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THE CLUSTERING OF RADIO SOURCES—I
THE THEORY OF POWER-SPECTRUM ANALYSIS

*Adrian Webster**

Mullard Radio Astronomy Observatory, Cavendish Laboratory, Cambridge

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SUMMARY

The theory of power-spectrum analysis of the clustering of points is described and developed as a sensitive and flexible test for the possible weak clustering of extragalactic sources.

1. INTRODUCTION

This paper is the first of a short series in which catalogues of radio sources are analysed for evidence of clustering of the sources. Most radio sources detected at metre and decimetre wavelengths in directions away from the galactic plane are distant objects with redshifts of the order of unity, and the fortunate absence of any disturbing foreground effect such as galactic obscuration makes it possible to study the source population in a statistically unbiased manner. As a result the study of the radio sources is an unrivalled method of investigating the large-scale structure of the Universe at late epochs. A number of analyses of the clustering of the sources have already been made, but it was thought worthwhile to carry out the present work on account of the special nature of the test employed. This test, called Power Spectrum Analysis (PSA), is a substantial improvement on previous tests by virtue both of its great statistical power and of its flexibility. It is powerful in that it is sensitive, and can detect weak clustering that other tests might miss, and it is flexible in that it is capable of revealing many types of observational effect in the data and of separating such effects from the true clustering of the sources.

The use of PSA to investigate the clustering of objects is not new in astronomy: in particular Peebles and his colleagues have used the method as the basis of an extensive investigation of the clustering of galaxies. The version of the PSA employed in the present work is not, however, the same as Peebles'. The first major difference is that the positions of the sources on the celestial sphere are projected on to a plane and then analysed using a Fourier PSA, as opposed to Peebles' method of performing a spherical harmonic PSA on the sphere itself: the benefit is that the Fourier analysis is much easier to compute. The second difference is that the test is modified in order to discriminate as sensitively as possible between weakly clustered and unclustered populations, rather than to provide an accurate description of the statistical properties of a population such as galaxies which is well known to exhibit clustering. This modification was made in the hope of settling once and for all the question of whether the radio sources are clustered or not. A third difference, but relatively minor, is in the treatment of

* Present address: Institute of Astronomy, Madingley Road, Cambridge.

surveys whose boundaries are of shapes not directly suited to PSA. In the present paper the mathematical results are gathered and developed, and in subsequent papers the results are applied to various catalogues of radio sources.

2. THE NATURE OF THE METHOD OF FOURIER POWER SPECTRUM ANALYSIS

Fourier PSA may be employed to investigate the statistical properties of a distribution of any number of points lying within a rectangular volume in a space of any number of dimensions. The method is to calculate, for a given distribution of points (a given 'constellation'), the values of a certain statistics which are functions of the Cartesian coordinates of the points; the values of these statistics are called collectively the power spectrum of the constellation. The values are compared with the probability distribution expected on some hypothesis as to the statistical nature of the distribution of the points to test whether the actual distribution of points is consistent with the hypothetical (Bartlett 1963, 1964).

Our present concern is simply to find out whether the sources are clustered or not, so the hypothesis which is employed to generate the probability distribution of the power spectrum terms is the following *NULL HYPOTHESIS: the radio sources are uniformly, independently, randomly distributed on the celestial sphere, in that the probability of finding one source within an infinitesimal element of surface is proportional to the solid angle of the element, is independent of the position of the element on the sky and is independent of the position of any other source.* If it should turn out that the power spectrum of the constellation of sources in a particular catalogue is consistent with the null hypothesis, it will be said that the PSA has revealed no evidence of clustering.

3. ONE-DIMENSIONAL FOURIER PSA

In the interests of clarity of presentation the one-dimensional test will first be described, and the results later extended to the two-dimensions needed to study the radio sources.

It is supposed that a constellation of m points is scattered along a line of finite length. The coordinate scale is chosen so that the length of the line is 2π units, with the origin at the centre of the line. A function $f(x)$ is defined by erecting a delta-function of strength π at the position of each point:

$$f(x) = \pi \sum_{j=1}^{m} \delta(x-x_j) \tag{1}$$

where x_j is the coordinate of the jth point. The Fourier PSA consists in investigating the properties of certain statistics, to be defined later, which are related to the coefficients c_u and s_u in the Fourier series expansion

$$f(x) = \tfrac{1}{2}c_0 + \sum_{u=1}^{\infty} (c_u \cos ux + s_u \sin ux). \tag{2}$$

The coefficients are evaluated as usual:

$$c_u = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos ux \, dx = \sum_{j=1}^{m} \cos ux_j$$

$$s_u = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin ux \, dx = \sum_{j=1}^{m} \sin ux_j. \tag{3}$$

For a given value of u, c_u and s_u contain all the information on the constellation on a scale size $\lambda_u \equiv 2\pi/u$. This information is, however, divided between c_u and s_u in a way which is of no particular interest when testing for clustering, so use is made instead of the statistic r_u^2 defined by

$$r_u^2 = c_u^2 + s_u^2 = \left(\sum_j \cos ux_j\right)^2 + \left(\sum_j \sin ux_j\right)^2 = \left|\sum_j \exp(iux_j)\right|^2. \qquad (4)$$

The probability distribution of r_u^2 is calculated from the null hypothesis:

$$p_j(x)\,dx = dx/2\pi, \quad -\pi < x \leqslant \pi \qquad (5)$$

where $p_j(x)\,dx$ is the probability that the position of the jth point in the constellation lies between x and $x + dx$. The quantity r_u^2 is seen from equations (4) and (5) to be the square of the distance of the finishing-point from the starting-point in a two-dimensional random walk of m steps of unit length, each step being taken in a random direction (uniformly distributed and independent of the direction of every other step).

The probability distribution for this random walk is (e.g. Watson 1922)

$$p(r_u^2)\,dr_u^2 = \frac{dr_u^2}{2}\int_0^\infty J_0{}^m(\omega)\,J_0(\omega r_u)\,\omega\,d\omega \qquad (6)$$

where J_0 is the Bessel function. This probability distribution is independent of u, so all the r_u^2 ($u = 1, 2, \ldots$) are identically distributed in this case. For any given number of points the Hankel transform in equation (6) could be evaluated on the computer to give the probability distribution to any required accuracy. When, however, m is as large as the number of radio sources in a typical catalogue ($m \sim 10^2$–10^4) this is not necessary because the probability distribution $p(r_u^2)$ tends rapidly towards its central limit. As $m \to \infty$, $J_0{}^m(\omega) \to \exp(-m\omega^2/4)$ so $p(r_u^2) \to (1/m)\exp(-r_u^2/m)$. At this stage the power spectrum statistics I_u are defined, as appropriately renormalized versions of the r_u^2:

$$I_u \equiv \frac{2}{m} r_u^2. \qquad (7)$$

The asymptotic probability distribution of the I_u as $m \to \infty$ is thus

$$p(I_u)\,dI_u = \tfrac{1}{2}\exp(-\tfrac{1}{2}I_u)\,dI_u \qquad (8)$$

which is identical to the distribution of χ_2^2, chi-square for two degrees of freedom. Much use will be made of this result.

4. THE EFFECT OF CLUSTERING

For the PSA to be useful as a test for clustering it must be shown that the statistics I_u are sensitive to clustering. The test would be useless if the distribution of the I_u were still approximately χ_2^2 even in the presence of strong clustering.

A simple and rather general model is taken for the clustering. It is supposed: that all sources are members of clusters; that there are exactly q sources in each cluster; that each source is randomly placed within its cluster; that the clusters are all of much the same width Δx ($\Delta x \ll 2\pi$); and that the centres of the clusters are distributed independently, uniformly and at random in the interval $-\pi < x \leqslant \pi$.

It is then possible to divide the I into three classes, depending on the relative values of u and $u' \equiv 2\pi/\Delta x$:

(i) $u \ll u'$, giving $\lambda_u \gg \Delta x$. These I_u correspond to sinusoids with wavelengths much greater than the size of a cluster, so all the sources in any given cluster give practically equal contributions to the sums in equation (4). Thus the distribution of $r_u{}^2$ in this case is given by the two-dimensional random walk of $M = m/q$ independent steps, each of length q units. The previous argument now shows that the variate (I_u/q) is distributed as equation (6), so (I_u/q) is distributed as χ_2^2 in the central limit as $M \to \infty$, with I_u defined as before. Therefore I_u is no longer distributed as χ_2^2, but is caused by the clustering to take values greater than on the null hypothesis. In particular the expectation value of I_u is q times greater than the value 2 expected for unclustered sources. It is not difficult to show that if q, the number of points per cluster, is itself a random variable the expectation value $E(I_u)$ is increased by a factor of $\langle q \rangle$, the average number of sources per cluster;

(ii) $u \gg u'$, giving $\lambda_u \ll \Delta x$. When there are many cycles of the sinusoid within each cluster the coherent behaviour of the clustered points is lost and the statistic I_u is again distributed as χ_2^2 because the sources were assumed to be scattered randomly within the clusters;

(iii) $u \sim u'$. In this intermediate range $E(I_u)$ changes smoothly from $2\langle q \rangle$ to 2. The exact functional dependence of $E(I_u)$ upon u depends on the detailed model assumed for the distribution of sources within the clusters.

