THE CLUSTER CATEGORY OF A CANONICAL ALGEBRA

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Abstract. We study the cluster category of a canonical algebra A in terms of the hereditary category of coherent sheaves over the corresponding weighted projective line $\mathbb{X}$. As an application we determine the automorphism group of the cluster category and show that the cluster-tilting objects form a cluster structure in the sense of Buan-Iyama-Reiten-Scott. The tilting graph of the sheaf category always coincides with the tilting or exchange graph of the cluster category. We show that this graph is connected if the Euler characteristic of $\mathbb{X}$ is non-negative, or equivalently, if A is of tame (domestic or tubular) representation type.

1. Introduction

Cluster categories $\mathcal{C}(H)$ of a hereditary algebra H were introduced in [4] as orbit categories $\mathrm{D^b}(H)/F^{\mathbb{Z}}$ of the derived category of H, where $F = \tau^- \circ [1]$. These categories have been extensively studied due to their remarkable connections to cluster algebras in the sense of Fomin-Zelevinsky [5]. In this paper we study the cluster category $\mathcal{C}(A)$ of a canonical algebra A. It was shown in [13] that also in this case $\mathcal{C}(A)$ is a triangulated category. For information on canonical algebras we refer to [6, 24].

The cluster categories studied here provide a new family of 2-Calabi-Yau triangulated categories [13] having a cluster structure in the sense of [3], see Theorem 3.1. These cluster categories give rise to a new class of cluster-tilted algebras. Our setting includes the cluster categories of tame hereditary algebras.

We give an important alternative description of the cluster category $\mathcal{C} = \mathcal{C}(A)$ in terms of the hereditary category $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ of coherent sheaves on the weighted projective line $\mathbb{X}$ attached to A, namely $\mathcal{C}$ can be obtained from $\mathcal{H}$ by "adding extra morphisms". Conversely, $\mathcal{H}$ can be recovered from $\mathcal{C}$ as the quotient $\mathcal{C}/\mathcal{I}$ by a suitable ideal, see Corollary 4.4.

Another main result is a lifting property which states that exact autoequivalences of the cluster category can be lifted to exact autoequivalences of $\mathrm{D^b}(\mathcal{H})$, see Theorem 6.4. For this a detailed study of cluster tubes is crucial. We show that the automorphism group $\mathrm{Aut}(\mathcal{C})$ is canonically isomorphic to $\mathrm{Aut}(\mathrm{D^b}(\mathcal{H}))$ modulo the cyclic subgroup generated by F, see Theorem 7.1. If A is not tubular, then $\mathrm{Aut}(\mathcal{C})$ is canonically isomorphic to $\mathrm{Aut}(\mathcal{H})$. In the tubular case we establish a natural bijection between the coset space $\mathrm{Aut}(\mathcal{C})/\mathrm{Aut}(\mathcal{H})$ and $\mathbb{Q} \cup \{\infty\}$, see Theorem 7.5.

As another application the investigation of the automorphism group in the tubular case yields – combined with results of Hübner [10] – the connectedness of the tilting graph in the tubular case.

2. Basic Results

Throughout this article, k will denote an algebraically closed field.

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Definition of the cluster category. Let A be a canonical algebra of weight type $\boldsymbol{p} = (p_1, \ldots, p_t)$ and parameter sequence $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t)$, see [24, 6]. We denote by $\mathrm{mod}(A)$ the category of finitely generated (right) A-modules and by $\mathrm{D^b}(A)$ the bounded derived category of $\mathrm{mod}(A)$. Recall that the Auslander-Reiten translation τ, its inverse τ^- and the shift [1] are autoequivalences of $\mathrm{D^b}(A)$.

By [6], $\mathrm{D^b}(A)$ is equivalent to $\mathrm{D^b}(\mathcal{H})$, where $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ is the category of coherent sheaves on a weighted projective line $\mathbb{X}$ of type $(\boldsymbol{p}, \boldsymbol{\lambda})$. Note that $\mathcal{H}$ is a connected abelian hereditary category with Serre duality $\mathrm{D}\,\mathrm{Ext}^1_{\mathcal{H}}(X, Y) = \mathrm{Hom}_{\mathcal{H}}(Y, \tau X)$ for all $X, Y \in \mathcal{H}$, where τ is an autoequivalence of $\mathcal{H}$. Further $\mathcal{H}$ has a tilting object T such that $\mathrm{End}_{\mathcal{H}}(T) = A$.

We study the *cluster category* $\mathcal{C} = \mathcal{C}(\mathcal{H})$, defined as the *orbit category* $\mathrm{D^b}(\mathcal{H})/F^{\mathbb{Z}}$, where $F = \tau^- \circ [1]$, see [4]; it has the same objects as $\mathrm{D^b}(\mathcal{H})$, morphism spaces are given by $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(\mathcal{H})}(X, F^n Y)$ with the obvious composition. We denote by $\pi\colon\ \mathrm{D^b}(\mathcal{H}) \to \mathcal{C}$ the natural projection functor and call a triangulated structure on $\mathcal{C}$ *admissible* if π becomes an exact functor; the shift in $\mathcal{C}$ is then given by τ. By [13] such an admissible triangulated structure always exists.

Description of $\mathcal{C}$ in terms of $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ be the category with the same objects as $\mathcal{H}$ and $\mathbb{Z}_2$-graded morphism spaces $\mathrm{Hom}_{\widetilde{\mathcal{H}}}(X, Y) = \mathrm{Hom}_{\mathcal{H}}(X, Y) \oplus \mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Y)$. Then each morphism $f \in \mathrm{Hom}_{\widetilde{\mathcal{H}}}(X, Y)$ can be written as $f = f_0 + f_1$ for some $f_0 \in \mathrm{Hom}_{\mathcal{H}}(X, Y)$ of degree zero and some $f_1 \in \mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Y)$ of degree one. The composition is given by $(g_0 + g_1) \circ (f_0 + f_1) = g_0 f_0 + (\tau^- g_0 f_1 + g_1 f_0)$ using the Yoneda product. Since τ^- is an autoequivalence of $\mathcal{H}$ we obtain the following result.

Proposition 2.1. *The category $\mathcal{C}$ is equivalent to the category $\widetilde{\mathcal{H}}$.* □

In the sequel we shall identify $\mathcal{C}$ with $\widetilde{\mathcal{H}}$, that is, we view $\mathcal{C}$ as the category $\mathcal{H}$ with extra morphisms of degree one. Fixing an admissible triangulated structure, $\mathcal{C}$ becomes a 2-Calabi-Yau triangulated category, see [13]. We call a triangle in $\mathcal{C}$ *induced* if it is isomorphic to the image of an exact triangle in $\mathrm{D^b}(\mathcal{H})$ under the projection functor.

Remark 2.2. In $\mathcal{C}$ the composition of two morphisms of degree one is zero, hence the degree one morphisms form an ideal $\mathcal{I}$ contained in the radical $\mathrm{rad}_{\mathcal{C}}$ of $\mathcal{C}$. Clearly $\mathcal{H} \simeq \mathcal{C}/\mathcal{I}$, that is, $\mathcal{H}$ can be recovered from $\mathcal{C}$ provided that the above $\mathbb{Z}_2$-grading is known. We shall later see how the $\mathbb{Z}_2$-grading can be obtained intrinsically in terms of the category $\mathcal{C}$, see Corollary 4.4.

For the notion of a *cluster-tilting object* in $\mathcal{C}$ we refer to [4].

Proposition 2.3. *The cluster category $\mathcal{C}$ is a Krull-Remak-Schmidt category. The categories $\mathcal{C}$ and $\mathcal{H}$ have the same indecomposables, the same isomorphism classes of objects and the tilting objects in $\mathcal{H}$ are precisely the cluster-tilting objects in $\mathcal{C}$.*

Proof. We first observe that a morphism $f = f_0 + f_1$ is invertible in $\mathcal{C}$ if and only if f_0 is invertible in $\mathcal{H}$ since f_1 is radical. Therefore isomorphism classes coincide in both categories. Since $\mathrm{End}_{\mathcal{C}}(X)/\mathrm{rad}_{\mathcal{C}}(X, X) = \mathrm{End}_{\mathcal{H}}(X)/\mathrm{rad}_{\mathcal{H}}(X, X)$, the categories $\mathcal{C}$ and $\mathcal{H}$ have the same indecomposables and $\mathcal{C}$ is a Krull-Remak-Schmidt category. The last assertion follows from the definitions. □

The following result can be proved as in [4, Prop. 1.3].

Proposition 2.4. *The cluster category $\mathcal{C}$ has Auslander-Reiten triangles which coincide with the triangles induced by almost-split sequences in $\mathcal{H}$. Moreover, $\mathcal{H}$ and $\mathcal{C}$ have the same Auslander-Reiten quiver.* □

Corollary 2.5. *For objects X, Y of $\mathcal{C}$ the space $\operatorname{Hom}_{\mathcal{C}}(X,Y)_1$ of degree one morphisms is contained in the infinite radical $\operatorname{rad}^{\infty}_{\mathcal{C}}(X,Y)$.*

Proof. Since X and FY belong to distinct Auslander-Reiten components in $\mathcal{C}$ we have $\operatorname{Hom}_{\mathcal{C}}(X,Y)_1 = \operatorname{Hom}_{\mathrm{D^b}(\mathcal{H})}(X,FY) \subseteq \operatorname{rad}^{\infty}_{\mathrm{D^b}(\mathcal{H})}(X,FY)$, hence the result. □

Remark 2.6. There are further hereditary categories allowing a treatment by the techniques of this paper. If H is a connected hereditary algebra of infinite representation type, then $\mathrm{D^b}(H) = \mathrm{D^b}(\mathcal{H})$ for some hereditary category $\mathcal{H}$ for which τ is an autoequivalence (take $\mathcal{H} = \mathcal{I}[-1] \vee \mathcal{P} \vee \mathcal{R}$, where $\mathcal{P}, \mathcal{I}$ are the preprojective, respectively preinjective component and $\mathcal{R}$ consists of all regular components of $\operatorname{mod}(H)$). The tame hereditary case is covered by our setting.

