The Closest Elastic Tensor of Arbitrary Symmetry to an Elasticity Tensor of Lower Symmetry

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Abstract The closest tensors of higher symmetry classes are derived in explicit form for a given elasticity tensor of arbitrary symmetry. The mathematical problem is to minimize the elastic length or distance between the given tensor and the closest elasticity tensor of the specified symmetry. Solutions are presented for three distance functions, with particular attention to the Riemannian and log-Euclidean distances. These yield solutions that are invariant under inversion, i.e., the same whether elastic stiffness or compliance are considered. The Frobenius distance function, which corresponds to common notions of Euclidean length, is not invariant although it is simple to apply using projection operators. A complete description of the Euclidean projection method is presented. The three metrics are considered at a level of detail far greater than heretofore, as we develop the general framework to best fit a given set of moduli onto higher elastic symmetries. The procedures for finding the closest elasticity tensor are illustrated by application to a set of 21 moduli with no underlying symmetry.

Key words elastic symmetry · anisotropy · closest moduli · Reimannian metric · log-Euclidean

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1 Introduction

We address the question of finding the elastic moduli with a given material symmetry closest to an arbitrary set of elastic constants. There are several reasons for reducing a set of elastic constants in this way. One might desire to fit a data set to the *a priori* known symmetry of the material. Alternatively, a seismic simulation might be best understood in terms of a model of the earth as a layered transversely isotropic medium, even though core samples indicate local anisotropy of a lower symmetry. More commonly, one might simply want to reduce the model complexity by decreasing the number of elastic parameters. In each case a distance function measuring the difference between sets of elastic moduli is necessary to define an appropriate closest set. The most natural metric is the Euclidean norm in which the "length" $\|\mathbb{C}\|$ of a set of moduli[1] C_{ijkl} is $\|\mathbb{C}\| = (C_{ijkl}C_{ijkl})^{1/2}$. The Euclidean distance function is, however, not invariant under inversion of the elasticity tensor – one obtains a different result using compliance as compared with stiffness. Alternative distance functions have been proposed which do not have this failing. In particular, the Riemannian distance function of Moakher [1] and the log-Euclidean metric of Arsigny et al. [2] are invariant under inversion, and as such are preferred general elastic norms.

The problem of simplifying elastic moduli by increasing the elastic symmetry was apparently first considered by Gazis et al. [3]. They provided a general framework for defining and deriving the Euclidean projection, i.e., the closest elasticity using the Euclidean norm. Several examples were given, including the closest isotropic material, which agrees with the isotropic approximant derived by Fedorov [4]. Fedorov obtained isotropic moduli using a different criterion: the isotropic material that best approximates the elastic wave velocities of the given moduli by minimizing the difference in the orientation averaged acoustical tensors. It may be shown [5] that the generalization of Fedorov's criterion to other symmetries is satisfied by the Euclidean projection of the stiffness tensor onto the elastic symmetry considered. The Euclidean projection is also equivalent to operating on the given elasticity tensor with the elements of the transformation group of the symmetry in question [6]. This approach has been used by François et al. [7] to find the closest moduli of trigonal and other symmetries for a set of ultrasonically measured stiffnesses. The equivalence of the projection and group transformation averaging will be discussed in Section 5. The Euclidean projection approach has received attention in the geophysical community from modelers interested in fitting rock data to particular elastic symmetries [8–11]. The work of Helbig [9] is particularly comprehensive. He considers the problem both in terms of the 6×6 matrix notation [12, 13] and in terms of 21-dimensional vectors representing the elastic moduli. The latter approach has been developed further by Browaeys and Chevrot [14] who describe the projection operators for different elastic symmetries in the 21-dimensional viewpoint. Dellinger [15] presents an algorithm for finding the closest transversely isotropic medium by searching over all orientations of the symmetry axis. Dellinger et al. [16] proposed a distinct approach to the problem of finding closest elasticity, based on the idea of generalized rotation of the 21-dimensional elasticity vector.

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[1] Lower case Latin suffices take on the values 1, 2, and 3, and the summation convention on repeated indices is assumed.

Although the Euclidean projection is commonly used in applications, it suffers from the fundamental drawback alluded to earlier, i.e., it is not invariant under inversion of the elasticity tensor. The projection found using the elastic stiffness is different from that obtained using the compliance tensor. This fundamental inconsistency arises from the dual physical properties of the elasticity tensor/matrix and its inverse. The projection of one is clearly not the same as the inverse of the projection of the other, although both projections by definition possess the same elastic symmetry. While there are circumstances in which the Euclidean projection is preferred, e.g., for the generalized Fedorov problem of finding the best acoustical approximant of a given symmetry [5], there is a clear need for a consistent technique to define a "closest" elastic material of a given symmetry. The solution to this quandary is to use an elastic distance function that is invariant under inversion. Several have been proposed of which we focus on two, the Riemannian distance function due to Moakher [1] and the log-Euclidean length of Arsigny et al. [2], described in Section 3. Moakher [1] describes how the Riemannian distance function gives a consistent method for averaging elasticity tensors. The only other application so far of these invariant distance functions to elasticity is by Norris [17] who discusses the closest isotropic elasticity.

The outline of the paper is as follows. The three distance functions are introduced in Section 3 after a brief review of notation in Section 2. Section 4 defines the different types of elastic symmetry using the algebraic tensor decomposition of Walpole [19]. Euclidean projection is presented in Section 5, with results for particular elastic symmetries summarized in Appendix A. Before discussing the closest tensors using the logarithmic norms, methods to evaluate the exponential and other functions of elasticity tensors are first described in Section 6. Section 7 discusses the closest tensors using the logarithmic norms, with particular attention given to isotropy and cubic symmetry as the target symmetries. Numerical examples are given in Section 8, and final conclusions in Section 9.

2 Notation

Elasticity tensors relate stress $\mathbf{T}$ and infinitesimal strain $\mathbf{E}$ linearly according to

$$\mathbf{T} = \mathbb{C}\mathbf{E}, \qquad \mathbf{E} = \mathbb{S}\mathbf{T}, \tag{1}$$

where $\mathbb{C}$ and $\mathbb{S}$ denote the fourth-order stiffness and compliance tensors, respectively. They satisfy $\mathbb{C}\mathbb{S} = \mathbb{S}\mathbb{C} = \mathbb{I}$, the identity. Although we are concerned primarily with fourth-order tensors in three-dimensional space, calculation and presentation are sometimes better performed using second-order tensors in six-dimensions. Accordingly we define $\mathcal{S}(d, r)$ as the space of symmetric tensors of order r in $d-$dimensions. Elasticity tensors, denoted by $\mathbb{E}\text{la} \subset \mathcal{S}(3, 4)$, are positive definite, i.e., $\mathbb{A} \in \mathbb{E}\text{la}$ if $\langle \mathbf{B}, \ \mathbb{A}\mathbf{B} \rangle > 0$ for all nonzero $\mathbf{B} \in \mathcal{S}(3, 2)$. Components are defined relative to the basis triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$; thus, $\mathbf{a} = a_j\mathbf{e}_j$, $\mathbf{A} = A_{ij}\,\mathbf{e}_i \otimes \mathbf{e}_j$, and $\mathbb{A} = A_{ijkl}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$, where the summation convention is assumed over $1, 2, 3$ for lower case subscripts. Symmetry of second-order ($r = 2$) tensors implies $A_{ij} = A_{ji}$, while for $\mathbb{A} \in \mathcal{S}(3, 4)$ the elements satisfy

$$A_{ijkl} = A_{jikl} = A_{ijlk}, \qquad A_{ijkl} = A_{klij}. \tag{2}$$

The first pair of identities reflects the symmetry of the stress and the strain, while the last one is a consequence of the assumed existence of a strain energy function, and consequently elasticity tensors have at most 21 independent components.

Throughout the paper lower case Latin, upper case Latin and "ghostscript" indicate, respectively, three-dimensional vectors and tensors of order 2 and 4; e.g., vector $\mathbf{b}$, $\mathbf{A} \in \mathcal{S}(3, 2)$, $\mathbb{A} \in \mathcal{S}(3, 4)$. The basis vectors are assumed orthonormal, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, so that products of tensors are defined by summation over pairs of indices: $(\mathbf{AB})_{ij} = A_{ik} B_{kj}$ and $(\mathbb{A}\mathbb{B})_{ijkl} = A_{ijpq} B_{pqkl}$. The inner product for tensors is defined as

$$\langle u, \ v \rangle = \text{tr}(uv), \tag{3}$$

where $\text{tr}\,\mathbf{A} = A_{ii}$, and for elasticity tensors, $\text{tr}\,\mathbb{A} = A_{ijij}$. The norm of a tensor is

$$\|u\| \equiv \langle u, \ u \rangle^{1/2}. \tag{4}$$

We take advantage of the well known isomorphisms between $\mathcal{S}(3, 2)$ and $\mathcal{S}(6, 1)$, and between $\mathcal{S}(3, 4)$ and $\mathcal{S}(6, 2)$. Thus, fourth-order elasticity tensors in three dimensions are equivalent to second-order symmetric tensor of six dimensions [20], with properties $\mathbb{B}\mathbf{A} \leftrightarrow \widehat{\mathbf{B}}\widehat{\mathbf{a}}$, $\mathbb{A}\mathbb{B} \leftrightarrow \widehat{\mathbf{A}}\widehat{\mathbf{B}}$, and $\langle \mathbb{A}, \mathbb{B} \rangle = \langle \widehat{\mathbf{A}}, \widehat{\mathbf{B}} \rangle$. Vectors and second-order tensors in six dimension are distinguished by a hat, e.g., vector $\widehat{\mathbf{a}}$, $\widehat{\mathbf{A}} \in \mathcal{S}(6, 2)$. Components are defined relative to the orthonormal sextet $\{\widehat{\mathbf{e}}_I, I = 1, 2, \ldots, 6\}$, $\widehat{\mathbf{e}}_I \cdot \widehat{\mathbf{e}}_J = \delta_{IJ}$, by $\widehat{\mathbf{a}} = a_I \widehat{\mathbf{e}}_I$, $\widehat{\mathbf{A}} = \widehat{A}_{IJ}\widehat{\mathbf{e}}_I \otimes \widehat{\mathbf{e}}_J$, with the summation convention over $1, 2, \ldots, 6$ for capital subscripts. Also, $\text{tr}\,\widehat{\mathbf{A}} = \widehat{A}_{II}$. The connection between $\mathcal{S}(6, 1)$ and $\mathcal{S}(3, 2)$ is made concrete by relating the basis vectors:

$$
\begin{array}{ccccc}
\text{6 - vector} & & & & \text{dyadic} \\
\\
\begin{array}{cc} \widehat{\mathbf{e}}_1 & \widehat{\mathbf{e}}_4 \\ \widehat{\mathbf{e}}_2 & \widehat{\mathbf{e}}_5 \\ \widehat{\mathbf{e}}_3 & \widehat{\mathbf{e}}_6 \end{array} & \leftrightarrow & & \begin{array}{cc} \mathbf{e}_1 \otimes \mathbf{e}_1 & \frac{1}{\sqrt{2}}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \\ \mathbf{e}_2 \otimes \mathbf{e}_2 & \frac{1}{\sqrt{2}}(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) \\ \mathbf{e}_3 \otimes \mathbf{e}_3 & \frac{1}{\sqrt{2}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \end{array} & \tag{5}
\end{array}
$$

This implies a unique $\widehat{\mathbf{A}} \in \mathcal{S}(6, 2)$ for each $\mathbb{A} \in \mathcal{S}(3, 4)$, and vice versa. Let $\mathbb{C} \in \mathbb{E}\text{la}$ be the tensor of elastic stiffness, usually defined by the Voigt notation: $C_{ijkl} \equiv c_{IJ}$, where I or $J = 1, 2, 3, 4, 5, 6$ correspond to ij or $kl = 11, 22, 33, 23, 13, 12$, respectively, and c_{IJ} are the elastic moduli in the Voigt notation [21], i.e.,

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\
 & & c_{33} & c_{34} & c_{35} & c_{36} \\
 & & & c_{44} & c_{45} & c_{46} \\
 & S \quad Y \quad M & & & c_{55} & c_{56} \\
 & & & & & c_{66}
\end{pmatrix}. \tag{6}
$$

The isomorphism implies that the associated matrix $\widehat{\mathbf{C}} \in \mathcal{S}(6, 2)$ is positive definite and has elements

$$\widehat{\mathbf{C}} \equiv \begin{pmatrix} c_{11} & c_{12} & c_{13} & 2^{\frac{1}{2}}c_{14} & 2^{\frac{1}{2}}c_{15} & 2^{\frac{1}{2}}c_{16} \\ c_{12} & c_{22} & c_{23} & 2^{\frac{1}{2}}c_{24} & 2^{\frac{1}{2}}c_{25} & 2^{\frac{1}{2}}c_{26} \\ c_{13} & c_{23} & c_{33} & 2^{\frac{1}{2}}c_{34} & 2^{\frac{1}{2}}c_{35} & 2^{\frac{1}{2}}c_{36} \\ 2^{\frac{1}{2}}c_{14} & 2^{\frac{1}{2}}c_{24} & 2^{\frac{1}{2}}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ 2^{\frac{1}{2}}c_{15} & 2^{\frac{1}{2}}c_{25} & 2^{\frac{1}{2}}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ 2^{\frac{1}{2}}c_{16} & 2^{\frac{1}{2}}c_{26} & 2^{\frac{1}{2}}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{pmatrix}. \tag{7}$$

The spectral decomposition of elasticity tensors can be expressed in both the fourth-order and second-order notations [12, 20],

$$\mathbb{C} = \sum_{I=1}^{6} \Lambda_I \mathbf{N}_I \otimes \mathbf{N}_I \quad \leftrightarrow \quad \widehat{\mathbf{C}} = \sum_{I=1}^{6} \Lambda_I \hat{\mathbf{n}}_I \otimes \hat{\mathbf{n}}_I, \tag{8}$$

where Λ_I and $\hat{\mathbf{n}}_I$, $I = 1, 2, \ldots, 6$ are the eigenvalues and eigenvectors of the matrix $\widehat{\mathbf{C}}$, $\mathbf{N}_I$ are the associated dyadics, with $\langle \hat{\mathbf{n}}_I, \hat{\mathbf{n}}_J \rangle = \langle \mathbf{N}_I, \mathbf{N}_J \rangle = \delta_{IJ}$. Also, $\Lambda_I > 0$ by virtue of the positive definite nature of the strain energy.

The elastic moduli $\mathbb{C}$ can also be expressed as a vector with 21 elements [14],

$$\begin{aligned} X &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21})^t \\ &= \Big(c_{11}, c_{22}, c_{33}, \sqrt{2}c_{23}, \sqrt{2}c_{13}, \sqrt{2}c_{12}, 2c_{44}, 2c_{55}, 2c_{66}, 2c_{14}, 2c_{25}, 2c_{36}, \\ &\qquad 2c_{34}, 2c_{15}, 2c_{26}, 2c_{24}, 2c_{35}, 2c_{16}, 2\sqrt{2}c_{56}, 2\sqrt{2}c_{46}, 2\sqrt{2}c_{45} \Big)^t \\ &= \Big(\hat{c}_{11}, \hat{c}_{22}, \hat{c}_{33}, \sqrt{2}\hat{c}_{23}, \sqrt{2}\hat{c}_{13}, \sqrt{2}\hat{c}_{12}, \hat{c}_{44}, \hat{c}_{55}, \hat{c}_{66}, \sqrt{2}\hat{c}_{14}, \sqrt{2}\hat{c}_{25}, \sqrt{2}\hat{c}_{36}, \\ &\qquad \sqrt{2}\hat{c}_{34}, \sqrt{2}\hat{c}_{15}, \sqrt{2}\hat{c}_{26}, \sqrt{2}\hat{c}_{24}, \sqrt{2}\hat{c}_{35}, \sqrt{2}\hat{c}_{16}, \sqrt{2}\hat{c}_{56}, \sqrt{2}\hat{c}_{46}, \sqrt{2}\hat{c}_{45} \Big)^t. \end{aligned} \tag{9}$$

The $\sqrt{2}$'s ensure that the inner product preserves the norm of the elastic moduli, whether it is a tensor, matrix or vector. Thus, the norm of an elasticity tensor can be expressed in various ways depending on how $\mathbb{C}$ is represented,

$$\| \mathbb{C} \|^2 \equiv \langle \mathbb{C}, \ \mathbb{C} \rangle = C_{ijkl} C_{ijkl} = \hat{c}_{IJ} \hat{c}_{IJ} = \Lambda_I \Lambda_I = \| X \|^2. \tag{10}$$

3 Elastic Distance Functions

We consider three metrics for $\mathbb{E}$la: the Euclidean or Frobenius metric d_F, the log-Euclidean norm d_L [2] and the Riemannian metric d_R [1]. They are defined for any pair of elasticity tensors as

$$d_F(\mathbb{C}_1, \mathbb{C}_2) = \| \mathbb{C}_1 - \mathbb{C}_2 \|, \tag{11a}$$

$$d_L(\mathbb{C}_1, \mathbb{C}_2) = \|\text{Log}(\mathbb{C}_1) - \text{Log}(\mathbb{C}_2)\|, \tag{11b}$$

$$d_R(\mathbb{C}_1, \mathbb{C}_2) = \left\| \text{Log}(\mathbb{C}_1^{-1/2} \mathbb{C}_2 \mathbb{C}_1^{-1/2}) \right\|. \tag{11c}$$

The Riemannian distance d_R is related to the exponential map induced by the scalar product on the tangent space to $\mathbb{E}$la at $\mathbb{C}$ [1, 22]. The log-Euclidean metric d_L can be motivated by the following definition of tensor multiplication [2] :

$$\mathbb{C}_1 \odot \mathbb{C}_2 \equiv \exp\left(\text{Log}(\mathbb{C}_1) + \text{Log}(\mathbb{C}_2)\right). \tag{12}$$

The $\odot$ product preserves symmetry and positive definiteness, unlike normal multiplication. The three metrics have the usual attributes of a distance function d:

1. non-negative, $d(\mathbb{C}_1, \mathbb{C}_2) \geq 0$ with equality iff $\mathbb{C}_1 = \mathbb{C}_2$,
2. symmetric in the arguments, $d(\mathbb{C}_1, \mathbb{C}_2) = d(\mathbb{C}_2, \mathbb{C}_1)$,
3. invariant under a change of basis, $d(\mathbb{C}_1', \mathbb{C}_2') = d(\mathbb{C}_1, \mathbb{C}_2)$ for all proper orthogonal coordinate transformations $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \rightarrow \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$, and
4. it satisfies the triangle inequality, $d(\mathbb{C}_1, \mathbb{C}_3) \leq d(\mathbb{C}_1, \mathbb{C}_2) + d(\mathbb{C}_2, \mathbb{C}_3)$.

The Riemannian and log-Euclidean distances possess the additional properties that they are invariant under inversion and (positive) scalar multiplication

$$d_{L,R}(\mathbb{C}_1^{-1}, \mathbb{C}_2^{-1}) = d_{L,R}(\mathbb{C}_1, \mathbb{C}_2), \quad \text{and} \quad d_{L,R}(a\,\mathbb{C}, a\,\mathbb{C}) = d_{L,R}(\mathbb{C}_1, \mathbb{C}_2), \ a > 0. \tag{13}$$

The following inequality between the logarithmic distance functions is a consequence of the metric increasing property of the exponential [23]

$$d_L(\mathbb{C}_1, \mathbb{C}_2) \leq d_R(\mathbb{C}_1, \mathbb{C}_2). \tag{14}$$

We also have (see [23] for the inequality)

$$d_L(\mathbb{C}_1^b, \mathbb{C}_2^b) = |b|\,d_L(\mathbb{C}_1, \mathbb{C}_2), \quad b \in \mathbb{R} \tag{15a}$$

$$d_R(\mathbb{C}_1^b, \mathbb{C}_2^b) \leq b\,d_R(\mathbb{C}_1, \mathbb{C}_2), \qquad b \in [0, 1], \tag{15b}$$

and the Riemannian distance is invariant under congruent transformations, i.e.,

$$d_R(\mathbb{T}\,\mathbb{C}_1\,\mathbb{T}^t, \mathbb{T}\,\mathbb{C}_2\,\mathbb{T}^t) = d_R(\mathbb{C}_1, \mathbb{C}_2), \quad \forall\,\mathbb{T} \text{ invertible.} \tag{16}$$

Distance functions satisfying Eq. (13) are called bi-invariant, a property that makes the Riemannian and log-Euclidean distances consistent metrics for elasticity tensors. The Frobenius norm, not being invariant under inversion, gives different results depending on whether the stiffness or compliance is considered. Other metric functions may be considered. For instance, the Kullback–Leibler metric [1] is invariant under inversion and congruent transformation but does not satisfy the triangle inequality, which we take here as prerequisite for consideration as a distance function.

Each distance function has a geometrical interpretation. For instance, the midpoint between $\mathbb{C}_1$ and $\mathbb{C}_2$ is defined as the unique $\mathbb{C}_3$ such that $d(\mathbb{C}_1, \mathbb{C}_3) = d(\mathbb{C}_3, \mathbb{C}_2) = \frac{1}{2}d(\mathbb{C}_1, \mathbb{C}_2)$. Using d_F, the midpoint is the vector halfway between the $21-$vectors X_1 and X_2 as defined by Eq. (9), i.e., $\mathbb{C}_3 = \frac{1}{2}(\mathbb{C}_1 + \mathbb{C}_2)$. The midpoint using d_L is $\mathbb{C}_3 = (\mathbb{C}_1 \odot \mathbb{C}_2)^{1/2}$. The midpoint with d_R is [24] $\mathbb{C}_3 = \mathbb{C}_1(\mathbb{C}_1^{-1}\mathbb{C}_2)^{1/2}$ which can be expressed in several other ways, see [1]. More generally, the midpoint is the value of the geodesic $\mathbb{C}(t)$ at $t = \frac{1}{2}$, where

$$\mathbb{C}(t) = \begin{cases} (1-t)\,\mathbb{C}_1 + t\,\mathbb{C}_2, & \text{Frobenius,} \\ \exp\left((1-t)\,\mathrm{Log}(\mathbb{C}_1) + t\,\mathrm{Log}(\mathbb{C}_2)\right), & \text{log-Euclidean,} \\ \mathbb{C}_1\,\exp\left(t\,\mathrm{Log}(\mathbb{C}_1^{-1}\mathbb{C}_2)\right), & \text{Riemannian,} \end{cases} \qquad 0 \le t \le 1. \quad (17)$$

4 Elasticity Tensors of the Different Symmetry Classes

Elasticity tensors for the different symmetry classes are described in this section. The choice of representation of $\mathbb{E}$la is important for discussing the closest approximants, particularly for the bi-invariant metrics. It should preferably be independent of the coordinate system, although at the same time, practical application is normally in terms of the Voigt notation, so the format should not deviate too far from this. We begin with a review of alternative representations for elastic moduli.