In this outline the small end-effect due to some clusters lying partly inside and partly outside the surveyed region $-\pi < x \leq \pi$ has been ignored.

The asymptotic behaviour of the I_u has been derived, both for clustered and unclustered points, by approximating analytically the exact result in equation (6). This approach is thought to be clearer and more satisfactory than previous derivations in which the behaviour in the central limit was assumed rather than derived.

A most valuable feature of the above results is that they are independent of the density of sources on the line (provided only that $m \gg 1$). Thus the method is equally sensitive whether the individual clusters are isolated from their neighbours $(m/2\pi < q/\Delta x)$ or are piled densely one on top of another $(m/2\pi > q/\Delta x)$.

5. THE SENSITIVITY TO WEAK CLUSTERING

The PSA test for clustering would be weak if only one of the I were scrutinized. For example, if a certain constellation were to give $I_7 = 4{\cdot}0$ it is possible that the constellation exhibits clustering with $\langle q \rangle = 2{\cdot}0$ sources per cluster on some scale size $\lambda_c < 2\pi/7$ but it is also quite possible that the constellation is entirely unclustered with $\langle q \rangle = 1$, because 4·0 is not an unlikely value for a χ_2^2 variate. To increase the power of the test for clustering on some scale size λ_c it is clearly necessary to scrutinize all the I_u with $\lambda_u > \lambda_c$, and the simplest way to do this is to add them all together and consider the statistic Σ_I:

$$\Sigma_I \equiv \sum_{u=1}^{U} I_u \quad \text{when } U \text{ is such that } \lambda_U \sim \lambda_c. \tag{9}$$

The sum of U independent χ_2^2 variates is distributed as χ_{2U}^2, so the value of Σ_I for a given constellation may be compared with this distribution. Defining $\Sigma_\nu \equiv 2U$ and the clustering statistic Q for a given constellation as

$$Q \equiv \Sigma_I/\Sigma_\nu \tag{10}$$

the expectation value of Q in the presence of clustering on the scale size λ_c is seen to be $E(Q) = \langle q \rangle$, and the standard deviation of Q is, from the properties of the chi-square distribution, $\sigma(Q) = \langle q \rangle / \sqrt{U}$. It is clear from this last expression that U should be taken as large as possible in order that the test be most sensitive in discerning that in a given case $\langle q \rangle$ is slightly greater than unity rather than equal to unity.

In the above discussion of the properties of Σ_I and Q one important point was overlooked because of its complexity: the I_u are not independent variates even on the null hypothesis, so however accurately the individual terms are distributed as χ_2^2 their sum is not in fact distributed exactly as $\chi_{\Sigma\nu}^2$. The exact distribution of Σ_I is not known on account of the complexity of the independence of the I_u but fortunately the weakness of the interdependence results in $\chi_{\Sigma\nu}^2$ being a very good approximation in most circumstances. The interdependence is weak in that not only are any two I_u asymptotically independent as $m \to \infty$, but they are also uncorrelated for all values of m (see Appendix). The asymptotic independence means in practice that if U, the number of I_u considered, is smaller than m the distribution of Σ_I will be $\chi_{\Sigma\nu}^2$ to a good approximation, and the absence of correlation implies that even for $U > m$, although $\chi_{\Sigma\nu}^2$ is no longer a good approximation to the distribution of Σ_I, it is not a bad guide either because it has the same mean and variance as the true distribution (the lowest moment of the distribution of Σ_I which is affected by the interdependence is the third, which is increased). It is in any case always possible to find an unbiased estimate of the true distribution of Σ_I even when $U \gg m$ by a Monte Carlo method involving the repeated generation of a random constellation of m sources in the computer followed by PSA. Thus, whilst the effect of the lack of independence of the power spectrum statistics I_u on the distribution of Σ_I is an unsolved theoretical problem of some interest, in practice it is no great hindrance to the application of PSA to test for clustering.

The idea of adding together a number of PSA statistics to improve the test is not new (Bartlett 1963; Yu & Peebles 1969). The discussion in this section of the effects of the interdependence of the statistics is, however, more thorough than its predecessors, as it needs be in view of the subsequent applications in which the sum of an exceptionally large number of statistics is considered.

6. TWO-DIMENSIONAL FOURIER PSA OF A RADIO SOURCE CONSTELLATION

Peebles (Yu & Peebles 1969; Peebles 1973) has shown how to use spherical harmonics as the basis function for a PSA to investigate the clustering of points on the surface of a sphere. This method was not used here because calculation of the spherical harmonics is expensive of computer resources, and because the test is only applicable in its simple form when the whole celestial sphere has been surveyed whilst in practice the radio surveys cover less than 4π sr and have irregular boundaries.

The present method employs instead an appropriate equal area projection to project the surveyed region on to a flat plane, on which a Fourier PSA is performed. An equal area projection has the advantage that if the points are scattered over the surface of the sphere in a fashion consistent with the null hypothesis then their projections are similarly distributed on the plane: uniformly, independently at random. Further, if the points on the sphere are clustered then so are their projections on the plane clustered, although the shapes and sizes of the clusters may be somewhat distorted. The problem of the irregular boundaries of the projected survey area could be resolved by a method analogous to Peebles' but for simplicity the largest possible rectangle which lies within the projected survey area is studied: all points within this rectangle are subjected to the Fourier PSA whilst those outside are discarded.

The two-dimensional PSA terms I_{uv} are derived from the statistics r_{uv}^2 defined by analogy with equation (4) as

$$r_{uv}^2 = \left| \sum_j \exp i(ux_j + vy_j) \right|^2 \qquad (11)$$

where (x_j, y_j) are the coordinates of the jth point in the constellation $(-\pi < x_j \leqslant \pi; -\pi < y_j \leqslant \pi; j = 1, 2, \ldots, m)$. The value of u may be zero or any positive integer: if u is zero v may be any positive integer, whilst if u is non-zero v may be any positive or negative integer or zero. From the previous discussion it is clear that if the null hypothesis holds r_{uv}^2 is distributed as r_u^2 in equation (6), so the statistic

$$I_{uv} \equiv \frac{2}{m} \times r_{uv}^2 \qquad (12)$$

is asymptotically $(m \to \infty)$ distributed as χ_2^2.

In two-dimensions the wavelength of the sinusoid corresponding to I_{uv} needs a little care in its definition because the field over which the Fourier PSA is performed need not be square, only rectangular. Thus if on the plane the projected coordinates of the radio sources are (ξ_j, η_j) where $X/2 < \xi_j < X/2$, $Y/2 < \eta_j < Y/2$, with X and Y measured in the same units (e.g. degrees or radians), then the dimensionless variables used in the PSA are defined as $x_j = 2\pi\xi_j/X$, $y_j = 2\pi\eta_j/Y$ and the wavelength λ_{uv} is $\lambda_{uv} = 1/\sqrt{(u^2/X^2 + v^2/Y^2)}$.