Shape of the cluster category. Denote by $\mathcal{H}_0$ (resp. $\mathcal{H}_+$) the full subcategory of $\mathcal{H}$ consisting of the objects of finite length (resp. the vector bundles). We define $\mathcal{C}_0$ (resp. C_+) as the full subcategory of $\mathcal{C}$ given by the objects of $\mathcal{H}_0$ (resp. of $\mathcal{H}_+$). A full subcategory of $\mathcal{C}$ given by the objects of a tube in $\mathcal{H}$ will be called *cluster tube*.

The *slope* function μ assigns to each non-zero object X of $\mathcal{C}$ an element of $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ by $\mu(X) = \deg(X)/\operatorname{rk}(X)$, where deg and rk are the degree and the rank functions on $\mathcal{H}$, respectively, see [6]. Recall that the *Euler characteristic* $\chi_{\mathcal{H}} = 2 - \sum_{i=1}^{t}(1 - 1/p_i)$ determines the representation type: if $\chi_{\mathcal{H}} > 0$ (resp. $= 0, < 0$) then $\mathcal{H}$ is tame domestic (resp. tubular, wild), see [6].

In the tubular case for each $q \in \overline{\mathbb{Q}}$, we denote by $\mathcal{C}^{(q)}$ the full subcategory of $\mathcal{C}$ given by the additive closure of all indecomposables of slope q. In particular we have $\mathcal{C}^{(\infty)} = \mathcal{C}_0$. For $q < q'$ (resp. $q > q'$) each non-zero morphism from $\mathcal{C}^{(q)}$ to $\mathcal{C}^{(q')}$ is of degree zero (resp. one).

By definition a *tubular family* in $\mathcal{C}$ is a maximal family of pairwise orthogonal cluster tubes. If $\chi_{\mathcal{H}} \neq 0$ then $\mathcal{C}_0$ is the unique tubular family in $\mathcal{C}$. If $\mathcal{H}$ is tubular then the categories $\mathcal{C}^{(q)}$ (for $q \in \overline{\mathbb{Q}}$) are precisely the tubular families, see [17].

3. Relationship to Fomin-Zelevinsky mutations

Cluster structure. By Proposition 2.3 the cluster-tilting objects in $\mathcal{C}$ correspond to the tilting objects in $\mathcal{H}$. For a cluster-tilting object T we denote by Q_T the quiver of the endomorphism algebra $\operatorname{End}_{\mathcal{C}}(T)$. We call an object $E \in \mathcal{H}$ *exceptional* if it is indecomposable with $\operatorname{Ext}^1_{\mathcal{H}}(E,E) = 0$.

We know from [10] that in $\mathcal{H}$ for each indecomposable direct summand M of T, that is, $T = \overline{T} \oplus M$ there exists a unique exceptional object $M^* \not\simeq M$ in $\mathcal{H}$ such that $T^* = \overline{T} \oplus M^*$ is again a tilting object.

A 2-Calabi-Yau triangulated category $\mathcal{C}$ with finite dimensional Hom-spaces *admits a cluster structure* [3] if (i) for each cluster-tilting object T the quiver Q_T has no loop and no 2-cycle, and (ii) if $T = \overline{T} \oplus M$ and $T^* = \overline{T} \oplus M^*$ are two cluster-tilted objects with non-isomorphic indecomposables M and M^*, then Q_{T^*} is the Fomin-Zelevinsky mutation [5] of Q_T in the vertex corresponding to M.

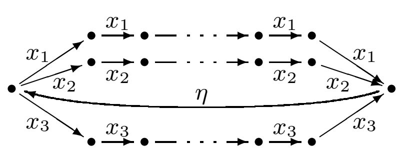
Theorem 3.1. *For any canonical algebra the category $\mathcal{C}$ admits a cluster structure.*

Proof. By [3, Thm. I.1.6] we only have to show condition (i). We know from [12] that in $\mathcal{C}$ there exist exact triangles $M^* \xrightarrow{u} B \xrightarrow{v} M \to M^*[1]$ and $M \xrightarrow{u'} B' \xrightarrow{v'} M^* \to M[1]$, where u, u' are minimal left and v, v' are minimal right $\operatorname{add}(\overline{T})$-approximations. By [10] one of them, say the first, is induced by an exact sequence in $\mathcal{H}$. As in [4, Lem. 6.13] we apply $\operatorname{Hom}_{\mathrm{D^b}(\mathcal{H})}(F^{-1}M, -)$ to the first sequence and get that each radical morphism $f : M \to M$ factors through the morphism v. Hence f is not irreducible in $\operatorname{End}_{\mathcal{C}}(T)$. This shows that there are no loops in Q_T.

To see that there are no 2-cycles in Q_T it is enough to show that B and B' have no common indecomposable summand. Assume that such a summand U exists. Since the first triangle is induced by an exact sequence from $\mathcal{H}$, the morphisms u, v in $\mathcal{C}$ are of degree zero. Denote by $j \in \mathrm{Hom}_{\mathcal{H}}(U, B')$ and by $p \in \mathrm{Hom}_{\mathcal{H}}(B', U)$ the canonical inclusion and projection, respectively. Since the endomorphism algebra $\mathrm{End}_{\mathcal{H}}(T)$ is triangular [8, Lem. IV.1.10], the non-zero morphisms $s = p \circ u'$ and $t = v' \circ j$ in $\mathcal{C}$ are of degree one. Thus s is given by a non-split short exact sequence. This sequence induces a triangle $\tau^- U \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{s} \tau^- U[1]$ in $\mathrm{D^b}(\mathcal{H})$, yielding by rotation an induced triangle $M \xrightarrow{s} U \xrightarrow{\tau f} \tau E \xrightarrow{\tau g} \tau M$ in $\mathcal{C}$. Since $s = p \circ u'$, by axiom [26, (TR3)] there exists a morphism $q \in \mathrm{Hom}_{\mathcal{C}}(M^*, \tau E)$ such that $q \circ v' = \tau f \circ p$. Therefore the degree zero morphism $\tau f = \tau f \circ p \circ j$ equals the degree one morphism $q \circ t$, showing that both are zero. This yields a contradiction, since $f \neq 0$. □

Cluster-tilted algebras. Considering the endomorphism rings of cluster-tilting objects we get a new class of algebras. For example, if A is a canonical algebra with $t = 3$ weights and if T is a tilting object in $\mathcal{H}$ for which $\mathrm{End}_{\mathcal{H}}(T) = A$ then $A_{\mathcal{C}} = \mathrm{End}_{\mathcal{C}}(T)$ is given as quotient of the path algebra of the following quiver Q_T modulo the ideal I generated by the elements described below (in Q_T the arm with arrows x_i contains p_i arrows).

Q_T :



I :
$x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$
$x_i^{p_i - a} \eta x_i^{a-1}$ for $a = 1, \ldots, p_i$. and $i = 1, 2, 3$.

If $t \leq 2$ then $A_{\mathcal{C}} = A$. For $t \geq 4$ the description of $A_{\mathcal{C}}$ is more complicated since the relations for the canonical algebra contain parameters.

4. Factorization

Degree one morphisms. We start with a general result.

Proposition 4.1. *Let X and Y be indecomposables in $\mathcal{C}_+$ and let $\mathcal{T}$ be a cluster tube from $\mathcal{C}_0$. Then each morphism $f\colon X \to Y$ of degree one factors through $\mathcal{T}$.*

Proof. By definition, we have $f = \eta$ with $\eta \in \mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Y)$. In $\mathrm{D^b}(\mathcal{H})$ the morphism $\eta : X \to \tau^- Y[1]$ factors through some object Z from $\mathcal{T}$, that is, $\eta = [X \xrightarrow{h} Z \xrightarrow{\eta'} \tau^- Y[1]]$. Note that $h \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$, $\eta' \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$, hence $\eta = \eta' h$ yields a factorization of f. □

The tubular case. We now turn to the tubular case. For $p, q \in \overline{\mathbb{Q}}$, we define the *slope interval* from p to q as

$$(p,q)_{\mathcal{C}} \colon = \begin{cases} (p,q), & \text{if } p < q, \\ (p,\infty] \cup (-\infty, q), & \text{if } p > q, \\ \overline{\mathbb{Q}} \setminus \{q\}, & \text{if } p = q, \end{cases}$$

where the intervals occurring on the right side are taken in $\overline{\mathbb{Q}}$. The idea behind the slope interval is that starting from p into the direction of bigger numbers all numbers are collected until q is reached. For further properties of $\overline{\mathbb{Q}}$ we refer to Section 7.

The following result states an important property of $\mathcal{C}$ which is analogous to the well-known factorization properties for the derived category of a tubular algebra.

Theorem 4.2. *We assume that $\mathcal{C}$ is the cluster category of a tubular category $\mathcal{H}$. Let $p, q \in \overline{\mathbb{Q}}$ and $f\colon X \to Y$ be a non-zero morphism in $\mathcal{C}$ with indecomposable objects $X \in \mathcal{C}^{(p)}$ and $Y \in \mathcal{C}^{(q)}$.*

(i) *Let $p \neq q$. Then f factors through each tube of $\mathcal{C}^{(r)}$ if and only if $r \in (p,q)_{\mathcal{C}}$.*
(ii) *Let $p = q$. Then f factors through each tube of $\mathcal{C}^{(r)}$ for each $r \neq q$ if and only if f lies in $\operatorname{rad}_{\mathcal{C}}^{\infty}$.*

Proof. Note first that each morphism $g \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}^{(s)}, \mathcal{C}^{(t)})$ has degree zero (resp. one) if $s < t$ (resp. $s > t$). Now if $p < q$ then $\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{H}}(X,Y)$ and by [17] the morphism f factors through each tube of $\mathcal{C}^{(r)}$ if $r \in (p,q)_{\mathcal{C}}$. For $r = p$ or $r = q$ clearly f does not factor through each tube of $\mathcal{C}^{(r)}$. In the remaining cases where $r \notin (p,q)_{\mathcal{C}}$, each composition $X \xrightarrow{g} Z \xrightarrow{h} Y$ with $Z \in \mathcal{C}^{(r)}$ has degree one and therefore $hg \neq f$.