4.1 Representation of Elasticity Tensors

There are many representations of $\mathbb{E}$la in addition to the standard Voigt notation of Eq. (6). The 21-dimensional vector format [14] is useful for some applications, including the Euclidean projection, as we show later. However, these two representations depend upon the coordinate system. Among the coordinate-free forms of $\mathbb{E}$la that can be identified in the literature we distinguish (a) spectral decomposition, (b) algebraic decomposition, (c) groups and reflection symmetries, (d) harmonic decomposition, (e) integrity bases.

The first, (a), spectral decomposition dates back to Kelvin [25]. The idea is simple: the elasticity tensor operates on the six-dimensional space of symmetric second-order tensors, and therefore has a six-dimensional spectral form. The associated eigenvalues are called the Kelvin moduli [12, 13]. Recent related developments are due to Rychlewski [12] and Mehrabadi and Cowin [13, 20], who independently rediscovered and extended Kelvin's approach, see also [26, 27]. The associated six-dimensional tensor representation will be used here for practical implementation. (b) tensor functions operating on $\mathbb{E}$la can be greatly simplified using the irreducible tensor algebra proposed by Walpole [19], and independently by Kunin [28]. This procedure provides the most efficient means of representing elastic tensors of a given symmetry class, especially those of high symmetry. We adopt this method to develop most of the results here. (c) the group of rotations associated with elastic symmetry provides an irreducible representation [29]. There are various related ways of considering elasticity tensors in terms of rotational group properties of tensors, e.g., based on Cartan decomposition [30], complex vectors and tensors [31, 32], and subgroups of $O(3)$ [33, 34]. These ideas are closely related to definitions of elastic symmetry in terms of a single symmetry element: reflection about a plane. The necessary algebra

and the relationship to the more conventional crystallographic symmetry elements are described in detail by Cowin and Mehrabadi [35]. Comparison between the rotation-based approach and that using symmetry planes is provided by Chadwick et al. [18]. (d) Backus [36] proposed a representation of $\mathbb{E}$la in terms of harmonic tensors. These are based on an isomorphism between the space of homogeneous harmonic polynomials of degree q and the space of totally symmetric tensors of order q. There has been considerable interest in Backus's representation [6, 37–39]. For instance, Baerheim and Helbig [40] provide an orthonormal decomposition of $\mathbb{E}$la in terms of harmonic tensors. (e) elasticity decomposition via integrity bases has been studied considerably [41–45]. An integrity basis is a set of polynomials, each invariant under the group of symmetry transformations, such that any polynomial function invariant under the group is expressible as a polynomial in elements of the integrity basis. For instance, Tu [42] used an integrity basis to construct five hierarchies of orthonormal tensor bases which span the space of elastic constants of all crystal systems. Any elastic tensor of order four possessing certain symmetry may be decomposed into a sum of tensors of increasing symmetry. The resulting decomposition has considerable similarity to the decomposition generated in Section 5 using Walpole's irreducible elements with Euclidean projection. Among other methods for decomposing elasticity tensors, we mention the scheme of Elata and Rubin [46] who use a set of six vectors related to a regular icosahedron to form a basis for $\mathbb{E}$la. This basis naturally splits a tensor into deviatoric and non-deviatoric parts.

We use Walpole's [19] algebraic representation as it provides a consistent and straightforward means to define projections of fourth-order tensors onto the given elastic symmetry. This representation is better for our purposes than the spectral decomposition, since it is independent of the elastic moduli (i.e., no distributors [13]) and depends only on the crystallographic orientation. It uses the notion of basis tensors, similar to but not the same as for a vector space, which makes it very suitable for Euclidean projection. We work mainly with fourth-order tensors directly, although the associated six-dimensional matrix notation is also provided. The latter is simpler for purposes of computation, e.g., many of the matrix operations are easily implemented using MATLAB. We begin with the highest elastic symmetry.

4.2 Isotropic System

A general isotropic fourth-order tensor is given by

$$\mathbb{A} = a\,\mathbb{J} + b\,\mathbb{K}, \tag{18}$$

where $\mathbb{J}$ and $\mathbb{K}$ are the two linearly independent symmetric tensors defined by

$$\mathbb{J} = \frac{1}{3}\mathbf{I} \otimes \mathbf{I}, \qquad \mathbb{K} = \mathbb{I} - \mathbb{J}. \tag{19}$$

The component forms follow from $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and $J_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$. Note that the tensors $\mathbb{J}$ and $\mathbb{K}$ sum to the identity,

$$\mathbb{I} = \mathbb{J} + \mathbb{K}, \tag{20}$$

and they satisfy the multiplication table

$$\mathbb{J}^2 = \mathbb{J}, \quad \mathbb{K}^2 = \mathbb{K}, \quad \mathbb{J}\mathbb{K} = \mathbb{K}\mathbb{J} = \mathbb{O}. \tag{21}$$

The Euclidean lengths are $\|\mathbb{J}\| = 1$, $\|\mathbb{K}\| = \sqrt{5}$. Tensors $\mathbb{J}$ and $\mathbb{K}$ form, respectively, one and five-dimensional Kelvin subspaces [13]. An important result (see [3, 4]) is that the closest isotropic elasticity tensor in the Euclidean sense is

$$\mathbb{C}_{\mathrm{iso}} = 3\kappa \, \mathbb{J} + 2\mu \, \mathbb{K}, \tag{22}$$

where

$$9\kappa = 3\langle \mathbb{C}, \, \mathbb{J} \rangle = \hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} + 2\hat{c}_{12} + 2\hat{c}_{13} + 2\hat{c}_{23},$$
$$30\mu = 3\langle \mathbb{C}, \, \mathbb{K} \rangle = 2(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} - \hat{c}_{12} - \hat{c}_{23} - \hat{c}_{31}) + 3(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66}). \tag{23}$$

Gazis et al. [3] obtained Eq. (22) using methods similar to those we will generalize in Section 5. Fedorov actually found κ and μ using an apparently different approach – the minimization of the mean square difference in the acoustical or Christoffel matrices for the original and isotropic systems. However, it can be shown [5] that the generalization of Fedorov's approach to other symmetries is identical to the Euclidean minimization. The moduli (23) will be derived later in the context of the general theory for Euclidean projection.

4.3 Cubic System

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three mutually orthogonal unit vectors that describe the three crystallographic directions of a cubic medium. We introduce the second-order tensors

$$\mathbf{U} = \frac{1}{\sqrt{2}}(\mathbf{a} \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{a}), \quad \mathbf{V} = \frac{1}{\sqrt{2}}(\mathbf{b} \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{b}), \quad \mathbf{W} = \frac{1}{\sqrt{2}}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}),$$
$$\mathbf{X} = \frac{1}{\sqrt{2}}(\mathbf{c} \otimes \mathbf{c} - \mathbf{a} \otimes \mathbf{a}), \quad \mathbf{Y} = \frac{1}{\sqrt{2}}(\mathbf{b} \otimes \mathbf{b} - \mathbf{c} \otimes \mathbf{c}), \quad \mathbf{Z} = \frac{1}{\sqrt{2}}(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}), \tag{24}$$

and fourth-order tensors $\mathbb{L}$ and $\mathbb{M}$ defined by them,

$$\mathbb{L} = \mathbf{U} \otimes \mathbf{U} + \mathbf{V} \otimes \mathbf{V} + \mathbf{W} \otimes \mathbf{W}, \qquad \mathbb{M} = \frac{2}{3}(\mathbf{X} \otimes \mathbf{X} + \mathbf{Y} \otimes \mathbf{Y} + \mathbf{Z} \otimes \mathbf{Z}). \tag{25}$$

The tensors $\mathbb{J}$, $\mathbb{L}$ and $\mathbb{M}$ sum up to the identity tensor, and $\mathbb{L}$ and $\mathbb{M}$ partition $\mathbb{K}$, i.e.,

$$\mathbb{I} = \mathbb{J} + \mathbb{L} + \mathbb{M}, \qquad \mathbb{K} = \mathbb{L} + \mathbb{M}. \tag{26}$$

The multiplication table for cubic tensors is

$$\mathbb{J}^2 = \mathbb{J}, \quad \mathbb{L}^2 = \mathbb{L}, \quad \mathbb{M}^2 = \mathbb{M}, \quad \mathbb{J}\mathbb{L} = \mathbb{L}\mathbb{J} = \mathbb{J}\mathbb{M} = \mathbb{M}\mathbb{J} = \mathbb{L}\mathbb{M} = \mathbb{M}\mathbb{L} = \mathbb{O}, \tag{27}$$

and the Euclidean lengths are $\|\mathbb{L}\| = \sqrt{3}$, $\|\mathbb{M}\| = \sqrt{2}$.

A general symmetric fourth-order tensor for a cubic media is given by a linear combination of the three linearly independent tensors $\mathbb{J}$, $\mathbb{L}$ and $\mathbb{M}$

$$\mathbb{A} = a\mathbb{J} + b\mathbb{L} + c\mathbb{M}. \tag{28}$$

The tensors $\mathbb{J}$, $\mathbb{L}$ and $\mathbb{M}$ form, respectively, one, three and two-dimensional Kelvin subspaces [13].

4.4 Transversely Isotropic System

Assume that the unit vector **c** characterizes the preferred direction of a transversely isotropic medium. Let **P** and **Q** be the second-order tensors

$$\mathbf{P} = \mathbf{c} \otimes \mathbf{c}, \quad \mathbf{Q} = \frac{1}{\sqrt{2}}(\mathbf{I} - \mathbf{c} \otimes \mathbf{c}), \tag{29}$$

and define the six linearly independent fourth-order elementary tensors

$$\mathbb{E}_1 = \mathbf{P} \otimes \mathbf{P}, \quad \mathbb{E}_2 = \mathbf{Q} \otimes \mathbf{Q}, \quad \mathbb{E}_3 = \mathbf{P} \otimes \mathbf{Q}, \quad \mathbb{E}_4 = \mathbf{Q} \otimes \mathbf{P},$$

$$\mathbb{F} = \mathbf{W} \otimes \mathbf{W} + \mathbf{Z} \otimes \mathbf{Z}, \qquad \mathbb{G} = \mathbf{U} \otimes \mathbf{U} + \mathbf{V} \otimes \mathbf{V}. \tag{30}$$

Both $\mathbb{E}_1$ and $\mathbb{E}_2$ are symmetric but the tensors $\mathbb{E}_3$ and $\mathbb{E}_4$ are not, although they are the transpose of one another and so their sum is a symmetric tensor. These asymmetric tensors are introduced because the set $\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 + \mathbb{E}_4\}$ is not closed under tensor multiplication. The multiplication table for the six elementary tensors is

$$
\begin{array}{c|cccccc}
 & \mathbb{E}_1 & \mathbb{E}_2 & \mathbb{E}_3 & \mathbb{E}_4 & \mathbb{F} & \mathbb{G} \\
\hline
\mathbb{E}_1 & \mathbb{E}_1 & \mathbb{O} & \mathbb{E}_3 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_2 & \mathbb{O} & \mathbb{E}_2 & \mathbb{O} & \mathbb{E}_4 & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_3 & \mathbb{O} & \mathbb{E}_3 & \mathbb{O} & \mathbb{E}_1 & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_4 & \mathbb{E}_4 & \mathbb{O} & \mathbb{E}_2 & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{F} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{F} & \mathbb{O} \\
\mathbb{G} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{G}
\end{array}
\tag{31}
$$

The tensors $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{F}$ and $\mathbb{G}$ sum up to the identity,

$$\mathbb{I} = \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{F} + \mathbb{G}. \tag{32}$$

The Euclidean lengths are $\|\mathbb{E}_1\| = \|\mathbb{E}_2\| = 1$, $\|\mathbb{E}_3 + \mathbb{E}_4\| = \|\mathbb{F}\| = \|\mathbb{G}\| = \sqrt{2}$.

A general symmetric fourth-order tensor of transversely isotropic symmetry is given by the five-parameter linear combination

$$\mathbb{A} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + f\,\mathbb{F} + g\,\mathbb{G}. \tag{33}$$

Note that the base tensors do not correspond to the Kelvin modes, although some can be identified as such, e.g., $\mathbb{G}$ is a two-dimensional Kelvin subspace [13]. The three bases tensors $\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3 + \mathbb{E}_4\}$ together define a two-dimensional subspace.

4.5 Tetragonal System

Nine base tensors are required: $\{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4, \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{G}\}$, where the new tensors are a symmetric pair $\mathbb{F}_1$, $\mathbb{F}_2$, and a pair that are mutual transposes, $\mathbb{F}_3$, $\mathbb{F}_4$,

$$\mathbb{F}_1 = \mathbf{W} \otimes \mathbf{W}, \qquad \mathbb{F}_2 = \mathbf{Z} \otimes \mathbf{Z}, \qquad \mathbb{F}_3 = \mathbf{W} \otimes \mathbf{Z}, \qquad \mathbb{F}_4 = \mathbf{Z} \otimes \mathbf{W}. \tag{34}$$

In comparison with the transversely isotropic system, $\mathbb{F}$ is replaced by the symmetric pair of tensors, i.e., $\mathbb{F} = \mathbb{F}_1 + \mathbb{F}_2$, and the decomposition, of the identity is

$$\mathbb{I} = \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{F}_1 + \mathbb{F}_2 + \mathbb{G}. \tag{35}$$

The multiplication table of the nine elementary tensors is

$$
\begin{array}{c|ccccccccc}
 & \mathbb{E}_1 & \mathbb{E}_2 & \mathbb{E}_3 & \mathbb{E}_4 & \mathbb{F}_1 & \mathbb{F}_2 & \mathbb{F}_3 & \mathbb{F}_4 & \mathbb{G} \\
\hline
\mathbb{E}_1 & \mathbb{E}_1 & \mathbb{O} & \mathbb{E}_3 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_2 & \mathbb{O} & \mathbb{E}_2 & \mathbb{O} & \mathbb{E}_4 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_3 & \mathbb{O} & \mathbb{E}_3 & \mathbb{O} & \mathbb{E}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{E}_4 & \mathbb{E}_4 & \mathbb{O} & \mathbb{E}_2 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\
\mathbb{F}_1 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{F}_1 & \mathbb{O} & \mathbb{F}_3 & \mathbb{O} & \mathbb{O} \\
\mathbb{F}_2 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{F}_2 & \mathbb{O} & \mathbb{F}_4 & \mathbb{O} \\
\mathbb{F}_3 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{F}_3 & \mathbb{O} & \mathbb{F}_1 & \mathbb{O} \\
\mathbb{F}_4 & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{F}_4 & \mathbb{O} & \mathbb{F}_2 & \mathbb{O} & \mathbb{O} \\
\mathbb{G} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{G}
\end{array}
\tag{36}
$$

Under multiplication the $\mathbb{E}_i$'s and $\mathbb{F}_i$'s decouple from one another and from $\mathbb{G}$. Furthermore, the algebra of the $\mathbb{F}_i$'s is similar to that of the $\mathbb{E}_i$'s. The Euclidean lengths of the new tensors are $\|\mathbb{F}_1\| = \|\mathbb{F}_2\| = 1$, $\|\mathbb{F}_3 + \mathbb{F}_4\| = \sqrt{2}$.

A general symmetric fourth-order tensor of tetragonal symmetry is given by the six-parameter combination

$$
\mathbb{A} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{F}_1 + q\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4) + g\,\mathbb{G}. \tag{37}
$$

Tetragonal symmetry is often represented by six rather than seven independent parameters. Fedorov [4] pointed out that transformation by rotation about the **c** axis by angle θ yields zero for the transformed coefficient r if (see Eq. (9.7) in [4])

$$
\tan 4\theta = r/(q - p). \tag{38}
$$

The transformation (38) depends on knowledge of the coefficients. However, it is assumed here that the only properties of the tetragonal symmetry available *a priori* are the orthogonal crystal axes **a**, **b** and **c**. The reduction from seven to six parameters can be achieved after the effective tetragonal material is found, but the axes of the final six-parameter material depend upon the initial moduli. Hence, it is important to retain the seven-parameter representation (37).

4.6 Trigonal System

Trigonal symmetry is characterized by three planes of reflection symmetry with normals coplanar and at 120° to one another. This symmetry class is also known as hexagonal [19] but we refer to it as trigonal since hexagonal symmetry is often used synonymously with transverse isotropy. It is advantageous to choose non-orthogonal basis vectors $\mathbf{a}'$, $\mathbf{b}'$, **c** with $\mathbf{a}'$ and $\mathbf{b}'$ normals to two of the planes and **c** perpendicular to them (the pair $\mathbf{a}'$ and $\mathbf{b}'$ may also be represented in terms of the orthonormal vectors via, for instance, $\mathbf{a}' = \mathbf{a}$, $\mathbf{b}' = -\frac{1}{2}\mathbf{a} + \frac{\sqrt{3}}{2}\mathbf{b}$). In order to describe symmetric fourth-order tensors of trigonal symmetry, following [19] we introduce the second-order tensors

$$
\mathbf{S} = \sqrt{\frac{2}{3}}(\mathbf{a}' \otimes \mathbf{b}' + \mathbf{b}' \otimes \mathbf{a}' + \mathbf{a}' \otimes \mathbf{a}'), \quad \mathbf{T} = \sqrt{\frac{2}{3}}(\mathbf{a}' \otimes \mathbf{b}' + \mathbf{b}' \otimes \mathbf{a}' + \mathbf{b}' \otimes \mathbf{b}'), \tag{39a}
$$

$$
\mathbf{U}' = \frac{1}{\sqrt{2}}(\mathbf{a}' \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{a}'), \qquad \mathbf{V}' = -\frac{1}{\sqrt{2}}(\mathbf{b}' \otimes \mathbf{c} + \mathbf{c} \otimes \mathbf{b}'), \tag{39b}
$$

the two symmetric fourth-order tensors

$$\mathbb{R}_1 = \frac{4}{3}\left(\mathbf{S}\otimes\mathbf{S} + \mathbf{T}\otimes\mathbf{T} - \frac{1}{2}\mathbf{S}\otimes\mathbf{T} - \frac{1}{2}\mathbf{T}\otimes\mathbf{S}\right), \tag{40a}$$

$$\mathbb{R}_2 = \frac{4}{3}\left(\mathbf{U}'\otimes\mathbf{U}' + \mathbf{V}'\otimes\mathbf{V}' - \frac{1}{2}\mathbf{U}'\otimes\mathbf{V}' - \frac{1}{2}\mathbf{V}'\otimes\mathbf{U}'\right), \tag{40b}$$

and two pairs of mutually transpose tensors:

$$\mathbb{R}_3 = \frac{4}{3}(\mathbf{S}\otimes\mathbf{U}' + \mathbf{T}\otimes\mathbf{V}' - \frac{1}{2}\mathbf{T}\otimes\mathbf{U}' - \frac{1}{2}\mathbf{S}\otimes\mathbf{V}'), \tag{41a}$$

$$\mathbb{R}_4 = \frac{4}{3}(\mathbf{U}'\otimes\mathbf{S} + \mathbf{V}'\otimes\mathbf{T} - \frac{1}{2}\mathbf{U}'\otimes\mathbf{T} - \frac{1}{2}\mathbf{V}'\otimes\mathbf{S}), \tag{41b}$$

and

$$\mathbb{R}_5 = \frac{2}{\sqrt{3}}(\mathbf{S}\otimes\mathbf{V}' - \mathbf{T}\otimes\mathbf{U}'), \quad \mathbb{R}_6 = \frac{2}{\sqrt{3}}(\mathbf{V}'\otimes\mathbf{S} - \mathbf{U}'\otimes\mathbf{T}). \tag{41c}$$

To make the $\mathbb{R}_i$'s closed under multiplication, we need to further introduce two skew-symmetric tensors

$$\mathbb{R}_7 = \frac{2}{\sqrt{3}}(\mathbf{T}\otimes\mathbf{S} - \mathbf{S}\otimes\mathbf{T}), \quad \mathbb{R}_8 = \frac{2}{\sqrt{3}}(\mathbf{U}'\otimes\mathbf{V}' - \mathbf{V}'\otimes\mathbf{U}'). \tag{42}$$

Thus, $\mathbb{E}_i\mathbb{R}_j = \mathbb{R}_j\mathbb{E}_i = \mathbb{O}$ for $i = 1, 2, \ldots, 4$ and $j = 1, 2, \ldots, 8$, and the multiplication table is[2]

$$\begin{array}{c|cccccccc}
 & \mathbb{R}_1 & \mathbb{R}_2 & \mathbb{R}_3 & \mathbb{R}_4 & \mathbb{R}_5 & \mathbb{R}_6 & \mathbb{R}_7 & \mathbb{R}_8 \\
\hline
\mathbb{R}_1 & \mathbb{R}_1 & \mathbb{O} & \mathbb{R}_3 & \mathbb{O} & \mathbb{R}_5 & \mathbb{O} & \mathbb{R}_7 & \mathbb{O} \\
\mathbb{R}_2 & \mathbb{O} & \mathbb{R}_2 & \mathbb{O} & \mathbb{R}_4 & \mathbb{O} & \mathbb{R}_6 & \mathbb{O} & \mathbb{R}_8 \\
\mathbb{R}_3 & \mathbb{O} & \mathbb{R}_3 & \mathbb{O} & \mathbb{R}_1 & \mathbb{O} & \mathbb{R}_7 & \mathbb{O} & \mathbb{R}_5 \\
\mathbb{R}_4 & \mathbb{R}_4 & \mathbb{O} & \mathbb{R}_2 & \mathbb{O} & \mathbb{R}_8 & \mathbb{O} & \mathbb{R}_6 & \mathbb{O} \\
\mathbb{R}_5 & \mathbb{O} & \mathbb{R}_5 & \mathbb{O} & -\mathbb{R}_7 & \mathbb{O} & \mathbb{R}_1 & \mathbb{O} & -\mathbb{R}_3 \\
\mathbb{R}_6 & \mathbb{R}_6 & \mathbb{O} & -\mathbb{R}_8 & \mathbb{O} & \mathbb{R}_2 & \mathbb{O} & -\mathbb{R}_4 & \mathbb{O} \\
\mathbb{R}_7 & \mathbb{R}_7 & \mathbb{O} & -\mathbb{R}_5 & \mathbb{O} & \mathbb{R}_3 & \mathbb{O} & -\mathbb{R}_1 & \mathbb{O} \\
\mathbb{R}_8 & \mathbb{O} & \mathbb{R}_8 & \mathbb{O} & -\mathbb{R}_6 & \mathbb{O} & \mathbb{R}_4 & \mathbb{O} & -\mathbb{R}_2 \\
\end{array} \tag{43}$$

The algebra of the $\mathbb{R}_i$'s is equivalent to that of 2×2 complex matrices [19]. We note that $\mathbb{R}_1 = \mathbb{F}$ and $\mathbb{R}_2 = \mathbb{G}$, and hence the decomposition for the identity tensor is

$$\mathbb{I} = \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{R}_1 + \mathbb{R}_2. \tag{44}$$

The Euclidean lengths of the new tensors are $\|\mathbb{R}_1\| = \|\mathbb{R}_2\| = \sqrt{2}$, $\|\mathbb{R}_3 + \mathbb{R}_4\| = \|\mathbb{R}_5 + \mathbb{R}_6\| = 2$.

