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The closed-universe recollapse conjecture

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Summary. It is widely believed that all expanding S^3 closed universes that satisfy the standard energy conditions recollapse to a second singularity. We show that this is false even for Friedmann universes: we construct an ever-expanding S^3 Friedmann universe in which the matter tensor satisfies the strong, weak and dominant energy conditions and the generic condition. We prove a general recollapse theorem for Friedmann universes: if the positive pressure criterion, dominant energy condition and matter regularity condition hold, then an S^3 Friedmann universe must recollapse. We show that all known vacuum solutions with Cauchy surface topology S^3 or $S^2 \times S^1$ recollapse, and we conjecture that this is a property of all vacuum solutions of Einstein's equations with such Cauchy surfaces. We consider a number of Kantowski–Sachs and Bianchi IX universes with various matter tensors, and formulate a new recollapse conjecture for matter-filled universes.

1 Introduction

The fact that the Universe may have undergone a period of exponential or power-law inflation during its early evolution because of a particular type of phase transition or the dominance of a self-interacting scalar field during this period (Guth 1981; Sato 1981; Barrow & Turner 1982; Linde 1984) has renewed interest in the question of whether closed general relativistic universes recollapse (Zel'dovich & Grishchuk 1984). It has been shown (Barrow & Tipler 1985, 1986) that the existence of a maximal hypersurface is a necessary and sufficient condition for the existence of both an initial and a final all-encompassing 'strong' singularity in a universe with a compact Cauchy surface satisfying the strong energy condition and a generic condition. A 'strong' singularity is one which, roughly speaking, crushes out of existence objects which hit it. Examples of 'strong' singularities are strong curvature singularities and crushing singularities; precise definitions of these types of singularities are given by Marsden & Tipler (1980). The restrictions on the singularities are required only for the 'necessary' part of the singularity theorem; i.e., a maximal hypersurface will give all-encompassing initial and final singularities, but only 'strong' initial and final singularities will have a maximal hypersurface between them.

Furthermore, it has been shown (Barrow & Tipler 1985, 1986) that only closed universes with either S^3 or $S^2 \times S^1$ spatial topology or more complicated hybrids formed by connected summations and special identifications of these two basic topologies actually admit maximal hypersurfaces. Thus closed universes with other topologies (e.g., the three-torus T^3) cannot recollapse to an all-encompassing final singularity. However, it is still an open question whether or not all closed universes with S^3 and $S^2 \times S^1$ topologies do recollapse.

It is this question that we wish to address in this paper. We shall use the phrases 'recollapsing universe' and 'recollapse' to mean a non-static space–time which has a maximal compact Cauchy hypersurface, and by 'all-encompassing initial singularity' we shall mean that all inextendible time-like curves have a length less than a constant L to the past of any Cauchy surface, where L may depend on the Cauchy surface but not on the particular curve. 'All-encompassing final singularity' is defined analogously. As stated above, a closed recollapsing universe begins and ends in an all-encompassing singularity if the strong energy and a generic condition hold. However, a recollapsing universe is distinct from the Wheeler universe defined by Marsden & Tipler (1980) and Barrow & Tipler (1985). A Wheeler universe begins and ends in all-encompassing singularities, but it could be that the singularity might not be strong, and hence a Wheeler universe might not have a maximal hypersurface. However, a Wheeler universe with a maximal hypersurface is a recollapsing universe if the strong energy and a generic condition hold.

We shall consider the Friedmann universe in Section 2, for it turns out that it illustrates many of the subtleties that can occur in the evolution of closed universes. Based on experience with the dust- and radiation-filled Friedmann models, it is widely believed that closed Friedmann universes with S^3 spatial topology recollapse if the weak and strong energy conditions and (possibly) the generic condition hold. However, we shall show that this belief is false: if the matter consists of non-interacting dust and a perfect fluid having pressure p and density μ with equation of state $p = -\frac{1}{3}\mu$ then an expanding S^3 Friedmann universe will expand forever, even though the weak, the strong and the dominant energy conditions and the generic condition hold. Furthermore, we shall show by explicit example that even restricting the pressure to be positive cannot ensure recollapse: unless the growth of positive pressure is restricted, an expanding S^3 Friedmann universe may never attain a maximal hypersurface but will hit a 'pressure' s.p. curvature singularity instead. This 'pressure' singularity is an example of a non-strong singularity.

