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The Clebsch-Gordan Coefficients

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A simpler method of deriving the Clebsch-Gordan coefficients is described. By the use of the new operators for angular momentum, introduced in a recent paper, the problem, which properly belongs to the domain of Algebra, is tackled here by the more convenient methods of Analysis. The natural occurrence of the hypergeometric function is an interesting feature of this treatment.

§ 1. Introduction

In a previous paper[1] (referred to hereafter as (I)) the one-variable operators

$$M_x \pm iM_y = e^{\pm i\varphi}(j \pm iD_\varphi),\ \ M_z = -iD_\varphi,\ \ D_\varphi = d/d\varphi \tag{1}$$

were shown to give a representation of angular momentum both for integral and half-integral values of j*, and it was correctly realised that their use might lead to an easier method of calculating the Clebsch-Gordan (briefly C-G) coefficients. An important step in this direction was taken by setting up a second order differential equation (eq. (4) below), whose solution would yield a general expression for these coefficients. As the very form of this equation discouraged any attempt at a solution, a recursion formula connecting three successive Fourier coefficients (which are identical with the C-G coefficients of the second kind as defined in § 2) was set up. But this formula was as intractable as the equation itself, and the possibility of getting any simplification seemed to be remote. Recently, however, it has been realised that the difficulties are illusory, and that eq. (4) is, in fact, one of the standard equations of Analysis in an apparently unrecognisable form. Furthermore, it has been possible to set up a system of $4j+2$ coupled equations, in a certain sense, more satisfactory than eq. (4). These new equations, being of the first order, can be solved immediately, and the solution involves hypergeometric functions, which naturally occur in the theory as formulated in the present paper. The entire theory can, therefore, be worked out from the known properties of these functions, and a general expression for the coefficients obtained without using the recurrence relations of Racah[2] or the group-theoretical method of Wigner[3]. With the help of the transformation formulae for hypergeometric functions this expression can, in fact, be written a variety of forms, of which only one will be given here. The reduction to the hypergeometric equation places a powerful tool at our disposal and may be of help in finding out interesting relationships.

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* The results of this paper are valid for half-integral quantum numbers as well.

§ 2. Clebsch–Gordan coefficients of the second kind

If the normalized spherical harmonics $Y_j^m(\theta, \varphi)$ are replaced by $e^{im\varphi}$ and the usual angular momentum operators by the operators (1), then the matrices assume the forms

$$(M_x+iM_y)_{m+1,m}=j-m, \quad (M_x-iM_y)_{m,m+1}=j+m+1, \quad (M_z)_{mm}=m$$

where, M_x, M_y, M_z are the components of angular momentum in units of $\hbar$. These are connected with the usual matrices $M_x^*\pm iM_y^*$, M_z^* by a similarity transformation $M=A^{-1}M^*A$, where A is a diagonal matrix with diagonal elements $A_{mm}=[(j-m)!(j+m)!]^{1/2}$. If $f(\varphi)=\sum_{m=-j}^{m=+j} a_m e^{im\varphi}$ is the solution of the reduced problem, then the actual solution in terms of spherical harmonics is $F(\theta,\varphi)=\sum_{m=-j}^{m=+j} A_{mm}a_m Y_j^m(\theta,\varphi)$ (see § 8 of (I)).

For a dymamical system with two angular momenta M_1 and M_2 the functions Ψ_{jm}, which simultaneously diagonalize $M_z=M_{z1}+M_{z2}$ and the total angular momentum $M^2=(M_1+M_2)^2$ are linear combinations of products of the type $Y_{j_1}^{m_1}(\theta_1,\varphi_1)Y_{j_2}^{m_2}(\theta_2,\varphi_2)$. The coefficients $\{m_1m_2|jm\}$ of these linear combinations are called the C-G coefficients. It will be convenient for us to call them "coefficients of the first kind." If the normalized eigenfunction Ψ_{jm} of the operator M^2 is multiplied by $[(j-m)!(j+m)!]^{1/2}$ and the replacements stated at the beginning of this section are made, we get a function

$$\Phi_{jm}=\sum_{m_1,m_2}(m_1m_2|jm)\exp(im_1\varphi_1+im_2\varphi_2)\,. \tag{2}$$