If $p > q$ then $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathrm{D^b}(\mathcal{H})}(X, \tau^{-}Y)$ factors through each tube of $\mathcal{H}^{(r)}$ (for $r \in (p,\infty)$) and each tube of $\mathcal{H}^{(r)}[1]$ (for $r \in (-\infty, q)$), thus through any cluster tube of $\mathcal{C}^{(r)}$ for $r \in (p,q)_{\mathcal{C}}$. The case $r = p$ or $r = q$ is similar as before and for the remaining slopes $r \notin (p,q)_{\mathcal{C}}$ each composition hg (h, g and Z as before) is zero as composition of morphisms of degree one. This proves assertion (i).

For (ii) we observe that the degree one part f_1 of f belongs to $\operatorname{Hom}_{\mathrm{D^b}(\mathcal{H})}(X, FY)$ hence also to the infinite radical of $\mathrm{D^b}(\mathcal{H})$. Thus f_1 factors through any tube of $\mathcal{H}^{(r)}[1]$ for $r \in (p,p)_{\mathcal{C}}$, and the assertion is true if $f_0 = 0$. If f_0 is non-zero then f_0 does not belong to the infinite radical of $\mathcal{H}$ and therefore f does not belong to the infinite radical of $\mathcal{C}$. This shows that X and Y lie in the same tube. The result follows. □

Recovering $\mathcal{H}$ from $\mathcal{C}$. We now focus on the problem of reconstructing the original category $\mathcal{H}$ from $\mathcal{C}$ (up to equivalence), assuming only intrinsic properties of the category $\mathcal{C}$.

We know already from Remark 2.2 that the morphisms of degree one form a two-sided ideal $\mathcal{I}$ such that $\mathcal{C}/\mathcal{I} \xrightarrow{\sim} \mathcal{H}$ canonically. We are now going to show how $\mathcal{I}$ can be recovered from $\mathcal{C}$ intrinsically. This is possible without any extra choice if $\mathcal{H}$ has Euler characteristic different from zero. In the tubular case the reconstruction will depend on the choice of a tubular family $\mathcal{C}^{(q)}$ in $\mathcal{C}$. By means of a suitable autoequivalence of $\mathrm{D^b}(\mathcal{H})$, see [17], we may then choose $q = \infty$, that is, $\mathcal{C}^{(q)} = \mathcal{C}_0$.

Proposition 4.3. *Regardless of the Euler characteristic, the ideal $\mathcal{I}$ of degree one morphisms is given on indecomposables X and Y by*

$$\mathcal{I}(X,Y) = \begin{cases} 0 & \textit{if } X \in \mathcal{C}_+,\ Y \in \mathcal{C}_0 \\ \operatorname{Hom}_{\mathcal{C}}(X,Y) & \textit{if } X \in \mathcal{C}_0,\ Y \in \mathcal{C}_+ \\ \operatorname{rad}_{\mathcal{C}}^{\infty}(X,Y) & \textit{if } X, Y \in \mathcal{C}_0 \\ \mathcal{F}(X,Y) & \textit{if } X, Y \in \mathcal{C}_+ \end{cases}$$

where $\mathcal{F}(X,Y)$ consists of all $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ factoring through $\mathcal{C}_0$.

Proof. Since $\operatorname{Hom}_{\mathcal{H}}(\mathcal{H}_0, \mathcal{H}_+) = 0 = \operatorname{Ext}^1_{\mathcal{H}}(\mathcal{H}_+, \mathcal{H}_0)$ the assertion follows for the first two cases. For the third case we use that $\operatorname{rad}_{\mathcal{H}}^{\infty}(\mathcal{H}_0, \mathcal{H}_0)$ equals zero. The last case is covered by Proposition 4.1. □

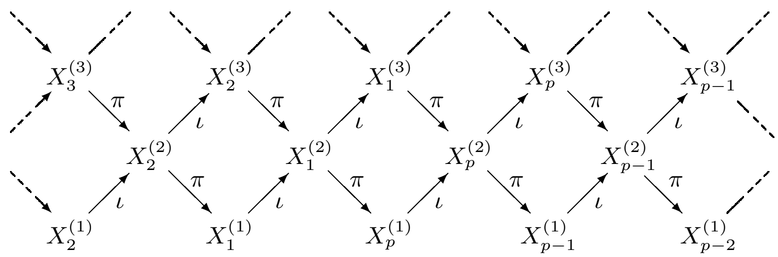
Corollary 4.4. *The category $\mathcal{H}$ can always be recovered from $\mathcal{C}$ as the quotient of $\mathcal{C}$ by a two-sided ideal $\mathcal{I}$. This ideal is unique if the Euler characteristic is non-zero. In the tubular case it only depends on the choice of a tubular family $\mathcal{C}^{(q)}$ in $\mathcal{C}$, $q \in \overline{\mathbb{Q}}$.* □

5. Cluster tubes

Notation. Let $\mathcal{T}$ be a cluster tube of rank p. We denote the indecomposable objects of $\mathcal{T}$ by $X_i^{(n)}$, $i \in \mathbb{Z}_p$ and n an integer ≥ 1, such that $\tau X_i^{(n)} = X_{i+1}^{(n)}$. Furthermore we choose irreducible morphisms $\iota_i^{(n)} \colon X_i^{(n)} \to X_i^{(n+1)}$ and $\pi_i^{(n)} \colon X_{i+1}^{(n+1)} \to X_i^{(n)}$ satisfying

$$\iota_i^{(n-1)} \pi_i^{(n-1)} = \pi_i^{(n)} \iota_{i+1}^{(n)} \tag{5.1}$$

for all $n \geq 1$ and $i \in \mathbb{Z}_p$. Here we used the convention $X_i^{(0)} = 0$ (and consequently $\iota_i^{(0)} = 0$, $\pi_i^{(0)} = 0$). Whenever possible we will skip the indices and just write ι and π. The situation is illustrated by the following figure.



Certain Yoneda products. We shall show the following result concerning the Yoneda product.

Lemma 5.1. *Let* $\iota_i^{(n)} \colon X_i^{(n)} \to X_i^{(n+1)}$.

(a) *For any object* Z *the linear map*

$$\mathrm{Ext}^1_{\mathcal{H}}(X_i^{(n+1)}, Z) \to \mathrm{Ext}^1_{\mathcal{H}}(X_i^{(n)}, Z), \quad \eta \mapsto \eta \iota_i^{(n)}$$

is surjective.

(b) *For* $Z = X_j^{(n)}$ *the linear map*

$$\mathrm{Ext}^1_{\mathcal{H}}(X_i^{(n+1)}, Z) \to \mathrm{Ext}^1_{\mathcal{H}}(X_i^{(n)}, Z), \quad \eta \mapsto \eta \iota_i^{(n)}$$

is bijective.

Proof. To simplify notations write $X = X_i^{(n)}$, $Y = X_i^{(n+1)}$ and $\iota = \iota_i^{(n)} \colon X \to Y$. For (a) apply $\mathrm{Hom}(-, Z)$ to the short exact sequence $0 \to X \xrightarrow{\iota} Y \xrightarrow{v} S \to 0$ to get

$$\mathrm{Ext}^1(S, Z) \xrightarrow{v^*} \mathrm{Ext}^1(Y, Z) \xrightarrow{\iota^*} \mathrm{Ext}^1(X, Z) \to 0, \tag{5.2}$$

where the last term $\mathrm{Ext}^2(S, Z)$ is zero because $\mathcal{H}$ is hereditary.

To see the injectivity in (b) apply Serre duality to (5.2) to get

$$\mathrm{D}\,\mathrm{Hom}(Z, \tau S) \xrightarrow{\mathrm{D}\, v^*} \mathrm{D}\,\mathrm{Hom}(Z, \tau Y) \xrightarrow{\mathrm{D}\, \iota^*} \mathrm{D}\,\mathrm{Hom}(Z, \tau X).$$

Since Z has length n whereas the uniserial object Y has length $n+1$ the image of each morphism $f \in \mathrm{Hom}(Z, \tau Y)$ lies in the unique maximal submodule $\mathrm{rad}\, Y$ and thus $v \circ f = 0$. This shows that $v^* \colon \mathrm{Hom}(Z, \tau Y) \to \mathrm{Hom}(Z, \tau S)$ is zero. $\square$

Automorphisms of cluster tubes. To investigate automorphisms of cluster tubes from $\mathcal{C}$ we start with a preliminary result.

Lemma 5.2. *The infinite radical* $\mathrm{rad}^{\infty}_{\mathcal{T}}$ *of a cluster tube* $\mathcal{T}$ *equals* $\mathcal{T} \cap \mathrm{rad}^{\infty}_{\mathcal{C}}$.

