* iff $\mu(g(A)) = \mu(A)$ for every $g \in G$ and Borel subset $A \subseteq X$. The G-invariant probability measure μ is *ergodic* iff for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. It is well known that the following two properties are equivalent:

(i) μ is ergodic.
(ii) If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \upharpoonright M$ is a constant function.

For later use, we shall now also recall the notions of a Kazhdan group and an amenable group. Let G be a lcsc group and let $\pi : G \to U(\mathcal{H})$ be a unitary representation of G on the separable Hilbert space $\mathcal{H}$. Then π *almost admits invariant vectors* if for every $\varepsilon > 0$ and every compact subset $K \subseteq G$, there exists a unit vector $v \in \mathcal{H}$ such that $||\pi(g).v - v|| < \varepsilon$ for all $g \in K$. We say that G is a *Kazhdan group* if for every unitary representation π of G, if π almost admits invariant vectors, then π has a non-zero invariant vector. If G is a connected semisimple $\mathbb{R}$-group, each of whose almost $\mathbb{R}$-simple factors has $\mathbb{R}$-rank at least 2, and Γ is a lattice in G, then Γ is a Kazhdan group. (For example, see Margulis [26] or Zimmer [34].) In particular, $SL_m(\mathbb{Z})$ is a Kazhdan group for each $m \geq 3$.

A countable (discrete) group G is *amenable* if there exists a finitely additive G-invariant probability measure $\nu : \mathcal{P}(G) \to [0,1]$ defined on every subset of G. During the proof of Theorem 2.3, we shall make use of the fact that if the countable G is soluble-by-finite, then G is amenable. (For example, see Wagon [33, Theorem 10.4].)

In the first three sections of this paper, we shall only discuss countable groups equipped with the discrete topology. In the later sections, we shall also need to consider various linear algebraic K-groups $G(K) \leqslant GL_n(K)$, where K is a local field. Throughout this paper, a field K is said to be *local* if K is a non-discrete locally compact field of characteristic 0. Each local field K is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p. If K is a local field and $G(K) \leqslant GL_n(K)$ is an algebraic K-group, then $G(K)$ is a lcsc group with respect to the *Hausdorff topology*; i.e. the topology obtained by restricting the natural topology on K^{n^2} to $G(K)$. Any topological notions concerning the group $G(K)$ will always refer to the Hausdorff topology.

Notation 1.4. Throughout this paper, $R(\mathbb{Q}^n)$ will denote the Borel set consisting of the groups $A \in S(\mathbb{Q}^n)$ of rank exactly n.

Theorem 1.5. *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n f(y)$. Then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\cong_n$-class.*

We shall prove Theorem 1.5 in Section 3.

Proof of Theorem 1.2. The case when $n = 1$ was dealt with in Hjorth [14]. So suppose that $n \geq 2$. Let $R(\mathbb{Q}^{n+1}, \mathbb{Z}^{n+1})$ be the Borel set consisting of those $A \in R(\mathbb{Q}^{n+1})$ such that $\mathbb{Z}^{n+1} \leqslant A$. Then $R(\mathbb{Q}^{n+1}, \mathbb{Z}^{n+1})$ is invariant under the action of the subgroup $SL_{n+1}(\mathbb{Z})$ of $GL_{n+1}(\mathbb{Q})$; and Hjorth [14] has shown that there exists an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset X of $R(\mathbb{Q}^{n+1}, \mathbb{Z}^{n+1})$ with the following properties:

(i) Each $A \in X$ is rigid.

(ii) There exists an $SL_{n+1}(\mathbb{Z})$-invariant nonatomic probability measure μ on X.

Furthermore, Adams-Kechris [3, Section 6] have shown that we can also suppose that:

(iii) μ is ergodic.

Suppose that $(\cong^*_{n+1}) \leq_B (\cong_n)$. Then there exists a Borel function $f : X \to S(\mathbb{Q}^n)$ such that $A \cong^*_{n+1} B$ iff $f(A) \cong_n f(B)$. By the ergodicity of μ, there exists an integer $1 \leq \ell \leq n$ and an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f(A)$ has rank ℓ for each $A \in X_0$. Let $S_\ell(\mathbb{Q}^n)$ be the Borel subset consisting of those $A \in S(\mathbb{Q}^n)$ such that A has rank exactly ℓ. Let $g : S_\ell(\mathbb{Q}^n) \to R(\mathbb{Q}^\ell)$ be a Borel function such that $C \cong_n D$ iff $g(C) \cong_\ell g(D)$ and let $h = (g \circ f) \restriction X_0$. Then $h : X_0 \to R(\mathbb{Q}^\ell)$ is a Borel function such that $A \cong^*_{n+1} B$ iff $h(A) \cong_\ell h(B)$. Clearly if A is $E^{X_0}_{SL_{n+1}(\mathbb{Z})}$-equivalent to B, then $A \cong^*_{n+1} B$ and so $h(A) \cong_\ell h(B)$. Hence by Theorem 1.5, there exists an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset $M \subseteq X_0$ with $\mu(M) = 1$ such that h maps M into a single $\cong_\ell$-class $\mathcal{C}$. But clearly $h^{-1}(\mathcal{C})$ consists of only countably many $E^{X_0}_{SL_{n+1}(\mathbb{Z})}$-classes, which contradicts the fact that μ is nonatomic. Hence $(\cong^*_{n+1}) \not\leq_B (\cong_n)$. □

The above argument also shows that the analogue of Theorem 1.3 holds for the isomorphism relations $\cong_n \restriction R(\mathbb{Q}^n)$ on the groups of rank exactly n. (To see that $(\cong_n \restriction R(\mathbb{Q}^n)) \leq_B (\cong_{n+1} \restriction R(\mathbb{Q}^{n+1}))$, notice that we can define a Borel reduction $f : R(\mathbb{Q}^n) \to R(\mathbb{Q}^{n+1})$ by $f(A) = A \oplus \mathbb{Q}$.)

The analogue of Theorem 1.3 also holds for the class of p-local torsion-free abelian groups, which is defined as follows. Throughout this paper, $\mathbb{P}$ will denote the set of primes. If $p \in \mathbb{P}$, then a group $A \in S(\mathbb{Q}^n)$ is said to be *p-local* iff $A = qA$ for every prime $q \neq p$; i.e. A is a $\mathbb{Z}_{(p)}$-module, where

$$\mathbb{Z}_{(p)} = \{a/b \in \mathbb{Q} \mid b \text{ is relatively prime to } p\}.$$

Let $S^{(p)}(\mathbb{Q}^n)$ and $R^{(p)}(\mathbb{Q}^n)$ be the Borel sets consisting of the p-local groups A such that $A \in S(\mathbb{Q}^n)$, $A \in R(\mathbb{Q}^n)$, respectively; and let $\cong^{(p)}_n$ be the restriction of the isomorphism relation to $S^{(p)}(\mathbb{Q}^n)$.

Theorem 1.6. *Let $p \in \mathbb{P}$ be a prime. Then $(\cong_n^{(p)}) <_B (\cong_{n+1}^{(p)})$ for all $n \geq 1$.*

Proof. It is easily checked that $|S^{(p)}(\mathbb{Q})| = \omega$ and that $|S^{(p)}(\mathbb{Q}^2)| = 2^\omega$; and clearly this implies that $(\cong_1^{(p)}) <_B (\cong_2^{(p)})$. So we can suppose that $n \geq 2$. Let $R^{(p)}(\mathbb{Q}^{n+1}, \mathbb{Z}_{(p)}^{n+1})$ be the Borel set consisting of those $A \in R^{(p)}(\mathbb{Q}^{n+1})$ such that $\mathbb{Z}_{(p)}^{n+1} \leqslant A$. Then $R^{(p)}(\mathbb{Q}^{n+1}, \mathbb{Z}_{(p)}^{n+1})$ is invariant under the action of the subgroup $SL_{n+1}(\mathbb{Z})$ of $GL_{n+1}(\mathbb{Q})$; and Hjorth [14] has shown that there exists an $SL_{n+1}(\mathbb{Z})$-invariant Borel subset X of $R^{(p)}(\mathbb{Q}^{n+1}, \mathbb{Z}_{(p)}^{n+1})$ with the following properties:

(i) There exists an $SL_{n+1}(\mathbb{Z})$-invariant nonatomic probability measure μ on X.
(ii) There exists an infinite subgroup $L \leqslant SL_{n+1}(\mathbb{Z})$ which acts freely on X.

Arguing as in Adams-Kechris [3, Section 6], we can also suppose that:

(iii) μ is ergodic.

Now arguing as in the proof of Theorem 1.2, we see that $(\cong_{n+1}^{(p)}) \not\leq_B (\cong_n)$. Hence $(\cong_n^{(p)}) <_B (\cong_{n+1}^{(p)})$. □

(Once again, it is easily seen that the analogue of Theorem 1.6 also holds for the isomorphism relations on p-local groups of rank exactly n.) Of course, the above proof is not completely satisfactory, since it gives no information concerning the complexity of the relation $\cong_2^{(p)}$. In particular, it leaves open the possibility that $\cong_2^{(p)}$ is smooth. This situation will be remedied in Section 5, where we shall show that $\cong_2^{(p)}$ is not treeable.

In an earlier version of this paper, I conjectured that if $p \neq q$ are distinct primes and $n \geq 2$, then the isomorphism relations $\cong_n^{(p)}$ and $\cong_n^{(q)}$ are incomparable with respect to Borel reducibility. In Thomas [30], this was proved in the case when $n \geq 3$. However, the case when $n = 2$ still remains open.

This paper is organised as follows. In Section 2, we shall discuss the notion of a cocycle of a group action and state the two cocycle reduction results which are needed in the proof of Theorem 1.5. In Sections 3 and 4, we shall consider the quasi-equality and quasi-isomorphism relations on $R(\mathbb{Q}^n)$. These relations were first introduced by Jónsson [17], who showed that much of the pathological behaviour of the class of finite rank torsion-free abelian groups disappears when the isomorphism relation is replaced by the coarser quasi-isomorphism relation. In Section 3, we shall initially prove the analogue of Theorem 1.5 for the quasi-isomorphism relation, modulo the result that the quasi-equality relation is hyperfinite, which will be proved in Section 4. Theorem 1.5 will then follow easily. In Section 5, we shall study the class of p-local torsion-free abelian groups of finite rank. In particular, we shall show that the isomorphism relation for the class of p-local groups of rank 2 is not treeable. Finally in Section 6, we shall prove our main cocycle reduction result.

Throughout this paper, we shall identify linear transformations $\pi : \mathbb{Q}^n \to \mathbb{Q}^n$ with the corresponding matrices $M_\pi \in \mathrm{Mat}_n(\mathbb{Q})$ with respect to the standard basis $e_1, \dots, e_n$. If $\pi \in \mathrm{Mat}_n(\mathbb{Q})$, then π^t denotes the transpose of π. If J is a ring, then J^* denotes the group of multiplicative units of J. If K is a field, then $\overline{K}$ denotes the algebraic closure of K.

2. Cocycles

In this section, we shall discuss the notion of a cocycle of a group action and state the two cocycle reduction results which are needed in the proof of Theorem 1.5. (Clear accounts of the theory of cocycles can be found in Zimmer [34] and Adams-Kechris [3]. In particular, Adams-Kechris [3, Section 2] provides a convenient introduction to the basic techniques and results in this area, written for the non-expert in the ergodic theory of groups.) Let G be a lcsc group and let X be a standard Borel G-space with an invariant probability measure μ.