The two-dimensional test is employed just as the one-dimensional. Many terms I_{uv} are summed to make the test most powerful. To investigate clustering on a scale size λ_c the quantities Σ_I and Σ_ν are defined:

$$\Sigma_I = \sum_{(u,v)} I_{uv}, \quad \Sigma_\nu = \sum_{(u,v)} 2 \qquad (13)$$

where the sums are taken over the same pairs of values of u and v such that $\lambda_{uv} > \lambda_c$. Then Σ_I is to a good approximation distributed as $\chi^2_{\Sigma\nu}$ on the null hypothesis, so if there is no significant clustering in the constellation under test the value of Σ_I should be consistent with this distribution. If clustering is found the clustering parameter $Q \equiv \Sigma_I/\Sigma_\nu$ may be evaluated as an estimate of $\langle q \rangle$, the average number of sources per cluster. In practice one generally has no preconceived ideas about what the scale of any clustering might be, so a histogram of $Q' \equiv \Sigma_I'/\Sigma_\nu'$ against $1/\lambda$ is drawn up and inspected, where the sums Σ' are defined as in equation (13) except that instead of including all terms with $0 < 1/\lambda < 1/\lambda_c$ the sums are taken over suitable successive constant intervals $\Delta_{1/\lambda}$ of reciprocal wavelength, so the nth bin of the histogram contains all I_{uv} with $(n-1)\,\Delta_{1/\lambda} < 1/\lambda_{uv} < n\Delta_{1/\lambda}$ where n is an integer. In this way any trend in the values of Q' is easily spotted, and may be used to pick out the characteristic reciprocal wavelength of any clustering. The values of Σ_I, Σ_ν and Q may then be evaluated for this scale size to test the significance of the clustering. All the remarks about the weak effects of the lack of independence of the power spectrum terms on the statistic Σ_I made in the one-dimensional case apply to the two-dimensional case also.

7. THE FLEXIBILITY OF THE TEST

The PSA test for clustering in two-dimensions is very flexible, chiefly because each I_{uv} is associated with a direction as well as with a wavelength. The directionality permits anticipated experimental irregularities of certain kinds to be eliminated, and unanticipated irregularities to be discovered. For example, a radio survey made by drift scans might result in an artificial corrugation in declination of the density of sources if the receiver gain varied from day to day. An equal area projection which keeps parallels of declination parallel on the plane and also parallel to the x-axis will cause the corrugations to be functions of y only, so the only terms affected are those with $u = 0$. These I_{0v} may be excluded from the sums for Σ_I and Σ_ν because they are expected to be spuriously large, or alternatively if the corrugation had not been expected it might have been discovered by inspecting the I_{0v}. By and large any anisotropic evidence of this kind for clustering of the radio sources should be viewed with some suspicion.

A second way in which the flexibility of the test is useful in critically evaluating evidence for clustering is that the variation of the power spectrum with limiting flux density may be investigated. Peebles (1973) has shown how this ‘scaling’ should affect a PSA of the distribution of galaxies: in brief, the clustering should shift to smaller scales as fainter galaxies are included because more distant clusters become accessible. A similar effect is perhaps to be expected for the radio sources, although it will not be as strong because the great spread of the radio luminosity distribution implies that even at the strongest flux densities the most powerful radio sources are visible at substantial redshifts ($z \sim 1$–3), after which cosmological effects cut off their flux, not the inverse-square law. For radio sources it would also be expected that Q, the estimated value of $\langle q \rangle$, should increase as fainter sources are included because fainter members of radio clusters whose bright members are already detected will progressively become accessible. Thus any evidence for clustering of bright sources but not of faint must again be viewed with suspicion.

8. COMPARISON OF PSA WITH OTHER TESTS FOR CLUSTERING

Two other tests have frequently been applied to examine possible clustering of points: nearest neighbour analysis (NNA) and binning analysis (BA). In the former each source in turn is used as a target and the distance to its nearest neighbouring source is measured. A histogram of all these nearest neighbours is drawn up and compared with the distribution expected for unclustered points. The BA consists in drawing up a two-dimensional grid on the surveyed area and counting the number of sources within each cell or bin. A chi-square test is then applied to check that the variance of the counts between bins tallies with that of the Poisson distribution for unclustered sources.

NNA is greatly inferior to PSA, both in flexibility and in power. There appears to be no way of applying it which checks on either the isotropy of the scaling of any clustering. The method is less powerful because only the nearest neighbour distance of a given point is investigated whilst the PSA uses information on the separation of all pairs of points. To show this, consider the one-dimensional PSA statistic Σ_I:

$$\frac{m}{2}\Sigma_I = \sum_{u=1}^{U}\left\{\left(\sum_j \cos ux_j\right)^2 + \left(\sum_j \sin ux_j\right)^2\right\}$$

$$= \sum_u \left\{ \sum_j \sum_k (\cos ux_j \cos ux_k + \sin ux_j \sin ux_k) \right\} = \sum_u \sum_j \sum_k \cos u(x_j - x_k)$$

$$= Um + \sum_u \sum_j \sum_{k \neq j} \cos u(x_j - x_k) = Um + 2 \sum_j \sum_{k<j} \sum_u \cos u(x_j - x_k)$$

$$= Um + \sum_j \sum_{k<j} F_U(x_j - x_k)$$

where

$$F_U(x) \equiv 2 \sum_{u=1}^{U} \cos ux = \frac{\sin (U+\frac{1}{2})\, x}{\sin \frac{1}{2}x} - 1.$$

Thus the part of Σ_I which depends on the x_j may be written as the sum over all pairs of points of a function of the separation of the pair. This function $F_U(x)$ has, for $U \gg 1$, a large positive peak at $|x| < \pi/U+\frac{1}{2}$ and elsewhere takes much smaller values, close to -1: this behaviour is illustrated in Fig. 1 for the cases $U = 5$ and $U = 20$. If the points are clustered on a scale $\Delta x \lesssim \pi/U+\frac{1}{2}$ *all* the points in the same cluster as the target point give substantial positive contributions to Σ_I, so the PSA test clearly has greater statistical power than the NNA. A similar result holds in two-dimensions.

PSA is particularly superior to NNA in the case in which clusters are densely piled on top of one another. It was shown earlier that this does not degrade the sensitivity of PSA in any way, but if the clusters are so densely stacked that the nearest neighbour of any target point is most likely to be a member of some other cluster the NNA becomes very insensitive to the clustering.

PSA is clearly superior to BA also, in that the single statistic produced by BA to be checked against the chi-square distribution admits of no checking for isotropy or scaling of clustering (Yu & Peebles 1969; Peebles 1973) and is of low statistical power. The last objection may be met in some degree by extending the BA to a multiple binning analysis (MBA) in which as many different grid sizes and shapes

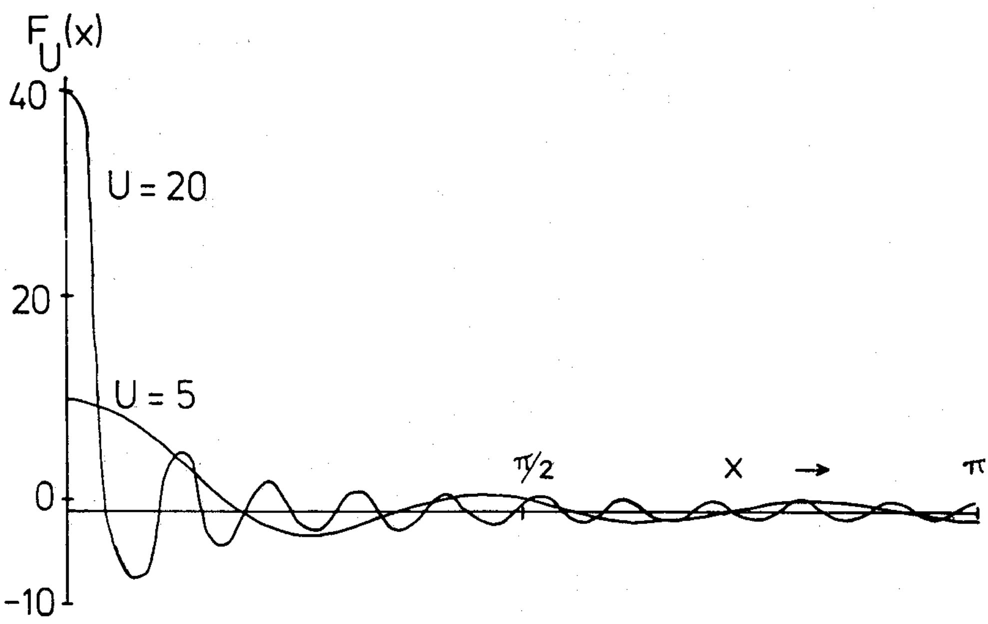

FIG. 1. *The function $F_U(x)$ for $U = 5$ and $U = 20$. These curves are presented to show the development of a narrow, positive spike near $|x| = 0$ as U is increased.*

as possible are investigated, but such an MBA appears never to have been applied to an astronomical problem, and in any case still suffers from a mixing of scales (Yu & Peebles 1969; Peebles 1973).