A general symmetric fourth-order tensor of trigonal symmetry is given by the linear combination

$$\mathbb{A} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{R}_1 + q\,\mathbb{R}_2 + r(\mathbb{R}_3 + \mathbb{R}_4) + s(\mathbb{R}_5 + \mathbb{R}_6). \tag{45}$$

---

[2] The multiplication table in [19] contains some typographical errors; specifically the elements (3,8), (4,5), (5,8), (6,3), (8,4) and (8,6) in the original need to be multiplied by -1.

4.7 Rhombic, Monoclinic and Triclinic Systems

The algebra for orthorhombic (equivalently orthotropic) symmetry is the same as that of 3×3 real matrices plus three real numbers. For monoclinic symmetry it is the same as the algebra of pairs of 4×4 and 2×2 real matrices. The lowest symmetry, triclinic or no symmetry, has algebra the same as that of 6×6 real matrices. These low symmetries are probably better defined in terms of their group properties, which also serves as efficient means for Euclidean projection, as described in Section 5.

4.8 Six-dimensional Representation

An equivalent six-dimensional matrix format is described for the various elementary tensors. For each symmetry class the six-dimensional tensors possess the same algebraic properties as the fourth-order tensors, with the same multiplication tables and Euclidean lengths.

The six-dimensional vectors $\{\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}\}$ represent the second-order tensors $\{\mathbf{a} \otimes \mathbf{a}, \mathbf{b} \otimes \mathbf{b}, \mathbf{c} \otimes \mathbf{c}\}$ that correspond to the axes $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, where according to Eq. (5),

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \Rightarrow \hat{\mathbf{a}} = \left(a_1^2,\ a_2^2,\ a_3^2,\ \sqrt{2}a_2a_3,\ \sqrt{2}a_3a_1,\ \sqrt{2}a_1a_2\right)^t, \quad \text{etc.} \quad (46)$$

The associated 6×6 base matrices are given next.

4.8.1 Isotropic System

The six-dimensional analogs of $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ are $\widehat{\mathbf{I}}$, the 6×6 identity matrix, and $\widehat{\mathbf{J}}$, $\widehat{\mathbf{K}}$, where

$$\widehat{\mathbf{J}} = \frac{1}{3}(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)(\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3)^t, \qquad \widehat{\mathbf{K}} = \widehat{\mathbf{I}} - \widehat{\mathbf{J}}. \quad (47)$$

4.8.2 Cubic System

The analogs of $\mathbb{L}$ and $\mathbb{M}$ are

$$\widehat{\mathbf{L}} = \widehat{\mathbf{I}} - \hat{\mathbf{a}}\hat{\mathbf{a}}^t - \hat{\mathbf{b}}\hat{\mathbf{b}}^t - \hat{\mathbf{c}}\hat{\mathbf{c}}^t, \qquad \widehat{\mathbf{M}} = \widehat{\mathbf{K}} - \widehat{\mathbf{L}}. \quad (48)$$

4.8.3 Transversely Isotropic System

Define the six-vectors

$$\hat{\mathbf{p}} = \hat{\mathbf{c}}, \quad \hat{\mathbf{q}} = \tfrac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}}), \quad \hat{\mathbf{z}} = \tfrac{1}{\sqrt{2}}(\hat{\mathbf{a}} - \hat{\mathbf{b}}),$$

$$\hat{\mathbf{w}} = \left(\sqrt{2}a_1b_1,\ \sqrt{2}a_2b_2,\ a\sqrt{2}_3b_3,\ (a_2b_3 + a_3b_2),\ (a_3b_1 + a_1b_3),\ (a_1b_2 + a_2b_1)\right)^t, \quad (49)$$

then the six-dimensional matrices for the tensors $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathbb{E}_4$, $\mathbb{F}$ and $\mathbb{G}$ are

$$\widehat{\mathbf{E}}_1 = \hat{\mathbf{p}}\hat{\mathbf{p}}^t, \quad \widehat{\mathbf{E}}_2 = \hat{\mathbf{q}}\hat{\mathbf{q}}^t, \quad \widehat{\mathbf{E}}_3 = \hat{\mathbf{p}}\hat{\mathbf{q}}^t, \quad \widehat{\mathbf{E}}_4 = \hat{\mathbf{q}}\hat{\mathbf{p}}^t,$$

$$\widehat{\mathbf{F}} = \hat{\mathbf{w}}\hat{\mathbf{w}}^t + \hat{\mathbf{z}}\hat{\mathbf{z}}^t, \quad \widehat{\mathbf{G}} = \widehat{\mathbf{L}} - \hat{\mathbf{w}}\hat{\mathbf{w}}^t. \quad (50)$$

4.8.4 Tetragonal System

The six-dimensional representation is as for TI, with the addition

$$\widehat{\mathbf{F}}_1 = \hat{\mathbf{w}}\hat{\mathbf{w}}^t, \qquad \widehat{\mathbf{F}}_2 = \hat{\mathbf{z}}\hat{\mathbf{z}}^t \qquad \widehat{\mathbf{F}}_3 = \hat{\mathbf{w}}\hat{\mathbf{z}}^t, \qquad \widehat{\mathbf{F}}_4 = \hat{\mathbf{z}}\hat{\mathbf{w}}^t. \tag{51}$$

The trigonal system can be treated in the same manner using equations similar to Eq. (46) for $\hat{\mathbf{a}}$ and Eq. (49) for $\hat{\mathbf{w}}$, but the details are omitted for brevity.

5 Euclidean Projection

Euclidean projection is an essential ingredient for determining the closest tensors in both the Euclidean and the log-Euclidean metrics. It is defined in abstract terms in the next subsection, with the remainder of the section devoted to explicit procedures for the projection.

5.1 Definition of the Projection

We wish to find the tensor $\mathbb{C}_{sym}$ of a specific symmetry class which minimizes the Euclidean distance $\| \mathbb{C} - \mathbb{C}_{sym} \|$ of an elasticity tensor of arbitrary symmetry, $\mathbb{C}$, from the particular symmetry. The solution is a Euclidean decomposition

$$\mathbb{C} = \mathbb{C}_{sym} + \mathbb{C}_{\perp sym}, \tag{52}$$

where $\mathbb{C}_{sym}$ possesses the symmetries appropriate to the symmetry class considered. The complement, or residue [3], is orthogonal to $\mathbb{C}_{sym}$,

$$\langle \mathbb{C}_{sym}, \mathbb{C}_{\perp sym} \rangle = 0, \tag{53}$$

and hence

$$\| \mathbb{C} \|^2 = \| \mathbb{C}_{sym} \|^2 + \| \mathbb{C}_{\perp sym} \|^2, \tag{54a}$$

$$\| \mathbb{C} - \mathbb{C}_{sym} \|^2 = \| \mathbb{C}_{\perp sym} \|^2. \tag{54b}$$

We illustrate the recursive nature of the Euclidean projection for the special case of isotropy. Any elasticity may be partitioned into orthogonal components as

$$\begin{aligned}
\mathbb{C} &= \mathbb{C}_{iso} + \mathbb{C}_{\perp iso} \\
&= \mathbb{C}_{iso} + \mathbb{C}_{cub/iso} + \mathbb{C}_{tet/cub} + \mathbb{C}_{ort/tet} + \mathbb{C}_{mon/ort} + \mathbb{C}_{\perp mon},
\end{aligned} \tag{55}$$

where

$$\mathbb{C}_{sym\,B/A} \equiv \mathbb{C}_{sym\,B} - \mathbb{C}_{sym\,A}, \tag{56}$$

with sym A of higher symmetry than sym B. Eq. (55) partitions $\mathbb{C}$ using the path from no symmetry to isotropy via cubic symmetry, see Figure 1. An alternative decomposition via hexagonal, gives

$$\mathbb{C} = \mathbb{C}_{iso} + \mathbb{C}_{hex/iso} + \mathbb{C}_{tet/hex} + \mathbb{C}_{ort/tet} + \mathbb{C}_{mon/ort} + \mathbb{C}_{\perp mon}. \tag{57}$$

The different paths in Figure 1 were identified by Gazis et al. [3] and discussed more recently in greater detail by Chadwick et al. [18]. The idea of sequential

Figure 1 The sequence of increasing elastic symmetries, from triclinic to isotropic. The number of independent elastic constants are listed. The *dashed boxes* for trigonal 6 and tetragonal 6 indicate that these are obtained from the lower symmetries by rotation, and do not represent new elastic symmetries. The true symmetries are the eight in *solid boxes* [18].
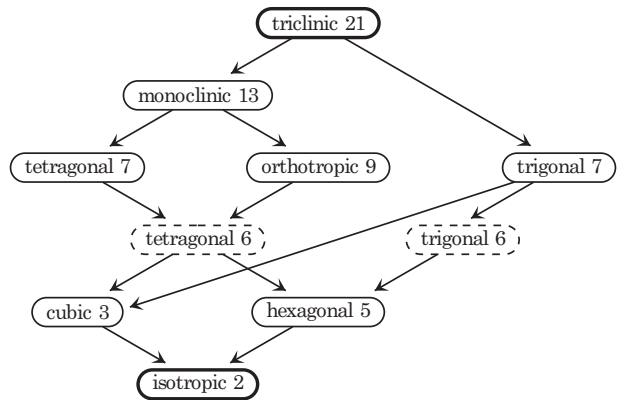
decomposition in the Euclidean norm is not new, e.g., [14, 42], but the explicit projection and complement have not been represented previously.

Details of the recursive Euclidean partition of $\mathbb{C}$ are given in the Appendix. For instance, Eq. (A.30) represents $\|\mathbb{C}\|^2$ using three different routes from triclinic to isotropic. In particular, the length of the isotropic projection is given by

$$\|\mathbb{C}_{\mathrm{iso}}\|^2 = 9\kappa^2 + 20\mu^2, \tag{58}$$

where κ and μ are defined in Eq. (23), and the distance of $\mathbb{C}$ from isotropy is given by

$$\|\mathbb{C}_{\perp\mathrm{iso}}\|^2 = \|\mathbb{C}\|^2 - 9\kappa^2 - 20\mu^2. \tag{59}$$

It appears that Fedorov [4] was the first to show that the distance from isotropy is minimized if the isotropic tensor is as given in Eq. (22).

The remainder of this section details three alternative projection methods using basis tensors, using symmetry planes or groups, and using 21-dimensional vectors. The projection operators are valid for all symmetries, but the method based on symmetry planes becomes more complicated for the higher symmetries. Conversely, the projection operator using basis tensors becomes simpler at the higher symmetries. The 21-dimensional vector procedure is straightforward but is coordinate dependent.

5.2 Projection Using a Tensor Basis

We assume that the set of tensors $\{\mathbb{V}_i \in \mathbb{E}\mathrm{la}\}$ form a linearly independent basis for the symmetry sym in the sense that any elasticity tensor of that symmetry may be expressed uniquely in terms of N linearly independent tensors $\mathbb{V}_1, \mathbb{V}_2, \ldots \mathbb{V}_N$, where $2 \leq N \leq 13$ is the dimension of the space for the material symmetry. We have seen the explicit form of several sets of basis tensors in Section 4, e.g., $N = 2$ for isotropic elasticity. $N = 13$ corresponds to monoclinic, which is the lowest symmetry apart from triclinic (technically $N = 21$) which is no symmetry. The N elements of the orthogonal basis are assumed to be independent of the elasticity itself, i.e., they do not require "elasticity distributors", which are necessary for the spectral decomposition of $\mathbb{E}\mathrm{la}$ [13]. They depend only on the choice of the planes and/or axes which define the group G of symmetry preserving transformations (see below). The precise form of the basis tensors is irrelevant, all that is required is that they be

linearly independent in the sense of elements of a vector space, and consequently any tensor with the desired symmetry can be expressed

$$\mathbb{C}_{\text{sym}} = \sum_{i=1}^{N} a_i \, \mathbb{V}_i \,. \tag{60}$$

Minimizing $d_F^2(\mathbb{C}, \mathbb{C}_{\text{sym}})$ with respect to the coefficients a_i implies

$$\frac{\partial}{\partial a_i} \left\| \mathbb{C} - \mathbb{C}_{\text{sym}} \right\|^2 = 2\langle \mathbb{C}, \, \mathbb{V}_i \rangle - 2\langle \mathbb{C}_{\text{sym}}, \, \mathbb{V}_i \rangle = 0. \tag{61}$$

Hence,

$$\sum_{j=1}^{N} a_j \langle \mathbb{V}_j, \, \mathbb{V}_i \rangle = \langle \mathbb{C}, \, \mathbb{V}_i \rangle. \tag{62}$$

Let **D** be the $N \times N$ symmetric matrix with elements

$$D_{ij} \equiv \langle \mathbb{V}_i, \, \mathbb{V}_j \rangle \,. \tag{63}$$

This is invertible by virtue of the linear independence of the basis tensors, and so

$$a_i = \sum_{j=1}^{N} D_{ij}^{-1} \langle \mathbb{V}_j, \, \mathbb{C} \rangle \,. \tag{64}$$

Noting that $\mathbb{C}_{\perp\text{sym}} = \mathbb{C} - \mathbb{C}_{\text{sym}}$, we have

$$\langle \mathbb{C}_{\perp\text{sym}}, \, \mathbb{C}_{\text{sym}} \rangle = \langle \mathbb{C} - \sum_{i=1}^{N} a_i \, \mathbb{V}_i, \, \sum_{j=1}^{N} a_j \, \mathbb{V}_j \rangle$$

$$= \sum_{j=1}^{N} a_j \langle \mathbb{C}, \, \mathbb{V}_j \rangle - \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \langle \mathbb{V}_i, \, \mathbb{V}_j \rangle = 0. \tag{65}$$

Hence, the partition (60) satisfies the fundamental projection property (52). It is a simple exercise to show that $\mathbb{C}_{\text{sym}}$ of Eq. (60) minimizes the Euclidean distance $\| \mathbb{C} - \mathbb{C}_{\text{sym}} \|$ iff the coefficients are given by Eq. (64).

The Euclidean projection can also be expressed in terms of tensors of order eight, P_{sym}, such that

$$\mathbb{C}_{\text{sym}} = P_{\text{sym}} \, \mathbb{C}, \qquad \Leftrightarrow \qquad \mathbb{C}_{ijkl}^{\text{sym}} = P_{ijklmnrs}^{\text{sym}} C_{mnrs}. \tag{66}$$

The projector can be expressed in terms of dyadics of basis tensors,

$$P_{\text{sym}} = \sum_{i,j=1}^{N} D_{ij}^{-1} \, \mathbb{V}_i \otimes \mathbb{V}_j \,. \tag{67}$$

We now apply these ideas to particular symmetries, focusing only on the higher symmetries since the lower ones (monoclinic, orthorhombic) are relatively trivial and can be handled using the projector of Eq. (71).

5.3 Projection Operators for Particular Symmetries

The projector for the symmetries described in Section 4 follow naturally from the vectorial formulation. In particular, we note that the **D** is diagonal in each case, and

$$P_{\mathrm{iso}} = \mathbb{J} \otimes \mathbb{J} + \frac{1}{5} \mathbb{K} \otimes \mathbb{K}, \tag{68a}$$

$$P_{\mathrm{cub}} = \mathbb{J} \otimes \mathbb{J} + \frac{1}{3} \mathbb{L} \otimes \mathbb{L} + \frac{1}{2} \mathbb{M} \otimes \mathbb{M}, \tag{68b}$$

$$P_{\mathrm{hex}} = \mathbb{E}_1 \otimes \mathbb{E}_1 + \mathbb{E}_2 \otimes \mathbb{E}_2 + \frac{1}{2} (\mathbb{E}_3 + \mathbb{E}_4) \otimes (\mathbb{E}_3 + \mathbb{E}_4)$$
$$+ \frac{1}{2} \mathbb{F} \otimes \mathbb{F} + \frac{1}{2} \mathbb{G} \otimes \mathbb{G}, \tag{68c}$$

$$P_{\mathrm{tet}} = \mathbb{E}_1 \otimes \mathbb{E}_1 + \mathbb{E}_2 \otimes \mathbb{E}_2 + \frac{1}{2} (\mathbb{E}_3 + \mathbb{E}_4) \otimes (\mathbb{E}_3 + \mathbb{E}_4)$$
$$+ \mathbb{F}_1 \otimes \mathbb{F}_1 + \mathbb{F}_2 \otimes \mathbb{F}_2 + \frac{1}{2} (\mathbb{F}_3 + \mathbb{F}_4) \otimes (\mathbb{F}_3 + \mathbb{F}_4) + \frac{1}{2} \mathbb{G} \otimes \mathbb{G}, \tag{68d}$$

$$P_{\mathrm{trig}} = \mathbb{E}_1 \otimes \mathbb{E}_1 + \mathbb{E}_2 \otimes \mathbb{E}_2 + \frac{1}{2} (\mathbb{E}_3 + \mathbb{E}_4) \otimes (\mathbb{E}_3 + \mathbb{E}_4) + \frac{1}{2} \mathbb{R}_1 \otimes \mathbb{R}_1$$
$$+ \frac{1}{2} \mathbb{R}_2 \otimes \mathbb{R}_2 + \frac{1}{4} (\mathbb{R}_3 + \mathbb{R}_4) \otimes (\mathbb{R}_3 + \mathbb{R}_4) + \frac{1}{4} (\mathbb{R}_5 + \mathbb{R}_6) \otimes (\mathbb{R}_5 + \mathbb{R}_6). \tag{68e}$$

Expressions can be given for the lower symmetries, but as discussed, the number of elementary tensors involved is much larger. The form of the projectors in Eq. (68) are independent of the crystal axes, and provide a coordinate-free scheme for Euclidean projection. They can be easily programmed using the six-dimensional matrix representation.

If sym A > sym B, we may convert P_{symB} into a form that yields $\mathbb{C}_{\mathrm{sym\,B}}$ and the complement $\mathbb{C}_{\mathrm{sym\,B/A}}$ of Eq. (56). A general procedure for achieving this uses the fact that basis tensors for a given symmetry are not unique, and accordingly, alternate forms of P_{sym} may be found. The decomposition

$$P_{\mathrm{sym\,B}} = P_{\mathrm{sym\,A}} + P_{\mathrm{sym\,B/A}}, \tag{69}$$

can be found by a Gram–Schmid type of process. As an example, let sym A= iso and sym B= cub, and consider the tensor basis for cubic symmetry $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\} = \{\mathbb{J}, \mathbb{K}, \mathbb{L} - a\,\mathbb{M}\}$, where a is a free parameter. Requiring that **D** is diagonal implies $a = 3/2$, and consequently we obtain

$$P_{\mathrm{cub}} = \mathbb{J} \otimes \mathbb{J} + \frac{1}{5} \mathbb{K} \otimes \mathbb{K} + \frac{1}{30} (2\,\mathbb{L} - 3\,\mathbb{M}) \otimes (2\,\mathbb{L} - 3\,\mathbb{M}) = P_{\mathrm{iso}} + P_{\mathrm{cub/iso}}. \tag{70}$$

The analogous partition for isotropy and transverse isotropy is described in Appendix A, with $P_{\mathrm{hex/iso}}$ given in Eq. (A.24).

5.4 Projection Using the Transformation Group

Let G be the group of transformations for the material symmetry. $G \subset SO(3)$, the space of three-dimensional special (or proper) orthogonal transformations. Under the change of basis associated with $\mathbf{Q} \in SO(3)$, an elasticity tensor transforms $\mathbb{C} \rightarrow \mathbb{C}'$ where $C'_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{pqrs}$. The projection is

$$C^{\text{sym}}_{ijkl} = \frac{1}{n} \sum_{m=1}^{n} Q^{(m)}_{ip} Q^{(m)}_{jq} Q^{(m)}_{kr} Q^{(m)}_{ls} C_{pqrs}, \tag{71}$$

where n is the order of the group G, and $\mathbf{Q}^{(m)}$, are elements of the group.