Definition 2.1. If H is a lcsc group, then a Borel function $\alpha : G \times X \to H$ is called a *cocycle* if for all $g, h \in G$,

$$\alpha(hg, x) = \alpha(h, g.x)\alpha(g, x) \quad \mu\text{-a.e.}(x).$$

If the above equation holds for all $g, h \in G$ and all $x \in X$, then α is said to be a *strict cocycle*. By Zimmer [34, Theorem B.9], if $\alpha : G \times X \to H$ is a Borel cocycle, then there exists a strict cocycle $\beta : G \times X \to H$ such that for all $g \in G$,

$$\beta(g, x) = \alpha(g, x) \quad \mu\text{-a.e.}(x).$$

The standard example of a cocycle arises in the following fashion. Suppose that Y is a standard Borel H-space and that H acts freely on Y. If $f : X \to Y$ is a Borel function such that $xE_G^X y$ implies $f(x)E_H^Y f(y)$, then we can define a strict Borel cocycle $\alpha : G \times X \to H$ by letting $\alpha(g, x)$ be the *unique* element of H such that

$$\alpha(g, x).f(x) = f(g.x).$$

Suppose now that $B : X \to H$ is a Borel function and that $f' : X \to Y$ is defined by $f'(x) = B(x).f(x)$. Then $xE_G^X y$ also implies that $f'(x)E_H^Y f'(y)$; and the corresponding cocycle $\alpha' : G \times X \to H$ satisfies

$$\alpha'(g, x) = B(g.x)\alpha(g, x)B(x)^{-1}$$

for all $g \in G$ and $x \in X$. This observation motivates the following definition.

Definition 2.2. Let H be a lcsc group. Then the cocycles $\alpha, \beta : G \times X \to H$ are *equivalent*, written $\alpha \sim \beta$, iff there exists a Borel function $B : X \to H$ such that for all $g \in G$,

$$\beta(g, x) = B(g.x)\alpha(g, x)B(x)^{-1} \quad \mu\text{-a.e.}(x).$$

The proof of Theorem 1.5 is based upon a number of cocycle reduction results; i.e. theorems which say that under suitable hypotheses on G and H, every cocycle $\alpha : \Gamma \times X \to H$ is equivalent to a cocycle β such that $\beta(\Gamma \times X)$ is contained in a "small" subgroup of H. Before we state our main cocycle reduction theorem, we need to review some notions from the theory of algebraic groups. Throughout this paper, Ω will denote a fixed algebraically closed field of characteristic 0 which contains $\mathbb{R}$ and all of the p-adic fields $\mathbb{Q}_p$. By an *algebraic group*, we shall always mean a subgroup G of some general linear group $GL_n(\Omega)$ which is Zariski closed in $GL_n(\Omega)$. If G can be defined using polynomials with coefficients in the subfield k of Ω, then we say that G is an algebraic k-group. For any subring $R \subseteq \Omega$, we define

$$GL_n(R) = \{(r_{ij}) \in GL_n(\Omega) \mid r_{ij} \in R \text{ and } \det(r_{ij})^{-1} \in R\};$$

and if $G \leqslant GL_n(\Omega)$ is an algebraic group, then we define

$$G(R) = G \cap GL_n(R).$$

In particular, if $R = k$ is a subfield of Ω and G is an algebraic k-group, then $G(k)$ is the group of all matrices in G with entries in k.

The following cocycle reduction theorem is a straightforward consequence of Zimmer's superrigidity theorem [34, Theorem 5.2.5] and the ideas of Adams-Kechris [3].

Theorem 2.3. *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure. Suppose that G is an algebraic $\mathbb{Q}$-group such that $\dim G < m^2 - 1$ and that $H \leqslant G(\mathbb{Q})$. Then for every Borel cocycle $\alpha : SL_m(\mathbb{Z}) \times X \to H$, there exists an equivalent cocycle γ such that $\gamma(SL_m(\mathbb{Z}) \times X)$ is contained in a finite subgroup of H.*

We shall prove Theorem 2.3 in Section 6. In the proof of Theorem 1.5, we shall also make use of the following cocycle reduction result, which is a special case of Hjorth-Kechris [15, Theorem 10.5]. (It should be pointed out that this particular special case is a consequence of Zimmer [34, Theorem 9.1.1].)

Theorem 2.4. *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that Y is a standard Borel space and that F is a hyperfinite equivalence relation on Y. If $f : X \to Y$ is a Borel function such that $x E^X_{SL_m(\mathbb{Z})} y$ implies $f(x) F f(y)$, then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single F-class.*

3. The quasi-isomorphism relation

In this section, we shall use our cocycle reduction theorems to prove Theorem 1.5. We shall begin by saying a few words about the strategy of the proof. So suppose that $m \geq 3$ and that X is a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $x E^X_{SL_m(\mathbb{Z})} y$ implies $f(x) \cong_n f(y)$. Because $GL_n(\mathbb{Q})$ does not act freely on $R(\mathbb{Q}^n)$, we are initially unable to define a corresponding cocycle $\alpha : SL_m(\mathbb{Z}) \times X \to GL_n(\mathbb{Q})$.

In Thomas [29], we were able to get around this difficulty, in the case when $n = 2$, by reducing to the situation where there exists a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $\mathrm{Aut}(f(x))$ is a *fixed* subgroup L of $GL_2(\mathbb{Q})$ for all $x \in X_0$. This meant that $\cong_2 \restriction f(X_0)$ was induced by a free action of the quotient group

$$H = N_{GL_2(\mathbb{Q})}(L)/L;$$

and so we could define a corresponding cocycle $\alpha : SL_m(\mathbb{Z}) \times X \to H$. However, this would not have been useful unless there existed a suitable cocycle reduction result for cocycles taking values in H; and for such a result to exist, it was necessary that H should be a "reasonably classical" group. Fortunately, using Król's analysis [22] of the automorphism groups of the torsion-free abelian groups of rank 2, we were able to explicitly enumerate the possibilities for L; and it turned out that we could make a further reduction to the situation when H was a slight variant of $PGL_2(\mathbb{Q})$.

Now suppose that $n > 2$. Then it seems likely that, using Arnold [4], we can once again reduce to the situation where there exists a Borel subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $\mathrm{Aut}(f(x))$ is a fixed subgroup L of $GL_n(\mathbb{Q})$ for all $x \in X_0$. Unfortunately, as n gets larger, the possibilities for L become more and more complex. In fact, by Corner's Theorem [7], L can be isomorphic to the group of units of an arbitrary reduced subring of $\mathrm{Mat}_d(\mathbb{Q})$, where $d = \lfloor n/2 \rfloor$; and it is far from clear whether $H = N_{GL_n(\mathbb{Q})}(L)/L$ will always be a "reasonably classical" group. In order to avoid these algebraic complications, we shall initially replace the isomorphism relation on $R(\mathbb{Q}^n)$ by the coarser relation of quasi-isomorphism. This relation was first introduced in Jónsson [17], where it was shown that the class of torsion-free abelian groups of finite rank has a better decomposition theory with respect to quasi-isomorphism than with respect to isomorphism. This decomposition theory will not concern us in this paper. Rather we shall exploit the fact that much of the number-theoretical complexity of a finite rank torsion-free abelian group is lost when we work with respect to quasi-isomorphism; and this turns out to be enough to ensure that the cocycles that arise in our analysis always take values in a "reasonably classical" group. As a bonus, we obtain that the problem of classifying the torsion-free abelian groups of finite rank up to quasi-isomorphism is also intractible. (However, we should point out that the shift from isomorphism to quasi-isomorphism comes at a cost. In Thomas [29], the proof yields an explicit decomposition of $\cong_2$ as a direct sum of amenable relations and orbit relations induced by free actions of homomorphic images of $GL_2(\mathbb{Q})$. It does not seem possible to extract an analogous decomposition of $\cong_n$ from the current proof.)

Definition 3.1. Suppose that $A, B \in R(\mathbb{Q}^n)$. Then A is said to be *quasi-contained* in B, written $A \prec_n B$, if there exists an integer $m > 0$ such that $mA \leqslant B$. If $A \prec_n B$ and $B \prec_n A$, then A and B are said to be *quasi-equal* and we write $A \approx_n B$.

Recall that if $A \in R(\mathbb{Q}^n)$ and $m > 0$, then $[A : mA] < \infty$. (For example, see [12, Exercise 92.5].) It follows that if $A, B \in R(\mathbb{Q}^n)$, then $A \approx_n B$ iff $A \cap B$ has finite index in both A and B.

Lemma 3.2. *$\approx_n$ is a countable Borel equivalence relation on $R(\mathbb{Q}^n)$.*

Proof. It is clear that $\approx_n$ is a Borel equivalence relation. Thus it is enough to show that if $A \in R(\mathbb{Q}^n)$, then there only exist countably many $B \in R(\mathbb{Q}^n)$ such that $A \approx_n B$. To see this, suppose that $A \approx_n B$ and let $r, s > 0$ be integers such that $rA \leqslant B$ and $sB \leqslant A$. Then $rA \leqslant B \leqslant (1/s)A$ and

$$[(1/s)A : rA] = [A : rsA] < \infty.$$

It follows that there are only countably many possibilities for B. □

Definition 3.3. Suppose that $A, B \in R(\mathbb{Q}^n)$. Then A and B are said to be *quasi-isomorphic*, written $A \sim_n B$, if there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx_n B$.

Using Lemma 3.2, it follows that each $\sim_n$-class consists of only countably many $\cong_n$-classes. In particular, $\sim_n$ is also a countable Borel equivalence relation on $R(\mathbb{Q}^n)$. Most of our effort in this section will be devoted to proving the following analogue of Theorem 1.5 for the quasi-isomorphism relation.

Theorem 3.4. *Let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $x E^X_{SL_m(\mathbb{Z})} y$ implies $f(x) \sim_n f(y)$. Then there exists an $SL_m(\mathbb{Z})$-invariant Borel subset M with $\mu(M) = 1$ such that f maps M into a single $\sim_n$-class.*

Arguing as in the proof of Theorem 1.2, we now easily obtain the following result.

Theorem 3.5. $(\sim_n) <_B (\sim_{n+1})$ *for all* $n \geq 1$.

(Of course, since $\sim_n$ is defined to be the quasi-isomorphism relation on the space $R(\mathbb{Q}^n)$ of groups of rank exactly n, it is necessary to explain why $(\sim_n) \leq_B (\sim_{n+1})$. To see this, let $f : R(\mathbb{Q}^n) \to R(\mathbb{Q}^{n+1})$ be the Borel map defined by $f(A) = A \oplus \mathbb{Q}$. Using Fuchs [12, Theorem 92.5], it follows easily that f is a Borel reduction from $\sim_n$ to $\sim_{n+1}$.) Before we begin the proof of Theorem 3.4, we shall show how to derive Theorem 1.5.