Proof. It suffices to show that each $u \in \mathcal{T} \cap \mathrm{rad}^{\infty}_{\mathcal{C}}$ belongs to $\mathrm{rad}^{\infty}_{\mathcal{T}}$. By [19] we may assume that $\mathcal{T}$ is a cluster tube in $\mathcal{C}_0$. For this it is enough to show that each composition $U_1 \to E \xrightarrow{v} U_2$ belongs to $\mathrm{rad}^{\infty}_{\mathcal{T}}$, where U_1, U_2 are indecomposables from $\mathcal{T}$ and E lies in $\mathcal{C}_+$. This uses that $\mathcal{C}_0$ consists of a family of pairwise orthogonal cluster tubes. Note that v belongs to $\mathrm{Hom}_{\mathcal{H}}(E, U_2)$ since $\mathrm{Ext}^1_{\mathcal{H}}(\mathcal{H}_+, \mathcal{H}_0) = 0$.

By Auslander-Reiten theory we get a representation $v = u_n \cdots u_1 v_n$ with radical morphisms u_i from $\mathcal{T}$ for each integer $n \geq 1$. This proves the claim. $\square$

The following result is crucial for our analysis of automorphism groups of the cluster category of a canonical algebra.

Proposition 5.3. *We fix an admissible triangulated structure on $\mathcal{C}$. Let $\mathcal{T} \subset \mathcal{C}$ be a cluster tube of rank $p \geq 1$. If $G\colon \mathcal{T} \to \mathcal{T}$ is an autoequivalence, sending induced triangles to exact triangles, and inducing the identity functor on $\mathcal{T}/\operatorname{rad}^{\infty}_{\mathcal{T}}$, then there is an isomorphism $\psi\colon 1 \xrightarrow{\sim} G$ of functors on $\mathcal{T}$ such that each $\psi_X\colon X \xrightarrow{\sim} G(X)$ has the form $\psi_X = 1_X + \eta_X$, where η_X is of degree one.*

Proof. The proof is done in several steps.
(1) *G is isomorphic to a functor G' which also induces the identity in $\mathcal{T}/\operatorname{rad}^{\infty}_{\mathcal{T}}$ and additionally satisfies $G(\iota_i^{(n)}) = \iota_i^{(n)}$ for any $n \geq 1$ and any $i \in \mathbb{Z}_p$.*

By assumption we have $G(\iota_i^{(n)}) = \iota_i^{(n)} + \xi_i^{(n)}$ for some $\xi_i^{(n)} \in \operatorname{Ext}^1_{\mathcal{H}}(X_i^{(n)}, \tau^- X_i^{(n+1)})$. We set $\eta_i^{(1)} = 0$ and define inductively, using Lemma 5.1 (a), elements $\eta_i^{(n+1)} \in \operatorname{Ext}^1_{\mathcal{H}}(X_i^{(n+1)}, \tau^- X_i^{(n+1)})$ such that

$$\eta_i^{(n+1)} \iota_i^{(n)} = \xi_i^{(n)} + \iota_i^{(n)} \eta_i^{(n)}. \tag{5.3}$$

Next we define isomorphisms $\psi_i^{(n)} = 1 + \eta_i^{(n)}\colon X_i^{(n)} \to X_i^{(n)}$ yielding $G(\iota_i^{(n)})\psi_i^{(n)} = \psi_i^{(n+1)} \iota_i^{(n)}$. Therefore, setting $G'(X) = X$ for any object X in $\mathcal{T}$ and $G'(f) = (\psi_j^{(n)})^{-1} \circ G(f) \circ \psi_i^{(m)}$ for any morphism $f\colon X_i^{(m)} \to X_j^{(n)}$ we obtain assertion (1).
(2) *The functor G' also satisfies $G'(\pi_i^{(n)}) = \pi_i^{(n)}$ for any $n \geq 1$ and any $i \in \mathbb{Z}_p$.*

Recall that $G'(\pi_i^{(n)}) = \pi_i^{(n)} + \xi_i^{(n)}$ for some $\xi_i^{(n)} \in \operatorname{Ext}^1_{\mathcal{H}}(X_{i+1}^{(n+1)}, \tau^- X_i^{(n)})$. By induction on n we shall show that $\xi_i^{(n)} = 0$. For $n = 1$ it follows from $\pi_i^{(1)} \iota_{i+1}^{(1)} = 0$ that $0 = G'(\pi_i^{(1)} \iota_{i+1}^{(1)}) = \left(\pi_i^{(1)} + \xi_i^{(1)}\right) \iota_{i+1}^{(1)}$. Hence by Lemma 5.1 (b), we obtain $\xi_i^{(1)} = 0$. Assuming inductively that $G'(\pi_i^{(n-1)}) = \pi_i^{(n-1)}$ for any i, we can apply G' to the identity (5.1) and obtain $G'(\pi_i^{(n)} \iota_{i+1}^{(n)}) = \pi_i^{(n)} \iota_{i+1}^{(n)}$. Therefore $\pi_i^{(n)} \iota_{i+1}^{(n)} = G(\pi_i^{(n)}) \iota_{i+1}^{(n)} = (\pi_i^{(n)} + \xi_i^{(n)}) \iota_{i+1}^{(n)}$ which implies $\xi_i^{(n)} \iota_{i+1}^{(n)} = 0$. By Lemma 5.1 (b) we get $\xi_i^{(n)} = 0$.
(3) *The functor G' is the identity functor.*

Let $\eta \in \operatorname{Ext}^1_{\mathcal{H}}(X_a^{(r)}, X_b^{(s)})$ be given by the sequence $0 \to X_b^{(s)} \xrightarrow{f} E \xrightarrow{g} X_a^{(r)} \to 0$, which gives rise to the induced triangle Δ forming the upper row of the following diagram. Since G and therefore also G' is exact on Δ, $G'(\Delta)$ is a triangle again. By axiom [26, (TR3)] there exists a morphism $\zeta\colon X_a^{(r)} \to X_a^{(r)}$ making the following diagram commutative.

$$\begin{tikzcd}
\Delta\colon & X_b^{(s)} \arrow[r, "f"] \arrow[d, equal] & E \arrow[r, "g"] \arrow[d, equal] & X_a^{(r)} \arrow[r, "\eta"] \arrow[d, dashed, "\zeta"] & X_b^{(s)}[1] \arrow[d, equal] \\
G'(\Delta)\colon & X_b^{(s)} \arrow[r, "f"'] & E \arrow[r, "g"'] & X_a^{(r)} \arrow[r, "G'(\eta)"'] & X_b^{(s)}[1]
\end{tikzcd}$$

Write $\zeta = \zeta_0 + \zeta_1$, the decomposition into different degrees. Then $g = g_0 = (\zeta g)_0 = \zeta_0 g$ implies $\zeta_0 = 1$ since g is surjective. Therefore it follows from $\eta = G'(\eta)\zeta$ that $\eta = G'(\eta) + G'(\eta)\zeta_1 = G'(\eta)$, where the last equation follows from Lemma 5.2 and the fact that the composition of two morphisms of degree one is zero. Hence the result. $\square$

6. Lifting of automorphisms

Non-zero prolongation. We shall need the following results.

Lemma 6.1. *For any non-zero $\eta \in \mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Y)$ there exists a morphism $h \in \mathrm{Hom}_{\mathcal{H}}(\tau^- Y, \tau X)$ such that $h\eta \neq 0$.*

Proof. Observe that $\eta \in \mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Y)$ is given by a non-split short exact sequence $\eta\colon 0 \to \tau^- Y \xrightarrow{a} F \xrightarrow{b} X \to 0$. Therefore b factors through b' in the almost-split sequence $\varepsilon_X : 0 \to \tau X \xrightarrow{a'} E \xrightarrow{b'} X \to 0$, that is, $b = b'c$ for some $c \in \mathrm{Hom}_{\mathcal{H}}(F, E)$ and consequently ca factors through a', say $ca = a'h$ for some $h \in \mathrm{Hom}_{\mathcal{H}}(\tau^- Y, \tau X)$. Therefore $h\eta = \varepsilon_X$ is non-zero as an almost-split sequence. □

Lemma 6.2. *Assume $f : X \to Y$ is a non-zero morphism in $\mathcal{H}_+$. Then there exists a morphism $h : Y \to Z$ in $\mathcal{H}_+$ such that $hf \neq 0$ and $\mathrm{Ext}^1_{\mathcal{H}}(X, \tau^- Z) = 0 = \mathrm{Ext}^1_{\mathcal{H}}(Y, \tau^- Z)$.*

Proof. By [6, Cor. 2.7] there exists an embedding $Y \hookrightarrow \bigoplus_{i=1}^r L_i$ into a direct sum of line bundles L_i. For each integer $n \geq 0$ we further obtain embeddings $L_i \to L_i^{(n)}$ with degree $\mu(L_i^{(n)}) \geq \mu(L_i) + n$ (we may take $L_i^{(n)} = L_i(n\vec{c})$). We thus obtain an embedding $h : Y \to Z$ with $Z = \bigoplus_{i=1}^r L_i^{(n)}$. Clearly $hf \neq 0$. By means of line bundle filtrations for X and Y the remaining assertions now follow as in [20, (S15)] if n is sufficiently large. □

Autoequivalences of $\mathcal{C}$ fixing all objects. The following two results are the key ingredients to determine the automorphism group of $\mathcal{C}$.

Proposition 6.3. *We fix an admissible triangulated structure on $\mathcal{C}$. If an autoequivalence G of $\mathcal{C}$ sends induced triangles to exact triangles and fixes all objects then G is isomorphic to the identity functor.*

Proof. Let $\mathcal{I}$ be the two-sided ideal of morphisms of degree one of $\mathcal{C}$. Since G is the identity on objects it follows from the description of $\mathcal{I}$ in Proposition 4.3 that $G(\mathcal{I}) = \mathcal{I}$. Hence G induces an autoequivalence σ of $\mathcal{C}/\mathcal{I} = \mathcal{H}$.

Observe that G is isomorphic to an autoequivalence G' which satisfies the following two properties:

(i) $G'(X) = X$ for any object $X \in \mathcal{C}$,
(ii) for each morphism f of degree zero there exists a morphism η_f of degree one such that $G'(f) = f + \eta_f$.