The order of the groups are as follows (see Table 6 of [35]): triclinic 1, monoclinic 2, trigonal 6, orthorhombic 4, tetragonal 8, transverse isotropy $\infty + 2$, and isotropy ∞^3. The projection defined by Eq. (71) was discussed by, among others, Gazis et al. [3], and has been applied to ultrasonic data by François et al. [47].

The six-dimensional version of the projection (71) is

$$\widehat{\mathbf{C}}_{\text{sym}} = \frac{1}{n} \sum_{m=1}^{n} \widehat{\mathbf{Q}}^{(m)} \widehat{\mathbf{C}} \widehat{\mathbf{Q}}^{(m)^t}, \tag{72}$$

where $\widehat{\mathbf{Q}}^{(m)} \in SO(6)$ is the orthogonal second-order tensor corresponding to $\mathbf{Q}^{(m)}$, satisfying $\widehat{\mathbf{Q}}\widehat{\mathbf{Q}}^t = \widehat{\mathbf{Q}}^t\widehat{\mathbf{Q}} = \widehat{\mathbf{I}}$. A necessary and sufficient condition that $\widehat{\mathbf{C}}_{\text{sym}}$ has the correct symmetry is that $\widehat{\mathbf{Q}}^{(j)}\widehat{\mathbf{C}}_{\text{sym}}\widehat{\mathbf{Q}}^{(j)^t}$, corresponding to the transformation $\mathbf{Q}^{(j)}$, is also a member of the symmetry class. Thus, for $j = 1, 2, \ldots, n$,

$$\widehat{\mathbf{Q}}^{(j)}\widehat{\mathbf{C}}_{\text{sym}}\widehat{\mathbf{Q}}^{(j)^t} = \frac{1}{n} \sum_{i=1}^{n} \widehat{\mathbf{Q}}^{(j)}\widehat{\mathbf{Q}}^{(i)}\widehat{\mathbf{C}}\widehat{\mathbf{Q}}^{(i)^t}\widehat{\mathbf{Q}}^{(j)^t} = \frac{1}{n} \sum_{k=1}^{n} \widehat{\mathbf{Q}}^{(k)}\widehat{\mathbf{C}}\widehat{\mathbf{Q}}^{(k)^t}. \tag{73}$$

The latter identity follows from $\widehat{\mathbf{Q}}^{(j)}\widehat{\mathbf{Q}}^{(i)} = \widehat{\mathbf{Q}}^{(k)}$ and the fact that the $\widehat{\mathbf{Q}}^{(i)}$ form a group of order n. This proves that $\mathbb{C}_{\text{sym}}$ defined in Eq. (71) is indeed an element of the symmetry class.

The projection formula (71) or (72) may be effected in several ways, e.g., by referring to the extensive lists of the 6×6 matrices $\widehat{\mathbf{Q}}^{(i)}$ provided by Cowin and Mehrabadi [35]. We note that each $\mathbf{Q}^{(i)}$ can be decomposed into a product of reflection operators [35]. The fundamental operator is $\mathbf{R}(\mathbf{n}) \equiv \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}$, where the unit vector $\mathbf{n}$ is normal to the plane. Chadwick et al. [18] provide a complete description of these sets and groups. It is also possible to represent the group of transformations in terms of elements each corresponding to a single rotation [18, 35]. The equivalence between reflection operator and rotations follows from the identity $\mathbf{Q}(\mathbf{e}, \pi) = -\mathbf{R}(\mathbf{e})$, where $\mathbf{Q}(\mathbf{n}, \theta) \in SO(3)$ defines rotation about $\mathbf{n}$ by angle θ.

We illustrate this general procedure with an example.

5.4.1 Example: Projection onto Monoclinic Symmetry

There is only one plane of symmetry [18], with normal $\mathbf{c}$, say, and $n = 2$ group elements $G = \{\mathbf{I}, -\mathbf{R}(\mathbf{c})\}$. Using Eq. (71) we have

$$C^{\text{mon}}_{ijkl} = \frac{1}{2} C_{ijkl} + \frac{1}{2} R_{ip}(\mathbf{c}) R_{jq}(\mathbf{c}) R_{kr}(\mathbf{c}) R_{ls}(\mathbf{c}) C_{pqrs}. \tag{74}$$

For instance, let $\mathbf{c} = \mathbf{e}_3$, and using the explicit form of $\widehat{\mathbf{R}}(\mathbf{e}_3)$ from Eq. (54) of [35] (where it is called $\widehat{\mathbf{R}}^{(3)}$),

$$\widehat{\mathbf{R}}(\mathbf{e}_3) = \text{diag}(1,\ 1,\ 1,\ -1,\ -1,\ 1)\,, \tag{75}$$

then Eq. (72) gives

$$\widehat{\mathbf{C}}_{\text{mon}} = \frac{1}{2}\widehat{\mathbf{C}} + \frac{1}{2}\widehat{\mathbf{R}}(\mathbf{e}_3)\widehat{\mathbf{C}}\widehat{\mathbf{R}}(\mathbf{e}_3) = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & \hat{c}_{16} \\ & \hat{c}_{22} & \hat{c}_{23} & 0 & 0 & \hat{c}_{26} \\ & & \hat{c}_{33} & 0 & 0 & \hat{c}_{36} \\ & & & \hat{c}_{44} & \hat{c}_{45} & 0 \\ & \text{S} \quad \text{Y} \quad \text{M} & & & \hat{c}_{55} & 0 \\ & & & & & \hat{c}_{66} \end{pmatrix}. \tag{76}$$

The example illustrates that the group projection method is practical if the order n of the group G is not large. This is the case for monoclinic, orthorhombic and perhaps trigonal systems, but even for these symmetries it is simpler to implement the algorithm by machine. It is not practical for isotropy and transverse isotropy, each with an infinity of symmetry planes. Application of Eq. (71) for cubic isotropy is unwieldy although has been performed [47]. The explicit orthogonal basis method of Eqs. (60) and (64) provides a simpler procedure for the higher symmetries.

5.5 Projections Using 21-dimensional Vectors

The projection is simplest for this representation of $\mathbb{E}$la because the elastic moduli are explicitly represented as a vector. The operator is a 21×21 matrix P_{sym}, and the closest elasticity is

$$X_{\text{sym}} = P_{\text{sym}}X, \tag{77}$$

where X is defined in Eq. (9). The various projection matrices can be read off from the formulas of Appendix A. Thus, the projector for cubic symmetry is

$$P_{\text{cub}} = \begin{pmatrix} p_{\text{cub}} & 0_{9\times12} \\ 0_{12\times9} & 0_{12\times12} \end{pmatrix}, \qquad p_{\text{cub}} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{78}$$

Browaeys and Chevrot [14] derived P_{sym} for isotropy, transverse isotropy (hexagonal symmetry), tetragonal, orthorhombic and monoclinic symmetry. They found that in all cases except monoclinic that the non-zero part of P_{sym} reduces to a 9×9 matrix

p_{sym}, as in Eq. (78). However, it should be noted that this is not true for the tetragonal projector derived here, see Eq. (A.6), for reasons discussed in Section 4.5. Nor it is true for the trigonal projector, see Eq. (A.10).

6 Exponential, Logarithm and Square Root of Symmetric Fourth-order Tensors

Unlike the Euclidean projection for the Frobenius norm, finding minimizers for the logarithmic and Riemannian distance functions involves evaluation of analytic functions with symmetric fourth-order tensor arguments. In this section we derive the necessary expressions with emphasis on exponential, logarithm and square root functions that will be used to obtain the closest elastic tensors in Section 7. For this purpose, let h denote the analytic function given by its power series

$$h(x) = \sum_{m=0}^{\infty} a_m x^m.$$

The higher symmetries are simpler, and are considered first.

6.1 Isotropic System

Using Eqs. (20) and (21), for $\mathbb{A}$ defined in Eq. (18) we have

$$h(\mathbb{A}) = \sum_{m=0}^{\infty} a_m (a\,\mathbb{J} + b\,\mathbb{K})^m = a_0\,\mathbb{I} + \sum_{m=1}^{\infty} a_m (a^m\,\mathbb{J} + b^m\,\mathbb{K})$$

$$= a_0\,\mathbb{I} + \sum_{m=0}^{\infty} a_m (a^m\,\mathbb{J} + b^m\,\mathbb{K}) - a_0(\mathbb{J} + \mathbb{K}) = h(a)\,\mathbb{J} + h(b)\,\mathbb{K}. \qquad (79)$$

It follows that

$$\exp \mathbb{A} = e^a\,\mathbb{J} + e^b\,\mathbb{K}. \qquad (80)$$

When $a \neq 0$ and $b \neq 0$, the symmetric tensor $\mathbb{A}$ is invertible and its inverse is

$$\mathbb{A}^{-1} = \frac{1}{a}\,\mathbb{J} + \frac{1}{b}\,\mathbb{K}. \qquad (81)$$

The tensor $\mathbb{A}$ is positive definite if $a > 0$ and $b > 0$, with square root given by

$$\mathbb{A}^{1/2} = \sqrt{a}\,\mathbb{J} + \sqrt{b}\,\mathbb{K}, \qquad (82)$$

and logarithm

$$\log \mathbb{A} = \ln a\,\mathbb{J} + \ln b\,\mathbb{K}. \qquad (83)$$

6.2 Cubic System

Using Eqs. $(26)_1$ and (27), for $\mathbb{A}$ defined in Eq. (28) we have

$$h(\mathbb{A}) = \sum_{m=0}^{\infty} a_0 (a\,\mathbb{J} + b\,\mathbb{L} + c\,\mathbb{M})^m$$

$$= a_0\,\mathbb{I} + \sum_{m=1}^{\infty} a_m (a^m\,\mathbb{J} + b^m\,\mathbb{L} + c^m\,\mathbb{M})$$

$$= a_0\,\mathbb{I} + \sum_{m=0}^{\infty} a_m (a^m\,\mathbb{J} + b^m\,\mathbb{L} + c^m\,\mathbb{M}) - a_0 (\mathbb{J} + \mathbb{L} + \mathbb{M})$$

$$= h(a)\,\mathbb{J} + h(b)\,\mathbb{L} + h(c)\,\mathbb{M}. \tag{84}$$

In particular,

$$\exp(\mathbb{A}) = e^a\,\mathbb{J} + e^b\,\mathbb{L} + e^c\,\mathbb{M}. \tag{85}$$

When $a \neq 0$, $b \neq 0$ and $c \neq 0$, the symmetric tensor $\mathbb{A}$ is invertible and its inverse is given by

$$\mathbb{A}^{-1} = \frac{1}{a}\,\mathbb{J} + \frac{1}{b}\,\mathbb{L} + \frac{1}{c}\,\mathbb{M}. \tag{86}$$

If $a > 0$, $b > 0$ and $c > 0$ then $\mathbb{A}$ is positive definite. Its square root is

$$\mathbb{A}^{1/2} = \sqrt{a}\,\mathbb{J} + \sqrt{b}\,\mathbb{L} + \sqrt{c}\,\mathbb{M}, \tag{87}$$

and its logarithm is given by

$$\text{Log }\mathbb{A} = \ln a\,\mathbb{J} + \ln b\,\mathbb{L} + \ln c\,\mathbb{M}. \tag{88}$$

6.3 Transversely Isotropic System

Using Eqs. (32) and (31), for $\mathbb{A}$ defined in Eq. (33) we have

$$h(\mathbb{A}) = \sum_{m=0}^{\infty} a_m \left[a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + f\,\mathbb{F} + g\,\mathbb{G} \right]^m$$

$$= a_0\,\mathbb{I} + \sum_{m=1}^{\infty} a_m \left[(a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4))^m + f^m\,\mathbb{F} + g^m\,\mathbb{G} \right]$$

$$= \sum_{m=0}^{\infty} a_m (a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4))^m + h(f)\,\mathbb{F} + h(g)\,\mathbb{G} - a_0 (\mathbb{F} + \mathbb{G})$$

$$= h(0)(\mathbb{E}_1 + \mathbb{E}_2 - \mathbb{I}) + h(a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4)) + h(f)\,\mathbb{F} + h(g)\,\mathbb{G}. \tag{89}$$

Contrary to the two previous classes of material symmetries, the upper left 4×4 multiplication table for the elementary tensors of a transversely isotropic medium is not diagonal. This fact makes the algebra of transversely isotropic tensors a bit more

difficult. Fortunately, Walpole [19] showed that the algebra for $\mathbb{E}_1$, $\mathbb{E}_2$ $\mathbb{E}_3$ and $\mathbb{E}_4$ is equivalent to the algebra of 2×2 matrices. The details are described in Appendix B.

For the sake of simplicity of notation, we use the symbol **a** to denote the quadruple (a, b, c, d) and introduce the functions

$$\alpha(\mathbf{a}) = \frac{1}{2}(a + b),$$

$$\beta(\mathbf{a}) = \frac{1}{2}(a - b),$$

$$\gamma(\mathbf{a}) = \frac{1}{2}\sqrt{(a - b)^2 + 4(c^2 + d^2)},$$

$$\delta(\mathbf{a}) = \sqrt{ab - c^2 - d^2} \quad (\text{if } ab - c^2 - d^2 \geq 0). \tag{90}$$

The variable d is not needed in this subsection, so we take $d = 0$. We used it to keep the number of auxiliary functions to a minimum.

The procedure is to diagonalize the compound tensor argument in Eq. (89),

$$\mathbb{B} \equiv a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) = (\alpha(\mathbf{a}) + \gamma(\mathbf{a}))\,\mathbb{E}_+ + (\alpha(\mathbf{a}) - \gamma(\mathbf{a}))\,\mathbb{E}_-, \tag{91}$$

where

$$\mathbb{E}_\pm = \frac{1}{2}(\mathbb{E}_1 + \mathbb{E}_2) \pm \frac{1}{2}[\cos\psi(\mathbf{a})(\mathbb{E}_1 - \mathbb{E}_2) + \sin\psi(\mathbf{a})(\mathbb{E}_3 + \mathbb{E}_4)], \tag{92}$$

with $\psi(\mathbf{a})$ defined by

$$\begin{cases} \cos\psi(\mathbf{a}) = \dfrac{\beta(\mathbf{a})}{\gamma(\mathbf{a})}, \quad \sin\psi(\mathbf{a}) = \dfrac{c}{\gamma(\mathbf{a})}, & \text{if } \gamma(\mathbf{a}) > 0, \\ \psi(\mathbf{a}) = 0, & \text{if } \gamma(\mathbf{a}) = 0. \end{cases}$$

The derivation is apparent from the results below, and explained in detail in Appendix B. The central feature of this decomposition is that $\mathbb{E}_\pm$ satisfy $\mathbb{E}_\pm \mathbb{E}_\pm = \mathbb{E}_\pm$, $\mathbb{E}_+ \mathbb{E}_- = \mathbb{O}$, $\mathbb{E}_+ + \mathbb{E}_- = \mathbb{E}_1 + \mathbb{E}_2$, and consequently any function, h, of the tensor $\mathbb{B}$ may be expressed

$$h(\mathbb{B}) = h(\alpha(\mathbf{a}) + \gamma(\mathbf{a}))\,\mathbb{E}_+ + h(\alpha(\mathbf{a}) - \gamma(\mathbf{a}))\,\mathbb{E}_- + h(0)(\mathbb{I} - \mathbb{E}_1 - \mathbb{E}_2), \tag{93}$$

and hence

$$h(\mathbb{A}) = h(\alpha(\mathbf{a}) + \gamma(\mathbf{a}))\,\mathbb{E}_+ + h(\alpha(\mathbf{a}) - \gamma(\mathbf{a}))\,\mathbb{E}_- + h(f)\,\mathbb{F} + h(g)\,\mathbb{G}. \tag{94}$$

Converting back to the $\mathbb{E}_i$'s gives

$$h(\mathbb{A}) = h^+(\mathbf{a})(\mathbb{E}_1 + \mathbb{E}_2) + h^-(\mathbf{a})\left[\beta(\mathbf{a})(\mathbb{E}_1 - \mathbb{E}_2) + c(\mathbb{E}_3 + \mathbb{E}_4)\right] + h(f)\,\mathbb{F} + h(g)\,\mathbb{G}, \tag{95}$$

where

$$h^+(\mathbf{a}) = \frac{h(\alpha(\mathbf{a}) + \gamma(\mathbf{a})) + h(\alpha(\mathbf{a}) - \gamma(\mathbf{a}))}{2},$$

and

$$h^-(\mathbf{a}) = \begin{cases} \dfrac{h(\alpha(\mathbf{a}) + \gamma(\mathbf{a})) - h(\alpha(\mathbf{a}) - \gamma(\mathbf{a}))}{2\gamma(\mathbf{a})}, & \text{if } \gamma(\mathbf{a}) > 0, \\ h'(\alpha(\mathbf{a})), & \text{if } \gamma(\mathbf{a}) = 0. \end{cases}$$

The results of Appendix B and above imply

$$\exp(\mathbb{A}) = \mathfrak{e}_1(\mathbf{a})\,\mathbb{E}_1 + \mathfrak{e}_2(\mathbf{a})\,\mathbb{E}_2 + \mathfrak{e}_3(\mathbf{a})(\,\mathbb{E}_3 + \mathbb{E}_4) + e^f\,\mathbb{F} + e^g\,\mathbb{G}, \qquad (96)$$

where

$$\mathfrak{e}_1(\mathbf{a}) = \exp(\alpha(\mathbf{a}))[\cosh(\gamma(\mathbf{a})) + \beta(\mathbf{a})\,\text{sinhc}(\gamma(\mathbf{a}))],$$

$$\mathfrak{e}_2(\mathbf{a}) = \exp(\alpha(\mathbf{a}))[\cosh(\gamma(\mathbf{a})) - \beta(\mathbf{a})\,\text{sinhc}(\gamma(\mathbf{a}))],$$

$$\mathfrak{e}_3(\mathbf{a}) = c\exp(\alpha(\mathbf{a}))\,\text{sinhc}(\gamma(\mathbf{a})), \qquad (97)$$

and sinhc($\cdot$) is the hyperbolic sine cardinal function defined by

$$\text{sinhc}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{\sinh(x)}{x} & \text{otherwise.} \end{cases} \qquad (98)$$

Note that sinhc(x) is continuous at 0.

When $ab - c^2 \neq 0$, $f \neq 0$ and $g \neq 0$, the symmetric tensor $\mathbb{A}$ is invertible and

$$\mathbb{A}^{-1} = \frac{1}{ab - c^2}\left[b\,\mathbb{E}_1 + a\,\mathbb{E}_2 - c(\,\mathbb{E}_3 + \mathbb{E}_4)\right] + \frac{1}{f}\,\mathbb{F} + \frac{1}{g}\,\mathbb{G}. \qquad (99)$$

$\mathbb{A}$ is positive definite if $a > 0$, $b > 0$, $ab - c^2 > 0$, $f > 0$ and $g > 0$, with square root

$$\mathbb{A}^{1/2} = \frac{1}{\sqrt{2(\alpha(\mathbf{a}) + \delta(\mathbf{a}))}}\left[(a + \delta(\mathbf{a}))\,\mathbb{E}_1 + (b + \delta(\mathbf{a}))\,\mathbb{E}_2 + c(\,\mathbb{E}_3 + \mathbb{E}_4)\right]$$

$$+ \sqrt{f}\,\mathbb{F} + \sqrt{g}\,\mathbb{G}, \qquad (100)$$

and logarithm

$$\text{Log } \mathbb{A} = \mathfrak{l}_1(\mathbf{a})\,\mathbb{E}_1 + \mathfrak{l}_2(\mathbf{a})\,\mathbb{E}_2 + \mathfrak{l}_3(\mathbf{a})(\,\mathbb{E}_3 + \mathbb{E}_4) + \ln f\,\mathbb{F} + \ln g\,\mathbb{G}, \qquad (101)$$

where

$$\mathfrak{l}_1(\mathbf{a}) = \ln \delta(\mathbf{a}) + \beta(\mathbf{a})\ell(\mathbf{a}),$$

$$\mathfrak{l}_2(\mathbf{a}) = \ln \delta(\mathbf{a}) - \beta(\mathbf{a})\ell(\mathbf{a}),$$

$$\mathfrak{l}_3(\mathbf{a}) = c\,\ell(\mathbf{a}), \qquad (102)$$

with the function $\ell(\cdot)$ defined by

$$\ell(\mathbf{a}) = \begin{cases} \dfrac{1}{\alpha(\mathbf{a})} & \text{if } \gamma(\mathbf{a}) = 0, \\ \dfrac{1}{2\gamma(\mathbf{a})}\ln\dfrac{\alpha(\mathbf{a}) + \gamma(\mathbf{a})}{\alpha(\mathbf{a}) - \gamma(\mathbf{a})} & \text{otherwise.} \end{cases} \qquad (103)$$

Note that for $\gamma(\mathbf{a}) \neq 0$ we have $\frac{1}{2\gamma(\mathbf{a})}\ln\frac{\alpha(\mathbf{a}) + \gamma(\mathbf{a})}{\alpha(\mathbf{a}) - \gamma(\mathbf{a})} = \frac{1}{\gamma(\mathbf{a})}\tanh^{-1}\frac{\gamma(\mathbf{a})}{\alpha(\mathbf{a})}$ and that the limit of this expression is $1/\alpha(\mathbf{a})$ as $\gamma(\mathbf{a})$ goes to zero.