Proof of Theorem 1.5. Suppose that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \cong_n f(y)$. Then obviously $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \sim_n f(y)$; and so by Theorem 3.4, there exists an $SL_m(\mathbb{Z})$-invariant Borel subset Y with $\mu(Y) = 1$ such that f maps Y into a single $\sim_n$-class $\mathcal{C}$. Since $\mathcal{C}$ is countable, there exists a Borel subset $Z \subseteq Y$ with $\mu(Z) > 0$ and a fixed group $A \in \mathcal{C}$ such that $f(x) = A$ for all $x \in Z$. Since μ is ergodic, the $SL_m(\mathbb{Z})$-invariant Borel subset $M = SL_m(\mathbb{Z}).Z$ satisfies $\mu(M) = 1$; and clearly f maps M into the $\cong_n$-class containing A. $\square$

For each $A \in R(\mathbb{Q}^n)$, let $[A]$ be the $\approx_n$-class containing A. During the proof of Theorem 3.4, we shall consider the action of $GL_n(\mathbb{Q})$ on the set of $\approx_n$-classes. In order to compute the setwise stabiliser in $GL_n(\mathbb{Q})$ of a $\approx_n$-class $[A]$, it is necessary to introduce the notions of a quasi-endomorphism and a quasi-automorphism. If $A \in R(\mathbb{Q}^n)$, then a linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-endomorphism* of A iff $\varphi(A) \prec_n A$. Equivalently, φ is a quasi-endomorphism of A iff there exists an integer $m > 0$ such that $m\varphi \in \mathrm{End}(A)$. It is easily checked that the collection $\mathrm{QE}(A)$ of quasi-endomorphisms of A is a $\mathbb{Q}$-subalgebra of $\mathrm{Mat}_n(\mathbb{Q})$ and that if $A \approx_n B$, then $\mathrm{QE}(A) = \mathrm{QE}(B)$. A linear transformation $\varphi \in \mathrm{Mat}_n(\mathbb{Q})$ is said to be a *quasi-automorphism* of A iff φ is a unit of the $\mathbb{Q}$-algebra $\mathrm{QE}(A)$. The group of quasi-automorphisms of A is denoted by $\mathrm{QAut}(A)$. By Exercise 6.1 [4], if $\psi \in \mathrm{End}(A)$, then $\psi \in \mathrm{QAut}(A)$ iff ψ is a monomorphism.

Lemma 3.6. *If* $A \in R(\mathbb{Q}^n)$*, then* $\mathrm{QAut}(A)$ *is the setwise stabiliser of* $[A]$ *in* $GL_n(\mathbb{Q})$.

Proof. First suppose that $\varphi \in \mathrm{QAut}(A)$. Then there exists an integer $m > 0$ such that $\psi = m\varphi \in \mathrm{End}(A)$. Clearly ψ is also a unit of $\mathrm{QE}(A)$ and so ψ is a monomorphism. Hence by [12, Exercise 92.5], $\psi(A)$ has finite index in A and so $\psi(A) \approx_n A$. Since $\psi(A) = m\varphi(A)$, it follows that $\psi(A) \approx_n \varphi(A)$. Thus $\varphi(A) \approx_n A$ and so φ stabilises $[A]$.

Conversely suppose that $\varphi \in GL_n(\mathbb{Q})$ stabilises $[A]$. Then $\varphi(A) \approx_n A$ and so there exists an integer $m > 0$ such that $m\varphi(A) \leqslant A$. Since $m\varphi \in \mathrm{End}(A)$ is a monomorphism, it follows that $m\varphi \in \mathrm{QAut}(A)$ and so $\varphi \in \mathrm{QAut}(A)$. $\square$

Now we are ready to begin the proof of Theorem 3.4. So let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $1 \leq n < m$ and that $f : X \to R(\mathbb{Q}^n)$ is a Borel function such that $xE^X_{SL_m(\mathbb{Z})}y$ implies $f(x) \sim_n f(y)$. Let $E = E^X_{SL_m(\mathbb{Z})}$ and for each $x \in X$, let $A_x = f(x) \in R(\mathbb{Q}^n)$. First notice that there are only countably many possibilities for the $\mathbb{Q}$-algebra $\mathrm{QE}(A_x)$. Hence there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed $\mathbb{Q}$-subalgebra S of $\mathrm{Mat}_n(\mathbb{Q})$ such that $\mathrm{QE}(A_x) = S$ for all $x \in X_1$. By the ergodicity of μ, we have that $\mu(SL_m(\mathbb{Z}).X_1) = 1$. In order to simplify notation, we shall assume that $SL_m(\mathbb{Z}).X_1 = X$. After slightly adjusting f if necessary, we can suppose that $\mathrm{QE}(A_x) = S$ for all $x \in X$. (More precisely, let $c : X \to X$ be a Borel function such that $c(x)Ex$ and $c(x) \in X_1$ for all $x \in X$. Then we can replace f with $f' = f \circ c$.) In particular, we have that $\mathrm{QAut}(A_x) = S^*$, the group of units of S, for each $x \in X$. Now suppose that x, $y \in X$ and that xEy. Then $A_x \sim_n A_y$ and so there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A_x) \approx_n A_y$. Notice that

$$\varphi S \varphi^{-1} = \varphi\, \mathrm{QE}(A_x)\varphi^{-1} = \mathrm{QE}(\varphi(A_x)) = \mathrm{QE}(A_y) = S$$

and so $\varphi \in N = N_{GL_n(\mathbb{Q})}(S)$. Clearly we also have that $\varphi([A_x]) = [A_y]$; and by Lemma 3.6, for each $x \in X$, the stabiliser of $[A_x]$ in $GL_n(\mathbb{Q})$ is $\mathrm{QAut}(A_x) = S^*$. Let $H = N/S^*$ and for each $\varphi \in N$, let $\overline{\varphi} = \varphi S^*$. Then we can define a Borel cocycle $\alpha : SL_m(\mathbb{Z}) \times X \to H$ by

$$\alpha(g, x) = \text{ the unique element } \overline{\varphi} \in H \text{ such that } \varphi([A_x]) = [A_{g.x}].$$

Lemma 3.7. *There exists an algebraic $\mathbb{Q}$-group G with $\dim G < m^2 - 1$ such that $H \leqslant G(\mathbb{Q})$.*

Proof. Recall that throughout this paper, Ω denotes a fixed algebraically closed field containing $\mathbb{R}$ and all of the p-adic fields $\mathbb{Q}_p$. Let

$$\Lambda = \Omega \otimes S \subseteq \mathrm{Mat}_n(\Omega)$$

be the associated Ω-algebra. Then Λ is an affine $\mathbb{Q}$-variety; and the Cayley-Hamilton Theorem implies that the group of units of Λ is given by

$$\Lambda^* = \{\varphi \in \Lambda \mid \det(\varphi) \neq 0\}.$$

Thus Λ^* is an algebraic $\mathbb{Q}$-group and $\Lambda^*(\mathbb{Q}) = S^*$. Furthermore, by [6, Proposition 1.7], $\Gamma = N_{GL_n(\Omega)}(\Lambda)$ is also an algebraic $\mathbb{Q}$-group and clearly $\Gamma(\mathbb{Q}) = N$. By [6, Theorem 6.8], $G = \Gamma/\Lambda^*$ is an algebraic $\mathbb{Q}$-group and

$$H = \Gamma(\mathbb{Q})/\Lambda^*(\mathbb{Q}) \leqslant G(\mathbb{Q}).$$

Finally note that

$$\dim G \leq \dim \Gamma \leq \dim GL_n(\Omega) = n^2 < m^2 - 1.$$

□

By Theorem 2.3, α is equivalent to a cocycle γ such that $\gamma(SL_m(\mathbb{Z}) \times X)$ is contained in a finite subgroup K of H. Let $B : X \to H$ be a Borel function such that:

(*) for all $g \in SL_m(\mathbb{Z})$, $\alpha(g, x) = B(g.x)\gamma(g, x)B(x)^{-1}$ μ-a.e.(x).

It is easily checked that if x satisfies (*) and xEy, then y also satisfies (*). To simplify notation, we shall assume that (*) holds for all $x \in X$. Now there exists a Borel subset $X_1 \subseteq X$ with $\mu(X_1) > 0$ and a fixed element $\overline{\psi} \in H$ such that $B(x) = \overline{\psi}$ for all $x \in X_1$. Since μ is ergodic, $\mu(SL_m(\mathbb{Z}).X_1) = 1$ and so we can also assume that X_1 intersects every $SL_m(\mathbb{Z})$-orbit on X. Let $c : X \to X$ be a Borel function such that $c(x)Ex$ and $c(x) \in X_1$ for each $x \in X$; and let $x_1 = c(x)$. Then for each $x \in X$,

$$\Psi(x) = \{\alpha(g, x_1) \mid g.x_1 \in X_1\} \subseteq \overline{\psi} K \overline{\psi}^{-1}$$

is a nonempty finite subset of H. Hence for each $x \in X$,

$$\begin{aligned}\Phi(x) &= \{\varphi([A_{x_1}]) \mid \overline{\varphi} = \alpha(g, x_1) \text{ for some } g.x_1 \in X_1\}\\ &= \{[A_{g.x_1}] \mid g.x_1 \in X_1\}\\ &= \{[A_y] \mid yEx \text{ and } y \in X_1\}\end{aligned}$$

is a nonempty finite set of $\approx_n$-classes; and clearly if xEy, then $\Phi(x) = \Phi(y)$. By the ergodicity of μ, we can suppose that there exists an integer $1 \leq k \leq |K|$ such that $|\Phi(x)| = k$ for all $x \in X$. Now let $x \mapsto (x_1, \dots, x_k)$ be a Borel function from X to X^k such that for each $x \in X$,

(a) $x_i E x$ and $x_i \in X_1$; and
(b) $\Phi(x) = \{[A_{x_1}], \dots, [A_{x_k}]\}$.

Finally let $\widetilde{f} : X \to R(\mathbb{Q}^n)^k$ be the Borel function defined by

$$\widetilde{f}(x) = (A_{x_1}, \dots, A_{x_k});$$

and let F be the countable Borel equivalence relation on $R(\mathbb{Q})^k$ defined by

$$(A_1, \dots, A_k)\, F\, (B_1, \dots, B_k) \quad \text{iff} \quad \{[A_1], \dots, [A_k]\} = \{[B_1], \dots, [B_k]\}.$$

To complete the proof of Theorem 3.4, we now require the following theorem, which will be proved in Section 4.

Theorem 3.8. *For each $n \geq 1$, the relation $\approx_n$ is hyperfinite.*

Using the fact that $\approx_n$ is hyperfinite, it follows easily that F is also hyperfinite. (For example, see [16, Section 1].) Notice that if xEy, then $\Phi(x) = \Phi(y)$ and so $\widetilde{f}(x) F \widetilde{f}(y)$. By Theorem 2.4, there exists an $SL_m(\mathbb{Z})$-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $\widetilde{f}$ maps M into a single F-class; and this implies that f maps M into a single $\sim_n$-class. This completes the proof of Theorem 3.4.

Finally we should point out that both of the following questions remain open.

Question 3.9. Is $(\cong_n) \leq_B (\sim_n)$ for $n \geq 2$?

Question 3.10. Is $(\sim_n) \leq_B (\cong_n)$ for $n \geq 2$?

In an earlier version of this paper, I pointed out that a negative answer to Question 3.9 would be especially interesting, since it was then unknown whether there existed a pair E, F of countable Borel equivalence relations on a standard Borel space X such that $E \subseteq F$ and $E \not\leq_B F$. Soon afterwards, Adams [2] proved that there exists a pair $E \subseteq F$ of countable Borel equivalence relations such that E and F are incomparable with respect to Borel reducibility.

4. The hyperfiniteness of the quasi-equality relation

In this section, we shall prove Theorem 3.8 which says that the quasi-equality relation $\approx_n$ is hyperfinite for each $n \geq 1$. The following lemma, which is due to Lady [24], will enable us to restrict our attention to the class of p-local groups. Recall that $\mathbb{Z}_{(p)}$ is the ring of rational numbers $a/b \in \mathbb{Q}$ such that b is relatively prime to p.