We shall nevertheless prove in Section 2 a general recollapse theorem for the S^3 Friedmann universe: if the *positive pressure criterion* of Collins & Hawking (1973) holds ($\Sigma_{i=1}^{3} p_i \geqslant 0$ for the principal pressures p_i) and the *dominant energy condition* holds and the matter tensor is regular in a sense defined below, then the S^3 Friedmann universe recollapses. (In the isotropic Friedmann universe case, $\Sigma_{i=1}^{3} p_i = 3p$, so this condition reduces to $p \geqslant 0$ in this instance.) No conditions on the initial and final singularities are imposed in this recollapse theorem. We show that the positive pressure criterion can be relaxed to the condition $p \geqslant (\varepsilon - 1/3)\mu$, where $\varepsilon > 0$ and μ is the mass density, and still recollapse must occur in S^3 Friedmann universes. Furthermore, recollapse will occur in S^3 Friedmann universes if the dominant energy condition is replaced by the requirement that $|p| < C\mu$, where C is some positive constant. There have been numerous incomplete proofs of recollapse for the S^3 Friedmann universe in the physics literature over the past 50 years, and we shall discuss the defects of these 'proofs'.

In Section 3 we shall extend our analysis of recollapse to the closed homogeneous but anisotropic cosmologies: the Bianchi and Kantowski–Sachs universe models. We find that all known vacuum exact solutions with S^3 or $S^2 \times S^1$ Cauchy surface topologies have maximal hypersurfaces, but there are models which obey the standard energy conditions but do not recollapse. Significantly, all the non-recollapsing models violate the positive pressure condition.

Finally, in Section 4 we shall consider all other known vacuum solutions with topology S^3 or $S^2 \times S^1$: all such solutions recollapse. Our examples suggest three conjectures giving sufficient conditions for the recollapse of closed universes, and we shall state these conjectures at the end of Section 4.

We shall not make any restriction on the equation of state for the matter, beyond requiring that the matter tensor T^{ab} be regular except at singularities where it is unbounded. We shall call this a matter regularity condition: if q is a point on the singular c-boundary of the Cauchy development (Penrose 1978) and if q is not a p.p. curvature singularity (Hawking & Ellis 1973) then the matter tensor T^{ab} defined in the Cauchy development has a continuous extension to q. Since any point in space–time can be regarded as being on the c-boundary of some (non-maximal) Cauchy development, this regularity requirement ensures that the time evolution does not stop because of continuity problems with the matter tensor but is unaccompanied by p.p. curvature singularities.

Our notation will be that of Hawking & Ellis (1973) unless otherwise noted. The Einstein equations will be

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi G T_{ab},$$

and the space–time signature will be $(-+++)$. The cosmological constant will be assumed to be zero. We shall assume that all manifolds are C^2 maximally extended.

2 The Friedmann universe

The S^3 closed Friedmann universe has the metric

$$ds^2 = -dt^2 + R^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{2.1}$$

where $d\Omega^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)$ is the metric of the unit three-sphere ($0 \leqslant \chi \leqslant \pi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$). The Einstein equations give us three relevant equations for the scale factor $R(t)$. These are (Misner, Thorne & Wheeler 1973) the constraint equation (sometimes called the Friedmann equation)

$$(R')^2 - \frac{8\pi G}{3} \mu R^2 = -k, \tag{2.2}$$

where the prime denotes differentation with respect to time t, the conservation of stress energy equation

$$\frac{d}{dt}(\mu R^3) = -\frac{p\, d(R^3)}{dt} \tag{2.3}$$

and finally the dynamical equation

$$\frac{2R''}{R} = -\left(\frac{R'}{R}\right)^2 - kR^{-2} - 8\pi G p. \tag{2.4}$$

For S^3 universes, the constant $k=+1$. It is well known that these equations are not independent and that equation (2.4) is implied by equations (2.2) and (2.3), but we shall find it convenient to use all three.