6.4 Tetragonal System

Using the decomposition (35) and the multiplication table (36), an analysis similar to that used for the transversely isotropic system implies that for a tetragonal tensor $\mathbb{A}$ as defined in Eq. (37),

$$h(\mathbb{A}) = h^+(\mathbf{a})(\mathbb{E}_1 + \mathbb{E}_2) + h^-(\mathbf{a})\left[\beta(\mathbf{a})(\mathbb{E}_1 - \mathbb{E}_2) + c(\mathbb{E}_3 + \mathbb{E}_4)\right]$$
$$+ h^+(\mathbf{p})(\mathbb{F}_1 + \mathbb{F}_2) + h^-(\mathbf{p})\left[\beta(\mathbf{p})(\mathbb{F}_1 - \mathbb{F}_2) + r(\mathbb{F}_3 + \mathbb{F}_4)\right] + h(g)\,\mathbb{G}. \quad (104)$$

As before, $\mathbf{a} = (a, b, c, d)$ (with $d = 0$) and $\mathbf{p} = (p, q, r, s)$ (with $s = 0$).
   Thus,

$$\exp(\mathbb{A}) = \mathfrak{e}_1(\mathbf{a})\,\mathbb{E}_1 + \mathfrak{e}_2(\mathbf{a})\,\mathbb{E}_2 + \mathfrak{e}_3(\mathbf{a})(\mathbb{E}_3 + \mathbb{E}_4)$$
$$+ \mathfrak{e}_1(\mathbf{p})\,\mathbb{F}_1 + \mathfrak{e}_2(\mathbf{p})\,\mathbb{F}_2 + \mathfrak{e}_3(\mathbf{p})(\mathbb{F}_3 + \mathbb{F}_4) + e^g\,\mathbb{G}. \quad (105)$$

The tensor $\mathbb{A}$ is invertible if $ab - c^2 \neq 0$, $pq - r^2 \neq 0$ and $g \neq 0$, and

$$\mathbb{A}^{-1} = \frac{1}{ab - c^2}\left[b\,\mathbb{E}_1 + a\,\mathbb{E}_2 - c(\mathbb{E}_3 + \mathbb{E}_4)\right]$$
$$+ \frac{1}{pq - r^2}\left[q\,\mathbb{F}_1 + p\,\mathbb{F}_2 - r(\mathbb{F}_3 + \mathbb{F}_4)\right] + \frac{1}{g}\,\mathbb{G}. \quad (106)$$

$\mathbb{A}$ is positive definite if $a > 0$, $b > 0$, $ab - c^2 > 0$, $p > 0$, $q > 0$, $pq - r^2 > 0$ and $g > 0$, with square root

$$\mathbb{A}^{1/2} = \frac{1}{\sqrt{2(\alpha(\mathbf{a}) + \delta(\mathbf{a}))}}\left[(a + \delta(\mathbf{a}))\,\mathbb{E}_1 + (b + \delta(\mathbf{a}))\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4)\right]$$
$$+ \frac{1}{\sqrt{2(\alpha(\mathbf{p}) + \delta(\mathbf{p}))}}\left[(p + \delta(\mathbf{p}))\,\mathbb{F}_1 + (q + \delta(\mathbf{p}))\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4)\right] + \sqrt{g}\,\mathbb{G}, \quad (107)$$

   and logarithm

$$\mathrm{Log}\,\mathbb{A} = \mathfrak{l}_1(\mathbf{a})\,\mathbb{E}_1 + \mathfrak{l}_2(\mathbf{a})\,\mathbb{E}_2 + \mathfrak{l}_3(\mathbf{a})(\mathbb{E}_3 + \mathbb{E}_4)$$
$$+ \mathfrak{l}_1(\mathbf{p})\,\mathbb{F}_1 + \mathfrak{l}_2(\mathbf{p})\,\mathbb{F}_2 + \mathfrak{l}_3(\mathbf{p})(\mathbb{F}_3 + \mathbb{F}_4) + \ln g\,\mathbb{G}. \quad (108)$$

6.5 Trigonal System

The decomposition (44) and the multiplication table (43) yield for $\mathbb{A}$ defined in Eq. (45)

$$h(\mathbb{A}) = h^+(\mathbf{a})(\mathbb{E}_1 + \mathbb{E}_2) + h^-(\mathbf{a})\left[\beta(\mathbf{a})(\mathbb{E}_1 - \mathbb{E}_2) + c(\mathbb{E}_3 + \mathbb{E}_4)\right]$$
$$+ h^+(\mathbf{p})(\mathbb{R}_1 + \mathbb{R}_2) + h^-(\mathbf{p})\left[\beta(\mathbf{p})(\mathbb{R}_1 - \mathbb{R}_2) + r(\mathbb{R}_3 + \mathbb{R}_4) + s(\mathbb{R}_5 + \mathbb{R}_6)\right],$$
$$(109)$$

where, as before, $\mathbf{a} = (a, b, c, d)$ (with $d = 0$) and $\mathbf{p} = (p, q, r, s)$ (here s need not be zero).
   Using these results we have

$$\exp(\mathbb{A}) = \mathfrak{e}_1(\mathbf{a})\,\mathbb{E}_1 + \mathfrak{e}_2(\mathbf{a})\,\mathbb{E}_2 + \mathfrak{e}_3(\mathbf{a})(\mathbb{E}_3 + \mathbb{E}_4)$$
$$+ \mathfrak{e}_1(\mathbf{p})\,\mathbb{R}_1 + \mathfrak{e}_2(\mathbf{p})\,\mathbb{R}_2 + \mathfrak{e}_3(\mathbf{p})(\mathbb{R}_3 + \mathbb{R}_4) + \mathfrak{e}_4(\mathbf{p})(\mathbb{R}_5 + \mathbb{R}_6), \quad (110)$$

where $\mathfrak{e}_i(\cdot)$, $i = 1, 2, 3$ are as defined in Eq. (97) and $\mathfrak{e}_4(\mathbf{p}) = s \exp(\alpha(\mathbf{p})) \, \text{sinhc}(\gamma(\mathbf{p}))$. The tensor $\mathbb{A}$ is invertible if $ab - c^2 \neq 0$ and $pq - r^2 - s^2 \neq 0$, with

$$
\mathbb{A}^{-1} = \frac{1}{ab - c^2} \left[b \, \mathbb{E}_1 + a \, \mathbb{E}_2 - c(\mathbb{E}_3 + \mathbb{E}_4) \right]
$$

$$
+ \frac{1}{pq - r^2 - s^2} \left[q \, \mathbb{R}_1 + p \, \mathbb{R}_2 - r(\mathbb{R}_3 + \mathbb{R}_4) - s(\mathbb{R}_5 + \mathbb{R}_6) \right]. \tag{111}
$$

$\mathbb{A}$ is positive definite if $a > 0$, $b > 0$, $ab - c^2 > 0$, $p > 0$, $q > 0$ and $pq - r^2 - s^2 > 0$, with square root

$$
\mathbb{A}^{1/2} = \frac{1}{\sqrt{2(\alpha(\mathbf{a}) + \delta(\mathbf{a}))}} \left[(a + \delta(\mathbf{a})) \, \mathbb{E}_1 + (b + \delta(\mathbf{a})) \, \mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) \right]
$$

$$
+ \frac{1}{\sqrt{2(\alpha(\mathbf{p}) + \delta(\mathbf{p}))}} \left[(p + \delta(\mathbf{p})) \, \mathbb{R}_1 + (q + \delta(\mathbf{p})) \, \mathbb{R}_2 + r(\mathbb{R}_3 + \mathbb{R}_4) + s(\mathbb{R}_5 + \mathbb{R}_6) \right], \tag{112}
$$

and logarithm

$$
\text{Log} \, \mathbb{A} = \mathfrak{l}_1(\mathbf{a}) \, \mathbb{E}_1 + \mathfrak{l}_2(\mathbf{a}) \, \mathbb{E}_2 + \mathfrak{l}_3(\mathbf{a})(\mathbb{E}_3 + \mathbb{E}_4)
$$

$$
+ \mathfrak{l}_1(\mathbf{p}) \, \mathbb{R}_1 + \mathfrak{l}_2(\mathbf{p}) \, \mathbb{R}_2 + \mathfrak{l}_3(\mathbf{p})(\mathbb{R}_3 + \mathbb{R}_4) + \mathfrak{l}_4(\mathbf{p})(\mathbb{R}_5 + \mathbb{R}_6), \tag{113}
$$

where $\mathfrak{l}_i(\cdot)$, $i = 1, 2, 3$ are as defined in Eq. (102) and $\mathfrak{l}_4(\mathbf{p}) = s\ell(\mathbf{p})$.

6.6 Rhombic, Monoclinic and Triclinic Systems

There are no practical analytical results for the exponential, logarithm and square root for these low symmetries. The purely numerical route is recommended.

7 The Closest Tensors Using Logarithmic Norms

We now consider the problem of finding the closest elasticity tensors using the Riemannian and the logarithmic distance functions. The solutions use the machinery developed in the previous Section for evaluating functions of tensors. We begin with the higher symmetries.

7.1 The Closest Isotropic Tensor Using the Riemannian Norm

We want to find the closest isotropic tensor $\mathbb{C}_R^{\text{iso}} = 3\kappa_R \, \mathbb{J} + 2\mu_R \mathbb{K}$ to a given elasticity tensor $\mathbb{C}$, i.e., find $\kappa_R > 0$ and $\mu_R > 0$ such that the Riemannian distance $d_R(\mathbb{C}, \mathbb{C}_R^{\text{iso}})$ is minimized, where d_R is defined in Eq. (11c). To find the optimality conditions, we note that minimizing the Riemannian distance is equivalent to minimizing the square of this distance, and recall the following result [24, Prop. 2.1]

$$
\frac{d}{dt} \text{tr} \left[\text{Log}^2 \, \mathbf{X}(t) \right] = 2 \, \text{tr} \left[\text{Log} \, \mathbf{X}(t) \mathbf{X}^{-1}(t) \frac{d}{dt} \mathbf{X}(t) \right].
$$

Here $\mathbf{X}(t)$ is a real matrix-valued function of the real variable t such that, for all t in its domain, $\mathbf{X}(t)$ is an invertible matrix which does not have eigenvalues on the closed negative real line. This result applies to fourth-order elasticity tensors via the isomorphism (7).

Using this general result implies two conditions,

$$\frac{\partial \, \mathrm{d}_R^2}{\partial \kappa_R}(\mathbb{C}, \, \mathbb{C}_R^{\mathrm{iso}}) = \frac{2}{\kappa_R} \, \mathrm{tr}\left[\mathrm{Log}(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}})\, \mathbb{J}\right], \tag{114a}$$

$$\frac{\partial \, \mathrm{d}_R^2}{\partial \mu_R}(\mathbb{C}, \, \mathbb{C}_R^{\mathrm{iso}}) = \frac{2}{\mu_R} \, \mathrm{tr}\left[\mathrm{Log}(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}})\, \mathbb{K}\right]. \tag{114b}$$

Hence, the optimality conditions for $\mathbb{C}_R^{\mathrm{iso}}$ are

$$\mathrm{tr}\left[\mathrm{Log}(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}})\, \mathbb{J}\right] = 0, \qquad \mathrm{tr}\left[\mathrm{Log}(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}})\, \mathbb{K}\right] = 0. \tag{115}$$

In view of the partition (20), addition of these two equations yields

$$\mathrm{tr}\left[\mathrm{Log}(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}})\right] = 0,$$

which can be rewritten as[3] (see Appendix B)

$$\det(\, \mathbb{C}^{-1}\, \mathbb{C}_R^{\mathrm{iso}}) = 1,$$

or, equivalently,

$$\det \mathbb{C} = \det \mathbb{C}_R^{\mathrm{iso}}. \tag{116}$$

This is the counterpart of the fact that, for the Euclidean distance, the closest isotropic tensor has the same trace as the given elasticity tensor.

7.1.1 The Closest Isotropic Tensor to a Given Cubic Tensor

For $\mathbb{C} = a\, \mathbb{J} + b\, \mathbb{L} + c\, \mathbb{M}$, the conditions $(115)_1$ and (116), after some algebra, reduce to

$$\ln \frac{3\kappa_R}{a} = 0, \qquad ab^3c^2 = 3\kappa_R(2\mu_R)^5, \tag{117}$$

which can readily be solved to yield

$$3\kappa_R = a, \qquad 2\mu_R = (b^3c^2)^{1/5}.$$

7.1.2 The Closest Isotropic Tensor to a Given Transversely Isotropic Tensor

For $\mathbb{C} = a\, \mathbb{E}_1 + b\, \mathbb{E}_2 + c(\, \mathbb{E}_3 + \mathbb{E}_4) + f\, \mathbb{F} + g\, \mathbb{G}$, the conditions $(115)_1$ and (116) yield

$$3\ln\delta(\mathbf{x}) - \ell(\mathbf{x})(\beta(\mathbf{x}) - 2\sqrt{2}z) = 0, \qquad (ab - c^2)f^2g^2 = 3\kappa_R(2\mu_R)^5, \tag{118}$$

---

[3] Here and throughout, the determinant of a fourth-order elasticity tensor $\mathbb{C}$ is $\det \mathbb{C} = \prod_{I=1}^{6} \Lambda_I$, where the Λ_I's are the eigenvalues of $\mathbb{C}$. From Eq. (8) we have $\det \mathbb{C} = \det \widehat{\mathbf{C}}$, where $\widehat{\mathbf{C}}$ is the associated six-dimensional second-order tensor.

where $\mathbf{x} = (x, y, z, 0)$ with

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix} = \mathbf{\Sigma}^t \mathbf{\Lambda} \begin{pmatrix} A & C \\ C & B \end{pmatrix} \mathbf{\Lambda} \mathbf{\Sigma}, \qquad \begin{pmatrix} A & C \\ C & B \end{pmatrix} = \mathbf{\Sigma} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \mathbf{\Sigma}^t, \qquad (119)$$

where

$$\mathbf{\Sigma} = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}, \quad \mathbf{\Lambda} = \mathrm{diag} \left(\frac{1}{\sqrt{3\kappa_R}}, \frac{1}{\sqrt{2\mu_R}} \right). \qquad (120)$$

Since $\mathbf{\Sigma}$ is an orthogonal transformation, it follows that $AB - C^2 = ab - c^2$, $A + B = a + b$, $xy - z^2 = (ab - c^2)/(6\kappa_R\mu_R)$ and $x + y = (a + b)/(6\kappa_R\mu_R)$.

Equation $(118)_2$ can be used to eliminate κ_R in Eq. $(118)_1$ and hence we obtain a single nonlinear equation for a single unknown. An appropriate choice for this unknown is the positive variable ξ defined by the ratio

$$\xi^2 = \frac{3\kappa_R}{2\mu_R} = \frac{1 + \nu_R}{1 - 2\nu_R},$$

where ν_R is the Poisson's ratio of the closest isotropic tensor with respect to the Riemannian distance. Then Eq. (118) yields the following equation for ξ

$$(\bar{A} - \bar{B}\xi^2) \ln \frac{\bar{A} + \bar{B}\xi^2 + R(\xi)}{\bar{A} + \bar{B}\xi^2 - R(\xi)} - R(\xi) \ln \frac{\xi^{4/3}}{\bar{A}\bar{B} - \bar{C}^2} = 0, \qquad (121)$$

where

$$R(\xi) = \sqrt{(\bar{A} - \bar{B}\xi^2)^2 + 4\bar{C}^2\xi^2},$$

and $\bar{A} = A/(\det \mathbb{C})^{1/6}$, $\bar{B} = B/(\det \mathbb{C})^{1/6}$, $\bar{C} = C/(\det \mathbb{C})^{1/6}$. Note here that for this type of symmetry $\det \mathbb{C} = (ab - c^2) f^2 g^2$.

The isotropic moduli therefore depend upon three unidimensional combinations of the transversely isotropic moduli, $\bar{A}$, $\bar{B}$ and $\bar{C}$. Once the solution ξ of Eq. (121) is found, the isotropic moduli are given by

$$3\kappa_R = (\det \mathbb{C})^{1/6} \xi^{5/3}, \qquad 2\mu_R = (\det \mathbb{C})^{1/6} \xi^{-1/3}. \qquad (122)$$

7.1.3 The Closest Isotropic Tensor to a Given Tetragonal Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{F}_1 + q\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4) + g\,\mathbb{G}$, the conditions $(115)_1$ and (116) yield

$$3 \ln \delta(\mathbf{x}) - \ell(\mathbf{x})(\beta(\mathbf{x}) - 2\sqrt{2}z) = 0, \qquad (ab - c^2)(pq - r^2)g^2 = 3\kappa_R(2\mu_R)^5, \quad (123)$$

where again $\mathbf{x} = (x, y, z, 0)$ is defined by Eq. (119). The solution is obtained in similar manner to that for transverse isotropy, thus, κ_R and μ_R are given by Eq. (122) but in this case $\det \mathbb{C} = (ab - c^2)(pq - r^2)g^2$.

7.1.4 The Closest Isotropic Tensor to a Given Trigonal Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{R}_1 + q\,\mathbb{R}_2 + r(\mathbb{R}_3 + \mathbb{R}_4) + s(\mathbb{R}_5 + \mathbb{R}_6)$, the conditions $(115)_1$ and (116) yield

$$3\ln\delta(\mathbf{x}) - \ell(\mathbf{x})(\beta(\mathbf{x}) - 2\sqrt{2}z) = 0, \quad \frac{1}{2}(ab - c^2)(pq - r^2 - s^2)^2 = 3\kappa_R(2\mu_R)^5, \quad (124)$$

with $\mathbf{x} = (x, y, z, 0)$ of Eq. (119). The isotropic elastic moduli are obtained in similar fashion as for of transverse isotropy, i.e., κ_R and μ_R are given by Eq. (122) but now $\det\mathbb{C} = \frac{1}{2}(ab - c^2)(pq - r^2 - s^2)^2$.

7.2 The Closest Cubic Tensor Using the Riemannian Norm

The optimality conditions that define the closest cubic tensor $\mathbb{C}_R^{\text{cub}} = 3\kappa_R\,\mathbb{J} + 2\mu_R\,\mathbb{L} + 2\eta_R\,\mathbb{M}$ to a given elasticity tensor $\mathbb{C}$ of lower symmetry, are obtained by minimizing the Riemannian distance $d_R(\mathbb{C}, \mathbb{C}_R^{\text{cub}})$:

$$\frac{\partial\,d_R^2}{\partial\kappa_R}(\mathbb{C}, \mathbb{C}_R^{\text{cub}}) = \frac{2}{\kappa_R}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\mathbb{C}_R^{\text{cub}})\,\mathbb{J}\right] = 0, \tag{125a}$$

$$\frac{\partial\,d_R^2}{\partial\mu_R}(\mathbb{C}, \mathbb{C}_R^{\text{cub}}) = \frac{2}{\mu_R}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\mathbb{C}_R^{\text{cub}})\,\mathbb{L}\right] = 0, \tag{125b}$$

$$\frac{\partial\,d_R^2}{\partial\eta_R}(\mathbb{C}, \mathbb{C}_R^{\text{cub}}) = \frac{2}{\eta_R}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\mathbb{C}_R^{\text{cub}})\,\mathbb{M}\right] = 0. \tag{125c}$$

As $\kappa_R > 0$, $\mu_R > 0$ and $\eta_R > 0$, these conditions are equivalent to

$$\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\mathbb{C}_R^{\text{cub}})\,\mathbb{V}\right] = 0, \qquad \mathbb{V} = \mathbb{J},\ \mathbb{L},\ \mathbb{M}. \tag{126}$$

In view of the partition $(26)_1$, combining these three equations yields, in the same manner as for Eq. (115),

$$\det\mathbb{C} = \det\mathbb{C}_R^{\text{cub}} = 3\kappa_R(2\mu_R)^3(2\eta_R)^2. \tag{127}$$

If $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{F}_1 + q\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4) + g\,\mathbb{G}$, i.e., is of tetragonal symmetry, then (126) $\mathbb{V} = \mathbb{J}$ reduces to

$$3\ln\delta(\tilde{\mathbf{x}}) - \ell(\tilde{\mathbf{x}})(\beta(\tilde{\mathbf{x}}) - 2\sqrt{2}\tilde{z}) = 0, \tag{128}$$

where $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z}, 0)$ with

$$\begin{pmatrix} \tilde{x} & \tilde{z} \\ \tilde{z} & \tilde{y} \end{pmatrix} = \boldsymbol{\Sigma}^t \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C} & \tilde{B} \end{pmatrix} \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^t \tilde{\boldsymbol{\Lambda}} \boldsymbol{\Sigma} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \boldsymbol{\Sigma}^t \tilde{\boldsymbol{\Lambda}} \boldsymbol{\Sigma}, \tag{129}$$

where $\boldsymbol{\Sigma}$ is as defined in Eq. (120) and

$$\tilde{\boldsymbol{\Lambda}} = \text{diag}\left(\frac{1}{\sqrt{3\kappa_R}}, \frac{1}{\sqrt{2\eta_R}}\right).$$

Again, since Σ is an orthogonal transformation, it follows that $\tilde{A}\tilde{B}-\tilde{C}^2=ab-c^2$, $\tilde{A}+\tilde{B}=a+b$, $\tilde{x}\tilde{y}-\tilde{z}^2=(ab-c^2)/(6\kappa_R\eta_R)$, and $\tilde{x}+\tilde{y}=(a+b)/(6\kappa_R\eta_R)$. Eq. (127) becomes

$$(ab-c^2)(pq-r^2)g^2 = 3\kappa_R(2\mu_R)^3(2\eta)^2. \tag{130}$$

Let ζ and σ be the nondimensional positive variables defined by

$$\zeta^2 = \frac{3\kappa_R}{2\eta_R}, \qquad \sigma^2 = \frac{\mu_R}{\eta_R},$$

then Eqs. (128) and (126) $\mathbb{V}=\mathbb{L}$ yield the following coupled two equations for ζ and σ

$$(\tilde{A}-\tilde{B}\zeta^2)\ln\frac{\tilde{A}+\tilde{B}\zeta^2+Q_1(\zeta)}{\tilde{A}+\tilde{B}\zeta^2-Q_1(\zeta)} - Q_1(\zeta)\ln\frac{\zeta^{4/3}\sigma^{-2}(\det\mathbb{C})^{1/3}}{ab-c^2} = 0, \tag{131a}$$

$$(p-q\sigma^2)\ln\frac{p+q\sigma^2+Q_2(\sigma)}{p+q\sigma^2-Q_2(\sigma)} - Q_2(\sigma)\ln\frac{\zeta^{-2}\sigma^{-1}(ab-c^2)}{g^2} = 0. \tag{131b}$$

where

$$Q_1(\zeta) = \sqrt{(\tilde{A}-\tilde{B}\zeta^2)^2+4\tilde{C}^2\zeta^2}, \qquad Q_2(\sigma) = \sqrt{(p-q\sigma)^2+4r^2\sigma^2}.$$

The three moduli of the closest cubic elasticity tensor are then given by

$$3\kappa_R = \zeta^{5/3}\sigma^{-1}(\det\mathbb{C})^{1/6}, \quad 2\mu_R = \zeta^{-1/3}\sigma(\det\mathbb{C})^{1/6}, \quad 2\eta_R = \zeta^{-1/3}\sigma^{-1}(\det\mathbb{C})^{1/6}.$$

7.3 The Closest Transversely Isotropic Tensor Using the Riemannian Norm

We want to find the closest, in the Riemannian metric, transversely isotropic tensor $\mathbb{C}_R^{\text{hex}} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3+\mathbb{E}_4) + f\,\mathbb{F} + g\,\mathbb{G}$ to a given elasticity tensor $\mathbb{C}$ of lower symmetry, i.e., find $a>0$, $b>0$ $f>0$, $g>0$ and c with $ab-c^2>0$ such that the Riemannian distance $d_R(\mathbb{C}, \mathbb{C}_R^{\text{hex}})$ is minimized.