The coefficients $(m_1m_2|jm)$ of this double Fourier series will be called "C-G coefficients of the second kind." Evidently the connection between the two kinds of coefficients is

$$\{m_1m_2|jm\}=(m_1m_2|jm)\left[\frac{(j_1-m_1)!\,(j_1+m_1)!\,(j_2-m_2)!\,(j_2+m_2)!}{(j-m)!\,(j+m)!}\right]^{1/2}$$

The introduction of the coefficients of the second kind is essential for the discussions to follow and results in considerable simplification of the mathematical treatment.

§ 3. The differential equations satisfied by Φ_{jm}

We now proceed to set up certain differential equations satisfied by the function Φ_{jm}, which is an unnormalized eigenfunction of the operator

$$\begin{aligned}M^2&=(M_1)^2+(M_2)^2+(M_{x1}+iM_{y1})(M_{x2}-iM_{y2})\\&\qquad+(M_{x1}-iM_{y1})(M_{x2}+iM_{y2})+2M_{z1}M_{z2}\\&=F_1+F_2+e^{i(\varphi_1-\varphi_2)}(j_1+iD_1)(j_2-iD_2)\\&\qquad+e^{-i(\varphi_1-\varphi_2)}(j_1-iD_1)(j_2+iD_2)-2D_1D_2\end{aligned}$$

belonging to the eigenvalue F, where,

$$F=j(j+1),\ F_1=j_1(j_1+1),\ F_2=j_2(j_2+1),\ D_1=\frac{\partial}{\partial\varphi_1},\ D_2=\frac{\partial}{\partial\varphi_2}.$$

Introducing the variables, $\xi=\varphi_1$, $\eta=\varphi_1-\varphi_2$, we have

$$M_x \pm iM_y = e^{\pm i\xi}[j_1 \pm iD_\xi \pm iD_\eta + e^{\mp i\eta}(j_2 \mp iD_\eta)], \quad M_z = -iD_\xi\,,$$
$$M^2 = F_1 + F_2 + e^{i\eta}(j_1 + iD_\xi + iD_\eta)(j_2 + iD_\eta) + e^{-i\eta}(j_1 - iD_\xi - iD_\eta)(j_2 - iD_\eta)$$
$$+ 2(D_\xi + D_\eta) D_\eta\,.$$

The differential equations are conveniently set up with the help of the commutation relations, $[M_z, M^2] = [M_x \pm iM_y, M^2] = 0$. In combination with the equation

$$[M^2 - F]\Phi_{jm} = 0 \tag{3}$$

they determine the functions Φ_{jm} except for an arbitrary factor, which may depend on j_1, j_2, j and sometimes on m. The relation $[M_z, M^2] \equiv -i[D_\xi, M^2] = 0$ implies that $m = m_1 + m_2$ is constant for a particular function Φ_{jm}. The double summation over m_1 and m_2 in the defining equation (2), therefore, reduces to a simple sum, and we have

$$\Phi_{jm} = e^{im\xi} \sum_{m_2} (m_1 m_2 | jm)\, e^{-im_2\eta} = e^{im_2\xi} \chi_{jm}\,.$$

As a consequence, the variable ξ drops out altogether from eq. (3), which reduces to

$$[e^{i\eta}(j_1 - m + iD_\eta)(j_2 + iD_\eta) + e^{-i\eta}(j_1 + m - iD_\eta)(j_2 - iD_\eta)$$
$$+ 2(im + D_\eta) D_\eta + F_1 + F_2 - F]\chi_{jm} = 0. \tag{4}$$

This important equation already occurs in (I). It determines the functions χ_{jm}, and therefore C–G coefficients, up to an arbitrary factor involving j_1, j_2, j, m.