9. SUMMARY OF RESULTS

In this paper the method of power spectrum analysis was developed as a test for possible weak clustering of the extragalactic radio sources, and the statistics Σ_I and Q were defined. It was shown that in the absence of clustering Σ_I should be distributed as a certain chi-square variate to a good approximation whilst in the presence of clustering it will take values greater than expected on this basis. In the latter case the statistic Q is a good estimate of the average number of points per cluster. It was further shown that the test is considerably more sensitive than any other which has been applied to astronomical objects and is also more flexible in that ways exist of checking that any significant effect is indicative of a true clustering of the sources rather than of some observational artefact.

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APPENDIX

To show that I_p is uncorrelated with I_q $(p \neq q)$ on the null hypothesis.

First the expectation value of r_p^2 is calculated:

$$r_p^2 = \left(\sum_{j1} \cos px_{j_1}\right)^2 + \left(\sum_{j2} \sin px_{j_2}\right)^2 = \sum_{j1}\sum_{j2} \exp ip[x_{j_1} - x_{j_2}]$$

therefore

$$\begin{aligned} E(r_p^2) &= \sum_{j1}\sum_{j2} E(\exp ip[x_{j_1} - x_{j_2}]) \\ &= \sum_{j1} E(\exp [0]) + \sum_{j1} \sum_{j2 \neq j1} E(\exp ipx_{j_1})\, E(\exp ipx_{j_2}) \\ &= m \end{aligned}$$

where use has been made of the null hypothesis (i) to separate the exponential of the two independent variates x_{j_1} and x_{j_2} $(j_1 \neq j_2)$, and (ii) to put $E(\exp ipx_j) = 0$.

The expectation value of the product $r_p^2 r_q^2$ is

$$E(r_p^2 r_q^2) = \sum_{j_1} \sum_{j_2} \sum_{j_3} \sum_{j_4} E(\exp i\{p[x_{j_1} - x_{j_2}] + q[x_{j_3} - x_{j_4}]\})$$

and for $p \neq q$ the only non-vanishing terms in the sum are those for which $j_1 = j_2$ and $j_3 = j_4$:

$$E(r_p^2 r_q^2) = \sum_{j_1} \sum_{j_3} 1 = m^2 = E(r_p^2)\, E(r_q^2).$$

Thus r_p^2 is uncorrelated with r_q^2, and it follows that I_p is uncorrelated with I_q. A similar calculation shows that I_p is uncorrelated with I_{qs} for all p, q and s, and that I_{pq} is uncorrelated with I_{st} (except for the trivial case $p = s$, $q = t$).

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THE CLUSTERING OF RADIO SOURCES—II
THE 4C, GB AND MC1 SURVEYS

Adrian Webster†

Mullard Radio Astronomy Observatory, Cavendish Laboratory, Cambridge

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SUMMARY

The 4C, GB and MC1 catalogues are subjected to power spectrum analysis to investigate possible clustering of the extragalactic radio sources. No evidence of clustering is found, although certain experimental effects are revealed. The results are compared and contrasted with previous work, and a stringent limit is put on the average number of radio sources in any hypothetical clusters.

1. INTRODUCTION

In this paper the results of a power spectrum analysis (PSA) of the 4C, GB and MC1 surveys are presented. The method of analysis was described in a previous paper (Webster 1976, hereinafter Paper I), to which the reader is referred for a definition of the symbols. The choice of surveys for analysis was motivated chiefly by a disagreement between previous workers on the fundamental issue of whether the sources are clustered or not. The 4C survey was selected because little evidence has previously been found in it for clustering (Holden 1966; Hughes & Longair 1967; Pearson 1974; Golden 1974), and the GB survey because of strong claims to the contrary (Maslowski, Machalski & Zieba 1973; Maslowski 1973; Machalski, Zieba & Maslowski 1974). The MC1 survey was selected partly as a control on the GB survey, to which it is similar in several important respects. It was pointed out in Paper I that the PSA is considerably more powerful and flexible than other tests which have been applied to the problem, so the present analysis was undertaken in the hope of settling the fundamental issue with some finality.

The PSA of the catalogues is carried out in Sections 2, 3 and 4. An interesting effect which is found in the 4C and GB surveys is analysed in Section 5 and shown to be instrumental in origin. After identification and removal of this instrumental effect the results lead straight to the conclusion that there is no good evidence at all in the three catalogues that the sources are distributed otherwise than completely at random. This conclusion is compared and reconciled with previous work on the 4C and GB surveys in Section 6, and in Section 7 the results are summarized and some inferences drawn.

2. THE 4C SURVEY

The 4C catalogue (Pilkington & Scott 1965; Gower, Scott & Wills 1967) contains 4843 sources in an area of 6·95 sr of the northern sky, down to a limiting

† Present address: Institute of Astronomy, Madingley Road, Cambridge.

flux density of $S_0 = 2{\cdot}0$ Jy at 178 MHz. The survey was made with an aperture synthesis interferometer of beamwidth $23' \times 31'$ sec (zenith angle), which gives an average of about 30 beam areas per source.

The surveyed region was divided into four areas for the analysis. Each area was a ' square ' of side 75°. The following equal area projection was used:

$$x_j = -\pi(\alpha_j - \alpha_0)\sqrt{\cos\delta_j}/\Delta\alpha; \quad -\pi < x_j \leq \pi, \quad j = 1, 2, \ldots m$$

$$y_j = 2\pi\left(\int_{\delta_-}^{\delta_j} \sqrt{\cos\delta}\, d\delta \Bigg/ \int_{\delta_-}^{\delta_+} \sqrt{\cos\delta}\, d\delta\right) - \pi; \quad -\pi < y_j \leq \pi$$

where (α_j, δ_j) are the celestial coordinates of the jth source in the catalogue, α_0 is the right ascension of the centre of the area, δ_- $(= -6°)$ and δ_+ $(= +69°)$ are the declination limits of the area, and $\Delta\alpha$ $(= 37°{\cdot}5 = 2^{\rm h}\, 30^{\rm m})$ is the half-width of the area at the equator. Thus the lengths of the sides of the chosen area are $X \sim 75°$, $Y = 75°$. This particular projection was chosen for two reasons. First the parallels of declination become lines of y = const., which enables the detection of any corrugation of the sensitivity of the survey in declination resulting from the observations being made by drift scans (Paper I, Section 7). And second the projection introduces little distortion and mixing of scales over most of the area: this may be seen from Fig. 1 in which a grid of lines of constant right ascension and declination is shown projected on to the (x, y)-plane.

The first line of Table I shows the central right ascension of each area. These four centres were chosen in order to give two areas (I and II) almost completely free of the interfering effects of bright sources. Area III was moderately affected, principally by Cas A and Cyg A at lower culmination, whilst area IV actually contained these sources together with the brightest part of the galactic plane. Because of this strong interference any peculiarity in the results from area IV which is not found in areas I and II is likely to be spurious.

The PSA was carried out on all the sources (i.e. with limiting flux density $S_0 = 2{\cdot}0$ Jy) in the four areas, and the results are shown in Fig. 2 as a plot of the

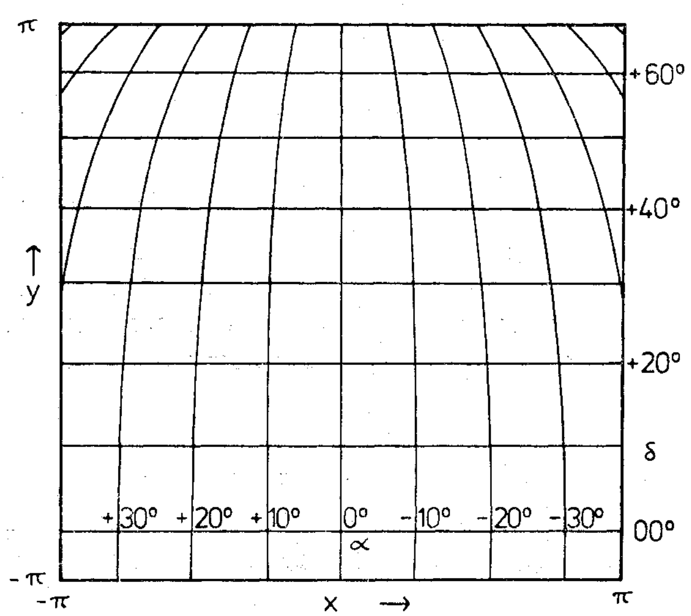

FIG. 1. *The equal-area projection employed for the 4C survey.*

TABLE I

The values of the PSA statistics from the four areas of the 4C survey

Area:	I	II	III	IV
1. α_0	04^h	16^h	10^h	22^h
2. Σ_I for $S_0 = 2$ Jy	943·3	991·1	876·5	962·1
3. Q for $S_0 = 2$ Jy	0·89	0·94	0·83	0·91
4. Σ_I for $S_0 = 3$ Jy	496·8	479·6	442·0	592·4
5. Q for $S_0 = 3$ Jy	0·98	0·94	0·87	1·17

run of Q' on $1/\lambda$: the average number of sources in each area was 976. No abnormally large value of I_{0v} was found in any area, showing that any corrugation of sensitivity in declination is small.