For isentropic perfect fluids with energy density μ and pressure p, the equation of state is

$$p = (\gamma - 1)\mu; \qquad \gamma \text{ constant}. \tag{2.5}$$

Putting (2.5) into (2.3) we obtain

$$\mu=\frac{3M}{8\pi G}R^{-3\gamma} \tag{2.6}$$

where M is a constant. With (2.6), equation (2.2) becomes

$$\left(\frac{R'}{R}\right)^2=\frac{M}{R^{3\gamma}}-\frac{1}{R^2}. \tag{2.7}$$

It is evident from (2.7) that $M>0$, and that the universe expands forever if $\gamma\leqslant 2/3$ (if $\gamma=2/3$, $M\geqslant 1$, since the left-hand side is positive).

All positive Friedmann universe stress-energy tensors are type I in the notation of Hawking & Ellis (1973). For type I matter fields, the weak energy condition, $T_{ab}W^aW^b\geqslant 0$ for all time-like vectors W^a, will hold if $\mu\geqslant 0$ and $\mu+p_i\geqslant 0$, $i=1, 2, 3$. Furthermore, the strong energy condition

$$(T_{ab}-\tfrac{1}{2}Tg_{ab})W^aW^b\geqslant 0$$

will hold if $\mu+p_i\geqslant 0$ and $\mu+\Sigma_{i=1}^3 p_i\geqslant 0$ for all time-like W^a. Thus if $\gamma=2/3$, i.e. if $p=-\tfrac{1}{3}\mu$, both the weak and strong energy conditions will hold (and so will the dominant energy condition $T_{00}\geqslant |T_{ab}|$ for every a, b). This example shows that the standard energy conditions are insufficient to give recollapse, even in the Friedmann universe case.

However, with $\gamma=2/3$, we have $R_{ab}W^aW^b=R_{tt}=0$ if W^a is the unit normal to a hypersurface of isotropy and homogeneity. Since $R_{tt}=3R^i_{tit}=-3R''/R$ in an orthonormal frame with the i labelling a space-like direction and t the time-like direction, the equality $R_{tt}=0$ means that $W^aW^bW_{[c}R_{d]ab[e}W_{f]}=0$ if W^a is the tangent to the geodesics normal to the hypersurfaces of isotropy and homogeneity. Thus the generic condition is violated in a Friedmann universe with $\mu=2/3$. Another way to see this is to note that (2.2) implies that R' is constant for $\mu=2/3$. Hence R'' and R_{tt} equal zero; such a Friedmann model expands with constant velocity from an initial singularity at $R=0$.

Nevertheless this model can easily be modified to satisfy the generic condition and yet still expand forever. Let us add to the $p=-\mu/3$ fluid dust satisfying $p=0$. The two fluids will satisfy the conservation equation (2.3) separately. The $M/R^{3\gamma}$ term in the Friedmann equation (2.2) will be replaced by $[M_A/R^2+M_D/R^3]$ where M_A and M_D are positive constants. We can choose M_A so that (2.2) becomes

$$(R')^2-\frac{M_D}{R}=+1 \tag{2.2a}$$

which is the Friedmann equation for dust, but with $k=-1$; clearly there is no collapse because R' must always have the same sign. The positive density of the dust will give strict inequalities for the weak and strong energy conditions, and will also ensure that the generic condition is satisfied. Thus a stronger condition than the generic condition is needed to prove the recollapse of closed matter-filled universes.

One other such condition is the positive pressure condition $\Sigma_{i=1}^3 p_i\geqslant 0$ where p_i is a principal pressure for type I matter. As mentioned above, this condition reduces to the condition $p\geqslant 0$ in the Friedmann models. We have the following theorem.

Theorem. If the positive pressure criterion, the matter regularity condition and the dominant energy condition hold, then a Friedmann universe with S^3 spatial topology expands from an initial singularity to a maximal hypersurface, and then recollapses to a final singularity.