Let us now examine the consequences of the pair of relations $[M_x \pm iM_y, M^2] = 0$, which imply that $(M_x \pm iM_y)\Phi_{jm}$ is a linear combination of the $2j+1$ eigenfunctions Φ_{jm} for different values of m. The occurrence of the factor $e^{\pm i\xi}$ in $M_x \pm M_y$ further ensures that this linear combination consists of a single term $\Phi_{jm\pm1}$. On suitably adjusting the arbitrary constant in Φ_{jm} and dropping the variable ξ, we have

$$[j_1 - m + iD_\eta + e^{-i\eta}(j_2 - iD_\eta)]\chi_{jm} = (j - m)\chi_{jm+1} \tag{5a}$$
$$[j_1 + m + 1 - iD_\eta + e^{i\eta}(j_2 + iD_\eta)]\chi_{jm+1} = (j + m + 1)\chi_{jm}\,. \tag{5b}$$

It is easily seen that they lead to eq. (4) and determine the functions χ_{jm} up to an arbitrary factor independent of m. This factor can be determined by normalizing the simplest of the functions Ψ_{jm}, namely, Ψ_{jj} or $\Psi_{j\,-j}$.

As it will not be necessary to consider more than one value of j at a time, it is better, at this stage, to drop the subscript j, and write χ_m for χ_{jm}, Φ_m for Φ_{jm}, etc. We notice that eq. (4) remains unchanged on changing the sign of m and passing on to the complex conjugate. An immediate consequence of this symmetry property is that $\chi^*_{-m} = C_m \chi_m$. Making the same changes in eq. (5a) and comparing with eq. (5b), we see that $C_m = C_{m-1} = C$ is independent of m. From $\chi_0^* = C\chi_0$ it then follows that $|C| = 1$. The exact value of C can be obtained by comparing χ_j and χ_{-j} given in eq. (9) below, and turns out out to be $C = (-1)^{j_1+j_2-j}$. Therefore,

$$\chi_{-m} = (-1)^{j_1+j_2-j}\chi_m^*, \quad (-m_1 - m_2 | j, -m) = (m_1 m_2 | jm)(-1)^{j_1+j_2-j}\,. \tag{6}$$

To bring eqs. (4), (5a), (5b) into more convenient forms and to show their connection with the hypergeometric equation we put $x=e^{-i\eta}$. This gives

$$\left[\frac{1}{x}(j_1-m+xD)(j_2+xD)+x(j_1+m-xD)(j_2-xD)+2(m-xD)xD\right.$$

$$\left.+F_1+F_2-F\right]\chi_m=0, \qquad D=d/dx, \tag{7}$$

$$L_1^m(\chi_m)\equiv[x(1-x)D+j_1-m+j_2x]\chi_m=(j-m)\chi_{m+1}, \tag{8a}$$

$$L_2^{m+1}(\chi_{m+1})\equiv[(1-x)D+j_1+m+1+j_2/x]\chi_{m+1}=(j+m+1)\chi_m. \tag{8b}$$

The eqs. (8a), (8b), being of the first order, can be solved easily. Putting $m=j$ in (8a) and $m=-j-1$ in (8b), we have

$$\chi_j=x^{-j_1+j}(1-x)^{j_1+j_2-j}, \qquad \chi_{-j}=x^{-j_2}(1-x)^{j_1+2-j}. \tag{9}$$

Next, put

$$\chi_m=x^{-j_1+m}(1-x)^{j_1+j_2-j}, \qquad v_m=x^{-j_2}(1-x)^{j_1+j_2-j}v_m. \tag{10}$$

This gives the following two alternative forms of the set of equations (8a), (8b) :

$$\left.\begin{aligned}&[(1-x)D+j-m]u_m=(j-m)u_{m+1},\\&[x(1-x)D-j_1+j_2+m+(j_1-j_2+j)x]u_m=(j+m)u_{m-1},\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}&[x(1-x)D+j_1-j_2-m+(-j_1+j_2+j)x]v_m=(j-m)v_{m+1},\\&[(1-x)D+j+m]v_m=(j+m)v_{m-1}.\end{aligned}\right\} \tag{12}$$