It was shown in Paper I that in the presence of clustering the expectation value of Q' is greater than unity up to some characteristic reciprocal wavelengh $1/\lambda_c$ and thereafter falls to unity, whilst in the absence of clustering the expectation value is unity for all reciprocal wavelengths. For areas I, II and III it is seen that there is no evidence at all of clustering: the values of Q' are not significantly greater than unity at any reciprocal wavelength, and there is no tendency for Q' to decrease as $1/\lambda$ increases. In fact the reverse is the case. The values of Q' are collectively significantly smaller than unity. Line 2 of Table I shows the values of the statistic Σ_I for the four areas, corresponding to the sum of all the power

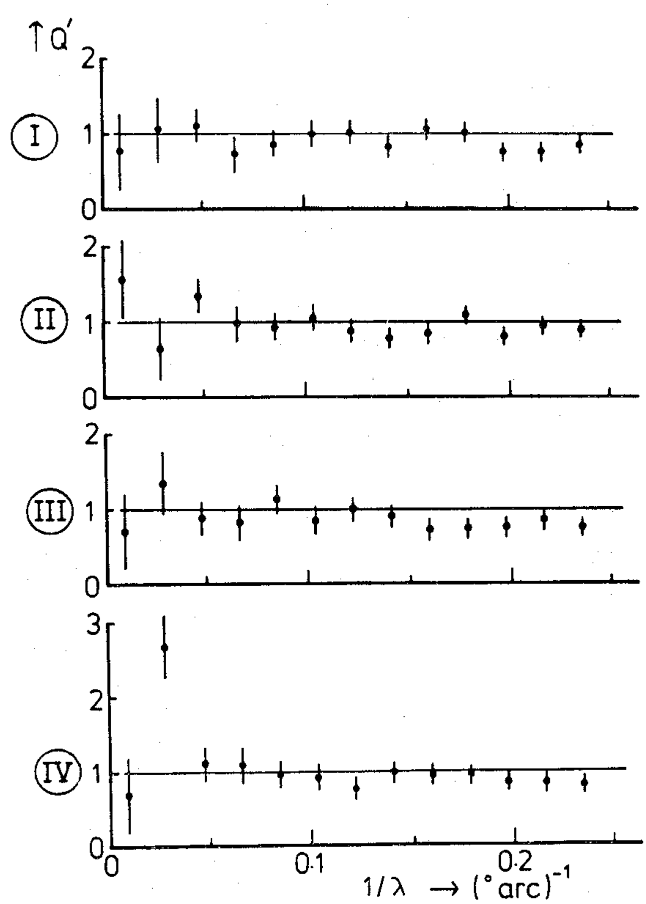

FIG. 2. *The run of Q' against $1/\lambda$ for the four areas of the 4C survey ($S_0 = 2$ Jy). The error bars in this and subsequent figures are $\pm 1\ \sigma$, based on the null hypothesis.*

spectrum terms I_{uv} included in Fig. 2. The value of Σ_ν is 1056 in each case. It was shown in Paper I that Σ_I should be distributed as $\chi'^2_{\Sigma_\nu}$ on the null hypothesis (i.e. in the absence of clustering), and the 2·5 percentile (2·5 %ile) of χ^2_{1056} is 965·9. Thus three of the four values of Σ_I are significantly small at this 95 per cent confidence level. Adding the four values Σ_I should give a χ^2_{4224} variate, and the sum (3773·0) is smaller than the 0·00005 %ile of χ^2_{4224} so the effect is highly significant.

The name 'anticlustering' is here coined for this phenomenon in which Σ_I is significantly *smaller* than expected on the null hypothesis. A discussion of the nature and cause of the anticlustering is to be found in Section 5. It is thought worthwhile to coin a new word where 'anticorrelated' would suffice because 'anticlustered' is so much more specific. 'Correlated' indicates, in the present context, only that the positions of the sources are not independent but 'clustered' tells more closely in what fashion they are interdependent. 'Anticlustered' refers specifically to the converse of 'clustered'.

The values of the statistic Q for all the terms in each area are shown in line 3 of Table I: the formal standard deviation is $\sigma(Q) = 0{\cdot}04$ (Paper I, Section 5). The statistic Q, whose expectation value is unity for unclustered sources, is a measure of the anticlustering of the 4C sources. The average over all the areas is $0{\cdot}89 \pm 0{\cdot}02$.

The run of Q' on $1/\lambda$ in area IV is somewhat different from the run in the other areas in that a long-wavelength point ($1/\lambda = 0{\cdot}025$ deg^{-1}) is abnormally large. This effect is just what would be expected from the strong interference in this area, and since the interference-free areas do not show it we must conclude that it does not indicate a true clustering of the sources.

The PSA was repeated using only those sources with $S \geqslant 3{\cdot}0$ Jy, to see whether the distribution of sources changed in any way with the limiting flux density but little change was found: the only appreciable effect was that the anticlustering decreased somewhat. The run of Q' on $1/\lambda$ was drawn down to scale sizes $\lambda = 6^\circ{\cdot}2$. In areas I and II the values of Q' were scattered about unity, in area III some anticlustering was evident, whilst area IV showed the interference as before. Lines 4 and 5 of Table I show the values of Σ_I and Q: $\Sigma_\nu = 508$ in this case, the average number of sources per area was 469, and $\sigma(Q) = 0{\cdot}06$. The 2·5 and 97·5 %iles of χ^2_{508} are 445·5 and 570·5 respectively so the area I and II values of Σ_I are consistent with the null hypothesis, the area III value is too small and the area IV is too great. The average value of Q for all four areas is 0·99, and for the first three areas is $0{\cdot}93 \pm 0{\cdot}04$. Thus, if area IV is excluded on account of the interference there is marginal evidence of anticlustering of the bright sources. No evidence for clustering is apparent.

In summary, the PSA has revealed no evidence at all of clustering of the radio sources in the 4C catalogue.

3. THE GB SURVEY

The GB survey (Maslowski 1972) was made at 1·4 GHz with the NRAO 300-ft transit telescope. The area surveyed was a thin strip subtending 0·16 sr ($07^h\,17^m < \alpha < 16^h\,23^m$, $+45^\circ{\cdot}8 < \delta < 51^\circ{\cdot}7$) and 1086 sources were found down to the nominal flux density limit of $S_0 = 0{\cdot}09$ Jy. The primary beamwidth of the antenna was $10'{\cdot}3 \times 11'{\cdot}1$ so there were on average about 19 beam areas per source and the survey was strongly confusion limited.

It is of particular interest that it has been claimed that the positions of the radio sources in the GB catalogue are significantly non-random and show clear evidence of clustering on solid angular scales between 0·005 and 0·08 sr (Maslowski, Machalski & Zieba 1973, hereinafter MMZ); other related non-random effects have also been claimed such as anisotropy of the slope of the source counts and of the distribution of mean flux density (Maslowski 1973; Machalski *et al.* 1974).

For the PSA the position of every source in the catalogue was projected on to the plane using Mercator's projection. The extra complication of an equal area projection was not worthwhile for such a narrow strip: at most only the $u = 0$ terms would be affected by the distortion of areas, and only by a very small amount. The power spectrum terms were calculated for wavelengths down to 1°·05. None of the I_{0v} took a particularly large value so neither the Mercator's projection nor any drift-scan sensitivity corrugation was very important.

The run of Q' against $1/\lambda$ is shown in Fig. 3. It is clear at once that there is no sign of clustering: at no reciprocal wavelength is Q' significantly greater than unity, nor do the values of Q' decrease as $1/\lambda$ increases. Quite the contrary is the case with strong anticlustering manifest over most of the range of wavelengths. In Table II, line 1 shows Σ_I, Σ_v and Q for all the terms in Fig. 3 (i.e. $\lambda \geqslant 1°{\cdot}05$) and line 2 for the first seven points ($\lambda \geqslant 1°{\cdot}53$), which show the anticlustering most clearly. The 2·5 %iles of $\chi^2_{\Sigma_v}$ are also shown. It is seen that in both cases the constellation shows highly significant anticlustering in that $\Sigma_I \ll 2{\cdot}5$ %ile of $\chi^2_{\Sigma_v}$.