Definition 4.1. If $A \in R(\mathbb{Q}^n)$ and $p \in \mathbb{P}$, then the *localisation of A at p* is defined to be $A_p = \mathbb{Z}_{(p)} \otimes A$.

Lemma 4.2. *If A, $B \in R(\mathbb{Q}^n)$, then $A \approx_n B$ iff the following two conditions are satisfied:*

(i) *$A_p \approx_n B_p$ for all primes $p \in \mathbb{P}$; and*
(ii) *$A_p = B_p$ for all but finitely many primes $p \in \mathbb{P}$.*

Definition 4.3. For each prime $p \in \mathbb{P}$, we define $\approx_n^{(p)}$ to be the restriction of the quasi-equality relation to the space $R^{(p)}(\mathbb{Q}^n)$ of p-local groups $A \in R(\mathbb{Q}^n)$.

Most of this section will be devoted to proving the following special case of Theorem 3.8.

Lemma 4.4. *For each prime $p \in \mathbb{P}$, the relation $\approx_n^{(p)}$ is smooth.*

Before proving Lemma 4.4, we shall show how to complete the proof of Theorem 3.8.

Proof of Theorem 3.8. Let E_0 be the Vitali equivalence relation on $2^{\mathbb{N}}$; and let E_0^* be the corresponding equivalence relation on $2^{\mathbb{P}\times\mathbb{N}}$, defined by xE_0^*y iff $x(p,n) = y(p,n)$ for all but finitely many pairs $(p,n) \in \mathbb{P}\times\mathbb{N}$. Then clearly $E_0^* \sim_B E_0$. For each prime $p \in \mathbb{P}$, since the relation $\approx_n^{(p)}$ is smooth, there exists an injective Borel map $g_p : R^{(p)}(\mathbb{Q}^n) \to 2^{\mathbb{N}}$ such that $A \approx_n^{(p)} B$ iff $g_p(A)E_0 g_p(B)$. Consider the Borel map $f : R(\mathbb{Q}^n) \to 2^{\mathbb{P}\times\mathbb{N}}$, defined by $f(A)(p,n) = g_p(A_p)(n)$. By Lemma 4.2, if A, $B \in R(\mathbb{Q}^n)$, then $A \approx_n B$ iff $f(A)E_0^* f(B)$. Hence $\approx_n$ is a hyperfinite equivalence relation. □

Thus it only remains to prove Lemma 4.4. For the rest of this section, we shall fix a prime $p \in \mathbb{P}$ and, to simplify the notation, we shall write $\approx$ instead of $\approx_n^{(p)}$. Throughout this section, we shall regard $\mathbb{Q}^n$ as an additive subgroup of the n-dimensional vector space $\mathbb{Q}_p^n$ over the field $\mathbb{Q}_p$ of p-adic numbers; and we shall extend the relation $\approx$ to the collection of all additive subgroups of $\mathbb{Q}_p^n$ by setting $C \approx D$ iff $C \cap D$ has finite index in both C and D. Let $\mathbb{Z}_p$ be the ring of p-adic integers. Following the approach of Kurosh [23] and Malcev [25], we shall now further localise each p-local group $A \in R^{(p)}(\mathbb{Q}^n)$ to a corresponding $\mathbb{Z}_p$-submodule $\widehat{A}$ of $\mathbb{Q}_p^n$. The motivation for this is that while A might have a very complex structure, $\widehat{A}$ will always decompose into a direct sum of copies of $\mathbb{Z}_p$ and $\mathbb{Q}_p$.

Definition 4.5. For each $A \in R^{(p)}(\mathbb{Q}^n)$, we define $\widehat{A} = \mathbb{Z}_p \otimes A$.

We shall regard each $\widehat{A}$ as a subgroup of $\mathbb{Q}_p^n$ in the usual way; i.e. $\widehat{A}$ is the subgroup consisting of all finite sums

$$\gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_t a_t,$$

where $\gamma_i \in \mathbb{Z}_p$ and $a_i \in A$ for $1 \le i \le t$. By [12, Lemma 93.3], there exist integers $0 \le k, \ell \le n$ with $k + \ell = n$ and elements v_i, $w_j \in \widehat{A}$ such that

$$\widehat{A} = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p w_j.$$

Definition 4.6. For each $A \in R^{(p)}(\mathbb{Q}^n)$, we define $V_A = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i$.

The following result clarifies the algebraic content of the quasi-equality relation on $R^{(p)}(\mathbb{Q}^n)$.

Theorem 4.7. *If A, $B \in R^{(p)}(\mathbb{Q}^n)$, then $A \approx B$ iff $V_A = V_B$.*

Theorem 4.7 is an immediate consequence of the following two lemmas.

Lemma 4.8. *If A, $B \in R^{(p)}(\mathbb{Q}^n)$, then $A \approx B$ iff $\widehat{A} \approx \widehat{B}$.*

Proof. First suppose that $\widehat{A} \approx \widehat{B}$. By [12, Lemma 93.2], we have that $\widehat{A} \cap \mathbb{Q}^n = A$ and $\widehat{B} \cap \mathbb{Q}^n = B$. Thus

$$[A : A \cap B] = [\widehat{A} \cap \mathbb{Q}^n : (\widehat{A} \cap \widehat{B}) \cap \mathbb{Q}^n] \leq [\widehat{A} : \widehat{A} \cap \widehat{B}] < \infty.$$

Similarly, $[B : A \cap B] < \infty$ and hence $A \approx B$.

Conversely suppose that $A \approx B$ and let $C = A \cap B$. Then $\widehat{C} \leqslant \widehat{A} \cap \widehat{B}$ and so it is enough to show that $\widehat{C}$ has finite index in both $\widehat{A}$ and $\widehat{B}$. To see this, let $F = A/C$ and consider the short exact sequence

$$0 \to C \to A \to F \to 0.$$

Then by [12, Theorem 60.2], the sequence

$$\mathbb{Z}_p \otimes C \to \mathbb{Z}_p \otimes A \to \mathbb{Z}_p \otimes F \to 0$$

is also exact. Let $F = \bigoplus_{r=1}^{s} C_r$ be a decomposition of F into a direct sum of finite cyclic groups C_r of order m_r. Then by [12, Section 59],

$$\mathbb{Z}_p \otimes F \cong \bigoplus_{r=1}^{s} \mathbb{Z}_p / m_r \mathbb{Z}_p.$$

Thus $\mathbb{Z}_p \otimes F$ is a finite group and so $\widehat{C}$ has finite index in $\widehat{A}$. Similarly $\widehat{C}$ has finite index in $\widehat{B}$. □

Lemma 4.9. *If A, $B \in R^{(p)}(\mathbb{Q}^n)$, then $\widehat{A} \approx \widehat{B}$ iff $V_A = V_B$.*

Proof. First suppose that $\widehat{A} \approx \widehat{B}$. Then for each $v \in V_A$, we have that

$$[\mathbb{Q}_p v : \widehat{B} \cap \mathbb{Q}_p v] = [\widehat{A} \cap \mathbb{Q}_p v : \widehat{A} \cap \widehat{B} \cap \mathbb{Q}_p v] \leq [\widehat{A} : \widehat{A} \cap \widehat{B}] < \infty.$$

Suppose that $v \in V_A \smallsetminus V_B$. Then $\widehat{B} \cap \mathbb{Q}_p v$ is a proper $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p v$ and so there exists a vector $0 \neq u \in \mathbb{Q}_p v$ such that $\widehat{B} \cap \mathbb{Q}_p v = \mathbb{Z}_p u$. But then

$$\mathbb{Q}_p v / (\widehat{B} \cap \mathbb{Q}_p v) \cong \mathbb{Q}_p / \mathbb{Z}_p \cong C(p^\infty),$$

which is a contradiction. Thus $V_A \leqslant V_B$. Similarly $V_B \leqslant V_A$ and so $V_A = V_B$.

Conversely suppose that $V_A = V_B = V$. Then there exists an integer $0 \leq \ell \leq n$ and elements $x_j \in \widehat{A}$, $y_j \in \widehat{B}$ such that

$$\widehat{A} = V \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p x_j$$

and

$$\widehat{B} = V \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p y_j.$$

Let $L_A = \bigoplus_{j=1}^{\ell} \mathbb{Z}_p x_j$ and $L_B = \bigoplus_{j=1}^{\ell} \mathbb{Z}_p y_j$. Then we can identify L_A, L_B with the corresponding $\mathbb{Z}_p$-submodules of the ℓ-dimensional $\mathbb{Q}_p$-vector space $W = \mathbb{Q}_p^n / V$.

Now there exists an integer $t \geq 0$ such that $p^t y_j \in L_A$ for each $1 \leq j \leq \ell$. It follows that

$$[L_B : L_A \cap L_B] \leq [L_B : p^t L_B] = p^{t\ell}.$$

Similarly $[L_A : L_A \cap L_B] < \infty$ and hence $\widehat{A} \approx \widehat{B}$. □

Now that we have dealt with the purely algebraic aspect of the quasi-equality relation on $R^{(p)}(\mathbb{Q}^n)$, we shall next consider its descriptive set-theoretic aspect. Recall that the vector space $\mathbb{Q}_p^n$ is a complete separable metric space with respect to the metric induced by the usual p-adic norm; and that each $\mathbb{Q}_p$-vector subspace $V \leqslant \mathbb{Q}_p^n$ is a closed subset of $\mathbb{Q}_p^n$ with respect to this metric. The *Effros Borel space* on $\mathbb{Q}_p^n$ is defined to be the set

$$F(\mathbb{Q}_p^n) = \{Z \subseteq \mathbb{Q}_p^n \mid Z \text{ is a closed subset of } \mathbb{Q}_p^n\}$$

equipped with the σ-algebra generated by the sets of the form

$$\{Z \in F(\mathbb{Q}_p^n) \mid Z \cap U \neq \emptyset\},$$

where U varies over the open subsets of $\mathbb{Q}_p^n$. By [20, Theorem 12.6], $F(\mathbb{Q}_p^n)$ is a standard Borel space. Thus to complete the proof of Theorem 3.8, we need only show that the map $A \mapsto V_A$ is a Borel map from $R^{(p)}(\mathbb{Q}^n)$ into $F(\mathbb{Q}_p^n)$. Furthermore, it is clear that the map $s : (\mathbb{Q}_p^n)^{\leq n} \to F(\mathbb{Q}_p^n)$, defined by

$$s(v_1, \dots, v_k) = \text{ the } \mathbb{Q}_p\text{-subspace spanned by } \{v_1, \dots, v_k\},$$

is Borel. (For example, this follows easily from [20, Exercise 12.14].) Hence we need only show that there exists a Borel map $b : R^{(p)}(\mathbb{Q}^n) \to (\mathbb{Q}_p^n)^{\leq n}$ such that $b(A)$ is a basis of V_A. (The details of the following construction of a basis $b(A)$ of V_A will be used in the next section, in which we study the structure of a "random" group $A \in R^{(p)}(\mathbb{Q}^n, \mathbb{Z}_{(p)}^n)$.)

Definition 4.10. Let $A \in R^{(p)}(\mathbb{Q}^n)$. Then a sequence $(a_1, \dots, a_\ell)$ of nonzero elements of A is said to be *p-independent* iff whenever $n_1, \dots, n_\ell \in \mathbb{Z}$ are such that

$$n_1 a_1 + \cdots + n_\ell a_\ell \in pA,$$

then p divides n_j for all $1 \leq j \leq \ell$. The sequence $(a_1, \dots, a_\ell)$ is said to be a *p-basis* iff $(a_1, \dots, a_\ell)$ is a maximal p-independent sequence.