A comparison with the relations between contiguous hypergeometric functions, namely,

$$(xD+a)F(a,b;c;x)=aF(a+1,b;c;x),$$

$$[x(1-x)D+c-a-bx]F(a,b;c;x)=(c-a)F(a-1,b;c;x)$$

now shows that the solution of the set (11) is

$$u_m=F(-j+m,\ -j_1+j_2-j;\ -2j;\ 1-x)$$

and the solution of the other set (12) is

$$v_m=F(-j-m,\ j_1-j_2-j;\ -2j;\ 1-x).$$

We can get two other forms of the solution by making use of the relation (6). These are

$$(1-x)^{-j_1-j_2+j}\chi_m=x^{-j_1+j}F(-j+m,\ j_1-j_2-j;\ -2j;\ (x-1)/x)$$

$$=x^{-j_2+j+m}F(-j-m,\ -j_1+j_2-j;\ -2j;\ (x-1)/x).$$

The four different froms can be obtained very simply from Kummer's connection formulae. Other forms may be obtained, for instance, from the relation

$$F(a,b;c;x)=\Gamma(c)\Gamma(c-a-b)/\{\Gamma(c-a)\Gamma(c-b)\}.\ \ F(a,b;a+b-c+1;1-x)$$

which is valid when at least one of the parameters a, b is a non-positive integer, and the denominator in the successive terms of the series on either side of the identity does not vanish earlier than the numerator. The factor involving Gamma functions can often be evaluated by a limiting process even if it is meaningless. Each new form of the solution leads to a different expression for the C-G coefficients. But it is usually difficult to establish the equivalence of the various expressions by direct algebraic reduction.

The foregoing discussions make it clear that eq. (4) or (7) is a hepergeometric equation in disguise, and can be reduced to it by either of the substitutions (10). Elimination of χ_m or χ_{m+1} from eqs. (8a), (8b) leads to the equations

$$[L_2^{m+1}L_1^m-(j-m)(j+m+1)]\chi_m=0, \text{ and } [L_1^{m-1}L_2^m-(j+m)(j-m+1)]\chi_m=0$$

and these must be identical with eq. (7). The entire set of coupled first order equations can, therefore, be replaced by the single second order equation, which may be taken to be the basic equation of the problem.

§ 4. Determination of the constant A_j

As Φ_m has a definite connection with the normalized function Ψ_m, the multiplying constant in χ_m cannot be arbitrary, but must have a definite dependence on j_1, j_2, j. To use Racah's notation we write

$$\chi_m=A_j(2j)!/(j_1+j_2-j)!\cdot x^{-j_1+m}(1-x)^{j_1+j_2-j}F(-j+m,\ -j_1+j_2-j\,;\ -2j\,;\ 1-x)\,.$$

The constant A_j is most easily determined by normalizing the functions Ψ_j or Ψ_{-j}, that is, from either of the relations

$$\sum_{m_2}\{m_1m_2|jj\}^2=\sum_{m_2}\{m_1m_2|j,-j\}^2=1. \tag{13}$$

The C-G coefficient $(m_1m_2|j_1j_2)$ is the coefficient of x^{m_2} in the expansion of χ_j:

$$(m_1m_2|jj)=A_j(2j)!(-1)^{j_1-m_1}/[(j_1-m_1)!(j_2-m_2)!]. \tag{14}$$

The coefficient of the first kind is, therefore,

$$\{m_1m_2|jj\}=A_j(-1)^{j_1-m_1}[(2j)!(j_1+m_1)!(j_2+m_2)!/\{(j_1-m_1)!(j_2-m_2)!\}]^{1/2}.$$

