The Coefficients of Capacity and the Mutual Attractions or Repulsions of Two Electrified Spherical Conductors when close together.

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Introduction.

In connection with spark systems of wireless telegraphy, a knowledge of the electrostatic energy stored between spherical electrodes at the instant of the disruptive discharge is of great value to the engineer. In order to increase this energy, without unduly increasing the applied potential difference, the electrodes have been placed in compressed or highly rarefied gases and in oils or other liquid dielectrics having great electric strength. In these cases the least distance between the electrodes may only be a small fraction of the radius of either, and so the computation of the electrostatic energy by the ordinary formulæ is so laborious that it is practically prohibitive.

By extending a mathematical theorem first given by Schlömilch,* the author has succeeded in greatly simplifying the computation of this energy. The formulæ given below are very easily evaluated when the spheres are close together, and hence, in conjunction with Kirchhoff's† final modification of his own formulæ, they give the complete practical solution of this important historical problem.

The formulæ obtained enable the attractive or repulsive forces between electrified spherical conductors to be easily calculated, however close the spheres are to one another. They have been employed to recalculate the latter portion of the table published by Kelvin.‡ This table has also been extended so as to make it more useful to physicists and electricians.

Mathematical Theorems.

By Schlömilch's method (*loc. cit.*) we can prove that

$$\sum_{s=1}^{s=\infty} \frac{1}{\epsilon^{(1+ns)x}-1} = \frac{F(n)-\log x}{nx} + \frac{1}{2}\left(\frac{1}{n}-\frac{1}{2}\right) - \frac{B_1 nx}{2\,!}A_1 - \frac{B_3 n^3 x^3}{4\,!}A_3$$
$$- \frac{B_{2m-1}(nx)^{2m-1}}{2m\,!}A_{2m-1}, \qquad (1)$$

* 'Zeitschrift für Mathematik und Physik,' vol. 6, p. 407, 1860.

† 'Annalen der Physik,' vol. 27, p. 673, 1886.

‡ 'Phil. Mag.,' April and August, 1853, or Reprint, p. 83.

approximately, provided that the last term is very small compared with unity.

In this formula B_1, B_3, B_5, ..., are Bernoulli's numbers,

$$F(n) = \frac{n}{2} + \frac{B_1}{2} n^2 - \frac{B_3}{4} n^4 + \frac{B_5}{6} n^6 - \ldots, \tag{2}$$

and
$$(-)^{m-1} A_{2m-1} = \frac{1}{2mn^{2m}} - \frac{1}{2n^{2m-1}} + \frac{(2m-1) B_1}{2!\, n^{2m-2}}$$
$$- \frac{(2m-1)(2m-2)(2m-3) B_3}{4!\, n^{2m-4}} + \ldots + (-)^{m-1} \frac{B_{2m-1}}{2m}. \tag{3}$$

It is easy to show by Stirling's theorem that

$$F(n) = -\Gamma'(1/n)/\Gamma(1/n) - \log n, \tag{4}$$

and hence, by the properties of gamma functions,

$$F(n) - F\left(\frac{n}{n-1}\right) = \pi \cot \frac{\pi}{n} - \log(n-1). \tag{5}$$

In many cases the value of $F(n)$ may be simply expressed. For instance, if γ denote Euler's constant, we have $F(1) = \gamma$, $F(2) = \gamma + \log 2$, $F(3) = \gamma + \pi/2\sqrt{3} + \frac{1}{2}\log 3$, $F(4) = \gamma + \pi/2 + \log 2$, etc. In general, however, the value of $F(n)$ has to be computed by a series formula. In our problem n is never less than unity, and so the following formula can always be used—

$$F(n) = \frac{n}{2} + \gamma + \frac{1}{n^2-1} + \frac{\pi}{2}\cot\frac{\pi}{n} - \log n + \frac{S_3 - 1}{n^2} + \frac{S_5 - 1}{n^4} + \ldots, \tag{6}$$

where $S_m = \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots$. The values of S_3, S_5, ..., are given (*e.g.*) in Dale's 'Mathematical Tables.'

From (3) we can show that when n is unity $A'_{2m-1} = B_{2m-1}/(2m)$, and when n is 2, $A''_{2m-1} = -B_{2m-1}(1-2^{-2m+1})/(2m)$. We can also show that for any value of n, other than 1 or 2, A_{2m-1} lies in value between A'_{2m-1} and A''_{2m-1}. Its value also is not altered when we write $1-1/n$ for $1/n$.

Writing t for $(n-1)/n^2$, we easily find that

$$A_1 = \frac{1}{12} - \frac{t}{2}, \qquad A_3 = \frac{1}{120} - \frac{t^2}{4}, \qquad A_5 = \frac{1}{252} - \frac{t^3}{6} - \frac{t^2}{12},$$

$$A_7 = \frac{1}{240} - \frac{t^4}{8} - \frac{t^3}{6} - \frac{t^2}{12}, \quad \text{and} \quad A_9 = \frac{5}{660} - \frac{t^5}{10} - \frac{t^4}{4} - \frac{3t^3}{10} - \frac{3t^2}{20}.$$

The author has verified by actual calculation that, to a seven figure accuracy at least, formula (1) is true, both when n is 1 and when n is 2, even when x is as great as $2 \log 2$, *i.e.* 1·386 nearly, and powers of x greater than the ninth are neglected.