Fix some $A \in R^{(p)}(\mathbb{Q}^n)$. Then we can clearly choose a p-basis $(a_1, \dots, a_\ell)$ of A in a Borel fashion. Let $P = \langle a_1, \dots, a_\ell \rangle$ be the subgroup generated by $\{a_1, \dots, a_\ell\}$. Then by [12, Section 32], A/P is p-divisible and so A/P is a divisible group. Thus $A/P = R \oplus T$, where T is the torsion subgroup and R is the direct sum of $k = n - \ell$ copies of $\mathbb{Q}$. Furthermore, by [12, Exercise 93.1], $\dim V_A = k$. Next, in a Borel fashion, we can choose a sequence $(z_1, \dots, z_k)$ of elements of A such that $(z_1 P, \dots, z_k P)$ is a basis of $R \cong \mathbb{Q}^k$. Finally we shall use $(z_1, \dots, z_k)$ and $(a_1, \dots, a_\ell)$ to construct a basis $(v_1, \dots, v_k)$ of V_A.

Fix some $1 \leq i \leq k$. Let $t \geq 1$ and suppose inductively that there exist integers $c_s^{(j)}$ for $1 \leq j \leq \ell$ and $1 \leq s \leq t$ such that the following conditions are satisfied.

(1) $0 \leq c_s^{(j)} < p$.

(2) There exists $d_t \in A$ such that

$$p^t d_t = z_i + n_t^{(1)} a_1 + \cdots + n_t^{(\ell)} a_\ell,$$

where $n_t^{(j)} = c_1^{(j)} + c_2^{(j)} p + \cdots + c_t^{(j)} p^{t-1}$.

Since R is divisible, there exists an element $d_{t+1} \in A$ and integers $c_{t+1}^{(1)}, \ldots, c_{t+1}^{(\ell)}$ such that

$$pd_{t+1} = d_t + c_{t+1}^{(1)} a_1 + \cdots + c_{t+1}^{(\ell)} a_\ell;$$

and after adjusting our choice of d_{t+1} if necessary, we can suppose that $0 \leq c_{t+1}^{(j)} < p$ for each $1 \leq j \leq \ell$. Clearly

$$p^{t+1} d_{t+1} = z_i + n_{t+1}^{(1)} a_1 + \cdots + n_{t+1}^{(\ell)} a_\ell.$$

Thus the induction can be completed. For $1 \leq j \leq \ell$, let

$$\gamma_j = c_1^{(j)} + c_2^{(j)} p + \cdots + c_t^{(j)} p^{t-1} + \cdots \in \mathbb{Z}_p.$$

Then the corresponding basis element of V_A is

$$v_i = z_i + \gamma_1 a_1 + \cdots + \gamma_\ell a_\ell \in \widehat{A}.$$

Since $z_1, \ldots, z_k$ are linearly independent over $\mathbb{Q}_p$, it follows that $v_1, \ldots, v_\ell$ are also linearly independent over $\mathbb{Q}_p$. Thus it is enough to check that $v_i \in p^t \widehat{A}$ for each $1 \leq i \leq k$ and $t \geq 1$. So fix some $1 \leq i \leq k$ and reconsider the element

$$v_i = z_i + \gamma_1 a_1 + \cdots + \gamma_\ell a_\ell \in \widehat{A}.$$

If $t \geq 1$, then

$$p^t d_t = z_i + n_t^{(1)} a_1 + \cdots + n_t^{(\ell)} a_\ell$$

and

$$v_i - p^t d_t = (\sum_{r=t+1}^{\infty} c_r^{(1)} p^{r-1}) a_1 + \cdots + (\sum_{r=t+1}^{\infty} c_r^{(\ell)} p^{r-1}) a_\ell.$$

Hence $p^t e_t = v_i$, where

$$e_t = d_t + (\sum_{r=t+1}^{\infty} c_r^{(1)} p^{r-(t+1)}) a_1 + \cdots + (\sum_{r=t+1}^{\infty} c_r^{(\ell)} p^{r-(t+1)}) a_\ell.$$

This completes the proof of Theorem 3.8.

5. p-local torsion-free abelian groups of finite rank

Fix some $n \geq 2$ and let $\mathcal{R} = R^{(p)}(\mathbb{Q}^n, \mathbb{Z}_{(p)}^n)$ be the standard Borel space consisting of those $A \in R(\mathbb{Q}^n)$ such that $\mathbb{Z}_{(p)}^n \leqslant A$. Then by [14, Lemma 4.11], $\mathcal{R} \subseteq R^{(p)}(\mathbb{Q}^n)$. Furthermore, $\mathcal{R}$ is invariant under the action of the subgroup $GL_n(\mathbb{Z}_{(p)})$ of $GL_n(\mathbb{Q})$; and Hjorth [14] has shown that there exists a $GL_n(\mathbb{Z}_{(p)})$-invariant probability measure μ on $\mathcal{R}$. Each $A \in \mathcal{R}$ has certain obvious automorphisms; namely, for each $s \in \mathbb{Z}_{(p)}^*$, we can define a corresponding automorphism of A by $a \mapsto sa$. So if we identify each $s \in \mathbb{Z}_{(p)}^*$ with the corresponding diagonal matrix $D_s \in GL_n(\mathbb{Q})$, we have that

$$\mathbb{Z}_{(p)}^* \leqslant \mathrm{Aut}(A) \leqslant GL_n(\mathbb{Q}).$$

Our main result in this section says that a "random" group $A \in \mathcal{R}$ has only these obvious automorphisms.

Theorem 5.1. $\mu(\{A \in \mathcal{R} \mid \mathrm{Aut}(A) = \mathbb{Z}_{(p)}^*\}) = 1.$

Using Theorem 5.1 and Kechris's result [21] that $PSL_2(\mathbb{Z}[1/q])$ is antitreeable for every prime $q \in \mathbb{P}$, we can now easily prove that $\cong_2^{(p)}$ is not treeable. (Recall that in [14], Hjorth proved that $\cong_n^{(p)}$ is not treeable for each $n \geq 3$.)

Definition 5.2. A countable group G is said to be *antitreeable* iff for every Borel action of G on a standard Borel space X, which is free and admits an invariant Borel probability measure, the correponding equivalence relation E_G^X is not treeable.

Corollary 5.3. $\cong_2^{(p)}$ *is not treeable.*

Proof. Let $q \in \mathbb{P}$ be a prime such that $q \neq p$ and let

$$G = PSL_2(\mathbb{Z}[1/q]) \leqslant PGL_2(\mathbb{Z}_{(p)}).$$

By Theorem 5.1, there exists a $GL_2(\mathbb{Z}_{(p)})$-invariant Borel subset $X \subseteq R^{(p)}(\mathbb{Q}^2, \mathbb{Z}_{(p)}^2)$ with $\mu(X) = 1$ such that $\mathrm{Aut}(A) = \mathbb{Z}_{(p)}^*$ for all $A \in X$. Now the μ-preserving action of $PGL_2(\mathbb{Z}_{(p)})$ on $R^{(p)}(\mathbb{Q}^2, \mathbb{Z}_{(p)}^2)$ induces a corresponding free action of $PGL_2(\mathbb{Z}_{(p)})$ on X; and hence there exists a μ-preserving free action of G on X. By [21, Theorem 7], G is antitreeable and so E_G^X is not treeable. Since $E_G^X \subseteq (\cong_2^{(p)} \restriction X)$, [14, Lemma 2.10] implies that $\cong_2^{(p)}$ is also not treeable. □

Before we begin the proof of Theorem 5.1, we shall review Hjorth's construction [14] of the measure μ on $\mathcal{R}$. Let $\mathcal{S}$ be the standard Borel space consisting of all subgroups of the quotient group $\mathbb{Q}^n/\mathbb{Z}_{(p)}^n$ and let $\pi : \mathbb{Q}^n \to \mathbb{Q}^n/\mathbb{Z}_{(p)}^n$ be the canonical surjection. Then we can define a Borel bijection $\mathcal{R} \to \mathcal{S}$ by $A \mapsto \pi(A)$. Let Γ be the dual group of $\mathbb{Q}^n/\mathbb{Z}_{(p)}^n$; i.e. the space of all homomorphisms $\psi : \mathbb{Q}^n/\mathbb{Z}_{(p)}^n \to \mathbb{R}/\mathbb{Z}$, equipped with the topology of pointwise convergence and the group operation of pointwise addition. Since $\mathbb{Q}^n/\mathbb{Z}_{(p)}^n$ is discrete, it follows that Γ is a compact group. Let ν be the Haar measure on Γ. Let $k : \Gamma \to \mathcal{S}$ be the Borel map assigning the subgroup

$$\ker(\psi) = \{h \in \mathbb{Q}^n/\mathbb{Z}_{(p)}^n \mid \psi(h) = 0\}$$

to each $\psi \in \Gamma$; and let $\widehat{\mu} = k\nu$ be the probability measure defined on $\mathcal{S}$ by

$$\widehat{\mu}(B) = \nu(\{\psi \in \Gamma \mid \ker(\psi) \in B\})$$

for each Borel subset $B \subseteq \mathcal{S}$. Then μ is the corresponding Borel probability measure on $\mathcal{R}$ induced by the Borel bijection $A \mapsto \pi(A)$.

During the proof of Theorem 5.1, we shall make use of the fact that Γ is naturally isomorphic to $\mathbb{Z}_p^n$. To see this, first notice that since $\mathbb{Q}^n/\mathbb{Z}_{(p)}^n \cong C(p^\infty)^n$, it follows that each $\psi \in \Gamma$ must take values in $\mathbb{Z}[1/p]/\mathbb{Z} \cong C(p^\infty)$. For each $v \in \mathbb{Q}^n$ and $a \in \mathbb{Z}[1/p]$, we shall denote the corresponding elements of $\mathbb{Q}^n/\mathbb{Z}_{(p)}^n$ and $\mathbb{Z}[1/p]/\mathbb{Z}$ by $[v]$, $[a]$, respectively. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Q}^n$. Then we can define an isomorphism $\psi \mapsto \widetilde{\psi}$ of Γ onto $\mathbb{Z}_p^n$ by

$$\widetilde{\psi} = (\beta_1, \dots, \beta_n),$$

where $\beta_i = b_1^{(i)} + b_2^{(i)}p + \cdots + b_t^{(i)}p^{t-1} + \cdots$ is the p-adic integer such that

$$\psi([e_i/p^t]) = \left[\frac{b_1^{(i)} + b_2^{(i)}p + \cdots + b_t^{(i)}p^{t-1}}{p^t}\right]$$

for each $t \geq 1$. We shall also denote the Haar measure on $\mathbb{Z}_p^n$ by ν.

For each $A \in \mathcal{R}$, let $V_A \leqslant \widehat{A} = \mathbb{Z}_p \otimes A$ be the $\mathbb{Q}_p$-vector space defined in Definition 4.6. Then $0 \leq \dim V_A \leq n$. We shall begin our analysis by determining the value of $\dim V_A$ for a "random" subgroup $A \in \mathcal{R}$.

Definition 5.4. Let $\mathcal{R}_1 \subset \mathcal{R}$ be the Borel subset consisting of those $A \in \mathcal{R}$ which satisfy the following conditions:

(i) For all $0 \neq a \in A$, there exists $t \geq 1$ such that $a \notin p^t A$.
(ii) $\dim V_A = n - 1$.

Lemma 5.5. $\mu(\mathcal{R}_1) = 1$.