The optimality conditions are

$$\frac{\partial\, d_R^2}{\partial a}(\mathbb{C}, \mathbb{C}_R^{\text{hex}}) = \frac{2}{ab-c^2}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\,\mathbb{C}_R^{\text{hex}})(b\,\mathbb{E}_1 - c\,\mathbb{E}_4)\right] = 0, \tag{132a}$$

$$\frac{\partial\, d_R^2}{\partial b}(\mathbb{C}, \mathbb{C}_R^{\text{hex}}) = \frac{2}{ab-c^2}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\,\mathbb{C}_R^{\text{hex}})(a\,\mathbb{E}_2 - c\,\mathbb{E}_3)\right] = 0, \tag{132b}$$

$$\frac{\partial\, d_R^2}{\partial c}(\mathbb{C}, \mathbb{C}_R^{\text{hex}}) = \frac{2}{ab-c^2}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\,\mathbb{C}_R^{\text{hex}})(-c\,\mathbb{E}_1 - c\,\mathbb{E}_2 + b\,\mathbb{E}_3 + a\,\mathbb{E}_4)\right] = 0, \tag{132c}$$

$$\frac{\partial\, d_R^2}{\partial f}(\mathbb{C}, \mathbb{C}_R^{\text{hex}}) = \frac{2}{f}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\,\mathbb{C}_R^{\text{hex}})\,\mathbb{F}\right] = 0, \tag{132d}$$

$$\frac{\partial\, d_R^2}{\partial g}(\mathbb{C}, \mathbb{C}_R^{\text{hex}}) = \frac{2}{g}\,\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\,\mathbb{C}_R^{\text{hex}})\,\mathbb{G}\right] = 0. \tag{132e}$$

As $a > 0$, $b > 0$, $ab - c^2 > 0$, $f > 0$ and $g > 0$, these five conditions are equivalent to

$$\text{tr}\left[\text{Log}(\mathbb{C}^{-1}\mathbb{C}_R^{\text{hex}})\,\mathbb{V}\right] = 0, \quad \mathbb{V} = b\,\mathbb{E}_1 - c\,\mathbb{E}_4, \; a\,\mathbb{E}_2 - c\,\mathbb{E}_3, \; \mathbb{E}_1 + \mathbb{E}_2, \; \mathbb{F}, \; \mathbb{G}. \quad (133)$$

Combining the conditions for $\mathbb{E}_1 + \mathbb{E}_2$, $\mathbb{F}$ and $\mathbb{G}$, and partition (32) implies the constraint

$$\det \mathbb{C} = \det \mathbb{C}_R^{\text{hex}}.$$

7.4 The Closest Tensors Using the Log-Euclidean Norm

The closest elasticity tensors according to the log-Euclidean metric follow by minimizing $\text{d}_L^2(\mathbb{C}, \mathbb{C}_{\text{sym}})$. The stationarity condition implies that $(\text{Log}\,\mathbb{C} - \text{Log}\,\mathbb{C}_{\text{sym}})$ is orthogonal (in the Euclidean sense) to the symmetry class, and hence

$$\text{Log}\,\mathbb{C}_{\text{sym}} = P_{\text{sym}}\,\text{Log}\,\mathbb{C}, \quad (134)$$

where the Euclidean projector is defined in Eq. (67). We may therefore write

$$\mathbb{C}_{\text{sym}} = \exp\left(P_{\text{sym}}\,\text{Log}\,\mathbb{C}\right). \quad (135)$$

This may be evaluated for the particular symmetries using the explicit expressions for P_{sym} in Eq. (68), combined with the formulas for the logarithm and exponential of elasticity tensors in Section 6. We note in particular the Euclidean property,

$$\text{d}_L^2\left(\mathbb{C}, \mathbb{C}_{\text{sym A}}\right) = \text{d}_L^2\left(\mathbb{C}, \mathbb{C}_{\text{sym B}}\right) + \text{d}_L^2\left(\mathbb{C}_{\text{sym B}}, \mathbb{C}_{\text{sym A}}\right), \quad \text{sym A} \geq \text{sym B}. \quad (136)$$

We note the following identity, which is a consequence of the requirement that $(\text{Log}\,\mathbb{C} - \text{Log}\,\mathbb{C}^{\text{sym}})$ is orthogonal to each of the basis tensors,

$$\text{tr}\,[\,\mathbb{V}_i\,\text{Log}\,\mathbb{C}\,] = \text{tr}\,[\,\mathbb{V}_i\,\text{Log}\,\mathbb{C}_{\text{sym}}]. \quad (137)$$

In the remainder of this section we apply this to the particular cases of isotropy and cubic isotropy, for which we derive explicit formulas for the closest moduli.

7.4.1 The Closest Isotropic Tensor Using the Log-Euclidean Norm

There are two basis tensors and therefore the general conditions (137) become, using Eq. (83) with $\mathbb{C}_L^{\text{iso}} = 3\kappa_L\,\mathbb{J} + 2\mu_L\,\mathbb{K}$,

$$\ln 3\kappa_L = \text{tr}\,[\,\mathbb{J}\,\text{Log}\,\mathbb{C}], \qquad \ln 2\mu_L = \tfrac{1}{5}\,\text{tr}\,[\,\mathbb{K}\,\text{Log}\,\mathbb{C}]. \quad (138)$$

Hence, we obtain explicit formulas

$$3\kappa_L = \exp\left(\text{tr}\,[\,\mathbb{J}\,\text{Log}\,\mathbb{C}]\right), \qquad 2\mu_L = \exp\left(\tfrac{1}{5}\,\text{tr}\,[\,\mathbb{K}\,\text{Log}\,\mathbb{C}]\right). \quad (139)$$

Using $\mathbb{I} = \mathbb{J} + \mathbb{K}$ and Eq. (B.14), it follows that

$$3\kappa_L(2\mu_L)^5 = \det \mathbb{C}. \quad (140)$$

7.4.2 The Closest Isotropic Tensor to a Given Cubic Tensor

For $\mathbb{C} = a\,\mathbb{J} + b\,\mathbb{L} + c\,\mathbb{M}$, we use Eq. (88) and $\mathbb{K} = \mathbb{L} + \mathbb{M}$ to simplify the conditions (138). After some algebra we find [17]

$$3\kappa_L = a, \qquad 2\mu_L = (b^3 c^2)^{1/5}.$$

These moduli coincide with those obtained for the Riemannian norm. This is a consequence of the fact that when two tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ commute under multiplication we have $\mathrm{Log}(\mathbb{C}_1\,\mathbb{C}_2^{-1}) = \mathrm{Log}\,\mathbb{C}_1 - \mathrm{Log}\,\mathbb{C}_2$.

7.4.3 The Closest Isotropic Tensor to a Given Transversely Isotropic Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + f\,\mathbb{F} + g\,\mathbb{G}$, we use Eqs. (101), (138)$_1$ and (A.22)$_1$ to get

$$\ln 3\kappa_L = \frac{1}{2}\ln(ab - c^2) + \frac{b - a + 4\sqrt{2}c}{12\gamma(\mathbf{a})}\ln\frac{a + b + 2\gamma(\mathbf{a})}{a + b - 2\gamma(\mathbf{a})}. \tag{141}$$

Thus,

$$3\kappa_L = \sqrt{ab - c^2}\left(\frac{a + b + \sqrt{(a - b)^2 + 4c^2}}{a + b - \sqrt{(a - b)^2 + 4c^2}}\right)^{\frac{b - a + 4\sqrt{2}c}{6\sqrt{(a-b)^2+4c^2}}}, \tag{142}$$

and the shear modulus then follows from Eq. (140) as

$$2\mu_L = \left[\frac{1}{3\kappa_L}(ab - c^2)\,f^2 g^2\right]^{1/5}. \tag{143}$$

Note that the Eq. (141) for κ_L can be cast in the form of Eq. (122)$_1$, where ξ satisfies an equation similar to Eq. (121),

$$(\bar{A} - \bar{B})\ln\frac{\bar{A} + \bar{B} + R(1)}{\bar{A} + \bar{B} - R(1)} - R(1)\ln\frac{\xi^{10/3}}{\bar{A}\bar{B} - \bar{C}^2} = 0. \tag{144}$$

7.4.4 The Closest Isotropic Tensor to a Given Tetragonal Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{F}_1 + q\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4) + g\,\mathbb{G}$, we again use Eqs. (101), (138)$_1$ and the identity (A.22)$_1$ to derive Eq. (142) for κ_L. The shear modulus is

$$2\mu_L = \left[\frac{1}{3\kappa_L}(ab - c^2)(pq - r^2)g^2\right]^{1/5}. \tag{145}$$

7.4.5 The Closest Isotropic Tensor to a Given Trigonal Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{R}_1 + q\,\mathbb{R}_2 + r(\mathbb{R}_3 + \mathbb{R}_4) + s(\mathbb{R}_5 + \mathbb{R}_6)$, the condition (142) for κ_L is again recovered, while μ_L follows from

$$2\mu_L = \left[\frac{1}{3\kappa_L}(ab - c^2)(pq - r^2 - s^2)^2\right]^{1/5}. \tag{146}$$

7.4.6 The Closest Cubic Tensor Using the Log-Euclidean Norm

Proceeding in the same way as for the isotropic case, in this case with three basis tensors, $\mathbb{C}_L^{\text{cub}} = 3\kappa_L \mathbb{J} + 2\mu_L \mathbb{L} + 2\eta_L \mathbb{M}$, we find that the closest cubic tensor has explicit solution

$$3\kappa_L = \exp\left(\text{tr}\,[\,\mathbb{J}\,\text{Log}\,\mathbb{C}\,]\right),$$

$$2\mu_L = \exp\left(\frac{1}{3}\,\text{tr}\,[\,\mathbb{L}\,\text{Log}\,\mathbb{C}\,]\right),$$

$$2\eta_L = \exp\left(\frac{1}{2}\,\text{tr}\,[\,\mathbb{M}\,\text{Log}\,\mathbb{C}\,]\right). \tag{147}$$

The moduli satisfy the same determinant constraint as for the Riemannian norm, in this case

$$3\kappa_L(2\mu_L)^3(2\eta_L)^2 = \det\,\mathbb{C}. \tag{148}$$

7.4.7 The Closest Cubic Tensor to a Given Tetragonal Tensor

For $\mathbb{C} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + p\,\mathbb{F}_1 + q\,\mathbb{F}_2 + r(\mathbb{F}_3 + \mathbb{F}_4) + g\,\mathbb{G}$, the bulk modulus κ_L is again given by Eq. (142). We use

$$\mathbb{L} = \mathbb{F}_1 + \mathbb{G}, \tag{149}$$

to obtain

$$2\mu_L = (pq - r^2)^{1/6}\,g^{2/3}\,\left(\frac{p + q + \sqrt{(p-q)^2 + 4r^2}}{p + q - \sqrt{(p-q)^2 + 4r^2}}\right)^{\frac{p-q}{2\sqrt{(p-q)^2 + 4r^2}}}. \tag{150}$$

The shear modulus η_L follows from

$$2\eta_L = \left[\frac{(ab - c^2)(pq - r^2)g^2}{3\kappa_L(2\mu_L)^3}\right]^{1/2}. \tag{151}$$

7.4.8 The Closest Transversely Isotropic Tensor Using the Log-Euclidean Norm

In this case there are five basis tensors, $\mathbb{C}_L^{\text{hex}} = a\,\mathbb{E}_1 + b\,\mathbb{E}_2 + c(\mathbb{E}_3 + \mathbb{E}_4) + f\,\mathbb{F} + g\,\mathbb{G}$. Using Eqs. (101) and (137), we have

$$f = \exp\left(\frac{1}{2}\,\text{tr}\,[\,\mathbb{F}\,\text{Log}\,\mathbb{C}\,]\right), \qquad g = \exp\left(\frac{1}{2}\,\text{tr}\,[\,\mathbb{G}\,\text{Log}\,\mathbb{C}\,]\right), \tag{152}$$

and the remaining three moduli follow from the identities

$$\mathfrak{l}_1(\mathbf{a}) = \text{tr}\,[\,\mathbb{E}_1\,\text{Log}\,\mathbb{C}\,], \quad \mathfrak{l}_2(\mathbf{a}) = \text{tr}\,[\,\mathbb{E}_2\,\text{Log}\,\mathbb{C}\,], \quad \mathfrak{l}_3(\mathbf{a}) = \frac{1}{2}\,\text{tr}\,[(\mathbb{E}_3 + \mathbb{E}_4)\,\text{Log}\,\mathbb{C}\,]. \tag{153}$$

Thus,

$$
\begin{cases}
a = \delta \frac{\cosh(\theta + \phi \cosh \theta)}{\cosh \theta}, \ b = \delta \frac{\cosh(\theta - \phi \cosh \theta)}{\cosh \theta}, \ c = \delta \frac{\sinh(\phi \cosh \theta)}{\cosh \theta}, & \text{if } \phi \neq 0, \\
a = \delta e^{\psi}, \quad b = \delta e^{-\psi}, \quad c = 0, & \text{if } \phi = 0,
\end{cases}
\tag{154}
$$

where

$$
\phi = \frac{1}{2} \operatorname{tr} \left[(\mathbb{E}_3 + \mathbb{E}_4) \operatorname{Log} \mathbb{C} \right], \qquad \psi = \frac{1}{2} \operatorname{tr} \left[(\mathbb{E}_1 - \mathbb{E}_2) \operatorname{Log} \mathbb{C} \right], \tag{155a}
$$

$$
\delta = \exp \left(\frac{1}{2} \operatorname{tr} \left[(\mathbb{E}_1 + \mathbb{E}_2) \operatorname{Log} \mathbb{C} \right] \right), \qquad \theta = \log \left[\frac{\psi}{\phi} + \left(\frac{\psi^2}{\phi^2} + 1 \right)^{1/2} \right]. \tag{155b}
$$

8 Application and Numerical Examples

We illustrate the methods developed above by considering an example of a general elasticity tensor, with no assumed symmetry. This type of data raises a problem typically encountered, i.e., find the optimal orientation of the symmetry axes in addition to finding the closest elastic tensors for a given symmetry axes or planes. We first present the data, and the isotropic approximations, and then consider the question of orientation.

8.1 Example

A complete set of 21 elastic constants were determined ultrasonically by François et al. [7]. The reader is referred to their paper for details of the measurement technique. The raw moduli are

$$
\mathbf{C} = \begin{pmatrix}
243 & 136 & 135 & 22 & 52 & -17 \\
136 & 239 & 137 & -28 & 11 & 16 \\
135 & 137 & 233 & 29 & -49 & 3 \\
22 & -28 & 29 & 133 & -10 & -4 \\
52 & 11 & -49 & -10 & 119 & -2 \\
-17 & 16 & 3 & -4 & -2 & 130
\end{pmatrix} \quad \text{(GPa)}. \tag{156}
$$

We first search for the presence of symmetry planes, which are an indicator of underlying symmetry. Following Cowin and Mehrabadi [35], define **A** and **B** by

$$
A_{ij} = C_{ijkk}, \qquad B_{ij} = C_{ikjk}. \tag{157}
$$

If **A** and **B** have no common eigenvectors then there are no planes of reflection symmetry and the material has no effective elastic symmetry [48]. We find that the two sets of eigenvectors are not coincident, and the smallest angle between any pair from the two sets of eigenvectors is 16°. Materials with symmetry higher than orthorhombic have five or fewer distinct eigenvalues Λ_I [20]. For the given moduli, we find $\Lambda_I = 47, 79, 244, 285, 312, 512$, consistent with the material having no symmetry plane.

We next consider isotropic approximations to the 21 moduli. The Euclidean and log-Euclidean approximations follow from Eqs. (68a) and (139), respectively. The

numerical procedure for finding the closest, in the Riemannian norm, isotropic tensor $\mathbb{C}_R^{\text{iso}} = 3\kappa_R \mathbb{J} + 2\mu_R \mathbb{K}$ is as follows. Define the function

$$F(3\kappa_R, 2\mu_R) = \text{tr}\left[\text{Log}(\mathbb{C}^{-1}(3\kappa_R \mathbb{J} + 2\mu_R \mathbb{K})) \mathbb{J}\right].$$

By the equal determinant rule (116), we have $3\kappa_R(2\mu_R)^5 = \det \mathbb{C}$, and therefore we need only solve the equation for one of κ_R or μ_R. In practice, we solve

$$F(3\kappa_R, (\tfrac{\det \mathbb{C}}{3\kappa_R})^{1/5}) = 0,$$

using Newton's method.

The isotropic approximations are

	κ	μ
Euclidean, $\mathbb{C}$	170.11	96.87
Euclidean, $\mathbb{S}$	169.33	55.81
log-Euclidean	169.84	75.91
Riemannian	169.69	75.92

(158)

The Euclidean projection is obviously not invariant under inversion. Another way to see this is to consider the product

$$\widehat{\mathbf{S}}_{\text{iso}}\widehat{\mathbf{C}}_{\text{iso}} \approx 1.00\widehat{\mathbf{I}} + 0.74\widehat{\mathbf{K}}, \tag{159}$$

where $\widehat{\mathbf{C}}_{\text{iso}}$ and $\widehat{\mathbf{S}}_{\text{iso}}$ are the isotropic projections for the stiffness $\widehat{\mathbf{C}}$ and compliance $\widehat{\mathbf{S}} = \widehat{\mathbf{C}}^{-1}$, respectively. The product (159) is an isotropic tensor, as expected, but not the identity. The analogous product $\widehat{\mathbf{S}}_{\text{sym}}\widehat{\mathbf{C}}_{\text{sym}}$ of the Euclidean stiffness and compliance projections becomes closer to the identity as the symmetry is reduced, although not uniformly. As one measure, we note that the parameter $\det(\widehat{\mathbf{S}}_{\text{sym}}\widehat{\mathbf{C}}_{\text{sym}})$, which is unity for the full triclinic matrices, takes the values 4.2, 4.6, 6.6, 4.5, 8.1, 5.3, 15.8 for mon, ort, trig, tet, hex, cub and iso, respectively.

8.2 Orientation Effects

This set of moduli present the general problem of finding the optimal orientation of the symmetry axes, for given symmetries. This can be done by a "brute force" approach of searching over all possible orientations of the basis vectors. In practice this is achieved using Euler angles $(\theta_1, \theta_2, \theta_3)$ to transform from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \to \{\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ by first rotating about the $\mathbf{e}_3$ axis by θ_1, then about the intermediate $\mathbf{e}_1''$ axis by θ_2, and finally about the $\mathbf{e}_3'$ axis by θ_3. The moduli transform as $\mathbf{C} \to \mathbf{C}'$, and for each triple $(\theta_1, \theta_2, \theta_3)$ the closest elasticity tensors $\mathbb{C}_{\text{sym}}$ of different symmetries are found. Numerically, we fix the symmetries by reference to the original basis, with $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Let d be any of the three distance functions, and define

$$\rho_{\text{sym}}(\theta_1, \theta_2, \theta_3) = \frac{\text{d}^2\left(\mathbb{C}_{\text{sym}},\ \mathbb{C}_{\text{iso}}\right)}{\text{d}^2\left(\mathbb{C},\ \mathbb{C}_{\text{iso}}\right)}. \tag{160}$$

Thus, $0 < \rho_{\text{sym}} \leq 1$, with equality only if the rotated $\mathbf{C}'$ has symmetry sym. The preliminary analysis above indicates the absence of any symmetry so we expect ρ_{sym} to be less than unity. We also define

$$\rho_{\text{sym}}^* = \max_{\theta_1, \theta_2, \theta_3} \rho_{\text{sym}}(\theta_1, \theta_2, \theta_3). \tag{161}$$

Table 1 ρ_{sym} calculated using the Euclidean projection of the stiffness $\mathbb{C}$

d_F, $\mathbb{C}$	ρ_{cub}	ρ_{hex}	ρ_{tet}	ρ_{ort}	ρ_{trig}	ρ_{mon}
cub	*0.91*	0.60	0.91	0.92	0.61	0.94
hex	0.85	*0.60*	0.91	0.87	0.62	0.93
tet	0.01	0.45	*0.92*	0.88	0.47	0.94
ort	0.03	0.54	0.90	*0.94*	0.58	0.95
trig	0.12	0.20	0.21	0.20	*0.95*	0.22
mon	0.02	0.07	0.35	0.81	0.08	*0.98*

The diagonal elements in italics are the values of ρ_{sym}^* for the symmetries indicated in the left column. The other numbers in each row are the values of ρ_{sym} evaluated at the same orientation as ρ_{sym}^*.

Orientations at which $\rho_{\text{sym}} = \rho_{\text{sym}}^*$ are candidates for symmetry axes that best approximate the moduli.