Proof. By (2.2) an S^3 Friedmann universe must have $\mu>0$. Furthermore, by (2.4) and with $p\geqslant 0$ we have $R''<0$, and so $R'=0$ is not allowed except at the maximal hypersurface. Suppose that $R'>0$ for all t in (t_0, t_f), where t_0 is some initial time and t_f is the limit of the future time development. Then $p\geqslant 0$ and (2.3) give $d(\mu R^3)/dt\leqslant 0$, which in turn implies $d(\mu R^3)/dR\leqslant 0$. This equality yields $\mu R^3\leqslant m$, where m is a positive constant. Therefore we have

$$0<\mu<m/R^3. \tag{2.8}$$

However, (2.2) gives the inequality

$$\frac{8\pi G}{3}\mu R^2\geqslant 1. \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$\frac{8m\pi G/3}{R}\leqslant 1. \tag{2.10}$$

Inequality (2.10) implies that $R\to+\infty$ as $t\to t_f$ is impossible. We now show that $R(t)\to R_0<+\infty$ as $t\to t_f$ is also impossible. The dominant energy condition $\mu\geqslant p$ and (2.2) give the inequality

$$\left(\frac{R'}{R}\right)^2+\frac{k}{R^2}\geqslant\frac{8\pi Gp}{3} \tag{2.11}$$

which, when inserted into (2.4), yields the right-hand inequality in

$$-\frac{k}{2R}\geqslant R''\geqslant-\frac{2[(R')^2+k]}{R}. \tag{2.12}$$

The left-hand inequality in (2.12) is obtained from the positive pressure criterion $p\geqslant 0$ and (2.4). Since R'' is bounded as $R\to R_0<+\infty$, both p and R'' must approach definite limits as $R\to R_0$, by the matter regularity condition, since R_0 is not a p.p. curvature singularity. Thus if $R\to R_0<+\infty$ as $t\to t_f<+\infty$ then R, R' and R'' will all approach, contradicting extendability, finite limits; therefore t_f must be $+\infty$. However, this contradicts (2.12). Thus we must have $R'=0$ for some $t=t_{max}$. Since $R''<0$ at this time and at all future times, R must decrease to $R=0$ in finite time. Obtaining $R'=0$ from the assumption $R'<0$ is similar. This proves the theorem.

The conditions of this theorem can be relaxed in several ways. For example, the positive pressure criterion can be replaced by the condition $p\geqslant(\varepsilon-1/3)\mu$, where ε is some positive constant. For an expanding universe, this condition and (2.3) give the inequality $0\leqslant\mu\leqslant m/R^{2+3\varepsilon}$, which is sufficient to rule out the possibility $R\to+\infty$ by an argument similar to that in the proof of the theorem. Furthermore, the inequality (2.12) is replaced by the inequality

$$-4\pi G\varepsilon R\geqslant R''\geqslant-\frac{2[(R')^2+k]}{R} \tag{2.13}$$

which will ensure the existence of a C^2 continuation to $R'=0$.

The only role that the dominant energy condition and the matter regularity condition play in the proof of the Friedmann recollapse theorem is to ensure the existence of the continuation of the manifold to the maximal hypersurface $R'=0$. If the dominant energy condition is not imposed (or the somewhat more general condition $|p|\leqslant C\mu$, where C is a positive constant), an s.p. curvature singularity (Hawking & Ellis 1973) could develop before the maximal hypersurface is reached. For example, consider the S^3 Friedmann universe in which the scale factor $R(t)$ varies as

$$R(t)=1+t-\tfrac{2}{3}(1-t)^{3/2} \tag{2.14}$$

in the domain $0<t<1$, with the pressure and density being defined by (2.4) and (2.2). For t near 1, the pressure will be positive, and

$$R'(t)=1+(1-t)^{1/2}>0 \tag{2.15}$$

$$R''(t)=-\tfrac{1}{2}(1-t)^{-1/2}<0. \tag{2.16}$$

However, $R''\rightarrow-\infty$ as $t\rightarrow 1$, so this Friedmann model cannot be extended to a maximal hypersurface. Since in this model $p\rightarrow+\infty$ as $t\rightarrow 1$, there is a singularity in the pressure, but the variables (μ, R, R') remain finite. This singularity is neither a strong curvature singularity nor a crushing singularity. Thus the dominant energy condition and the matter regularity condition can be replaced by a requirement that all singularities are sufficiently strong to give a divergence of R' at the singularity (requiring the singularities to be either strong curvature or crushing would give this divergence), and this condition on the strength of the singularity would be sufficient to guarantee the existence of the maximal hypersurface in the C^2 maximally extended Friedmann manifold.