Proof. Let $\Gamma_1 \subseteq \Gamma$ be the Borel subset consisting of those ψ such that for each $z \in \mathbb{Z}^n_{(p)}$, there exists $t \geq 1$ such that $\psi([z/p^t]) \neq 0$. Then it is easily checked that $\nu(\Gamma_1) = 1$. We shall show that

$$\{A \in \mathcal{R} \mid \pi(A) = \ker \psi \text{ for some } \psi \in \Gamma_1\} \subseteq \mathcal{R}_1.$$

So let $\psi \in \Gamma_1$ and let $\pi(A) = \ker \psi$. Then it is clear that A satisfies condition 5.4(i). Let $r \geq 0$ be the greatest integer such that $a = e_n/p^r \in A$. By Exercise 93.1 [12], in order to prove that $\dim V_A = n-1$, it is enough to show that a is a p-basis of A. Suppose not. Then there exists an element $b = z/p^s \in A$, where $z \in \mathbb{Z}^n_{(p)}$ and $s \geq 0$, such that (a, b) is a p-independent sequence. Since $\pi(A) = \ker \psi$, we must have that:

(1) $\psi([e_n/p^r]) = \psi([z/p^s]) = 0$;
(2) there exist integers $0 < m_1, m_2 < p$ such that

$$\psi([e_n/p^{r+1}]) = [m_1/p] \quad \text{and} \quad \psi([z/p^{s+1}]) = [m_2/p].$$

Let $0 < \ell < p$ satisfy $\ell m_1 + m_2 \equiv 0 \pmod{p}$. Then

$$\psi(\ell[e_n/p^{r+1}] + [z/p^{s+1}]) = 0$$

and so $(\ell a + b)/p \in A$, which contradicts the hypothesis that (a, b) is a p-independent sequence. $\square$

Let $\psi \in \Gamma_1$ and let $A \in \mathcal{R}_1$ satisfy $\pi(A) = \ker \psi$. We shall next discuss the relationship between $\widetilde{\psi} = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_p^n$ and the $\mathbb{Q}_p$-space V_A. For each integer $1 \leq i \leq n$, let

$$\beta_i = b_1^{(i)} + b_2^{(i)} p + \cdots + b_t^{(i)} p^{t-1} + \cdots$$

and let $r_i \geq 0$ be the greatest integer such that $x_i = e_i/p^{r_i} \in A$. Then x_n is a p-basis of A; and for each $1 \leq i \leq n$, we have that

$$\alpha_i = \beta_i/p^{r_i} = a_1^{(i)} + a_2^{(i)} p + \cdots + a_t^{(i)} p^{t-1} + \cdots \in \mathbb{Z}_p^*$$

is a p-adic unit and

$$\psi([x_i/p^t]) = \left[\frac{a_1^{(i)} + a_2^{(i)} p + \cdots + a_t^{(i)} p^{t-1}}{p^t} \right].$$

For each $1 \leq i \leq n-1$, let

$$\gamma_i = -\alpha_i/\alpha_n = c_1^{(i)} + c_2^{(i)} p + \cdots + c_t^{(i)} p^{t-1} + \cdots \in \mathbb{Z}_p^*.$$

Then

$$\sum_{r=1}^{t} a_r^{(i)} p^{r-1} + (\sum_{r=1}^{t} c_r^{(i)} p^{r-1})(\sum_{r=1}^{t} a_r^{(n)} p^{r-1}) \equiv 0 \pmod{p^t}$$

for each $1 \leq i \leq n-1$; and so

$$\psi([x_i/p^t] + (\sum_{r=1}^{t} c_r^{(i)} p^{r-1})[x_n/p^t]) = 0.$$

Thus $x_i + (\sum_{r=1}^{t} c_r^{(i)} p^{r-1})x_n \in p^t A$ for each $t \geq 1$. Hence, arguing as in Section 4, we see that

$$(x_1 + \gamma_1 x_n, \ldots, x_{n-1} + \gamma_{n-1} x_n)$$

is a basis of V_A.

Now we shall begin our study of the structure of $\mathrm{Aut}(A)$ for $A \in \mathcal{R}_1$. First note that the natural action of $GL_n(\mathbb{Q})$ on $\mathbb{Q}^n$ extends canonically to an action on $\mathbb{Q}_p^n$. Furthermore, if $\pi \in \mathrm{Aut}(A) \leqslant GL_n(\mathbb{Q})$, then $\pi(\widehat{A}) = \widehat{A}$ and so $\pi(V_A) = V_A$. It will also be useful to consider the corresponding *contragredient representation* of $GL_n(\mathbb{Q})$ on the dual space $D = \mathrm{Hom}(\mathbb{Q}_p^n, \mathbb{Q}_p)$, defined by

$$(\pi.f)(v) = f(\pi^{-1} v)$$

for $\pi \in GL_n(\mathbb{Q})$, $f \in D$ and $v \in \mathbb{Q}_p^n$. Recall that the linear transformation $f \mapsto \pi.f$ is represented by the matrix $(\pi^{-1})^t$ relative to the dual basis $\widehat{e}_1, \ldots, \widehat{e}_n$. (For example, see [8, Section 43].) For subspaces $U \leqslant \mathbb{Q}_p^n$ and $W \leqslant D$, let

$$U^\perp = \{f \in D \mid f(u) = 0 \text{ for all } u \in U\}$$

and

$$W^\perp = \{u \in \mathbb{Q}_p^n \mid f(u) = 0 \text{ for all } f \in W\}.$$

Notice that if $\pi \in GL_n(\mathbb{Q})$ satisfies $\pi(U) = U$, then $\pi(U^\perp) = U^\perp$.

Let $\psi \in \Gamma_1$ and let $A \in \mathcal{R}_1$ satisfy $\pi(A) = \ker \psi$. Suppose that $\pi \in \mathrm{Aut}(A) \leqslant GL_n(\mathbb{Q})$. Then $\pi(V_A) = V_A$ and so $\pi(V_A^\perp) = V_A^\perp$. Since $\dim V_A = n-1$, we have that $\dim V_A^\perp = 1$. Let $0 \neq f \in V_A^\perp$. Then $\pi.f = \lambda f$ for some eigenvalue $0 \neq \lambda \in \overline{\mathbb{Q}} \cap \mathbb{Q}_p$ of the matrix $(\pi^{-1})^t$. Let $E \leqslant D$ be the eigenspace corresponding to λ. We shall show that if $A \in \mathcal{R}_1$ is "sufficiently random", then $E = D$ and so $(\pi^{-1})^t = \lambda I$. Hence $\pi = \lambda I$; and since A is not p-divisible, we must have that $\lambda \in \mathbb{Z}_{(p)}^*$.

So suppose that $\dim E = e < n$. Since $\lambda \in \overline{\mathbb{Q}} \cap \mathbb{Q}_p$, it follows that E has a basis consisting of functions $f_1, \ldots, f_e$ such that for each $1 \leq k \leq e$,

$$f_k = r_1^{(k)} \widehat{e}_1 + \cdots + r_n^{(k)} \widehat{e}_n$$

for some $r_i^{(k)} \in \overline{\mathbb{Q}} \cap \mathbb{Q}_p$. This implies that $E^\perp \cap (\overline{\mathbb{Q}} \cap \mathbb{Q}_p)^n$ is an $(n-e)$-dimensional vector space over $\overline{\mathbb{Q}} \cap \mathbb{Q}_p$. In particular, since $E^\perp \leqslant (V_A^\perp)^\perp = V_A$, it follows that there exists a nonzero vector $0 \neq v \in V_A \cap \overline{\mathbb{Q}}^n$. Once again, for each $1 \leq i \leq n$, let $x_i = e_i/p^{r_i}$, where $r_i \geq 0$ is the greatest integer such that $e_i \in p^{r_i} A$. Then $v = q_1 x_1 + \cdots + q_n x_n$ for some $q_i \in \overline{\mathbb{Q}} \cap \mathbb{Q}_p$. Since $(x_1 + \gamma_1 x_n, \ldots, x_{n-1} + \gamma_{n-1} x_n)$ is a basis of V_A, there exist $\theta_j \in \mathbb{Q}_p$ for $1 \leq j \leq n-1$ such that

$$q_1 x_1 + \cdots + q_n x_n = \theta_1(x_1 + \gamma_1 x_n) + \cdots + \theta_{n-1}(x_{n-1} + \gamma_{n-1} x_n).$$

It follows that $\theta_j = q_j$ for $1 \leq j \leq n-1$ and hence

$$q_n = q_1 \gamma_1 + \cdots + q_{n-1} \gamma_{n-1}.$$

Since $\gamma_j = -\alpha_j/\alpha_n$ for $1 \leq j \leq n-1$, we obtain that

$$q_1 \alpha_1 + \cdots + q_n \alpha_n = 0.$$

Finally since $\alpha_i = \beta_i/p^{r_i}$ for $1 \leq i \leq n$, we see that $\beta_1, \ldots, \beta_n$ are linearly independent over $\overline{\mathbb{Q}} \cap \mathbb{Q}_p$. Hence in order to complete the proof of Theorem 5.1, we need only show that if

$$Y = \{(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_p^n \mid \beta_1, \ldots, \beta_n \text{ are linearly dependent over } \overline{\mathbb{Q}} \cap \mathbb{Q}_p\},$$

then $\nu(Y) = 0$. To see this, note that for each $\overline{q} = (q_1, \ldots, q_n) \in (\overline{\mathbb{Q}} \cap \mathbb{Z}_p)^n$, the closed subgroup

$$H_{\overline{q}} = \{(\beta_1, \ldots, \beta_n) \in \mathbb{Z}_p^n \mid q_1\beta_1 + \cdots + q_n\beta_n = 0\}$$

has infinite index in $\mathbb{Z}_p^n$ and so $\nu(H_{\overline{q}}) = 0$. Hence

$$\nu(Y) = \nu(\bigcup\{H_{\overline{q}} \mid \overline{q} \in (\overline{\mathbb{Q}} \cap \mathbb{Z}_p)^n\}) = 0.$$

In the remainder of this section, we shall restrict our attention to the class of p-local groups of rank 2; and we shall point out some connections between this material and the main open problems on treeable countable Borel equivalence relations. We shall begin by presenting an alternative realisation (up to Borel bireducibility) of the isomorphism relation $\cong_2^{(p)}$ on $S^{(p)}(\mathbb{Q}^2)$.

Definition 5.6. For each $p \in \mathbb{P}$, let $E^{(p)}$ be the orbit equivalence relation induced by the Borel action of $GL_2(\mathbb{Q})$ on $\mathbb{Q}_p \cup \{\infty\}$ as a group of fractional linear transformations; i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .\gamma = \frac{a\gamma + b}{c\gamma + d}$$

for each $\gamma \in \mathbb{Q}_p \cup \{\infty\}$.

Theorem 5.7. $(\cong_2^{(p)}) \sim_B E^{(p)}$ *for each* $p \in \mathbb{P}$.

Proof. Let $B^{(p)}(\mathbb{Q}^2)$ be the Borel set consisting of those $A \in S^{(p)}(\mathbb{Q}^2)$ which satisfy the following conditions:

(i) $A \in R^{(p)}(\mathbb{Q}^2)$.
(ii) $\dim V_A = 1$.