Tables 1 and 2 list the results using the Euclidean projection for the stiffness and compliance, respectively. Table 3 gives the results for the log-Euclidean distance. The results in Tables 1, 2 and 3 were obtained using $60 \times 60 \times 60$ discretized Euler angles. In general, larger values give an indication of the proximity of the symmetry. These numbers taken together suggests that the material is not well approximated by hex, but that cub and certainly tet are reasonable candidates for approximating symmetries. The analogous log-Euclidean computations, given in Table 3, reinforce this view. Also, $\rho_{\text{cub}}^* = 0.95$ for the Riemannian norm, using the method below. We focus on the cubic approximation for the remainder of this section.

8.3 Cubic Approximations

We note some properties of the optimally oriented cubic approximations. First, the decomposition (70) for the Euclidean cubic projection has an interesting implication. Noting that

$$2\mathbb{L} - 3\mathbb{M} = 5\mathbb{J} + 2\mathbb{K} - 5\left(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b} + \mathbf{c} \otimes \mathbf{c} \otimes \mathbf{c} \otimes \mathbf{c}\right), \quad (162)$$

and using Eq. (58), we may write the cubic length

$$\|\mathbb{C}_{\text{cub}}\|^2 = 9\kappa^2 + 20\mu^2 + \tfrac{5}{6}\left(3\kappa + 4\mu - c_{aa} - c_{bb} - c_{cc}\right)^2, \quad (163)$$

Table 2 ρ_{sym} calculated using the Euclidean projection of the compliance $\mathbb{C}^{-1}$

d_F, $\mathbb{S}$	ρ_{cub}	ρ_{hex}	ρ_{tet}	ρ_{ort}	ρ_{trig}	ρ_{mon}
cub	*0.82*	0.54	0.83	0.97	0.54	0.99
hex	0.18	*0.71*	0.90	0.83	0.72	0.97
tet	0.82	0.35	*0.96*	0.97	0.35	0.98
ort	0.82	0.35	0.96	*0.97*	0.35	0.98
trig	0.21	0.32	0.33	0.42	*0.84*	0.43
mon	0.82	0.54	0.83	0.97	0.54	*0.99*

The elements are determined in the same manner as in Table 1.

Table 3 ρ_{sym} calculated using the log-Euclidean distance function, determined in the same manner as in Table 1

d_L	ρ_{cub}	ρ_{hex}	ρ_{tet}	ρ_{ort}	ρ_{trig}	ρ_{mon}
cub	*0.92*	0.68	0.94	0.96	0.69	0.96
hex	0.87	*0.69*	0.93	0.92	0.70	0.96
tet	0.11	0.47	*0.95*	0.71	0.48	0.96
ort	0.09	0.20	0.37	*0.96*	0.21	0.98
trig	0.26	0.43	0.43	0.44	*0.94*	0.47
mon	0.91	0.60	0.92	0.96	0.62	*0.99*

In particular the diagonal elements are ρ_{sym}^*.

where κ and μ are Fedorov's isotropic moduli, and $c_{aa} = \langle \mathbb{C}, \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a} \rangle = C_{ijkl} a_i a_j a_k a_l$, etc. Consider $\| \mathbb{C}_{\text{cub}} \|$ as a function of the orientation of the cube axes. Since κ and μ are isotropic invariants, and $c_{aa} > 0, c_{bb} > 0, c_{cc} > 0$ on account of the positive definite nature of $\mathbb{C}$, it follows that the largest length occurs when $c_{aa} + c_{bb} + c_{cc}$ achieves it smallest value. This implies that *the best cubic approximation in the Euclidean sense occurs in the coordinate system with smallest value of* $(c'_{11} + c'_{22} + c'_{33})$.

The closest cubic material for log-Euclidean distance function is that which minimizes $d_L(\mathbb{C}, \mathbb{C}_{\text{cub}})$. Using the Euclidean property, see Eq. (136), that

$$d_L^2(\mathbb{C}, \mathbb{C}_{\text{cub}}) = d_L^2(\mathbb{C}, \mathbb{C}_{\text{iso}}) - d_L^2(\mathbb{C}_{\text{cub}}, \mathbb{C}_{\text{iso}}), \tag{164}$$

and the fact that $d_L(\mathbb{C}, \mathbb{C}_{\text{iso}})$ is unchanged under rotation, it follows that the optimal $\mathbb{C}_{\text{cub}}$ *maximizes* $d_L(\mathbb{C}_{\text{cub}}, \mathbb{C}_{\text{iso}})$. Let μ_{Li} denote the isotropic modulus, from Eq. (139)$_2$, then since the bulk modulus κ_L is the same for isotropy and cubic symmetry, it follows that

$$d_L(\mathbb{C}_{\text{cub}}, \mathbb{C}_{\text{iso}}) = \sqrt{\tfrac{15}{2}} \left| \log \frac{\mu_L}{\mu_{Li}} \right|. \tag{165}$$

Therefore, the optimal orientation is that for which $\frac{1}{2}(\frac{\mu_L}{\mu_{Li}} + \frac{\mu_{Li}}{\mu_L})$ achieves its largest value.

Alternatively, if we define the Euclidean cubic approximation for $\mathbb{C}$ as $\mathbb{C}_F^{\text{cub}} = 3\kappa_F \mathbb{J} + 2\mu_F \mathbb{L} + 2\eta_F \mathbb{L}$, then $\kappa_F = \kappa$ and the optimal μ_F and η_F maximize $|\mu_F - \eta_F|$ subject to the additive constraint $\frac{3}{5}\mu_F + \frac{2}{5}\eta_F = \mu$, where μ is the isotropic (Fedorov) shear modulus. By comparison, the optimal log-Euclidean moduli maximize $|\log \mu_L - \log \eta_L|$ subject to the constraint $\frac{3}{5}\log \mu_L + \frac{2}{5}\log \eta_L = \log \mu$.

The closest cubic tensor in the Riemannian norm, $\mathbb{C}_R^{\text{cub}} = 3\kappa_R \mathbb{J} + 2\mu_R \mathbb{L} + 2\eta_R \mathbb{M}$, is obtained using a numerical scheme similar to that for the isotropic case. Define the functions

$$F_1(3\kappa_R, 2\mu_R, 2\eta_R) = \text{tr}\,[\text{Log}(\mathbb{C}^{-1}(3\kappa_R \mathbb{J} + 2\mu_R \mathbb{L} + 2\eta_R \mathbb{M}))\,\mathbb{J}],$$

$$F_2(3\kappa_R, 2\mu_R, 2\eta_R) = \text{tr}\,[\text{Log}(\mathbb{C}^{-1}(3\kappa_R \mathbb{J} + 2\mu_R \mathbb{L} + 2\eta_R \mathbb{M}))\,\mathbb{L}].$$

Using the condition (127) of equality of determinants we can eliminate one of the three unknowns, and therefore only need to solve two equations for κ_R and μ_R

$$F_1\left(3\kappa_R, 2\mu_R, \left(\frac{\det \mathbb{C}}{3\kappa_R(2\mu_R)^3}\right)^{1/2}\right) = 0, \qquad F_2\left(3\kappa_R, 2\mu_R, \left(\frac{\det \mathbb{C}}{3\kappa_R(2\mu_R)^3}\right)^{1/2}\right) = 0.$$

These equations are solved by the two-dimensional Newton's method.

The numerical search for the optimal cubic approximation indicates that the orientation which yields the maximum ρ_{cub}^* coincides for the three distance functions for this set of moduli. The optimal cubic moduli are as follows

$$
\begin{array}{l|ccc}
 & \kappa & \mu & \eta \\
\hline
\text{Eucl, } \mathbb{C} & 170.1 & 139.7 & 32.6 \\
\text{Eucl, } \mathbb{S} & 169.3 & 135.1 & 29.7 \\
\text{log-Eucl} & 169.7 & 137.5 & 31.2 \\
\text{Riemannian} & 169.8 & 138.1 & 30.9
\end{array}
\tag{166}
$$

The Euclidean moduli of the first row are in agreement with François et al. [7], who used a method based on transformation groups to effect the projection. Note that the Euclidean projection for compliance again gives a different answer than for stiffness. Also, the bulk modulus for the log-Euclidean distance is the same as for the isotropic approximation, see Eq. (158).

This example illustrates the practical application of all three distance functions. The procedure is similar for materials in which the closest or best symmetry for approximation might be, for instance, transversely isotropic. The Euclidean projection follow from Eq. (68c) and the log-Euclidean moduli from Eqs. (152) and (153). The closest transversely isotropic tensor in the Riemannian norm can be found in a similar manner as above – in this case a system of four nonlinear equations for four unknowns is obtained, which can be solved by Newton's method.

9 Conclusions

Three distance functions for elastic tensors have been introduced and their application to elasticity examined. The Euclidean distance function is the most commonly used, but it lacks the property of invariance under inversion. Two distance functions with this important property have been described in detail: the log-Euclidean and the Riemannian norms. For each of the three distance functions we have developed a coordinate-independent procedure to find the closest elasticity tensor of a given symmetry to a set of moduli of a lower symmetry.

For the Euclidean distance function we have described a projection scheme using basis tensors, similar to vector space projections. Explicit forms for the projection operator have been given for isotropic, cubic, transversely isotropic, tetragonal and trigonal symmetries. The projection method was compared with the group transformation scheme and with the 21-dimensional approach. We also described the form of the projection operator using 6×6 matrices, which is easily implemented on a computer, and derived explicit expressions for the tensor complements/residues, and for the lengths of the projections. This allows one to decompose, in a Pythagorean sense, the elastic stiffness or compliance, although with different results for each.

Detailed and practical results have been presented for applying the logarithmic based distance functions to the same problem of approximating using a higher elastic symmetry. Both the log-Euclidean and the Riemannian distance functions are invariant under inversion. As such they offer an unambiguous method for determining the closest moduli of a given symmetry, providing unique results regardless of whether stiffness or compliance is considered.

Calculation of the logarithmic distance functions requires evaluation of the logarithm, square root and exponential of tensors. Semi-analytical procedures for obtaining these for the most important, higher, elastic symmetries have been derived. For the first time, practical expressions are available for computing the logarithmic distances for symmetries higher than orthotropic, that is: isotropic, cubic, hexagonal (transversely isotropic), tetragonal, and trigonal. The numerical example considered illustrates how a data set with no symmetry can be reduced to find the optimally fitting cubic moduli. The Euclidean/Frobenius norm yields different results based on whether stiffness or compliances is use, but the log-Euclidean and the Riemannian distance functions provide unique answers.

Appendix

A The Closest Tensors Using the Euclidean Norm

Results for the Euclidean projection are summarized. We first give the explicit form of the projection for the various symmetries.

A.1 Summary of the Euclidean Projections

The projections onto the symmetry classes are expressed in terms of the elements of the six-dimensional second-order tensor $\widehat{\mathbf{C}}$ of Eq. (7). For the purpose of subsequent calculations, we take $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The symmetry classes are characterized by a distinct direction, the direction of the monoclinic symmetry plane, the direction perpendicular to the plane spanned by the normals to the planes of reflection symmetry for trigonal and tetragonal symmetry, the axis of transversely isotropic symmetry, a cube axis. In each case we take this direction as $\mathbf{e}_3$.

A.1.1 Monoclinic and Orthorhombic Symmetry

$$\widehat{\mathbf{C}}_{\mathrm{mon}} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & \hat{c}_{16} \\ & \hat{c}_{22} & \hat{c}_{23} & 0 & 0 & \hat{c}_{26} \\ & & \hat{c}_{33} & 0 & 0 & \hat{c}_{36} \\ & & & \hat{c}_{44} & \hat{c}_{45} & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \hat{c}_{55} & 0 \\ & & & & & \hat{c}_{66} \end{pmatrix}, \tag{A.1}$$

$$\widehat{\mathbf{C}}_{\mathrm{ort}} = \begin{pmatrix} \hat{c}_{11} & \hat{c}_{12} & \hat{c}_{13} & 0 & 0 & 0 \\ & \hat{c}_{22} & \hat{c}_{23} & 0 & 0 & 0 \\ & & \hat{c}_{33} & 0 & 0 & 0 \\ & & & \hat{c}_{44} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \hat{c}_{55} & 0 \\ & & & & & \hat{c}_{66} \end{pmatrix}. \tag{A.2}$$

A.1.2 Tetragonal Symmetry

$\mathbb{C}_{\text{tet}}$ as defined in Eq. (37) becomes in 6×6 notation

$$\widehat{\mathbf{C}}_{\text{tet}} = \begin{pmatrix} \frac{1}{2}(b+q) & \frac{1}{2}(b-q) & \frac{1}{\sqrt{2}}c & 0 & 0 & \frac{r}{\sqrt{2}} \\ & \frac{1}{2}(b+q) & \frac{1}{\sqrt{2}}c & 0 & 0 & -\frac{r}{\sqrt{2}} \\ & & a & 0 & 0 & 0 \\ & & & g & 0 & 0 \\ & \text{SYM} & & & g & 0 \\ & & & & & p \end{pmatrix}, \tag{A.3}$$

and the norm is

$$\| \mathbb{C}_{\text{tet}} \|^2 = a^2 + b^2 + 2c^2 + p^2 + q^2 + 2r^2 + 2g^2. \tag{A.4}$$

Performing the inner products using Eq. (64), with $N = 7$ and $\{ \mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4, \mathbb{V}_5, \mathbb{V}_6, \mathbb{V}_6 \} = \{ \mathbb{E}_1, \mathbb{E}_2, (\mathbb{E}_3 + \mathbb{E}_4), \mathbb{F}_1, \mathbb{F}_2, (\mathbb{F}_3 + \mathbb{F}_4), \mathbb{G} \}$ gives

$$a = \hat{c}_{33}, \quad b = \frac{1}{2}(\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{12}), \quad c = \frac{1}{\sqrt{2}}(\hat{c}_{13} + \hat{c}_{23}), \quad g = \frac{1}{2}(\hat{c}_{44} + \hat{c}_{55}),$$

$$p = \hat{c}_{66}, \quad q = \frac{1}{2}(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{12}), \quad r = \frac{1}{\sqrt{2}}(\hat{c}_{16} - \hat{c}_{26}). \tag{A.5}$$

In summary, the projection is

$$\widehat{\mathbf{C}}_{\text{tet}} = \begin{pmatrix} \frac{1}{2}(\hat{c}_{11} + \hat{c}_{22}) & \hat{c}_{12} & \frac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & 0 & 0 & \frac{1}{2}(\hat{c}_{16} - \hat{c}_{26}) \\ & \frac{1}{2}(\hat{c}_{11} + \hat{c}_{22}) & \frac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & 0 & 0 & \frac{1}{2}(\hat{c}_{26} - \hat{c}_{16}) \\ & & \hat{c}_{33} & 0 & 0 & 0 \\ & & & \frac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & 0 & 0 \\ & \text{SYM} & & & \frac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & 0 \\ & & & & & \hat{c}_{66} \end{pmatrix}. \tag{A.6}$$

A.1.3 Trigonal Symmetry

$\mathbb{C}_{\text{trig}}$ as defined in Eq. (45) becomes

$$\widehat{\mathbf{C}}_{\text{trig}} = \begin{pmatrix} \frac{1}{2}(b+p) & \frac{1}{2}(b-p) & \frac{1}{\sqrt{2}}c & \frac{1}{\sqrt{2}}r & \frac{1}{\sqrt{2}}s & 0 \\ & \frac{1}{2}(b+p) & \frac{1}{\sqrt{2}}c & -\frac{1}{\sqrt{2}}r & -\frac{1}{\sqrt{2}}s & 0 \\ & & a & 0 & 0 & 0 \\ & & & q & 0 & -s \\ & \text{SYM} & & & q & r \\ & & & & & p \end{pmatrix}, \tag{A.7}$$

and the norm is

$$\| \mathbb{C}_{\text{tet}} \|^2 = a^2 + b^2 + 2c^2 + 2p^2 + 2q^2 + 4r^2 + 4s^2, \tag{A.8}$$

where

$$a = \hat{c}_{33}, \quad b = \tfrac{1}{2}(\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{12}), \quad c = \tfrac{1}{\sqrt{2}}(\hat{c}_{13} + \hat{c}_{23}), \quad p = \hat{c}_{66},$$
$$q = \tfrac{1}{2}(\hat{c}_{44} + \hat{c}_{55}), \quad r = \tfrac{1}{2\sqrt{2}}(\hat{c}_{14} - \hat{c}_{24}) + \tfrac{1}{2}\hat{c}_{56}, \quad s = \tfrac{1}{2\sqrt{2}}(\hat{c}_{15} - \hat{c}_{25}) - \tfrac{1}{2}\hat{c}_{46}. \tag{A.9}$$

In this case,

$$\widehat{\mathbf{C}}_{\text{trig}} = \begin{pmatrix} \hat{c}_{11}^* & \hat{c}_{11}^* - \hat{c}_{66}^* & \tfrac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & \tfrac{1}{\sqrt{2}}\hat{c}_{56}^* & -\tfrac{1}{\sqrt{2}}\hat{c}_{46}^* & 0 \\ & \hat{c}_{11}^* & \tfrac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & -\tfrac{1}{\sqrt{2}}\hat{c}_{56}^* & \tfrac{1}{\sqrt{2}}\hat{c}_{46}^* & 0 \\ & & \hat{c}_{33} & 0 & 0 & 0 \\ & & & \tfrac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & 0 & \hat{c}_{46}^* \\ & \text{SYM} & & & \tfrac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & \hat{c}_{56}^* \\ & & & & & \hat{c}_{66}^* \end{pmatrix}, \tag{A.10}$$

where

$$\hat{c}_{11}^* = \frac{1}{8}\left(3\hat{c}_{11} + 3\hat{c}_{22} + 2\hat{c}_{12} + 2\hat{c}_{66}\right), \tag{A.11a}$$

$$\hat{c}_{66}^* = \frac{1}{4}\left(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{12} + 2\hat{c}_{66}\right), \tag{A.11b}$$

$$\hat{c}_{46}^* = \frac{1}{2}\left(\hat{c}_{46} - \frac{1}{\sqrt{2}}\hat{c}_{15} + \frac{1}{\sqrt{2}}\hat{c}_{25}\right), \tag{A.11c}$$

$$\hat{c}_{56}^* = \frac{1}{2}\left(\hat{c}_{56} + \frac{1}{\sqrt{2}}\hat{c}_{14} - \frac{1}{\sqrt{2}}\hat{c}_{24}\right). \tag{A.11d}$$

A.1.4 Transverse Isotropy

With $\mathbb{C}_{\text{hex}}$ defined as in Eq. (33), we have

$$\widehat{\mathbf{C}}_{\text{hex}} = \begin{pmatrix} \tfrac{1}{2}(b + f) & \tfrac{1}{2}(b - f) & \tfrac{1}{\sqrt{2}}c & 0 & 0 & 0 \\ & \tfrac{1}{2}(b + f) & \tfrac{1}{\sqrt{2}}c & 0 & 0 & 0 \\ & & a & 0 & 0 & 0 \\ & & & g & 0 & 0 \\ & \text{SYM} & & & g & 0 \\ & & & & & f \end{pmatrix}, \tag{A.12}$$

and

$$\| \mathbb{C}_{\text{hex}} \|^2 = a^2 + b^2 + 2c^2 + 2f^2 + 2g^2. \tag{A.13}$$

The projection onto the basis tensors $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4, \mathbb{V}_5\} = \{\mathbb{E}_1, \mathbb{E}_2, (\mathbb{E}_3 + \mathbb{E}_4),$ $\mathbb{F}, \mathbb{G}\}$, yields a, b, c and g as given in Eq. (A.5) and

$$f = \frac{1}{2}(p + q) = \frac{1}{4}(\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{66} - 2\hat{c}_{12}). \tag{A.14}$$

Thus,

$$
\widehat{\mathbf{C}}_{\text{hex}} = \begin{pmatrix} \hat{c}_{11}^* & \hat{c}_{11}^* - \hat{c}_{66}^* & \frac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & 0 & 0 & 0 \\ & \hat{c}_{11}^* & \frac{1}{2}(\hat{c}_{13} + \hat{c}_{23}) & 0 & 0 & 0 \\ & & \hat{c}_{33} & 0 & 0 & 0 \\ & & & \frac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & 0 & 0 \\ & \text{SYM} & & & \frac{1}{2}(\hat{c}_{44} + \hat{c}_{55}) & 0 \\ & & & & & \hat{c}_{66}^* \end{pmatrix}. \tag{A.15}
$$

A.1.5 Cubic Symmetry

The 6-D matrix associated with $\mathbb{C}_{\text{cub}}$ defined in Eq. (28) is

$$
\widehat{\mathbf{C}}_{\text{cub}} = \begin{pmatrix} \frac{1}{3}(a+2c) & \frac{1}{3}(a-c) & \frac{1}{3}(a-c) & 0 & 0 & 0 \\ & \frac{1}{3}(a+2c) & \frac{1}{3}(a-c) & 0 & 0 & 0 \\ & & \frac{1}{3}(a+2c) & 0 & 0 & 0 \\ & & & b & 0 & 0 \\ & \text{SYM} & & & b & 0 \\ & & & & & b \end{pmatrix}, \tag{A.16}
$$

with length

$$
\| \mathbb{C}_{\text{cub}} \|^2 = a^2 + 3b^2 + 2c^2 . \tag{A.17}
$$

The Euclidean projection onto cubic symmetry follows from Eq. (64) with $N = 3$ and $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\} = \{\mathbb{J}, \mathbb{L}, \mathbb{M}\}$, and gives

$$
a = \frac{1}{3}(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} + 2\hat{c}_{12} + 2\hat{c}_{13} + 2\hat{c}_{23}) \tag{A.18a}
$$

$$
b = \frac{1}{3}(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66}) , \tag{A.18b}
$$

$$
c = \frac{1}{3}(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} - \hat{c}_{12} - \hat{c}_{23} - \hat{c}_{31}). \tag{A.18c}
$$

Note that $a = 3\kappa$ where κ defined in Eq. $(23)_1$. In summary, the projection is

$$
\widehat{\mathbf{C}}_{\text{cub}} = \begin{pmatrix} \hat{c}_{11}^C & \hat{c}_{12}^C & \hat{c}_{12}^C & 0 & 0 & 0 \\ & \hat{c}_{11}^C & \hat{c}_{12}^C & 0 & 0 & 0 \\ & & \hat{c}_{11}^C & 0 & 0 & 0 \\ & & & \hat{c}_{66}^C & 0 & 0 \\ & \text{SYM} & & & \hat{c}_{66}^C & 0 \\ & & & & & \hat{c}_{66}^C \end{pmatrix}, \tag{A.19}
$$

where

$$
\hat{c}_{11}^C = \frac{1}{3}(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33}), \quad \hat{c}_{12}^C = \frac{1}{3}(\hat{c}_{12} + \hat{c}_{13} + \hat{c}_{23}), \quad \hat{c}_{66}^C = \frac{1}{3}(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66}). \tag{A.20}
$$

A.1.6 Isotropy

With $\mathbb{C}_{\text{iso}}$ as given by Eq. (22), we have

$$
\widehat{\mathbf{C}}_{\text{iso}} = \begin{pmatrix}
\kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\
& \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\
& & \kappa + \frac{4}{3}\mu & 0 & 0 & 0 \\
& & & 2\mu & 0 & 0 \\
& \text{SYM} & & & 2\mu & 0 \\
& & & & & 2\mu
\end{pmatrix},
\tag{A.21}
$$

where κ and μ are defined in Eq. (23). These follow from the general projection formula (64) with $N = 2$ and $\{\mathbb{V}_1, \mathbb{V}_2\} = \{\mathbb{J}, \mathbb{K}\}$.