Indeed, by [19, Prop. 2.1], the autoequivalence σ is isomorphic to the identity functor, say $\varphi\colon \sigma \xrightarrow{\sim} 1_{\mathcal{H}}$. Clearly, φ_X is a degree zero isomorphism in $\mathcal{C}$ for each object $X \in \mathcal{C}$ and therefore we can define G' on morphisms by setting $G'(f) = \varphi_Y \circ G(f) \circ \varphi_X^{-1}$ for $f\colon X \to Y$.

By Proposition 5.3 we get an isomorphism $\psi\colon 1_{\mathcal{C}_0} \to G'|_{\mathcal{C}_0}$, where the degree zero part of ψ_X is the identity on X. Changing G' by means of ψ we may additionally assume that the restriction of G' to $\mathcal{C}_0$ is the identity functor. Due to the special form of ψ, see Proposition 5.3, the functor G' still keeps property (ii).

Simplifying notation we write G instead of G'. We show now that $G(f) = f$ for each $f : X \to Y$ in the remaining cases: (a) $X \in \mathcal{C}_+$, $Y \in \mathcal{C}_0$, (b) $X \in \mathcal{C}_0$, $Y \in \mathcal{C}_+$ and (c) $X \in \mathcal{C}_+$, $Y \in \mathcal{C}_+$.

In case (a) this follows from the fact that there are no morphisms of degree one from $\mathcal{C}_+$ to $\mathcal{C}_0$.

To prove case (b) we assume $G(f) \neq f$. Since there are no morphisms of degree zero from X to Y, Lemma 6.1 yields a morphism $h : Y \to \tau X$ such that $h(f -$

$G(f)) \neq 0$. Then $G(hf) = hf$ since G is the identity on $\mathcal{C}_0$. On the other hand $G(hf) = hG(f)$ by case (a), yielding the contradiction $h(f - G(f)) = 0$.

Concerning (c) we first assume that f has degree one. By Proposition 4.1 we get a factorization $f = \eta h$ for some $h \in \mathrm{Hom}_{\mathcal{H}}(X, Z)$ and $\eta \in \mathrm{Ext}^1_{\mathcal{H}}(Z, \tau^- Y)$ for some $Z \in \mathcal{C}_0$. We thus get $G(f) = f$ by (a) and (b). Next we assume that f is a degree zero morphism with $f - G(f) \neq 0$. By Lemma 6.2 we find $Z \in \mathcal{C}_+$ and a morphism $h : Y \to Z$ such that $h(f - G(f)) \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(X, Z)_1 = 0 = \mathrm{Hom}_{\mathcal{C}}(Y, Z)_1$. Then again $G(hf) = hf$ and $G(h) = h$, hence $h(f - G(f)) = 0$, a contradiction. We have shown that G is the identity functor on $\mathcal{C}$. □

A lifting property. We first observe that each exact autoequivalence u of $\mathrm{D^b}(\mathcal{H})$ induces canonically an autoequivalence $u_{\mathcal{C}}$ of $\mathcal{C}$ which makes the diagram

$$\begin{tikzcd} \mathrm{D^b}(\mathcal{H}) \arrow[r, "u"] \arrow[d, "\pi"'] & \mathrm{D^b}(\mathcal{H}) \arrow[d, "\pi"] \\ \mathcal{C} \arrow[r, "u_{\mathcal{C}}"'] & \mathcal{C} \end{tikzcd}$$

commutative and sends induced triangles to induced triangles. The setting induces a canonical homomorphism $\pi_* : \mathrm{Aut}(\mathrm{D^b}(\mathcal{C})) \to \mathrm{Aut}(\mathcal{C}_{\mathrm{ind}})$, $u \mapsto u_{\mathcal{C}}$, where $\mathrm{Aut}(\mathcal{C}_{\mathrm{ind}})$ denotes the semigroup of isomorphism classes of autoequivalences of the category $\mathcal{C}$, commuting with the translation [1] and preserving induced triangles. It follows from the next proposition that $\mathrm{Aut}(\mathcal{C}_{\mathrm{ind}})$ actually is a group, called the automorphism group of $\mathcal{C}_{\mathrm{ind}}$.

Theorem 6.4. *Let $\mathcal{C}$ be the cluster category of $\mathcal{H}$ equipped with an admissible triangulated structure. Then each autoequivalence G of $\mathcal{C}$, sending induced triangles to exact triangles, lifts to an exact autoequivalence u of the derived category* $\mathrm{D^b}(\mathcal{H})$*, that is, G is isomorphic to $u_{\mathcal{C}}$. In particular, G sends induced triangles to induced triangles.*

Proof. We recall that $\mathcal{C}$ has a unique tubular family if the Euler characteristic of $\mathcal{H}$ is non-zero and therefore $G(\mathcal{C}_0) = \mathcal{C}_0$. In the tubular case we can assume the same, changing G by an automorphism of $\mathrm{D^b}(\mathcal{H})$, if necessary: Note that the automorphism group of $\mathrm{D^b}(\mathcal{H})$ acts transitively on the set of tubular families of $\mathcal{C}$ by means of π_*. Since $G^{-1}(\mathcal{C}_0) = \mathcal{C}^{(q)}$ for some q, we find an automorphism u of $\mathrm{D^b}(\mathcal{H})$ with $u_{\mathcal{C}}(\mathcal{C}_0) = \mathcal{C}^{(q)}$, and consider $Gu_{\mathcal{C}}$ instead of G.

The property $G(\mathcal{C}_0) = \mathcal{C}_0$ now implies that $G(\mathcal{I}) = \mathcal{I}$ for the ideal $\mathcal{I}$ of degree one morphisms of $\mathcal{C}$. Consequently G induces an autoequivalence γ of $\mathcal{H} = \mathcal{C}/\mathcal{I}$. We then extend γ to the exact autoequivalence $u = \mathrm{D^b}(\gamma)$ of $\mathrm{D^b}(\mathcal{H})$ and consider $u_{\mathcal{C}}$. By construction $G\,u_{\mathcal{C}}^{-1}$ is exact on induced triangles and fixes the objects of $\mathcal{C}$, hence G is isomorphic to $u_{\mathcal{C}}$ by Proposition 6.3, proving the claim. □

Corollary 6.5. *The homomorphism π_*:* $\mathrm{Aut}(\mathrm{D^b}(\mathcal{H})) \to \mathrm{Aut}(\mathcal{C}_{\mathrm{ind}})$ *is surjective.*□

By $\mathcal{C}_{\mathrm{kel}}$ we denote the cluster category $\mathcal{C}$, equipped with the triangulated structure defined in [13]. We are going to show that exact autoequivalences of $\mathrm{D^b}(\mathcal{H})$ induce exact autoequivalences of the category $\mathcal{C}_{\mathrm{kel}}$.

Lemma 6.6. *Each exact autoequivalence G of* $\mathrm{D^b}(\mathcal{H}) = \mathrm{D^b}(A)$*, A canonical, is standard in the sense of* [23, Def. 3.4].

Proof. It follows from [23, Cor. 3.5] that there is a standard autoequivalence G' such that $G(X) \simeq G'(X)$ for all objects X in $\mathrm{D^b}(A)$. From [19, Prop. 2.1] we get $G \simeq G'$, and hence G is standard itself. □

Proposition 6.7. *An autoequivalence of $\mathcal{C}$ is exact in $\mathcal{C}_{\mathrm{kel}}$ if and only if it is exact on induced triangles. Moreover,* $\mathrm{Aut}(\mathcal{C}_{\mathrm{kel}}) = \mathrm{Aut}(\mathcal{C}_{\mathrm{ind}})$.

Proof. We first note that each autoequivalence of $\mathrm{D^b}(A)$ which is standard in the sense of [23, Def. 3.4] is also standard in the sense of [13, 9.8]. By Lemma 6.6 Keller's result [13, 9.4] implies that each exact autoequivalence of $\mathrm{D^b}(\mathcal{H})$ induces an exact autoequivalence of $\mathcal{C}_{\mathrm{kel}}$. Both assertions now follow from Theorem 6.4. $\square$

Remark 6.8. Each autoequivalence G of $\mathcal{C}$ which is exact with respect to an admissible triangulated structure belongs to $\mathrm{Aut}(\mathcal{C}_{\mathrm{ind}}) = \mathrm{Aut}(\mathcal{C}_{\mathrm{kel}})$. It is not clear whether the converse inclusion also holds.

7. The automorphism group of the cluster category

We now continue our study of the automorphism group $\mathrm{Aut}(\mathcal{C})$ where the cluster category $\mathcal{C} = \mathcal{C}(\mathcal{H})$ is equipped with the triangulated structure due to Keller [13]. Note that F defines a central element in $\mathrm{Aut}(\mathrm{D^b}(\mathcal{H}))$.

Theorem 7.1. *The automorphism group* $\mathrm{Aut}(\mathcal{C})$ *is canonically isomorphic to the quotient* $\mathrm{Aut}(\mathrm{D^b}(\mathcal{H}))/\langle F\rangle$.