Claim 5.8. $S^{(p)}(\mathbb{Q}^2) \smallsetminus B^{(p)}(\mathbb{Q}^2)$ *is countable.*

Proof of Claim 5.8. It is easily checked that $S^{(p)}(\mathbb{Q}^2) \smallsetminus R^{(p)}(\mathbb{Q}^2)$ is countable; and Theorem 4.7 implies that there are only countably many $A \in R^{(p)}(\mathbb{Q}^2)$ such that $\dim V_A \neq 1$. □

It follows easily that $(\cong_2^{(p)}) \sim_B (\cong_2^{(p)} \restriction B^{(p)}(\mathbb{Q}^2))$. Next we shall show that $(\cong_2^{(p)} \restriction B^{(p)}(\mathbb{Q}^2)) \leq_B E^{(p)}$. To see this, let e_1, e_2 be the standard basis of the vector space $\mathbb{Q}_p^2$. Then for each $A \in B^{(p)}(\mathbb{Q}^2)$, there exists a unique element $\gamma_A \in \mathbb{Q}_p \cup \{\infty\}$ such that

$$V_A = \langle\ \gamma_A e_1 + e_2\ \rangle;$$

and an easy calculation shows that if $\pi \in GL_2(\mathbb{Q})$, then

$$V_{\pi(A)} = \pi(V_A) = \langle\ (\pi.\gamma_A)e_1 + e_2\ \rangle.$$

(Of course, $\langle\ \infty e_1 + e_2\ \rangle$ should be interpreted as $\langle\ e_1\ \rangle$.) Let $f : B^{(p)}(\mathbb{Q}^2) \to \mathbb{Q}_p \cup \{\infty\}$ be the Borel map defined by $f(A) = \gamma_A$. Then we have just seen that if $A \cong_2^{(p)} B$, then $f(A) E^{(p)} f(B)$. Conversely, suppose that $f(A) E^{(p)} f(B)$; say, $\pi.\gamma_A = \gamma_B$. Then $V_{\pi(A)} = V_B$ and so by Theorem 4.7, $\pi(A) \approx B$. By [12, Exercise 32.5], each p-basis of A can be lifted to a p-basis of A/pA. It follows that $|A/pA| = p$ and so [12, Proposition 92.1] implies that $A \cong_2^{(p)} B$.

Finally we shall show that $E^{(p)} \leq_B (\cong_2^{(p)} \restriction B^{(p)}(\mathbb{Q}^2))$. For each $\gamma \in \mathbb{Q}_p \cup \{\infty\}$, let $v_\gamma = \gamma e_1 + e_2$ and let $w_\gamma = p^r e_2$, where $p^r\gamma \in \mathbb{Z}_p^*$. (For $\gamma \in \{0, \infty\}$, we set $w_0 = e_1$ and $w_\infty = e_2$.) Let $g : \mathbb{Q}_p \cup \{\infty\} \to B^{(p)}(\mathbb{Q}^2)$ be the Borel map defined by

$$g(\gamma) = [\mathbb{Q}_p v_\gamma \oplus \mathbb{Z}_p w_\gamma] \cap \mathbb{Q}^2.$$

Then it is easily checked that $V_{g(\gamma)} = \mathbb{Q}_p v_\gamma$. It follows that if γ_1, $\gamma_2 \in \mathbb{Q}_p \cup \{\infty\}$, then $\gamma_1 E^{(p)} \gamma_2$ iff $g(\gamma_1) \cong_2^{(p)} g(\gamma_2)$. □

Now let F_2 be the free group on two generators. In [19], Kechris asked whether for every countable Borel equivalence relation E, the following statements are equivalent:

(a) E is non-hyperfinite.
(b) There is a free action of F_2 on a standard Borel space X, which admits an invariant probability measure, such that $E_{F_2}^X \leq_B E$.

Note that the relation $E_{F_2}^X$ in clause (b) is necessarily treeable and non-hyperfinite. Thus a negative answer to the following question would also give a counterexample to the preceeding question.

Question 5.9. Does there exist a non-hyperfinite treeable countable Borel equivalence relation E such that $E \leq_B (\cong_2^{(p)})$?

Another important open question asks whether there exists a (necessarily treeable) countable Borel equivalence relation E such that $E_0 <_B E <_B E_{T\infty}$. Of course, this question would receive a positive answer if we could find two treeable countable Borel equivalences which were incomparable with respect to Borel reducibility. We shall conclude this section by presenting some possible candidates.

Definition 5.10. For each $p \in \mathbb{P}$, let $E_T^{(p)}$ be the orbit equivalence relation induced by the Borel action of $GL_2(\mathbb{Z})$ on $\mathbb{Q}_p \cup \{\infty\}$ as a group of fractional linear transformations.

Theorem 5.11. *For each $p \in \mathbb{P}$, the countable Borel equivalence relation $E_T^{(p)}$ is treeable and non-hyperfinite.*

Proof. Since the subgroup $\{\pm I\}$ of $GL_2(\mathbb{Z})$ fixes each element of $\mathbb{Q}_p \cup \{\infty\}$, we can also regard $E_T^{(p)}$ as the orbit equivalence relation of the induced action of $PGL_2(\mathbb{Z})$ on $\mathbb{Q}_p \cup \{\infty\}$. Suppose that $\gamma \in \mathbb{Q}_p \cup \{\infty\}$ is fixed by some nonidentity element $\varphi \in PGL_2(\mathbb{Z})$. Then an easy calculation shows that $\gamma \in \overline{\mathbb{Q}} \cup \{\infty\}$. In particular, $PGL_2(\mathbb{Z})$ acts freely on the complement of a countable subset of $\mathbb{Q}_p \cup \{\infty\}$. By Jackson-Kechris-Louveau [16, Proposition 3.4], $E_T^{(p)}$ is treeable.

By Jackson-Kechris-Louveau [16, Proposition 1.7], in order to show that $E_T^{(p)}$ is non-hyperfinite, it is enough to show that there exists a $PGL_2(\mathbb{Z})$-invariant probability measure on $\mathbb{Q}_p \cup \{\infty\}$. As in the proof of Theorem 5.1, let $\mathcal{R} = R^{(p)}(\mathbb{Q}^2, \mathbb{Z}_{(p)}^2)$ be the standard Polish $GL_n(\mathbb{Z}_{(p)})$-space consisting of those $A \in R(\mathbb{Q}^2)$ such that $\mathbb{Z}_{(p)}^n \leqslant A$; and let $\mathcal{R}_1 \subset \mathcal{R}$ be the Borel subset consisting of those $A \in \mathcal{R}$ which satisfy the following conditions:

(i) For all $a \in A$, there exists $t \geq 1$ such that $a \notin p^t A$.
(ii) $\dim V_A = 1$.

By Lemma 5.5, there exists a $GL_n(\mathbb{Z}_{(p)})$-invariant probability measure μ on $\mathcal{R}_1$. As in the proof of Theorem 5.7, let $f : \mathcal{R}_1 \to \mathbb{Q}_p \cup \{\infty\}$ be the Borel map defined by $f(A) = \gamma_A$, where $\gamma_A \in \mathbb{Q}_p \cup \{\infty\}$ is the unique element such that

$$V_A = \langle\, \gamma_A e_1 + e_2 \,\rangle.$$

Let $\widetilde{\mu} = f\mu$ be the probability measure defined on $\mathbb{Q}_p \cup \{\infty\}$ by

$$\widetilde{\mu}(B) = \mu(f^{-1}(B))$$

for each Borel subset $B \subseteq \mathbb{Q}_p \cup \{\infty\}$. Since $f(\pi(A)) = \pi.f(A)$ for all $A \in \mathcal{R}_1$ and $\pi \in GL_2(\mathbb{Z})$, it follows that $\widetilde{\mu}$ is $PGL_2(\mathbb{Z})$-invariant. □

Conjecture 5.12. *If $p \neq q$ are distinct primes, then $E_T^{(p)} \nleq_B E_T^{(q)}$.*

Of course, it is also possible to define a Borel action of $GL_2(\mathbb{Z})$ on $\mathbb{R} \cup \{\infty\}$ as a group of fractional linear transformations. However, Jackson-Kechris-Louveau [16] have pointed out that, in this case, the induced orbit equivalence relation is hyperfinite.

6. A cocycle reduction result

In this section, we shall prove Theorem 2.3. As we mentioned earlier, this result is a straightforward consequence of Zimmer's superrigidity theorem [34, Theorem 5.2.5] and the ideas of Adams-Kechris [3]. First we need to recall some notions from valuation theory. (A clear account of this material can be found in Margulis [26, Chapter I].) Let F be an algebraic number field; i.e. a finite extension of the field $\mathbb{Q}$ of rational numbers. Let $\mathcal{R}$ be the set of all non-equivalent valuations of F and let $\mathcal{R}_\infty \subset \mathcal{R}$ be the set of archimedean valuations. For each $\nu \in \mathcal{R}$, let F_ν be the completion of F relative to ν. If $\nu \in \mathcal{R}_\infty$, then $F_\nu = \mathbb{R}$ or $F_\nu = \mathbb{C}$; and if $\nu \in \mathcal{R} \smallsetminus \mathcal{R}_\infty$, then F_ν is a totally disconnected local field; i.e. a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers for some prime p. In particular, each F_ν is a local field.

Let $S \subseteq \mathcal{R}$ be a set of valuations of F. Then an element $x \in F$ is said to be *S-integral* iff $|x|_\nu \leq 1$ for each non-archimedean valuation $\nu \notin S$. The set of all S-integral elements is a subring of F, which will be denoted by $F(S)$. Furthermore, F is the union of the subrings $F(S)$, where S ranges over the finite sets of valuations of the field F.

The proof of Theorem 2.3 will be based upon the following cocycle reduction theorem.

Theorem 6.1. *Let $m \geq 3$ and let Y be a standard Borel $SL_m(\mathbb{R})$-space with an invariant ergodic probability measure. Let K be a local field and let T be a simple algebraic K-group such that $\dim T < m^2 - 1$. Then for every Borel cocycle $\alpha : SL_m(\mathbb{R}) \times Y \to T(K)$, there exists an equivalent cocycle γ such that $\gamma(SL_m(\mathbb{R}) \times Y)$ is contained in a compact subgroup of $T(K)$.*

Proof. If we add the hypothesis that α is not equivalent to a cocycle β taking values in a subgroup of the form $L(K)$ for some proper algebraic K-subgroup L of T, then Theorem 6.1 is an immediate consequence of Zimmer [34, Theorem 5.2.5]. But arguing as in the proof of Adams-Kechris [3, Theorem 3.5], we can easily reduce our analysis to the case when this extra hypothesis holds. □

In the proof of Theorem 2.3, we shall also make use of the notions of an *induced action* and an *induced cocycle*, which are defined as follows. (A fuller account of these notions can be found in Adams-Kechris [3, Section 2].) Suppose that G is a lcsc group and that Γ is a closed subgroup of G. Let G/Γ be the set of left cosets of Γ in G. By the Effros-Mackey cross section theorem [28, Theorem 5.4.2], there exists a Borel transversal T for G/Γ; i.e. a Borel subset $T \subseteq G$ which meets every coset in a unique element. Fix such a Borel transversal with $1 \in T$. Then we can identify T with G/Γ by identifying t with $t\Gamma$; and then the action of G on G/Γ induces a corresponding Borel action of G on T, defined by

$$g.t = \text{ the unique element in } T \cap gt\Gamma.$$

The associated strict cocycle $\rho : G \times T \to \Gamma$ is defined by

$$\begin{aligned} \rho(g,t) &= \text{ the unique } \varphi \in \Gamma \text{ such that } (g.t)\varphi = gt \\ &= (g.t)^{-1}gt. \end{aligned}$$