These results (that the catalogue is overall strongly anticlustered, shows no evidence of clustering in any range of wavelengths and does not show the trend of Q' against $1/\lambda$ expected for clustered sources) are surprising in view of the earlier claims that the catalogue shows evidence of significant clustering. This disagreement will be discussed in Section 6 but first the results of a PSA of the bright GB sources is presented because the most significant clustering effects previously claimed were for bright sources. This PSA was carried out in the same way as the

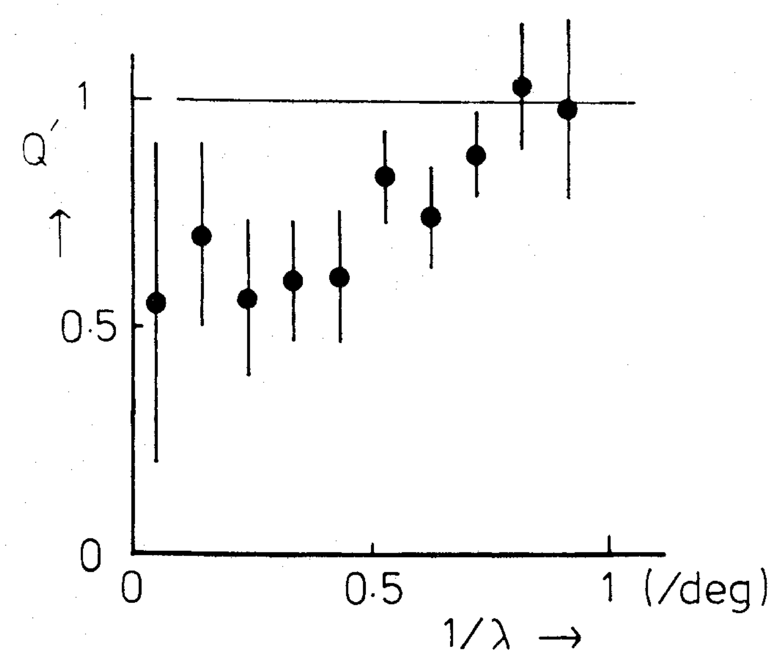

FIG. 3. *The run of Q' against $1/\lambda$ for the entire GB survey (S_0 = 0·09 Jy).*

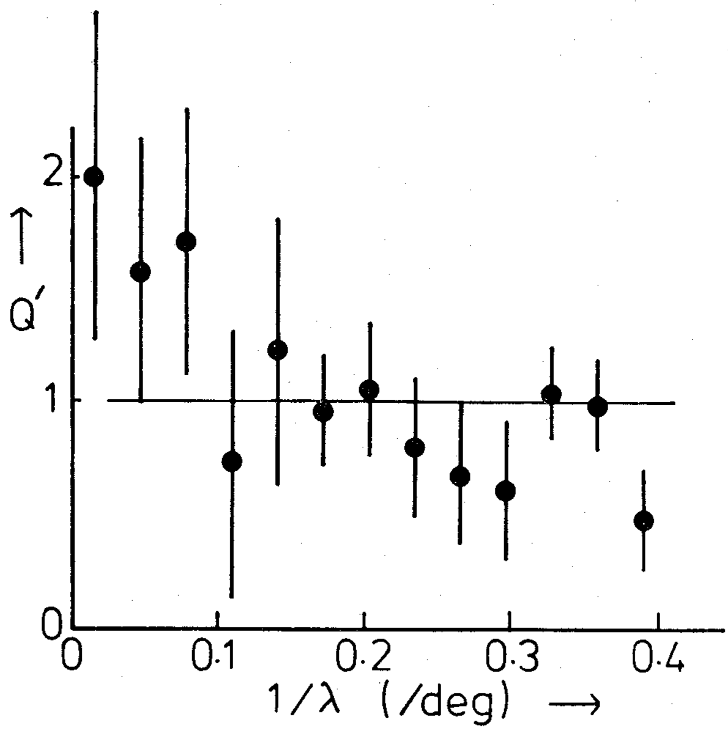

FIG. 4. *The run of Q′ against* $1/\lambda$ *for the bright GB sources* (S_0 = 0·25 *Jy*).

first except that only sources brighter than 0·25 Jy were included. The run of Q' on $1/\lambda$ is shown in Fig. 4. This time there is evidence of clustering, albeit weak evidence: Q' is greater than unity for $1/\lambda \leqslant 0{\cdot}094$ deg^{-1} and decreases at greater reciprocal wavelengths. The PSA statistics are shown in lines 3 and 4 of Table II for $\lambda \geqslant 2^\circ{\cdot}47$ and $\lambda \geqslant 10^\circ{\cdot}7$ respectively. It is seen that for $\lambda \geqslant 10^\circ{\cdot}7$, corresponding to the first three points in Fig. 4, there is evidence for clustering with about 1·7 sources per cluster. However, this clustering is of marginal statistical significance because the 97·5 %ile of χ^2_{16} has the value 28·85, and the value of Σ_I is a little smaller than this. At reciprocal wavelengths greater than about 0·1 deg^{-1} the values of Q' fall to less than unity, indicating that the anticlustering is still present. From line 3 it is seen that in total, on scale sizes down to $2^\circ{\cdot}5$ the anticlustering dominates the clustering to give a value of Q smaller than 1·0, although not significantly smaller.

In summary there is evidence of clustering of the bright GB sources which is, however, not quite significant at the 95 per cent confidence level. It is now of interest critically to examine this evidence to check that it has the characteristics expected for true celestial clustering. Unfortunately it is not possible to examine the isotropy of the clustering because the long wavelengths ($\lambda \geqslant 10^\circ{\cdot}7$) which

TABLE II

The values of the PSA statistics for the GB survey

	λ_{min}(°)	S_0 (Jy)	Σ_ν	Σ_I	Q	$\sigma(Q)$	2·5 %ile of $\chi^2_{\Sigma\nu}$
1.	1·05	0·09	1088	849·7	0·78	0·04	996·6
2.	1·53	0·09	708	492·8	0·70	0·05	634·3
3.	2·47	0·25	294	264·2	0·90	0·08	246·5
4.	10·7	0·25	16	27·5	1·72	0·35	—

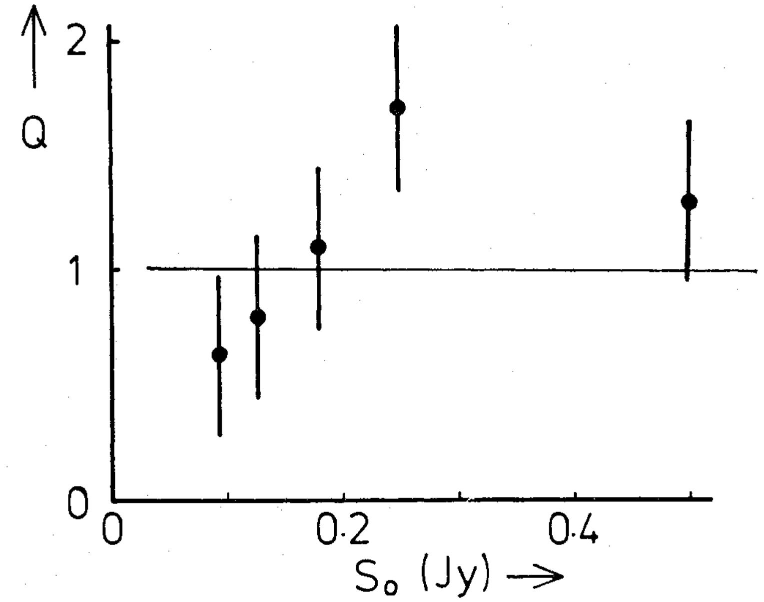

FIG. 5. *The dependence of Q on S_0 for the long-wavelength ($\lambda \geqslant 10°.7$) effect found in the GB sources.*

exhibit the effect can only be found amongst the I_{u0} terms: the survey is too narrow in declination for any other I_{uv} to have such a long wavelength. It is possible though to examine the dependence of Q on S_0, the limiting flux density, by repeating the PSA for various values of S_0. The results of this procedure are shown in Fig. 5. The terms included in the sums Σ_I and Σ_ν from which Q is calculated are in each case those with $\lambda \geqslant 10°.7$. It was pointed out in Paper I, Section 7, that Q is expected to increase as S_0 decreases if the clustering is truly a property of the distribution of sources in space, but it is clear from the Figure that the clustering effect in the GB sources shows the opposite behaviour. It is not easy to understand how this behaviour can be the result of true clustering without making unlikely *ad hoc* assumptions about the sources. If the conventional view is accepted that the sources are mostly at cosmological distances then it is necessary to assume that the most luminous of them are clustered whilst the least luminous in the same region of space are smoothly spread. If, on the other hand, it is supposed that the clustered, bright sources are intrinsically faint members of a nearby group, then that group must be a rare object because sources with the same luminosity but a little further away do not show the clustering.