Even the dominant energy condition by itself would be insufficient to obtain recollapse in the absence of our matter regularity condition on the matter tensor. If regularity did not hold, then the pressure p might oscillate between finite bounds without attaining any definite limit as $t\rightarrow t_f$, and this sort of oscillatory singularity would stop the time evolution before a maximal hypersurface was reached. Since the density μ is finite everywhere except where R' diverges, we could also eliminate the pressure singularity (in the Friedmann case at least) by requiring that the pressure be a function of the density alone and that $p(\mu)$ be finite for a finite μ. Thus the dominant energy and the matter regularity condition can be replaced by this restriction on the equation of state.

Our Friedmann recollapse theorem shows that it is possible to obtain a recollapse theorem without making any requirements on the singularity, but to accomplish this it is necessary to use all the Einstein field equations; the first-order Friedmann equation alone is not sufficient. The fact that all field equations must be used is what makes obtaining recollapse theorems hard in space–times more complex than Friedmann. Our analysis shows that the Friedmann case is equally subtle.

The first general recollapse theorem for the S^3 Friedmann universe, using only the positive pressure criterion, was published by Tolman & Ward (1932) (see also Tolman 1931, 1932, 1934). Their proof was essentially equivalent to our proof of the first part of the above theorem. However, they were concerned only with showing that an upper bound existed for the scale factor, and they overlooked the possibility that the evolution of the universe could cease before recollapse occurred. It is not possible to obtain a proof of recollapse under their hypotheses; the positive pressure criterion alone is not sufficient.

Unfortunately, their theorem seems to have been accepted uncritically (Robertson 1933). Claims that $\mu>0$ and $p\geqslant 0$ or $\mu+p>0$ and $\mu+3p>0$ are sufficient to obtain recollapse in the Friedmann's S^3 universe can be found even in Ellis (1971).

3 Recollapse in the closed homogeneous anisotropic cosmologies

It is known that in Newtonian theory the conditions required for a pressure-free region to collapse anisotropically under gravity are less strict than those needed to guarantee isotropic collapse (Barrow & Silk 1981); there are just so many more ways for an anisotropic collapse to occur. The same is true in general relativity.

There are three known classes of closed homogeneous anisotropic universes with topology S^3 or $S^2\times S^1$: (1) the Kantowski–Sachs universes, (2) the Taub universe and (3) the general Bianchi type IX universes (Kramer *et al.* 1980). The Taub and Kantowski–Sachs models admit exact solutions and the type IX universe reduces to the Taub solution in the special case of axial symmetry. The Kantowski–Sachs models admit $S^2\times S^1$ topology; the others admit S^3.

It can be shown using Einstein's equations that if an open ever-expanding Kantowski–Sachs model (actually of Bianchi type III) containing with zero pressure and positive density μ^* possesses a metric

$$ds^2=-dt^2+A^2(t)\,dr^2+B^2(t)(d\theta^2+\sinh^2\theta\,d\phi^2), \tag{3.1}$$

then the closed $S^2\times S^1$ Kantowski–Sachs universe with metric

$$ds^2=-dt^2+A^2(t)\,dr^2+B^2(t)(d\theta^2+\sin^2\theta\,d\phi^2) \tag{3.2}$$

with the *same* functions $A(t)$ and $B(t)$ as in (3.1) solves the Einstein equations if the stress-energy tensor has density and principal pressures given by

$$(\mu, p_1, p_2, p_3)=\left(\mu^*+\frac{2}{B^2},\ -\frac{2}{B^2},\ 0,\ 0\right). \tag{3.3}$$

By this method, which is similar to our trick in the Friedmann case (equation 2.2a) of constructing a closed universe from an open universe, we have constructed an example of a closed anisotropic universe which obeys the strong and weak energy conditions (with a strict inequality, and so also the generic condition) and yet does not collapse. As in the Friedmann case, this means that in the anisotropic case also it is necessary to impose stronger conditions on the matter content of space–time if we are to obtain recollapse of closed universes. Two obvious possibilities which would eliminate the recollapse counterexample (3.3) are either the requirement that all principal pressures are non-negative or, more weakly, the positive pressure condition.