Now suppose that X is a standard Borel Γ-space with an invariant ergodic probability measure μ. Suppose also that Γ is a *lattice* in G; i.e. Γ is a discrete subgroup of G and the action of G on G/Γ admits an invariant probability measure. Let ν be the corresponding invariant probability measure on T. Then the induced action of G on the standard Borel space $Y = X \times T$ is defined by

$$g.(x,t) = (\rho(g,t).x, g.t);$$

and it is easily checked that $\mu \times \nu$ is an invariant ergodic probability measure on Y. (See Adams-Kechris [3, Section 2].) Finally given any strict cocycle $\beta : \Gamma \times X \to H$, the corresponding induced cocycle $\widehat{\beta} : G \times Y \to H$ is defined by

$$\widehat{\beta}(g,(x,t)) = \beta(\rho(g,t),x).$$

In the proof of Theorem 2.3, these notions will be used in the case when $\Gamma = SL_m(\mathbb{Z})$ and $G = SL_m(\mathbb{R})$. We shall suppress explicit mention of the Borel transversal T and instead write

$$Y = X \times (SL_m(\mathbb{R})/SL_m(\mathbb{Z})).$$

We are now ready to begin the proof of Theorem 2.3. So let $m \geq 3$ and let X be a standard Borel $SL_m(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that G is an algebraic $\mathbb{Q}$-group such that $\dim G < m^2 - 1$ and that $H \leqslant G(\mathbb{Q})$. Let $\alpha : SL_m(\mathbb{Z}) \times X \to H$ be a Borel cocycle. Then we can view α as a cocycle taking values in $G(\overline{\mathbb{Q}})$. Furthermore, after passing to a finite ergodic extension of X if necessary, we can suppose that G is connected. (For example, see [3, Propositions 2.5 and 2.6].) Let R be the soluble radical of G. Then G/R is a connected semisimple algebraic $\mathbb{Q}$-group; and, in particular, G/R has a finite centre A/R. Let $\overline{G} = G/A$. Then there exist simple algebraic $\overline{\mathbb{Q}}$-groups $T_1, \dots, T_k$ such that $\overline{G} = T_1 \times \cdots \times T_k$. Now consider the Borel cocycle

$$\overline{\alpha} : SL_m(\mathbb{Z}) \times X \to \overline{G}(\overline{\mathbb{Q}}) = T_1(\overline{\mathbb{Q}}) \times \cdots \times T_k(\overline{\mathbb{Q}})$$

defined by $\overline{\alpha} = \pi \circ \alpha$, where $\pi : G(\overline{\mathbb{Q}}) \to \overline{G}(\overline{\mathbb{Q}})$ be the canonical surjection. Since $SL_m(\mathbb{Z})$ is a Kazhdan group, [35, Lemma 2.2] implies that $\overline{\alpha}$ is equivalent to a cocycle $\overline{\beta}$ taking values in a finitely generated subgroup Λ of $\overline{G}(\overline{\mathbb{Q}})$. So there exists an algebraic number field F and a finite set S of valuations of F such that:

(i) each T_i is an algebraic F-group; and
(ii) $\Lambda \leqslant \overline{G}(F(S)) = T_1(F(S)) \times \cdots \times T_k(F(S))$.

Clearly we can suppose that S contains the set $\mathcal{R}_\infty$ of archimedean valuations of F. It follows that if $\overline{G}(F(S))$ is identified with its image under the diagonal embedding into

$$\overline{G}_S = \prod_{\nu \in S} \overline{G}(F_\nu) = \prod_{i=1}^{k} \prod_{\nu \in S} T_i(F_\nu),$$

then $\overline{G}(F(S))$ is a discrete subgroup of $\overline{G}_S$. (For example, see [26, Section I.3.2].) By Zimmer [34, Theorem B.9], we can suppose that $\overline{\beta} : SL_m(\mathbb{Z}) \times X \to \Lambda$ is a strict cocycle. Consider the induced Borel action of $SL_m(\mathbb{R})$ on $Y = X \times (SL_m(\mathbb{R})/SL_m(\mathbb{Z}))$ and let $\widehat{\beta} : SL_m(\mathbb{R}) \times Y \to \Lambda$ be the corresponding induced cocycle. For each $\nu \in S$ and $1 \leq i \leq k$, let $p_{\nu i} : \overline{G}_S \to T_i(F_\nu)$ be the canonical projection; and, viewing $\widehat{\beta}$ as a cocycle into $\overline{G}_S$, let $\widehat{\beta}_{\nu i} : SL_m(\mathbb{R}) \times Y \to T_i(F_\nu)$ be the Borel cocycle defined by $\widehat{\beta}_{\nu i} = p_{\nu i} \circ \widehat{\beta}$. By Theorem 6.1, $\widehat{\beta}_{\nu i}$ is equivalent to a cocycle taking values in a compact subgroup $C_{\nu i}$ of $T_i(F_\nu)$. It follows that $\widehat{\beta}$ is equivalent to a cocycle taking values in the compact subgroup $C = \prod_{i=1}^{k} \prod_{\nu \in S} C_{\nu i}$ of $\overline{G}_S$. By Adams-Kechris [3, Proposition 2.4], there exists $\overline{g} \in \overline{G}_S$ and a cocycle $\widehat{\gamma} : SL_m(\mathbb{R}) \times Y \to \Lambda$ such that $\widehat{\gamma} \sim \widehat{\beta}$ and $\widehat{\gamma}$ takes values in the finite subgroup $\overline{B} = \Lambda \cap \overline{g} C \overline{g}^{-1}$ of Λ; and hence by Adams-Kechris [3, Proposition 2.3], there exists a cocycle $\overline{\gamma} : SL_m(\mathbb{Z}) \times X \to \Lambda$ such that $\overline{\gamma} \sim \overline{\beta}$ and $\overline{\gamma}$ also takes values in $\overline{B}$. Let $B = \pi^{-1}(\overline{B})$. Then B is a soluble-by-finite subgroup of $G(\overline{\mathbb{Q}})$; and since $\overline{\alpha} \sim \overline{\gamma}$, it follows that α is equivalent to a cocycle $\tilde{\gamma} : SL_m(\mathbb{Z}) \times X \to G(\overline{\mathbb{Q}})$ taking values in B. Since $SL_m(\mathbb{Z})$ is a Kazhdan group and B is a countable amenable group, Zimmer [34, Theorem 9.1.1] implies that $\tilde{\gamma}$ is equivalent to a cocycle taking values in a finite subgroup of B. Applying Adams-Kechris [3, Proposition 2.4] once more, it follows that α is equivalent to a cocycle $\gamma : SL_m(\mathbb{Z}) \times X \to H$ taking values in a finite subgroup of H. This completes the proof of Theorem 2.3.

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References

[1] S. R. Adams, *Trees and amenable equivalence relations*, Ergodic Theory and Dynamical Systems **10** (1990), 1–14. MR **91d:**28041

[2] S. Adams, *Containment does not imply Borel reducibility*, in: S. Thomas (Ed.), *Set Theory: The Hajnal Conference*, DIMACS Series vol. 58, American Mathematical Society, 2002, pp. 1–23.

[3] S. R. Adams and A. S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), 909–943. MR **2001g:**03086

[4] D. M. Arnold, *Finite Rank Torsion-Free Abelian Groups and Rings*, Lecture Notes in Mathematics **931**, Springer-Verlag, New York, 1982. MR **84d:**20002

[5] R. Baer, *Abelian groups without elements of finite order*, Duke Math. Journal **3** (1937), 68–122.

[6] A. Borel, *Linear Algebraic Groups: Second Enlarged Edition*, Graduate Texts in Mathematics **126**, Springer-Verlag, 1991. MR **92d:**20001

[7] A. L. S. Corner, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. **13** (1963), 687–710. MR **27:**3704

[8] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley-Interscience, 1962. MR **26:**2519

[9] R. Dougherty, S. Jackson and A. S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193–225. MR **94c:**03066

[10] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289–324. MR **58:**28261a

[11] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, Journal of Symbolic Logic **54** (1989), 894–914. MR **91f:**03062

[12] L. Fuchs, *Infinite Abelian Groups*, Pure and Applied Mathematics Vol. 36, Academic Press, 1970. MR **41:**333

[13] L. Harrington, A. S. Kechris and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), 903–927. MR **91h:**28023

[14] G. Hjorth, *Around nonclassifiability for countable torsion-free abelian groups*, in Abelian groups and modules (Dublin, 1998), Trends Math., Birkhäuser, Basel, 1999, 269–292. MR **2001c:**20113

[15] G. Hjorth and A. S. Kechris, *Borel equivalence relations and classification of countable models*, Annals of Pure and Applied Logic **82** (1996), 221–272. MR **99m:**03073

[16] S. Jackson, A. S. Kechris, and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic **2** (2002), 1–80.

[17] B. Jónsson, *On direct decompositions of torsion-free abelian groups*, Math. Scand. **7** (1959), 361–371. MR **23:**A204

[18] A. S. Kechris, *Countable sections for locally compact group actions*, Ergodic Theory and Dynamical Systems **12** (1992), 283–295. MR **94b:**22003

[19] A. S. Kechris, *Amenable versus hyperfinite Borel equivalence relations*, J. Symbolic Logic **58** (1993), 894–907. MR **95f:**03081

[20] A. S. Kechris *Classical Descriptive Set Theory*, Graduate Texts in Mathematics **156**, Springer-Verlag, 1995. MR **96e:**03057

[21] A. S. Kechris, *On the classification problem for rank 2 torsion-free abelian groups*, J. Lond. Math. Soc. **62** (2000), 437–450. MR **2002h:**20078

[22] M. Król, *The automorphism groups and endomorphism rings of torsion-free abelian groups of rank two*, Dissertationes Math. Rozprawy Mat. **55** (1967), 1–76. MR **36:**1536

[23] A. G. Kurosh, *Primitive torsionsfreie abelsche Gruppen vom endlichen Range*, Ann. Math. **38** (1937), 175–203.

[24] E. L. Lady, *Extensions of scalars for torsion-free modules over Dedekind domains*, in Symposia Mathematica, Vol. XXIII, (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), Academic Press, 1979, 287–305. MR **82l:**13011

[25] A. I. Malcev, *Torsion-free abelian groups of finite rank* (Russian), Mat. Sbor. **4** (1938), 45–68.

[26] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Erg. der Math. und ihrer Grenz., **17**, Springer-Verlag, 1991. MR **92h:**22021

[27] R. S. Pierce, *Associative Algebras*, Graduate Texts in Mathematics **88**, Springer-Verlag, 1982. MR **84c:**16001

[28] S. M. Srivastava, *A Course on Borel Sets*, Graduate Texts in Mathematics **180**, Springer-Verlag, 1998. MR **99d:**04002

[29] S. Thomas, *On the complexity of the classification problem for torsion-free abelian groups of rank two*, to appear in Acta Math.

[30] S. Thomas, *The classification problem for p-local torsion-free abelian groups of finite rank*, preprint (2002).

[31] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for finitely generated groups*, J. Algebra **217** (1999), 352–373. MR **2002l:**20001

[32] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for fields of finite transcendence degree*, J. Pure and Appl. Algebra. **159** (2001), 347–363. MR **2002c:**03076

[33] S. Wagon, *The Banach-Tarski Paradox*, Encyclopedia of Mathematics and its Applications **24**, Cambridge University Press, 1985. MR **87e:**04007

[34] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Birkhäuser, 1984. MR **86j:**22014

[35] R. Zimmer, *Groups generating transversals to semisimple Lie group actions*, Israel J. Math. **73** (1991), 151–159. MR **93e:**22012

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019

E-mail address: sthomas@math.rutgers.edu