The long-wavelength effect in the bright GB sources is marginal and has a peculiar dependence on the limiting flux density, so it cannot be taken as good evidence of true clustering of the sources in space. It is thus concluded that this PSA of the GB catalogue has revealed no strong or clear evidence of clustering, and that by far the main effect shown by the catalogue is strong anticlustering. The origin of this anticlustering is discussed in Section 5 and the conclusion of MMZ that the GB sources are strongly clustered is discussed in Section 6.

4. THE MC1 SURVEY

MC1, the *First Molonglo radio source catalogue* (Davies, Little & Mills 1973), contains the positions and flux densities of 1545 sources down to a nominal limit of $S_0 = 0{\cdot}1$ Jy in the 0·21 sr defined by $-19°{\cdot}3 > \delta > -22°{\cdot}4$; $01^h\,08^m < \alpha < 16^h\,48^m$. The survey was made at 408 MHz with the Molonglo one-mile cross telescope; the beam was $2'{\cdot}7$ in diameter giving 330 beam areas per source so the survey was noise limited. The survey is analysed here because of its southern declination and because it is similar to the GB survey in the shape of the surveyed area and in the density of sources.

Only those sources in the catalogue which had flux densities listed were employed in the PSA. The width of the area was reduced by eliminating two strips of sky each $0°{\cdot}3$ wide at the nominal declination limits of the survey because it was found that the source density was a little irregular at the edges. This left 1369 sources, which were projected on to the plane by means of Mercator's projection. The PSA was performed down to a shortest wavelength of $1°{\cdot}1$.

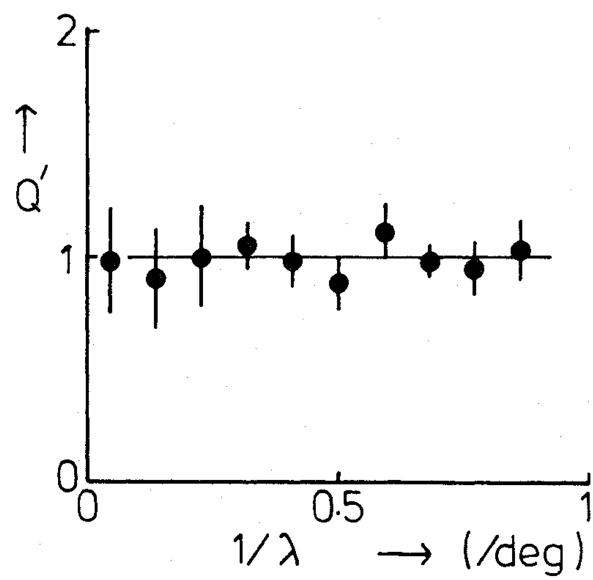

FIG. 6. *The run of Q' against $1/\lambda$ for the MC1 survey* ($S_0 = 0{\cdot}1$ *Jy*).

It was found that $I_{01} = 17{\cdot}14$ which is much too large for a chance value of a χ_2^2 variate, so this term was eliminated from the analysis. The term takes a value about 100 times greater than can be accounted for by the use of Mercator's projection so it must have some other explanation, such as the drift-scan effect.

The run of Q' against $1/\lambda$ is shown in Fig. 6. It is apparent that the points are scattered at random about the value 1·0 and there is no evidence either of clustering or anticlustering. The power spectrum statistics for all the terms in the Figure are $\Sigma_I = 1390{\cdot}5$, $\Sigma_\nu = 1402$ and $Q = 0{\cdot}99 \pm 0{\cdot}04$.

The PSA was repeated for the bright sources ($S_0 = 0{\cdot}40$ Jy) in the catalogue. There were 651 such sources in the restricted area. The run of Q' against $1/\lambda$ showed no evidence of clustering or anticlustering. The statistics for all terms down to a wavelength of $1°{\cdot}8$ were $\Sigma_I = 655{\cdot}6$, $\Sigma_\nu = 624$ and $Q = 1{\cdot}05 \pm 0{\cdot}06$, so no significant clustering or anticlustering is evident.

It is inferred from this analysis that there is no evidence that the sources in the MC1 catalogue are distributed in any fashion other than independently, uniformly and at random.

5. THE ANTICLUSTERING

There would be far-reaching consequences for cosmology and the theory of formation of galaxies if the anticlustering found in the 4C and GB catalogues could be shown to be a property of the distribution of radio sources in space. It is therefore necessary to investigate the cause of the anticlustering, and in particular to check that it is not some instrumental effect.

The anticlustering represents a smoother-than-random distribution of the sources on the sky, so an obvious possibility for such an instrumental effect is the confusion or blending caused by the finite angular width of the antenna response pattern. This causes, amongst other effects, faint sources close to brighter sources to be missed, so a random distribution of sources might show anticlustering by virtue of the instrumental exclusion of random close neighbours. Fortunately this hypothesis is susceptible to a simple test. The ' confusion anticlustering ' is expected to increase as the angular size of the telescope beam is increased relative to the mean angular separation of the sources on the sky, so the anticlustering should depend on b, the average number of sources per beam area in each survey. The quantity $A \equiv 1-Q$ is a measure of the anticlustering of the surveys: its expectation value is zero for a completely random distribution of sources and it increases with the anticlustering. Fig. 7 shows A plotted against b for the three surveys at their nominal flux density limits: the 4C $S_0 = 3$ Jy and GB $S_0 = 0{\cdot}25$ Jy results are also indicated. The area IV data were excluded from the 4C points on account of the interference. It is clear from the Figure that A is a monotonically increasing function of b, so there is very good evidence that the anticlustering in the catalogues is principally the result of confusion.

It does not seem possible that the anticlustering is a property of the sources themselves. If it were, one would expect the strength of the anticlustering to be correlated only with general properties of the source distribution such as the

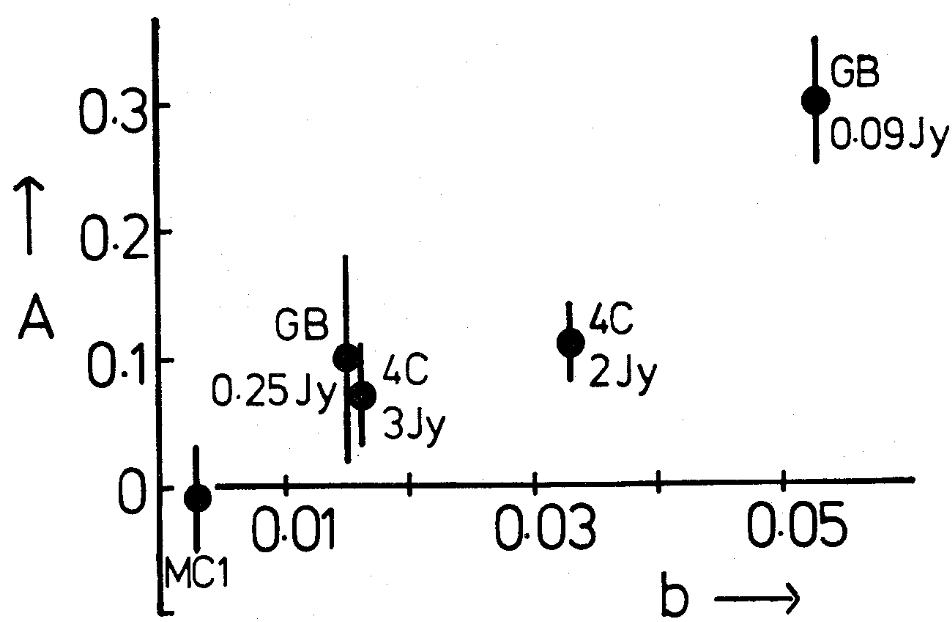

FIG. 7. *The anticlustering parameter A plotted against the confusion parameter b. The 4C and GB surveys are represented twice and are labelled by S_0, the limiting flux density in each case.*

density of sources or the frequency of observation, but no such correlation exists. The strong correlation with the instrumental parameter b indicates clearly that the anticlustering is instrumental.