Putting $n=1$ in (1) we get Schlömilch's Theorem. Schlömilch has verified by actual calculation that when x is log 2·5, *i.e.* 0·916 nearly, his formula, neglecting powers of x beyond the ninth, practically gives the numerical value of the series correctly to 10 significant figures. He has also shown that when x is not greater than log (10/9), *i.e.* 0·105 nearly, the formula

$$\sum_{1}^{\infty}\frac{1}{\epsilon^{sx}-1}=\frac{\gamma-\log x}{x}+\tfrac{1}{4}-\frac{x}{144} \tag{7}$$

gives a seven figure accuracy. To obtain the same accuracy by Clausen's Theorem, which was utilised and extended by Kirchhoff (*loc. cit.*), 13 terms would have to be taken, and the calculation of the later terms is very laborious.

The Capacity Coefficients of Two Spheres.

Let us consider the case of two spherical conductors whose radii are a and b respectively, and let c be the distance between their centres. If the charges and potentials of the spheres be q_1, q_2 and v_1, v_2 respectively, we have

$$\left.\begin{aligned} q_1 &= k_{1,1}v_1+k_{1,2}v_2 \\ \text{and}\qquad q_2 &= k_{2,2}v_2+k_{1,2}v_1 \end{aligned}\right\} \tag{8}$$

where $k_{1,1}$ and $k_{1,2}$ are the capacity coefficients of the two spheres. The values of these quantities in terms of a, b, and c are given by the following equations* :—

$$k_{1,1}=\lambda\sum_{s=0}^{s=\infty}\frac{1}{\sinh(\alpha+s\omega)}, \tag{9}$$

$$-k_{1,2}=\lambda\sum_{1}^{\infty}\frac{1}{\sinh s\omega}, \tag{10}$$

$$k_{2,2}=\lambda\sum_{0}^{\infty}\frac{1}{\sinh(\beta+s\omega)}, \tag{11}$$

where $$4c^2\lambda^2=(c+a+b)(c-a-b)(c+a-b)(c-a+b), \tag{12}$$

$$\sinh\alpha=\lambda/a,\qquad \sinh\beta=\lambda/b,\quad\text{and}\quad \sinh\omega=\lambda c/ab. \tag{13}$$

We also have

$$\omega=\alpha+\beta\quad\text{and}\quad \alpha=\log\left(\frac{\lambda}{a}+\sqrt{1+\frac{\lambda^2}{a^2}}\right). \tag{14}$$

When $c-a-b$ is small, it is necessary to alter the formulæ (9), (10) and

* *Cf.* Maxwell, vol. 1, § 173.

(11) by means of (1) so as to lessen the labour involved in the computation. From (9) we have

$$\frac{k_{1,1}}{2\lambda} = \sum_0^\infty \frac{1}{\epsilon^{\alpha+s\omega}-1} - \sum_0^\infty \frac{1}{\epsilon^{2\alpha+2s\omega}-1}$$

$$= \frac{F(n)-\log(\omega/2n)}{2\omega} + \frac{B_1\omega}{2!}A_1 + \frac{7B_3\omega^3}{4!}A_3 + \frac{31B_5\omega^5}{6!}A_5$$

$$+\frac{127\,B_7\omega^7}{8!}A_7 + \frac{511\,B_9\omega^9}{10!}A_9, \quad (15)$$

very approximately, where $n=\omega/\alpha$.

Finding the corresponding formula for $k_{2,2}/2\lambda$ and subtracting it from (15), we get, by (5),

$$k_{1,1}-k_{2,2} = \frac{\pi\lambda}{\alpha+\beta}\cot\frac{\pi\alpha}{\alpha+\beta}, \quad (16)$$

very approximately.

When the spheres are so close together that λ is small compared with a or b, $\alpha=\lambda/a$ and $\beta=\lambda/b$, and hence

$$k_{1,1}-k_{2,2} = \frac{\pi ab}{a+b}\cot\frac{\pi b}{a+b}. \quad (17)$$

This formula* has been obtained previously for the case of spheres in contact.

From (10) also we find that

$$-\frac{k_{1,2}}{2\lambda} = \frac{\gamma-\log(\omega/2)}{2\omega} + \frac{\omega}{144} + \frac{7\,\omega^3}{86400} + \frac{31\,\omega^5}{7620480}$$

$$+\frac{127\,\omega^7}{290304000} + \frac{511\,\omega^9}{3161410560}, \quad (18)$$

very approximately.

The Attractions or Repulsions between the Spheres.