Proof. The homomorphism $\pi_*\colon \mathrm{Aut}(\mathrm{D^b}(\mathcal{H})) \to \mathrm{Aut}(\mathcal{C})$, $u \mapsto u_{\mathcal{C}}$ is surjective by Corollary 6.5. Clearly F lies in the kernel of π_*. Suppose now that $u \in \mathrm{Aut}(\mathrm{D^b}(\mathcal{H}))$ satisfies $u_{\mathcal{C}} = 1$. Then, for any indecomposable object $X \in \mathrm{D^b}(\mathcal{H})$, we must have $u(X) \simeq F^{n_X}(X)$ for some integer n_X. In particular, $u(X)$ belongs to $\mathcal{H}_0[n_X]$ for each indecomposable X from $\mathcal{H}_0$. Since $u(\mathcal{H}_0)$ is a tubular family in $\mathrm{D^b}(\mathcal{H})$ it then follows that $u(\mathcal{H}_0) = \mathcal{H}_0[n]$ for a fixed integer n. By [19, Prop. 4.2 and 6.2] this in turn implies that $u(\mathcal{H}) = \mathcal{H}[n]$. Hence u is isomorphic to F^n on objects of $\mathrm{D^b}(\mathcal{H})$, implying that u is isomorphic to F^n as a functor by [19, Prop. 2.1]. $\square$

The tubular situation. We assume that $\mathcal{H}$ is tubular and first give an invariant description of the set $\overline{\mathbb{Q}}$ of slopes for $\mathcal{H}$. We define $\mathbb{W}$ as the set of all rank one direct summands of the free abelian group $R = \mathrm{rad}\ \mathrm{K}_0(\mathcal{H})$ of rank two, where R consists of all elements of the Grothendieck group fixed under the automorphism induced by τ. We consider R to be equipped with the Euler form.

Let p be the least common multiple of the weights of $\mathcal{H}$. Fixing w in $\mathbb{W}$ the (additive closure of the) subcategory consisting of all indecomposable objects $X = \pi(Y)$ of $\mathcal{C}$ such that $\sum_{j=1}^{p}[\tau^j Y]$ belongs to w is a tubular family $\mathcal{C}^{(w)}$ in $\mathcal{C}$ and each tubular family in $\mathcal{C}$ has this form. This follows from [17, Thm. 4.6]. In this way we get a bijection $\mathbb{W} \xrightarrow{\sim} \overline{\mathbb{Q}}$. From [19] we know that the group M of $\mathbb{Z}$-linear automorphism of R preserving the Euler form is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. We consider $M/\{\pm 1\}$ as the automorphism group of $\mathbb{W}$ and hence $\mathrm{Aut}(\mathbb{W}) \simeq \mathrm{PSL}_2(\mathbb{Z})$. We call $\mathbb{W}$ the *rational circle*.

Lemma 7.2. *There is a natural surjective homomorphism* $\mu : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(\mathbb{W})$ *such that* $\rho = \mu(G)$ *satisfies* $G(\mathcal{C}^{(w)}) = \mathcal{C}^{(\rho(w))}$ *for each automorphism* G *of* $\mathcal{C}$.

Proof. By Corollary 6.5 an automorphism G of $\mathcal{C}$ lifts to an automorphism u of $\mathrm{D^b}(\mathcal{H})$. Since u induces an automorphism of the Grothendieck group, preserving the Euler form, it induces an automorphism $u_{\mathbb{W}}$ of $\mathbb{W}$. Note that the automorphism $F = \tau^-[1]$ induces the identity on $\mathbb{W}$. By Theorem 7.1 two liftings of G differ by a power of F, hence induce the same element $G_{\mathbb{W}} = u_{\mathbb{W}}$ of $\mathrm{Aut}(\mathbb{W})$. This defines the homomorphism μ satisfying the above formula. By [19, Thm. 6.3] surjectivity of μ follows. $\square$

Remark 7.3. For the preceding arguments it is essential to relate the cluster category with the Grothendieck group of the derived category $\mathrm{D^b}(\mathcal{H})$ by means of a lifting property. The Grothendieck group $\mathrm{K}_0(\mathcal{C})$ of the cluster category, investigated in [2], cannot be used for the present purpose since essential information on the radical is lost under the canonical projection $\mathrm{K}_0(\mathcal{H}) \to \mathrm{K}_0(\mathcal{C})$.

We recall from [19] that the automorphism group $\mathrm{Aut}(\mathbb{X})$ of a weighted projective line $\mathbb{X}$ consists of all members of $\mathrm{Aut}(\mathcal{H})$ fixing the structure sheaf. This group is finite if $\mathbb{X}$ has at least three weights, in particular if $\mathcal{H}$ is tubular. Further the subgroup $\mathrm{Pic}_0(\mathbb{X})$ of $\mathrm{Aut}(\mathcal{H})$ consists of all shift functors $E \mapsto E(\vec{x})$ of $\mathcal{H}$ which preserve slopes. This group is always finite abelian.

Proposition 7.4. *Assume $\mathcal{H}$ is tubular. Then there is an exact sequence*

$$1 \to \mathrm{Pic}_0(\mathbb{X}) \rtimes \mathrm{Aut}(\mathbb{X}) \to \mathrm{Aut}(\mathcal{C}) \xrightarrow{\mu} \mathrm{Aut}(\mathbb{W}) \to 1.$$

Proof. Indeed, we have the following commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \langle F \rangle \arrow[r, "\sim"] \arrow[d] & \langle t \rangle \arrow[d] & \\
1 \arrow[r] & \mathrm{Pic}_0(\mathbb{X}) \rtimes \mathrm{Aut}(\mathbb{X}) \arrow[r] \arrow[d, equal] & \mathrm{Aut}(\mathrm{D^b}(\mathcal{H})) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathbb{Y}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Pic}_0(\mathbb{X}) \rtimes \mathrm{Aut}(\mathbb{X}) \arrow[r] & \mathrm{Aut}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathbb{W}) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

where the vertical exact sequence on the right is taken from the proof of [19, Thm. 5.1]. By Theorem 7.1 and [19, Thm. 6.3] the central vertical and horizontal sequences are also exact, yielding the claim. $\square$

Comparison of $\mathrm{Aut}(\mathcal{H})$ **and** $\mathrm{Aut}(\mathcal{C})$**.** There is a natural homomorphism $j : \mathrm{Aut}(\mathcal{H}) \to \mathrm{Aut}(\mathcal{C})$ sending an automorphism u of $\mathcal{H}$ to the automorphism $u_{\mathcal{C}}$ of $\mathcal{C}$ induced by $\mathrm{D^b}(u)$. By Theorem 7.1 j is injective. In the following we identify $\mathrm{Aut}(\mathcal{H})$ with its image in $\mathrm{Aut}(\mathcal{C})$.

Theorem 7.5. $\mathrm{Aut}(\mathcal{H})$ *consists of those automorphisms G of $\mathcal{C}$ with $G(\mathcal{C}_0) = \mathcal{C}_0$. Moreover,*

(i) *If $\chi_{\mathcal{H}} \neq 0$ we have* $\mathrm{Aut}(\mathcal{H}) = \mathrm{Aut}(\mathcal{C})$.

(ii) *If $\mathcal{H}$ is tubular, we have a canonical bijection ν:* $\mathrm{Aut}(\mathcal{C})/\mathrm{Aut}(\mathcal{H}) \xrightarrow{\sim} \mathbb{W}$, *where $G \cdot \mathrm{Aut}(\mathcal{H})$ is mapped to w with $G(\mathcal{C}_0) = \mathcal{C}^{(w)}$. The bijection is compatible with the natural left* $\mathrm{Aut}(\mathcal{C})$*-actions on both sets.*

Proof. The first assertion and (i) follow from Theorem 6.4. Concerning (ii) we note that $\mathrm{Aut}(\mathcal{C})$ acts transitively on $\mathbb{W}$ by means of Lemma 7.2. The bijectivity follows. $\square$

Remark 7.6. With a little extra work, concerning Proposition 5.3 and using [15, Cor. 3.3] instead of [19], the results of our paper extend to the case of an arbitrary base field as long as canonical algebras in the sense of Ringel [24] or weighted projective lines are concerned. In the more general situation, dealing with canonical algebras in the sense of [25], equivalently with categories of coherent sheaves on exceptional curves [16], the uniqueness result Proposition 6.3 does not carry over by [14], giving the setting a different shape.

8. The tilting graph

In this section $\mathcal{H}$ denotes the category of coherent sheaves on a weighted projective line and $\mathcal{C}$ the corresponding cluster category, equipped with an admissible triangulated structure. We are proving in this section that the tilting (or exchange) graphs for $\mathcal{H}$ and $\mathcal{C}$ agree and, moreover, are connected if the Euler characteristic is non-negative.

We start with a result due to Hübner [10, Prop. 5.14].

Proposition 8.1. *Let $T = E \oplus U$ be a basic tilting object in $\mathcal{H}$ with E indecomposable. Then there exists exactly one exceptional object E^* in $\mathcal{H}$ such that E^* is not isomorphic to E and $T' = E^* \oplus U$ is also a tilting object.*

Moreover, exactly one of the spaces $\mathrm{Ext}^1_{\mathcal{H}}(E, E^*)$ and $\mathrm{Ext}^1_{\mathcal{H}}(E^*, E)$ is non-zero, and then one-dimensional over k. □

The object T' is called *mutation* of T in E. Moreover, if a tilting object T' can be obtained by a sequence of mutations from a tilting object T, then we say that T' and T are *connected by mutations*. If E is a source (resp. sink) in the quiver of $\mathrm{End}(T)$ then the mutation is given by APR-tilt (resp. APR-cotilt), see [1].

The *tilting graph* $\Gamma_{\mathcal{H}}$ of $\mathcal{H}$ has as vertices the isomorphism classes of basic (or multiplicity-free) tilting objects T of $\mathcal{H}$. Two vertices, represented by basic tilting objects T and T' are connected by an edge if and only if they differ by exactly one indecomposable summand. By Proposition 2.3 this graph may be identified with the tilting (or exchange) graph $\Gamma_{\mathcal{C}}$ of the cluster category $\mathcal{C} = \mathcal{C}(\mathcal{H})$, compare [4]. Note that this graph is hence independent of the choice of the admissible triangulated structure.