The MC1 survey, with its complete lack of instrumental anticlustering shows straight away the chief property of the source distribution: the sources are scattered at random. It is possible also to investigate the true clustering of the 4C and GB sources despite the instrumental anticlustering by the following method, which consists in calculating the dependence of A on b expected from confusion, and using this law to extrapolate back to $b = 0$.

Consider a constellation of m sources scattered at random in the $(2\pi)^2$ area of the (x, y)-plane. The probability that a given target source has another source nearby, within a small circle of area a centred on the target, $(a \ll (2\pi)^2)$, is $a(m-1)/(2\pi)^2 \sim am/(2\pi)^2$. For this constellation $E(Q) = \langle q \rangle = 1.0$, but a constellation with clustering could be generated by adding more points within the small area a around some of the existing points. In particular, if a new point is added close to any given target point with probability $am/(2\pi)^2$, and this is repeated for all m target points in the original constellation, the new constellation will have $E(Q) = 1 + am/(2\pi)^2$ for wavelengths greater than $a^{1/2}/2\pi$. Confusion may be envisaged as having precisely the opposite effect by creating a zone of avoidance of area a around each target, so the expectation value of Q will be less than unity by the same amount: $E(Q) = 1 - am/(2\pi)^2$ so $E(A) = am/(2\pi)^2$. The number of sources per beam area is b, so if the zone of avoidance has an area of ρ beam areas it follows finally that $E(A) = \rho b$. This is the required equation giving the dependence of A on b. The derivation is strictly only accurate in the case in which a is infinitesimal, but in practice it is a good approximation provided $b \ll 1$, a condition which holds even for the GB survey.

It is clear from Fig. 7 that the values of A from the 4C survey are consistent with $A = \rho b$ with $\rho = 3.7 \pm 0.8$, and the GB with $\rho = 6.0 \pm 1.0$ (it is not surprising that these values are a little different, given the very different telescopes and data reduction methods). Thus the important conclusion may be drawn that all three surveys analysed here give results consistent with the hypothesis that the sources are scattered independently, uniformly and at random.

By fitting a law of the form $A = \rho b + \beta$ to the 4C and GB points the value of $1 - \beta$ may be used as a numerical estimate of the number of sources per celestial cluster with the anticlustering allowed for. 4C gives 0.97 ± 0.08 and GB gives 0.98 ± 0.11 sources per cluster.

6. COMPARISON WITH PREVIOUS WORK

(i) *The 4C survey*

The only previous results with which the present are directly comparable are those of Holden (1966), who found from an analysis of the distribution of the neighbours of each source that the sources are scattered at random on scales of $0°.5$ to $4°$ when confusion is taken into account, and from a binning analysis that the distribution is isotropic on scales of 5–60°. The present work confirms these results entirely. Golden (1974) found no evidence of clustering of 4C sources on scales of 10–30°, but his results are not directly comparable because they were derived from considerations of the slope of the source counts in different areas, rather than the positions of individual sources. Pearson (1974) found the sources to be isotropic on very large angular scales inaccessible to the present analysis, and Hughes & Longair (1967) found from the raw 4C interferometer output that the faint sources, too weak to be isolated individually and catalogued, showed no evidence of anisotropy on scales of 10–60°.

It appears from all these analyses of the 4C survey that the radio sources are isotropically distributed on all angular scales, whatever limiting flux density is chosen.

(ii) *The GB survey*

The present results differ substantially from those of the previous analysis of MMZ. The main effects claimed by MMZ are:

(i) That when divided into nine bins of right ascension, each 1^h wide, a binning analysis (BA) shows the bright sources to be non-randomly distributed at significance levels of between 5 and 1 per cent;

(ii) That a BA employing four carefully-selected areas 2^h wide shows an even greater significance;

(iii) The histograms of the nearest-neighbour separations from the four areas chosen in (ii) differ by more than expected on a chance basis, at significance levels of about 0·1 per cent for the bright sources.

In order to investigate points (i) and (ii) a multiple binning analysis (MBA) of the GB sources was performed. The right ascension range of the GB survey was divided into n bins, with n ranging from 2 to 30 in unit steps, and a BA applied in each case; for each value of n the BA was repeated with the bins shifted by half the width of a bin in order to extract as much information as possible. The whole MBA was repeated for S_0 = 0·09, 0·13, 0·18, 0·25 and 0·5 Jy in turn. For S_0 = 0·09 and 0·13 Jy the variance in the number of counts from bin to bin was smaller than expected on a chance basis for the overwhelming majority of values of n, which reflects the anticlustering found in the PSA. Individual configurations (i.e. choices of n, shifted or unshifted) showed considerable scatter about this average behaviour, but the anticlustering was clear. For S_0 = 0·18, 0·25 and 0·5 Jy the average variance was greater than expected by chance, but not very much so; again this reflects the weak effect found in the PSA. Individual configurations were again scattered about the average behaviour, and in particular the point for $n = 9$ chosen by MMZ in (i) showed a greater effect than the average, though within the typical scatter. The marginal clustering plus the noise on this one point give a significant-looking result. This clearly shows the inferiority of BA compared with PSA; by unluckily happening on an unrepresentative value of n, MMZ were led to overestimate the significance of a marginal effect. Much the same is true of the choice $n = 4$ in (ii): it is unrepresentative. In this case, however, the choice was not unlucky but was the result of a procedural impropriety: it is unacceptable to search by eye for an extreme configuration and then to show by BA that indeed it is unlikely without allowing carefully for the initial subjective selection.

The same must be said for the nearest-neighbour analysis in (iii)—no allowance was made for the initial careful choice of areas. Moreover there is an unfortunate fallacy in the test MMZ employed to compare the histograms of separations from the different areas. The test is admissible if the separations counted in the histograms are independent, random events, but the separations are not independent. The nearest-neighbour separation of a source is not an independent quantity if some of the sources in its locality have already had their nearest-neighbour separations investigated. One particular aspect of this is that if source A is the nearest neighbour of source B, then quite often source B will be the nearest neighbour of source A. Both of these separations in this case enter the same bin of the histogram for their area and thus increase the variance in that bin over what would be expected for single, independent entries. The increase of variance makes a pair of such histograms typically appear more different than expected on the false assumption that the entries are independent and this, combined with the subjective choice of areas, caused MMZ to overestimate the marginal clustering of the strong GB sources and even to find significant evidence of clustering of the sources down to $S_0 = 0{\cdot}13$ Jy (which are in fact anticlustered).

The previous claim that ' isotropy in the distribution of radio sources over the GB region of the sky is out of the question no matter whether the bright sources or the weaker ones are considered ' seems to be without foundation.

7. SUMMARY AND DISCUSSION

The general conclusion to be drawn from this analysis of the distribution of radio sources on the celestial sphere is that there is no good evidence in any of the surveys considered that the sources are distributed in any fashion other than independently, uniformly at random. The anticlustering found in two of the catalogues was shown to be an instrumental effect, and on allowing for the effect the true distribution of sources was seen to be random. The anomalous long-wavelength behaviour found in area IV of the 4C survey was also an instrumental effect. Whilst possible clustering of the bright sources in the GB catalogue was found it was shown to be of marginal statistical significance and to depend on the limiting flux density in a way which was not readily intelligible as a property of the distribution of sources in space.

A general point may be made about the usefulness of various radio surveys in studies of isotropy: strongly confusion-limited surveys are of less value than noise-limited surveys on account of the confusion anticlustering. Whilst it is possible to allow for the anticlustering the statistical accuracy of the result is poorer than would have been the case for a comparable confusion-free survey.

It is hoped to discuss the cosmological significance of the lack of clustering in a future paper, so for the present just one point will be made on this topic: the radio sources are not merely unclustered, they are remarkably unclustered. No other present-day constituent of the Universe has this property. For example the value of $\langle q \rangle$ for stars in galaxies is $\sim 10^{11}$ and for galaxies in clusters ~ 30. Even for the ' weak ' superclustering of clusters of galaxies, $\langle q \rangle \sim 3$. In contrast $\langle q \rangle$ for radio sources is within a few percent of $1{\cdot}00$, the value for totally unclustered objects. Taken with the fact that most of the sources in the catalogues are at substantial redshifts ($z \sim 1$–3) it must be inferred that on very large linear scales (100–5000 Mpc) the Universe is not clumped or clustered in any way, and this has far-reaching consequences for the theory of the origin and growth of primordial density fluctuations.

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