It can also be shown using the autonomous form of the Kantowski–Sachs field equations (Collins 1977) that all Kantowski–Sachs initial data lying close to isotropy do not lead to recollapse if the matter content is perfect fluid with $p\leqslant-\mu/3$.

In order to explore the consequences of such pressure restrictions we now consider the more general case of the closed S^3 Bianchi type IX universes. Although exact solutions are not available for these universes unless they are axisymmetric we can determine some circumstances under which they do not recollapse. We take any diagonal vacuum Bianchi type VIII (open-universe) solution with metric (Kramer *et al.* 1980)

$$ds^2=-dt^2+\gamma_{\alpha\beta}(t)\sigma^\alpha\wedge\sigma^\beta, \qquad \alpha,\beta=1,2,3, \tag{3.4}$$

where σ^α are differential forms invariant under the Bianchi type VIII group of motions and where the spatial metric $\gamma_{\alpha\beta}(t)$ is

$$\gamma_{\alpha\beta}(t)=\mathrm{diag}\,[a^2(t),\ b^2(t),\ c^2(t)]. \tag{3.5}$$

These type VIII vacuum solutions all expand indefinitely as $t\to\infty$. Now it can be shown that the same spatial metric (3.5) will also solve the Einstein equations for a non-vacuum diagonal Bianchi type IX closed universe if and only if the Bianchi IX universe has density and principal pressures given by

$$(\mu, p_1, p_2, p_3)=(b^{-2}+c^{-2},\ -b^{-2}-c^{-2},\ c^{-2}-b^{-2},\ b^{-2}-c^{-2}). \tag{3.6}$$

Therefore closed S^3 Bianchi type IX universes with stress-energy tensor (3.6) expand forever. We see that the sum of the principal pressures is negative and $\mu+\Sigma_{i=1}^3 p_i=0$ for (3.6). In the axially symmetric case ($b=c$, and so $\mu=-p_1, p_2=p_3=0$) the type IX metric reduces to the Taub (1951)

Table 1. Summary of the closed spatially homogeneous models discussed in the text.

Cosmological model	Principal pressures p_1	p_2	p_3	$\Sigma_{i=1}^{3} p_i$	Energy conditions $\mu+\Sigma_{i=1}^{3} p_i$	$\mu-\lvert p_i\rvert$	Recollapse to singularity
Friedmann, $k=+1$, $-\mu<p<-\mu/3$	−	−	−	−	≤ 0	+	No
Friedmann, $k=+1$, $-\mu/3<p<0$	−	−	−	−	+	+	Yes
Friedmann, $k=+1$, $p=0$ and $p=-\mu/3$	−	−	−	−	+	+	No
Friedmann, $k=+1$ equation (2.14)[a]	+	+	+	+	+	−	No
Kantowski–Sachs S^3 and $S^2\times S^1$ [b]	−	0	0	−	+	+	No
Bianchi type IX S^3	−	+(−)	−(+)	−	0	≥ 0	No
Axisymmetric IX S^3	−	0	0	−	0	0	No
Axisymmetric IX S^3 electromagnetic field	−	+	+	+	+	0	Yes

[a] The special Friedmann model has indefinite signs of the pressures and energy conditions. We have listed the signs holding as $t\to 1$ in (2.14); as discussed in the text this model experiences a singularity before a maximal hypersurface can develop.
[b] Collins (1977) gives conditions, $0\leq p\leq\mu$, $p=p(\mu)$, $0\leq dp/d\mu\leq 1$ sufficient for recollapse.

solution and it is possible to find $a(t)$ and $b(t)$ explicitly as functions of time. As in the previous cases the strong and weak energy conditions hold.

We note that there also exist exact closed axisymmetric type IX solutions containing electromagnetic fields (Brill 1964) but these all recollapse. Such examples obey the condition that the sum of the principal pressure is positive, but not all pressures are positive and so the stipulation that all the principal pressures be positive is not necessary for recollapse to occur.

The results from these examples and those of the previous section are summarized in Table 1.

It is worth noting that the stress-energy tensor (3.6) reduces, in the axisymmetric Taub universes, to the line source $\mu=-p_1$, $p_2=p_3=0$, which is the stress-energy tensor of a static vacuum string (Vilenkin 1985) in the weak-field limit. Another physically motivated anisotropic stress-energy tensor which violates the strong energy condition and so prevents closed universes with S^3 or $S^2\times S^1$ topologies from recollapsing is that of a vacuum domain wall (Vilenkin 1985) for which $\mu=-p_1=-p_2$ and $p_3=0$. Such a stress is gravitationally repulsive.