A.2 The Space Between Isotropy and Transverse Isotropy

We use a Gram–Schmid approach to define an orthogonal subspace between isotropy and transverse isotropy (hexagonal symmetry). Noting that

$$
\mathbb{J} = \frac{1}{3}\left[\mathbb{E}_1 + 2\mathbb{E}_2 + \sqrt{2}(\mathbb{E}_3 + \mathbb{E}_4)\right], \quad \mathbb{K} = \frac{1}{3}\left[2\mathbb{E}_1 + \mathbb{E}_2 - \sqrt{2}(\mathbb{E}_3 + \mathbb{E}_4) + 3\mathbb{F} + 3\mathbb{G}\right], \tag{A.22}
$$

we introduce

$$
\mathbb{L}_1 = \mathbb{G} - \mathbb{F}, \tag{A.23a}
$$

$$
\mathbb{L}_2 = 2\mathbb{E}_1 - 2\mathbb{E}_2 + \frac{1}{\sqrt{2}}(\mathbb{E}_3 + \mathbb{E}_4), \tag{A.23b}
$$

$$
\mathbb{L}_3 = 8\mathbb{E}_1 + 4\mathbb{E}_2 - 4\sqrt{2}(\mathbb{E}_3 + \mathbb{E}_4) - 3\mathbb{F} - 3\mathbb{G}. \tag{A.23c}
$$

Thus, $\langle \mathbb{L}_i, \mathbb{L}_j \rangle = 0$, $i \neq j$, and $\langle \mathbb{L}_i, \mathbb{J} \rangle = 0$, $\langle \mathbb{L}_i, \mathbb{K} \rangle = 0$, $i = 1, 2, 3$. Hence $\{\mathbb{V}_i, i = 1, 2, \ldots 5\} = \{\mathbb{J}, \mathbb{K}, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3\}$ form a basis for Hex, with $\mathbf{D} = \text{diag}(1, 5, 4, 9, 180)$. Eq. (67) gives

$$
P_{\text{hex}} = \mathbb{J} \otimes \mathbb{J} + \frac{1}{5}\mathbb{K} \otimes \mathbb{K} + \frac{1}{4}\mathbb{L}_1 \otimes \mathbb{L}_1 + \frac{1}{9}\mathbb{L}_2 \otimes \mathbb{L}_2 + \frac{1}{180}\mathbb{L}_3 \otimes \mathbb{L}_3
$$

$$
= P_{\text{iso}} + P_{\text{hex/iso}}. \tag{A.24}
$$

Thus, we have the explicit decomposition into orthogonal complements $\mathbb{C}_{\text{hex}} = \mathbb{C}_{\text{iso}} + \mathbb{C}_{\text{hex/iso}}$, where

$$
\mathbb{C}_{\text{hex/iso}} = d_1\mathbb{L}_1 + d_2\mathbb{L}_2 + d_3\mathbb{L}_3, \qquad \left\|\mathbb{C}_{\text{hex/iso}}\right\|^2 = 4d_1^2 + 9d_2^2 + 180d_3^2, \tag{A.25}
$$

with

$$
d_1 = \frac{1}{2}(g - f), \quad d_2 = \frac{1}{9}(2a - 2b + \sqrt{2}c), \quad d_3 = \frac{1}{90}(4a + 2b - 4\sqrt{2}c - 3f - 3g), \tag{A.26}
$$

and $a, \ldots, g$ are given by A.5 and A.14. Eliminating the latter gives

$$d_1 = \frac{1}{8}\left[2(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{12}) - (\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{66})\right], \tag{A.27a}$$

$$d_2 = \frac{1}{9}\left[\hat{c}_{13} + \hat{c}_{23} + 2\hat{c}_{33} - (\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{12})\right], \tag{A.27b}$$

$$d_3 = \frac{1}{360}\left[\hat{c}_{11} + \hat{c}_{22} + 14\hat{c}_{12} + 16(\hat{c}_{33} - \hat{c}_{13} - \hat{c}_{23}) - 6(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66})\right]. \tag{A.27c}$$

A.3 Complements

Some orthogonal complements or residues [3] between the symmetry classes are presented here. These lead to the explicit expressions for elastic lengths in the next subsection, Eqs. (A.30) and (A.31). Thus,

$$\widehat{\mathbf{C}}_{\perp\mathrm{mon}} = \begin{pmatrix} 0 & 0 & 0 & \hat{c}_{14} & \hat{c}_{15} & 0 \\ & 0 & 0 & \hat{c}_{24} & \hat{c}_{25} & 0 \\ & & 0 & \hat{c}_{34} & \hat{c}_{35} & 0 \\ & & & 0 & 0 & \hat{c}_{46} \\ & \mathrm{S\ Y\ M} & & & 0 & \hat{c}_{56} \\ & & & & & 0 \end{pmatrix}, \tag{A.28a}$$

$$\widehat{\mathbf{C}}_{\mathrm{mon/ort}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \hat{c}_{16} \\ & 0 & 0 & 0 & 0 & \hat{c}_{26} \\ & & 0 & 0 & 0 & \hat{c}_{36} \\ & & & 0 & \hat{c}_{45} & 0 \\ & \mathrm{S\ Y\ M} & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \tag{A.28b}$$

$$\widehat{\mathbf{C}}_{\mathrm{mon/tet}} = \begin{pmatrix} \frac{1}{2}(\hat{c}_{11} - \hat{c}_{22}) & 0 & \frac{1}{2}(\hat{c}_{13} - \hat{c}_{23}) & 0 & 0 & \frac{1}{2}(\hat{c}_{16} + \hat{c}_{26}) \\ & \frac{1}{2}(\hat{c}_{22} - \hat{c}_{11}) & \frac{1}{2}(\hat{c}_{23} - \hat{c}_{13}) & 0 & 0 & \frac{1}{2}(\hat{c}_{16} + \hat{c}_{26}) \\ & & 0 & 0 & 0 & \hat{c}_{36} \\ & & & \frac{1}{2}(\hat{c}_{44} - \hat{c}_{55}) & \hat{c}_{45} & 0 \\ & & \mathrm{SYM} & & \frac{1}{2}(\hat{c}_{55} - \hat{c}_{44}) & 0 \\ & & & & & 0 \end{pmatrix}, \tag{A.28c}$$

$$\widehat{\mathbf{C}}_{\mathrm{ort/hex}} = \begin{pmatrix} r_0 + \frac{1}{2}(\hat{c}_{11} - \hat{c}_{22}) & -r_0 & \frac{1}{2}(\hat{c}_{13} - \hat{c}_{23}) & 0 & 0 & 0 \\ & r_0 + \frac{1}{2}(\hat{c}_{22} - \hat{c}_{11}) & \frac{1}{2}(\hat{c}_{23} - \hat{c}_{13}) & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & \frac{1}{2}(\hat{c}_{44} - \hat{c}_{55}) & 0 & 0 \\ & & \mathrm{SYM} & & \frac{1}{2}(\hat{c}_{55} - \hat{c}_{44}) & 0 \\ & & & & & -2r_0 \end{pmatrix}, \tag{A.28d}$$

$$\widehat{\mathbf{C}}_{\text{tet/hex}} = \begin{pmatrix} r_0 & -r_0 & 0 & 0 & 0 & \frac{1}{2}(\hat{c}_{16} - \hat{c}_{26}) \\ & r_0 & 0 & 0 & 0 & \frac{1}{2}(\hat{c}_{26} - \hat{c}_{16}) \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \text{S} & \text{Y} & \text{M} & & 0 & 0 \\ & & & & & -2r_0 \end{pmatrix}, \tag{A.28e}$$

$$\widehat{\mathbf{C}}_{\text{tet/cub}} = \begin{pmatrix} r_1 & -2r_2 & r_2 & 0 & 0 & \frac{1}{2}(\hat{c}_{16} - \hat{c}_{26}) \\ & r_1 & r_2 & 0 & 0 & \frac{1}{2}(\hat{c}_{26} - \hat{c}_{16}) \\ & & -2r_1 & 0 & 0 & 0 \\ & & & r_3 & 0 & 0 \\ \text{S} & \text{Y} & \text{M} & & r_3 & 0 \\ & & & & & -2r_3 \end{pmatrix}, \tag{A.28f}$$

$$\widehat{\mathbf{C}}_{\text{cub/iso}} = \begin{pmatrix} 2r & -r & -r & 0 & 0 & 0 \\ & 2r & -r & 0 & 0 & 0 \\ & & 2r & 0 & 0 & 0 \\ & & & -2r & 0 & 0 \\ \text{S} & \text{Y} & \text{M} & & -2r & 0 \\ & & & & & -2r \end{pmatrix}, \tag{A.28g}$$

where

$$r_0 = \frac{1}{8}(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{12} - 2\hat{c}_{66}), \qquad r_1 = \frac{1}{6}(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{33}), \tag{A.29a}$$

$$r_2 = \frac{1}{6}(\hat{c}_{13} + \hat{c}_{23} - 2\hat{c}_{12}), \qquad r_3 = \frac{1}{6}(\hat{c}_{44} + \hat{c}_{55} - 2\hat{c}_{66}), \tag{A.29b}$$

$$r = \frac{1}{15}\left(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} - \hat{c}_{12} - \hat{c}_{13} - \hat{c}_{23} - \hat{c}_{44} - \hat{c}_{55} - \hat{c}_{66}\right). \tag{A.29c}$$

Also, $\widehat{\mathbf{C}}_{\text{hex/iso}}$ follows from Eq. (A.27) and (A.25) but is not given explicitly because of its length, and neither are the complements involving $\widehat{\mathbf{C}}_{\text{trig}}$.

A.4 The Euclidean Distances

Three projections of the elastic length are as follows

$$\begin{aligned} \|\mathbb{C}\|^2 &= \|\mathbb{C}_{\text{iso}}\|^2 + \left\|\mathbb{C}_{\text{cub/iso}}\right\|^2 + \left\|\mathbb{C}_{\text{tet/cub}}\right\|^2 + \left\|\mathbb{C}_{\text{mon/tet}}\right\|^2 + \|\mathbb{C}_{\perp\text{mon}}\|^2 \\ &= \|\mathbb{C}_{\text{iso}}\|^2 + \left\|\mathbb{C}_{\text{hex/iso}}\right\|^2 + \left\|\mathbb{C}_{\text{tet/hex}}\right\|^2 + \left\|\mathbb{C}_{\text{mon/tet}}\right\|^2 + \|\mathbb{C}_{\perp\text{mon}}\|^2 \\ &= \|\mathbb{C}_{\text{iso}}\|^2 + \left\|\mathbb{C}_{\text{hex/iso}}\right\|^2 + \left\|\mathbb{C}_{\text{ort/hex}}\right\|^2 + \left\|\mathbb{C}_{\text{mon/ort}}\right\|^2 + \|\mathbb{C}_{\perp\text{mon}}\|^2. \end{aligned} \tag{A.30}$$

These can be explicitly calculated using

$$\left\|\widehat{\mathbf{C}}_{\perp\text{mon}}\right\|^2 = 2\left(\hat{c}_{14}^2 + \hat{c}_{15}^2 + \hat{c}_{24}^2 + \hat{c}_{25}^2 + \hat{c}_{34}^2 + \hat{c}_{35}^2 + \hat{c}_{46}^2 + \hat{c}_{56}^2\right), \tag{A.31a}$$

$$\left\|\widehat{\mathbf{C}}_{\text{mon/ort}}\right\|^2 = 2\left(\hat{c}_{16}^2 + \hat{c}_{26}^2 + \hat{c}_{36}^2 + \hat{c}_{45}^2\right), \tag{A.31b}$$

$$\left\|\widehat{\mathbf{C}}_{\text{ort/hex}}\right\|^2 = \frac{1}{8}\left(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{12} - 2\hat{c}_{66}\right)^2 + \frac{1}{2}(\hat{c}_{11} - \hat{c}_{22})^2 + (\hat{c}_{13} - \hat{c}_{23})^2 + \frac{1}{2}(\hat{c}_{44} - \hat{c}_{55})^2, \tag{A.31c}$$

$$\left\|\widehat{\mathbf{C}}_{\text{mon/tet}}\right\|^2 = \frac{1}{2}(\hat{c}_{11} - \hat{c}_{22})^2 + (\hat{c}_{13} - \hat{c}_{23})^2 + \frac{1}{2}(\hat{c}_{44} - \hat{c}_{55})^2 + (\hat{c}_{16} + \hat{c}_{26})^2 + 2\hat{c}_{45}^2 + 2\hat{c}_{36}^2, \tag{A.31d}$$

$$\left\|\widehat{\mathbf{C}}_{\text{tet/cub}}\right\|^2 = \frac{1}{6}(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{33})^2 + \frac{1}{3}(\hat{c}_{13} + \hat{c}_{23} - 2\hat{c}_{12})^2$$
$$+ \frac{1}{6}(\hat{c}_{44} + \hat{c}_{55} - 2\hat{c}_{66})^2 + (\hat{c}_{16} - \hat{c}_{26})^2, \tag{A.31e}$$

$$\left\|\widehat{\mathbf{C}}_{\text{cub/iso}}\right\|^2 = \frac{2}{15}\left(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} - \hat{c}_{12} - \hat{c}_{13} - \hat{c}_{23} - \hat{c}_{44} - \hat{c}_{55} - \hat{c}_{66}\right)^2, \tag{A.31f}$$

$$\left\|\widehat{\mathbf{C}}_{\text{tet/hex}}\right\|^2 = \frac{1}{8}\left(\hat{c}_{11} + \hat{c}_{22} - 2\hat{c}_{12} - 2\hat{c}_{66}\right)^2 + (\hat{c}_{16} - \hat{c}_{26})^2 \tag{A.31g}$$

$$\left\|\widehat{\mathbf{C}}_{\text{hex/iso}}\right\|^2 = \frac{1}{16}\left[2(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{12}) - (\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{66})\right]^2$$
$$+ \frac{1}{9}\left[\hat{c}_{13} + \hat{c}_{23} + 2\hat{c}_{33} - (\hat{c}_{11} + \hat{c}_{22} + 2\hat{c}_{12})\right]^2$$
$$+ \frac{1}{720}\left[\hat{c}_{11} + \hat{c}_{22} + 14\hat{c}_{12} + 16(\hat{c}_{33} - \hat{c}_{13} - \hat{c}_{23}) - 6(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66})\right]^2, \tag{A.31h}$$

and Eqs. (23) and (58) give

$$\|\mathbb{C}_{\text{iso}}\|^2 = \frac{1}{9}\left(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} + 2\hat{c}_{12} + 2\hat{c}_{13} + 2\hat{c}_{23}\right)^2$$
$$+ \frac{1}{45}\left[2(\hat{c}_{11} + \hat{c}_{22} + \hat{c}_{33} - \hat{c}_{12} - \hat{c}_{23} - \hat{c}_{31}) + 3(\hat{c}_{44} + \hat{c}_{55} + \hat{c}_{66})\right]^2. \tag{A.32}$$

All versions of the elastic length in Eq. (A.30) are the sum of 21 positive numbers, the same as the raw form of the squared length in any rectangular coordinate system using the Voigt elements. However, the three sets of 21 positive numbers in Eq. (A.30) contain far more information about the underlying elastic symmetry of the material.

B Exponential, Logarithm and Square Root of Hermitian Matrices

Let A be the 2×2 Hermitian matrix

$$A = \begin{bmatrix} a & c + id \\ c - id & b \end{bmatrix}, \tag{B.1}$$

where a, b, c and d are real numbers. The matrix can be decomposed into the sum of an isotropic part and a deviatoric part,

$$A = A_{\text{iso}} + A_{\text{dev}} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta & c + id \\ c - id & -\beta \end{bmatrix}, \tag{B.2}$$

where $\alpha = (a+b)/2$, $\beta = (a-b)/2$ and $\gamma = \sqrt{\beta^2 + c^2 + d^2}$. Note that A_{dev} vanishes if and only if $\gamma = 0$, i.e., $a = b$ and $c = d = 0$, and that $A_{\mathrm{dev}}^2 = \gamma^2 I$. The eigenvalues of A are real and given by

$$\lambda_{\pm} = \alpha \pm \gamma. \tag{B.3}$$

For convenience we introduce the matrix

$$A' = \begin{cases} I & \text{if} \quad \gamma = 0, \\ \frac{1}{\gamma} A_{\mathrm{dev}} & \text{if} \quad \gamma > 0, \end{cases} \tag{B.4}$$

so that $A'^2 = I$. The exponential of A is then

$$\exp A = e^{\alpha}(\cosh \gamma I + \sinh A') = \begin{cases} e^{\alpha} I & \text{if} \quad \gamma = 0, \\ e^{\alpha}\left(\cosh \gamma I + \dfrac{\sinh \gamma}{\gamma} A_{\mathrm{dev}}\right) & \text{if} \quad \gamma > 0. \end{cases} \tag{B.5}$$

When $ab - c^2 \neq 0$ the matrix A is invertible and its inverse is

$$A^{-1} = \frac{1}{ab - c^2 - d^2}\begin{bmatrix} b & -(c+id) \\ -(c-id) & a \end{bmatrix} = \frac{1}{ab - c^2 - d^2}(\alpha I - \gamma A')$$

$$= \frac{1}{ab - c^2 - d^2}(\alpha I - A_{\mathrm{dev}}). \tag{B.6}$$

If $a > 0$, $b > 0$ and $ab - c^2 - d^2 > 0$, then A is positive definite. Let $\delta = \sqrt{ab - c^2 - d^2}$, then the positive-definite square root of A is

$$A^{1/2} = \frac{1}{\sqrt{2(\alpha + \delta)}}\begin{bmatrix} a+\delta & c+id \\ c-id & b+\delta \end{bmatrix} = \frac{1}{\sqrt{2(\alpha + \delta)}}\left[(\alpha + \delta) I + A_{\mathrm{dev}}\right], \tag{B.7}$$

and the inverse of its square root is given by

$$A^{-1/2} = \frac{1}{\delta\sqrt{2(\alpha + \delta)}}\begin{bmatrix} b+\delta & -(c+id) \\ -(c-id) & a+\delta \end{bmatrix} = \frac{1}{\delta\sqrt{2(\alpha + \delta)}}\left[(\alpha + \delta) I - A_{\mathrm{dev}}\right]. \tag{B.8}$$

The logarithm is

$$\mathrm{Log}\, A = \ln \delta I + \ln \sqrt{\frac{\alpha + \gamma}{\alpha - \gamma}} A' = \begin{cases} \ln \delta I & \text{if} \quad \gamma = 0, \\ \ln \delta I + \dfrac{1}{\gamma}\ln \sqrt{\dfrac{\alpha + \gamma}{\alpha - \gamma}} A_{\mathrm{dev}} & \text{if} \quad \gamma > 0. \end{cases} \tag{B.9}$$

More generally, for any analytic function f we have

$$f(A) = \begin{cases} f(\alpha) I & \text{if} \quad \gamma = 0, \\ \dfrac{f(\alpha + \gamma) + f(\alpha - \gamma)}{2} I + \dfrac{f(\alpha + \gamma) - f(\alpha - \gamma)}{2\gamma} A_{\mathrm{dev}} & \text{if} \quad \gamma > 0. \end{cases} \tag{B.10}$$

The Cayley–Hamilton theorem states that

$$\det(A) I - \mathrm{tr}(A) A + A^2 = 0. \tag{B.11}$$

Therefore, any analytic function of A can be written as a linear combination of I, A and A^2, or, if A is invertible, as a linear combination of I, A and A^{-1}. Using the fact

that $2A_{\text{dev}} = A - \delta^2 A^{-1}$, we have the alternative expressions for the exponential, logarithm and square roots

$$\exp A = e^\alpha \left[\cosh \gamma \, I + \frac{\sinh \gamma}{2\gamma} (A - \delta^2 A^{-1}) \right], \tag{B.12a}$$

$$\text{Log } A = \ln \delta I + \frac{1}{4\gamma} \ln \frac{\alpha + \gamma}{\alpha - \gamma} \left(A - \delta^2 A^{-1} \right), \tag{B.12b}$$

$$A^{1/2} = \frac{1}{\sqrt{2(\alpha + \delta)}} \left[\delta I + A \right], \tag{B.12c}$$

$$A^{-1/2} = \frac{1}{\sqrt{2(\alpha + \delta)}} \left[I + \delta A^{-1} \right]. \tag{B.12d}$$

We recall that for any $n \times n$ matrix M we have

$$\det(\exp M) = e^{\text{tr } M}. \tag{B.13}$$

Therefore, when $M = \text{Log } P$ for some matrix P we get

$$\det P = e^{\text{tr}(\text{Log } P)}. \tag{B.14}$$

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