Lemma 8.2. *Let T be a cluster-tilting object in $\mathcal{C}$ and $\langle T \rangle$ its connected component in $\Gamma_{\mathcal{C}}$. If σ is an automorphism of $\mathcal{C}$ such that $\sigma(T)$ belongs to $\langle T \rangle$ then $\sigma^{\pm 1}(\langle T \rangle) = \langle T \rangle$.*

Proof. Note that $\sigma(\langle T \rangle)$ is connected having a non-empty intersection with $\langle T \rangle$. □

By a *wing* in $\mathcal{H}_0$ we understand the full subcategory $\mathcal{W}$ consisting of all objects having a finite filtration whose factors belong to a *proper τ-segment* $S, \tau S, \tau^2 S, \ldots, \tau^{r-1} S$ of simple objects of an exceptional tube of $\mathcal{H}_0$. Proper means that not all simple objects of the tube belong to the segment. By construction each wing $\mathcal{W}$ with r simples is equivalent to the k-linear representations of the linear quiver $1 \to 2 \to 3 \to \cdots \to r$. Each tilting object B in $\mathcal{W}$ is called a *branch*. Such a branch has r indecomposable (non-isomorphic) direct summands and always contains a simple object and also the *root* R of $\mathcal{W}$, defined as the unique indecomposable of maximal length r.

Assume $\mathcal{H}$ has weight type $(p_1, \ldots, p_t)$. In the notation of [6], the line bundles $\mathcal{O}(\vec{x})$, with $0 \leq \vec{x} \leq \vec{c}$, form a tilting object T_{can} in $\mathcal{H}$ whose endomorphism algebra is the canonical algebra attached to $\mathcal{H}$. We call T_{can} the *standard canonical configuration* and $\mathcal{O}(\vec{x}_i), \mathcal{O}(2\vec{x}_i), \ldots, \mathcal{O}((p_i - 1)\vec{x}_i)$, $i = 1, \ldots, t$, the *i-th arm* of T_{can}. Note that $\mathcal{H}$ has exactly t exceptional tubes consisting of sheaves of finite length. In the i-th exceptional tube of rank p_i there is exactly one simple object S_i satisfying $\mathrm{Hom}_{\mathcal{H}}(\mathcal{O}, S_i) \neq 0$. Moreover, in the same tube there exists a sequence of exceptional objects and epimorphisms

$$B_i : S_i^{[p_i - 1]} \twoheadrightarrow S_i^{[p_i - 2]} \twoheadrightarrow \cdots \twoheadrightarrow S_i^{[1]} = S_i$$

where $S_i^{[j]}$ has length j and top S_i. The direct sum of $\mathcal{O}$, $\mathcal{O}(\vec{c})$ and all the $S_i^{[j]}$, $i = 1, \ldots, t$; $j = 1, \ldots, p_i - 1$, forms another tilting object of $\mathcal{H}$, called the *standard squid* T_{sq}. Further B_i is called the *i-th branch* of T_{sq}.

We recall that the *Picard group* $\mathrm{Pic}(\mathbb{X})$ of $\mathbb{X}$ (or $\mathcal{H}$) is the subgroup of the automorphism group of $\mathcal{H}$ consisting of all shift functors $E \mapsto E(\vec{x})$, see [6].

Proposition 8.3 ([10], Prop. 5.30)**.** *The standard squid T_{sq} belongs to the connected component $\Delta = \langle T_{can} \rangle$ of T_{can} in $\Gamma_{\mathcal{H}}$. Moreover Δ is preserved under the operations of* $\mathrm{Pic}(\mathbb{X})$*, in particular under Auslander-Reiten translation.*

Proof. For the convenience of the reader we include a proof. To obtain a sequence of mutations transforming T_{can} into T_{sq} we consider the following part

$$\mathcal{O} \to \mathcal{O}(\vec{x}_h) \to \mathcal{O}(2\vec{x}_h) \to \ldots \to \mathcal{O}((p_h-1)\vec{x}_h) \to \mathcal{O}(\vec{c})$$

of T_{can}. Forming successively mutations in $\mathcal{O}((p_h-1)\vec{x}_h), \ldots, \mathcal{O}(\vec{x}_h)$ we replace first $\mathcal{O}((p_h-1)\vec{x}_h)$ by S_h, then $\mathcal{O}((p_h-2)\vec{x}_h)$ by $S_h^{[2]}$, and so on, and finally $\mathcal{O}(\vec{x}_h)$ by $S_h^{[p_h-1]}$. Dealing with all the arms, we thus obtain T_{sq} from T_{can}.

We next show that Δ is preserved under shift by $-\vec{x}_h$, where $h = 1, \ldots, t$. The branch B_h of T_{sq} forms a tilting object in the *wing* $\mathcal{W}_h$ generated by the τ-segment $S_h, \tau S_h, \ldots, \tau^{p_i-2}S_h$. Since $\mathcal{W}_h$ is equivalent to the category of k-linear representations of the quiver $1 \to 2 \to \cdots \to (p_h-1)$, the tilting graph of $\mathcal{W}_h$ is connected. Without changing the component of T_{sq}, we may thus replace the linear branch B_h by the branch $S_h^{[p_h-1]}, \tau S_h^{[p_h-2]}, \ldots, \tau S_h$, while keeping the other indecomposable summands of T_{sq}.

Next we form a sequence of four mutations, replacing first $S_h^{[p_h-1]}$ by $\mathcal{O}(\vec{c}-\vec{x}_h)$, next $\mathcal{O}$ by $\mathcal{O}(2\vec{c}-\vec{x}_h)$, then $\mathcal{O}(\vec{c})$ by $\tau S_h^{[p_h-1]}$ and finally $\mathcal{O}(2\vec{c}-\vec{x}_h)$ by $\mathcal{O}(-\vec{x}_h)$. By way of example we verify the second last step, which is the only one where the mutation is not given by an APR-tilt or APR-cotilt. In the endomorphism ring of the tilting object T, consisting of $\mathcal{O}(\vec{c}-\vec{x}_h)$, $\mathcal{O}(\vec{c})$, $\mathcal{O}(2\vec{c}-\vec{x}_h)$, B_j $(j \neq h)$ and $B'_h = \{\tau S_h, \ldots, \tau S^{[p_h-2]}\}$, there is a unique irreducible map (up to scalars) starting in $\mathcal{O}(\vec{c})$, namely $\mathcal{O}(\vec{c}) \xrightarrow{x_h^{p_h-1}} \mathcal{O}(2\vec{c}-\vec{x}_h)$. Passage to the cokernel yields $\tau S_h^{p_h-1}$, the exceptional object replacing $\mathcal{O}(\vec{c})$.

To sum up, we have shown that $T_{sq}(-\vec{x}_h)$ lies in the component of T_{sq}. By Lemma 8.2 this shows that $\langle T_{sq}\rangle$ is stable under the operations of $\mathrm{Pic}(\mathbb{X})$. □

For a tilting object T we write $T = T_+ \oplus T_0$ for the summands $T_+ \in \mathcal{H}_+$ and $T_0 \in \mathcal{H}_0$ and we call T_0 the *torsion part* of T.

Proposition 8.4. *Let T be a tilting object in $\mathcal{H}$ with non-zero torsion part T_0. Then there exist tilting objects T' and T'' in the component $\langle T\rangle$ where T' lies in $\mathcal{H}_+$ and where the torsion part of T'' is an exceptional simple object.*

Proof. By [18] the torsion part of T has a decomposition $T_0 = \bigoplus_{j\in J} B_j$ where each B_j is a branch in a wing $\mathcal{W}_j$ and, moreover, the wings $\mathcal{W}_j$ are pairwise Hom- and Ext-orthogonal. Fixing an index j, we may assume by a sequence of mutations inside $\mathcal{W}_j$ that the root R_j of $\mathcal{W}_j$ is a sink in the endomorphism ring of T. Mutation (APR-cotilt) at R_j then replaces R_j by the kernel term R_j^* of an exact sequence $0 \to R_j^* \to \bigoplus_{i\in K} T_i \to R_j \to 0$, where each T_i is an indecomposable summand of T different from R_j. Due to connectedness of $\mathrm{End}(T)$, at least one T_i, $i \in K$, has positive rank. It follows that R_j^* has positive rank. The claim now follows by induction. □

Proposition 8.5. *Assume that $\chi_{\mathcal{H}} > 0$. For any two tilting bundles T and T' in $\mathcal{H}$ there exists a mutation sequence of tilting bundles linking T and T'.*

Proof. Note that the Auslander-Reiten quiver Γ of $\mathcal{H}_+$ has shape $\mathbb{Z}\Delta$ where Δ is extended Dynkin. *Step 1.* We show first that each tilting bundle admits a mutation sequence of tilting bundles $T = T^{(1)}, T^{(2)}, \ldots, T^{(r)}$, where $\mathrm{End}(T^{(r)})$ is hereditary, accordingly $T^{(r)}$ yields a slice in Γ. If the quiver of $\mathrm{End}(T)$ has no relations, then $\mathrm{End}(T)$ is hereditary, and we are done. Otherwise we choose an ordering of the indecomposable direct summands $T_1, \ldots, T_n$ of T such that (a) $\mathrm{Hom}(T_j, T_i) = 0$ for $j > i$ and (b) the index s is minimal among $1, \ldots, n$ such that a relation of $\mathrm{End}(T)$ starts in T_s. The full subquiver $T_1, \ldots, T_s$ will not contain any cycle since otherwise $\mathrm{End}(T)$ would be wild, contradicting $\chi_{\mathcal{H}} > 0$. Without changing the component $\langle T\rangle$, by invoking suitable APR-tilts in sources and APR-cotilts in sinks of $T_1,\ldots,T_{s-1}$, we may then assume that T_s is a source in the quiver of $\mathrm{End}(T)$. Now mutation (APR-tilt) in T_s yields a complement T_s^*, replacing T_s, given by an exact sequence $0\to T_s^*\to\bigoplus_{q\in J}T_q\to T_s\to 0$, where the T_q are indecomposable summands of T not isomorphic to T_s. Passing to ranks we obtain:

$$\begin{aligned}\mathrm{rk}(T_s)+\mathrm{rk}(T_s^*) &= \sum_{s\to q}\mathrm{rk}(T_q)\\ 2\,\mathrm{rk}(T_s) &= \sum_{s\to q}\mathrm{rk}(T_q)-\sum_{s\cdots r}\mathrm{rk}(T_r),\end{aligned}$$

where the first formula is obvious and the second one, with summation over a set of minimal relations starting at s in the second sum, expresses Hübner's rank additivity on tilting objects in $\mathcal{H}$, see [11] or [21]. We thus obtain

$$\mathrm{rk}(T_s^*)-\mathrm{rk}(T_s)=\sum_{s\cdots r}\mathrm{rk}(T_r)>0.$$

Since for $\chi_{\mathcal{H}}>0$ there exists an upper bound on the ranks of indecomposable bundles, this procedure allows only a finite number of repetitions, finally yielding a tilting bundle whose endomorphism ring has no relations. This finishes the proof of the first step.