4 Inhomogeneous examples and the new recollapse conjectures

The examples described in the preceding two sections show that it is not necessary for the sum of the principal pressures to be positive in order for closed universes to avoid recollapse, yet they suggest that non-negativity is sufficient to guarantee recollapse with S^3 or $S^2\times S^1$ topologies. The known examples of inhomogeneous closed universes which recollapse include the S^3 and $S^2\times S^1$ Gowdy vacuum universes (Gowdy 1971, 1974, 1975) and matter-filled universes with $\mu=p$ perfect fluid sources (Carmelia, Charach & Malin 1981), the zero-pressure S^3 models of Tolman (Tolman 1934; Zel'dovich & Grishchuk 1984; Bonnor 1985), and the zero-pressure S^3 and $S^2\times S^1$ Szekeres models (Szekeres 1975; Bonnor & Tomimura 1976; Barrow & Silk 1981). The Tolman and Szekeres dust models are not particularly instructive in this respect since they are essentially Newtonian and do not contain any of the true gravitational degrees of freedom which the Bianchi type IX universes contain.

However, such inhomogeneous dust models provide examples of shell-crossing singularities, and such singularities can prevent the attainment of maximal hypersurfaces even in closed universes which begin and end in all-encompassing singularities (Eardley & Smarr 1979; Barrow & Tipler 1979; Marsden & Tipler 1980). It is generally thought that such singularities are unphysical, but in practice their occurrence must be ruled out by explicit assumption, for instance by requiring that all singularities be either strong curvature or crushing. Shell-crossing singularities are a problem only in inhomogeneous matter-filled universes, so probably no restrictions on the nature of the singularity are necessary to obtain recollapse in vacuum closed universes or in homogeneous closed universes. We also note that, in non-vacuum space–times in which the strong energy condition and the positive pressure criterion hold, the version of the generic condition needed to force recollapse from a maximal hypersurface [the hypersurface generic condition (Marsden & Tipler 1980)] is automatically satisfied if the cosmological constant is zero. This version of the generic condition is also satisfied in a vacuum space–time with an S^3 or $S^2\times S^1$ Cauchy surface topology. [It is a violation of the hypersurface generic condition that prevents the collapse of the Einstein static universe; see Marsden & Tipler (1980) for a discussion of this point.]

We have restricted attention to cosmologies whose C^2 maximal extensions are globally hyperbolic, but it would have been sufficient to require merely that we restrict attention to the maximal C^2 Cauchy development from a space-like partial Cauchy surface with topology S^3 or $S^2\times S^1$. Thus, in the following statements, by 'globally hyperbolic' we mean the globally hyperbolic portion of such a space–time.

In the light of our findings we propose the following three new conjectures as precise statements to be proved or shown to be false by counterexamples. These new conjectures refine earlier statements in Marsden & Tipler (1980) and Barrow & Tipler (1985, 1986).

Conjecture 1. All globally hyperbolic closed vacuum universes with S^3 or $S^2\times S^1$ spatial topology expand from an all-encompassing initial singularity to a maximal hypersurface and recollapse to an all-encompassing final singularity.

Conjecture 2. All globally hyperbolic spatially homogeneous closed universes with S^3 or $S^2\times S^1$ spatial topology and with stress-energy tensors which obey (1) the strong energy condition, (2) the positive pressure criterion, (3) the dominant energy condition and (4) the matter regularity condition expand from an all-encompassing initial singularity to a maximal hypersurface and recollapse to an all-encompassing final singularity.

Conjecture 3. All globally hyperbolic closed universes with S^3 or $S^2\times S^1$ spatial topology and with stress-energy tensors which obey (1) the strong energy condition and (2) the positive pressure criterion begin in an all-encompassing initial singularity and end in an all-encompassing final singularity. Hence, if all singularities are strong-curvature or crushing, then the universe expands from the all-encompassing initial singularity to a maximal hypersurface and recollapses to the all-encompassing final singularity.

Acknowledgments

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