The relation (13) now gives*, on writing t for j_1-m_1,

$$A_j^{-2}=\sum_t\frac{(2j)!(2j_1-t)!(-j+j_2+j+t)!}{t!(j_1+j_2-j-t)!}=\frac{(2j)!(2j_1)!(-j_1+j_2+j)!}{(j_1+j_2-j)!}\times$$
$$\times F(-j_1-j_2+j,-j+j_2+j\ +1\,;\ -2j_1\,;\ 1).$$

Gauss's formula, $F(a,\ b;\ c;\ 1)=\Gamma(c)\Gamma(c-a-b)/\{\Gamma(c-a)\Gamma(c-b)\}$, then gives

$$A_j=[(2j+1)(j_1+j_2-j)!]^{1/2}/[(j_1-j_2+j)!(-j_1+j_2+j)!(j_1+j_2+j+1)!]^{1/2}.$$

This method of evaluation of A_j is perhaps simpler.

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* Here, as in all formulae of this paper, the summation index takes all integral values consistent with the factorial notation, the factorial of a negative number being meaningless.

§ 5. General formulae for the coefficients

From the various expressions for χ_m it is easy to obtain general formulae for the C-G coefficients. In the second form they are identical with the coefficient of x^{m_2} in the expansion of χ_m. To get a formula for the coefficients we select, at random, any particular from of the solution, say,

$$\chi_m = A_j(2j)!/(j_1+j_2-j)!\cdot x^{-j_2}(1-x)^{j_1+j_2-j}F(-j-m,\, j_1-j_2-j\,;\, -2j\,;\, 1-x)$$
$$= \frac{A_j(-j_1+j_2+j)!(j+m)!}{(j_1+j_2-j)!}\times$$
$$\times\sum_{s,t} x^{-j_2+s}(-1)^{t+s}\frac{(2j-t)!(j_1+j_2-j+t)!}{t!(j+m-t)!(-j_1+j_2+j-t)!s!(j_1+j_2-j+t-s)!}$$

whence,

$$(m_1m_2|jm) = \frac{A_j(-j_1+j_2+j)!(j+m)!}{(j_1+j_2-j)!(j_2+m_2)!}$$
$$\sum_t (-1)^{j_2+m_2+t}\frac{(2j-t)!(j_1+j_2-j+t)!}{t!(j+m-t)!(-j_1+j_2+j-t)!(j_1-j-m_2+t)!}. \qquad (15)$$

The correctness of this formula can be tested by seeing if it satisfies the two recurrence relations,

$$(j-m)\,(m_1m_2+1|j\,m+1) = (m_1-1\;m_2+1|jm)\,(j_1-m_1+1) + (m_1m_2|jm)\,(j_2-m_2)$$
$$(j+m)\,(m_1m_2-1|j\,m-1) = (m_1+1\;m_2-1|jm)\,(j_1+m_1+1) + (m_1m_2|jm)\,(j_2+m_2),$$

which follow from eqs. (5a), (5b), and are substantially the same as Racah's relations[2] (3) and (5). A simple calculation shows that it satisfies the second of the two relations. Instead of trying to verify that it satisfies the other relation also it is much easier to show that it gives the correct initial conditions, that is, the expression (14) for $(m_1m_2|jj)$. That it does so is immediately seen by writing the expression, obtained from (15), in the from

$$(m_1m_2|jj) = \frac{A_j(-j_1+j_2+j)!(2j)!}{(j_1+j_2-j)!(j_2+m_2)!}\sum_t(-1)^{j_2+m_2+t}\frac{(j_1+j_2-j+t)!}{t!(-j_1+j_2+j-t)!(j_1-j-m_2+t)!}$$
$$= \frac{A_j(2j)!(-1)^{j_2+m_2}}{(j_1+j_2-j)!(j_2+m_2)!}\left[D^{j_2+m_2}\left[(1-x)^{-j_1+j_2+j}x^{j_1+j_2-j}\right]\right]_{x=1}.$$

References

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