Step 2. Let T' and T'' be tilting bundles in $\mathcal{H}$. By step 1 there is are mutation sequences of tilting bundles transforming T' and T'' into slices $\bar{T}'$ and $\bar{T}''$, respectively. It is well known (and easy to see) that any two slices of Γ are connected by a mutation sequence in Γ consisting of BGP-reflections. This proves the claim. □

Proposition 8.6. *Assume that $\chi_{\mathcal{H}}\geq 0$. Then each tilting object T with a non-zero torsion part belongs to the connected component $\langle T_{can}\rangle$.*

Proof. Since the torsion part T_0 of T is non-zero we may assume by Proposition 8.4 that T_0 is an exceptional simple object S. Invoking Proposition 8.3 we may further assume that $\mathrm{Hom}(\mathcal{O},S)=k$. Hence $T=T_+\oplus S$ where T_+ is a bundle. Mutation of T_{can} at $R=\mathcal{O}((p_i-1)\vec{x}_i)$ replaces R by S and yields a tilting object $T'=T'_{can}\oplus S$ in $\langle T_{can}\rangle$ with the same torsion part as T. By [7] the right perpendicular category $\mathcal{H}'=S^\perp$, formed in $\mathcal{H}$, is naturally equivalent to a category of coherent sheaves of weight type $(p'_1,\ldots,p'_t)$ where $p'_j=p_j$ for $j\neq i$ and $p'_i=p_i-1$ and such that T'_{can} equals the canonical configuration in $\mathcal{H}'$. Since $\chi_{\mathcal{H}}\geq 0$ the Euler characteristic of $\mathcal{H}'$ is strictly positive, hence Proposition 8.5 yields a mutation sequence $T_+=T^{(1)},T^{(2)},\ldots,T^{(s)}=T'_{can}$ of tilting bundles in $\mathcal{H}'$ connecting T_+ and T'_{can}. It follows that $T=T^{(1)}\oplus S$, $T^{(2)}\oplus S,\ldots,T^{(s)}\oplus S=T'$ is a mutation sequence connecting T and T'. We conclude that T belongs to $\langle T_{can}\rangle$. □

Note that the automorphism group $\mathrm{Aut}(\mathcal{C})$, where $\mathcal{C}=\mathcal{C}(\mathcal{H})$, naturally acts on the tilting graph $\Gamma_{\mathcal{C}}$.

Proposition 8.7. *The connected component $\langle T_{can}\rangle$ is preserved under the action of* $\mathrm{Aut}(\mathcal{C})$.

Proof. For $\chi_{\mathcal{H}}\neq 0$ this is immediate from Proposition 8.3 and Lemma 8.2. For $\chi_{\mathcal{H}}=0$ we need an additional argument. Let σ and ρ be automorphisms of $\mathrm{D^b}(\mathcal{H})$ such that σ and ρ act on slopes by $x\mapsto x+1$ and $x\mapsto x/(1+x)$, respectively; see [17] or [19]. We put $\phi=\rho\sigma^{1-p}$, where p is the least common multiple of the weight sequence. Then ϕ sends T_{can} to a tilting object $T'=\phi(T_{can})$ having maximal slope ∞, hence a non-zero torsion part. By Proposition 8.6 the tilting object T' belongs to the component T_{can}. Hence $\langle T_{can}\rangle$ is stable under the automorphism $\bar{\phi}$ of $\mathcal{C}$ induced by ϕ and, invoking Proposition 8.3, also under the action of the subgroup

$\langle \bar{\sigma}, \bar{\rho} \rangle$, which acts transitively on the rational circle $\mathbb{W}$. The claim now follows from Proposition 7.4. □

Theorem 8.8. *Assume $\mathcal{H}$ has Euler characteristic $\chi_{\mathcal{H}} \geq 0$. Then the tilting graph $\Gamma_{\mathcal{C}(\mathcal{H})}$ is connected.*

Proof. Let T be a tilting object in $\mathcal{H}$. We show that T belongs to the connected component $\langle T_{can} \rangle$.

Case $\chi_{\mathcal{H}} > 0$. By Proposition 8.4 we may assume that T is a tilting bundle. Then Proposition 8.5 proves that T belongs to $\langle T_{can} \rangle$, proving the claim in this case.

Case $\chi_{\mathcal{H}} = 0$. Let T be a tilting object in $\mathcal{H}$. By an automorphism ϕ in $\mathrm{D^b}(\mathcal{H})$ we achieve that $T' = \phi(T)$ has maximal slope ∞, hence by Proposition 8.6 belongs to $\langle T_{can} \rangle$. Since $\langle T_{can} \rangle$ is closed under the action of the automorphism group of $\mathcal{C}$, it follows that T belongs to $\langle T_{can} \rangle$. □

Remark 8.9. For $\chi_{\mathcal{H}} > 0$ connectedness of $\Gamma_{\mathcal{H}}$ is known for a long time, compare [9] and [4]. For the tubular weight type $(2, 2, 2, 2)$ connectedness of $\Gamma_{\mathcal{H}}$ has been shown by Barot and Geiß using combinatorial techniques (unpublished). It is conjectured that the tilting graph is also connected for $\chi_{\mathcal{H}} < 0$.

References

[1] M. Auslander, M. I. Platzeck and I. Reiten: *Coxeter functors without diagrams.* Trans. Amer. Math. Soc. **250** (1979), 1–46.

[2] M. Barot, D. Kussin and H. Lenzing, *The Grothendieck group of a cluster category.* J. Pure Appl. Algebra **212** (2008), 33–46.

[3] A. Buan, O. Iyama, I. Reiten and J. Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups.* Preprint (2007) `arXiv:math/0701557`.

[4] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics.* Adv. Math. **204** (2006), 572–618.

[5] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations.* J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[6] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras.* Springer Lecture Notes in Math. **1273**, 265-297, Springer-Verlag, Berlin, 1987.

[7] ———, *Perpendicular categories with applications to representations and sheaves.* J. Algebra 144 (1991), 273–343.

[8] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras.* Cambridge University Press, Cambridge, 1988.

[9] D. Happel and L. Unger, *On the set of tilting objects in hereditary categories.* Fields Institute Communications 45 (2005), 141–159.

[10] T. Hübner, *Exzeptionelle Vektorbündel und Reflektionen an Kippgarben über projektiven gewichteten Kurven.* Dissertation, Universität Paderborn, 1996.

[11] ———, *Rank additivity for quasi-tilted algebras of canonical type.* Colloq. Math. 75 (1998), 183–193.

[12] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules.* Invent. Math., to appear.

[13] B. Keller, *On triangulated orbit categories.* Documenta Math. **10** (2005), 551–581.

[14] D. Kussin, *Noncommutative curves of genus zero – related to finite dimensional algebras.* Mem. Amer. Math. Soc., to appear.

[15] ———, *Parameter curves for the regular representations of tame bimodules.* J. Algebra, to appear.

[16] H. Lenzing, *Representations of finite dimensional algebras and singularity theory*, Trends in ring theory (Miskolc, 1996) (V. Dlab et al., ed.), CMS Conf. Proc., vol. 22, Amer. Math. Soc., Providence, RI, 1998, pp. 71–97.

[17] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one, and representations of a tubular algebra.* CMS Conf. Proc. **14** (1992), 313–337.

[18] ———, *Tilting sheaves and concealed-canonical algebras.* CMS Conf. Proc. **14** (1992), 455–473.

[19] ———, *The automorphism group of the derived category for a weighted projective line.* Comm. Algebra **28** (2000), 1685–1700.

[20] H. Lenzing and J.A. de la Peña, *Concealed-canonical algebras and separating tubular families.* Proc. London Math. Soc. (3) **78** (1999), 513–540.

[21] H. Lenzing and I. Reiten, *Additive functions for quivers with relations.* Colloq. Math. 82 (1999), 85–103.

[22] H. Meltzer, *Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines.* Mem. Amer. Math. Soc. 171 (2004), no. 808, viii+139 pp.

[23] J. Rickard, *Derived equivalences as derived functors.* J. London Math. Soc. **43** (1991), 37–48.

[24] C. M. Ringel, *Tame algebras and integral quadratic forms.* Lecture Notes in Mathematics **1099**, Springer-Verlag, Berlin, 1984.

[25] ———, *The canonical algebras.* Topics in Algebra, Part I. Banach Center Publ., no. 26, PWN, 1990, With an appendix by William Crawley-Boevey, pp. 407–432.

[26] J.-L. Verdier, *Catégories dérivées (etat 0).* (french) in: P. Deligne ed.: Séminarie de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$. Lecture Notes in Mathematics **569**, Springer-Verlag, Berlin, 1977.

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