If the potentials of the spheres be maintained constant, Kelvin† proved that when they alter their positions owing to their mutual electric actions they move in such a way that the electrostatic energy of the system is increased by an amount exactly equal to the work done on the conducting spheres by the electric forces. If W be the electrostatic energy, we have

$$W = \tfrac{1}{2}k_{1,1}v_1^2 + \tfrac{1}{2}k_{2,2}v_2^2 + k_{1,2}v_1v_2,$$

and therefore

$$\frac{\partial W}{\partial x} = \tfrac{1}{2}v_1^2\frac{\partial k_{1,1}}{\partial x} + \tfrac{1}{2}v_2^2\frac{\partial k_{2,2}}{\partial x} + v_1v_2\frac{\partial k_{1,2}}{\partial x} = F,$$

* See Maxwell, vol. 1, § 175 and the references given there.

† Reprint, p. 466, Second Edition.

where $x=c-a-b=$ the least distance between the spheres, and F is the force between them. If F is negative, W increases as x diminishes, and therefore the force is attractive, but if F is positive the force is repulsive. The values of $\partial k_{1,1}/\partial x$, $\partial k_{2,2}/\partial x$ and $\partial k_{1,2}/\partial x$ can easily be found from the preceding formulæ.

Spheres at Microscopic Distances apart.

In this case, $x/(a+b)$ being supposed negligibly small compared with unity, we have $\lambda^2 = \frac{2ab}{a+b}x$, and $\omega^2 = \frac{2(a+b)}{ab}x$, and hence

$$\left.\begin{aligned} k_{1,1} &= \frac{ab}{a+b}\left\{F\left(1+\frac{a}{b}\right)+\tfrac{1}{2}\log\frac{2a(a+b)}{b}+\tfrac{1}{2}\log\frac{1}{x}\right\}, \\ k_{2,2} &= \frac{ab}{a+b}\left\{F\left(1+\frac{b}{a}\right)+\tfrac{1}{2}\log\frac{2b(a+b)}{a}+\tfrac{1}{2}\log\frac{1}{x}\right\}, \\ -k_{1,2} &= \frac{ab}{a+b}\left\{\gamma+\tfrac{1}{2}\log\frac{2ab}{a+b}+\tfrac{1}{2}\log\frac{1}{x}\right\}. \end{aligned}\right\} \quad (19)$$

If the difference of potential produced between the spheres by giving a charge $+q$ to one and $-q$ to the other be V, then q/V is defined* to be the capacity between the spheres.

This is the capacity that is generally considered by electrical engineers. It is easy to show† that its value is

$$\frac{k_{1,1}k_{2,2}-k_{1,2}^2}{k_{1,1}+k_{2,2}+2k_{1,2}},$$

and hence it can be found by (19).

The joint capacity of the two spheres when at the same potential is $k_{1,1}+k_{2,2}+2k_{1,2}$. It therefore equals

$$\frac{ab}{a+b}\left\{F\left(1+\frac{a}{b}\right)+F\left(1+\frac{b}{a}\right)-2\gamma+\log\frac{(a+b)^2}{ab}\right\}. \quad (20)$$

When $a=b$, this capacity is $2a\log 2$; when $a=2b$, it equals $a\log 3$; when $a=3b$, it equals $\frac{3}{4}a\log 4$, etc.

If the square of the difference of potential $(v_1-v_2)^2$ between the two spheres be not very small compared with $v_1^2+v_2^2$, the force F between them is attractive and is given by

$$F = \frac{ab(v_1-v_2)^2}{4(a+b)x}. \quad (21)$$

* Sir J. J. Thomson, 'Electricity and Magnetism,' Third Edition, p. 84.

† Russell, 'Alternating Currents,' vol. 1, p. 93.

The Case of Equal Spheres.

When the spheres are equal, $k_{1,1} = k_{2,2}$ and the formulæ become

$$k_{1,1} = \frac{\lambda}{\omega}\left\{2\cdot6566572 - \log\omega - \frac{\omega^2}{144} - \frac{49\,\omega^4}{345600} - \frac{961\,\omega^6}{121927680} - \frac{16129\,\omega^8}{18579456000} - \frac{261121\,\omega^{10}}{1618642206720}\right\}; \qquad (22)$$

$$-k_{1,2} = \frac{\lambda}{\omega}\left\{1\cdot2703628 - \log\omega + \frac{\omega^2}{72} + \frac{7\,\omega^4}{43200} + \frac{31\,\omega^6}{3810240} + \frac{127\,\omega^8}{145152000} + \frac{511\,\omega^{10}}{3161410560}\right\}; \qquad (23)$$

$$-\frac{\partial k_{1,1}}{\partial x} = -\frac{k_{1,1}}{\lambda^2}\left(\frac{c}{4} - \frac{\lambda}{\omega}\right) + \frac{1}{\omega^2}\left\{1 + \frac{\omega^2}{72} + \frac{49\,\omega^4}{86400} + \frac{961\,\omega^6}{20321280} + \frac{16129\,\omega^8}{2322432000} + \frac{261121\,\omega^{10}}{161864220672}\right\}; \qquad (24)$$

and

$$\frac{\partial k_{1,2}}{\partial x} = -\frac{-k_{1,2}}{\lambda^2}\left(\frac{c}{4} - \frac{\lambda}{\omega}\right) + \frac{1}{\omega^2}\left(1 - \frac{\omega^2}{36} - \frac{7\,\omega^4}{10800} - \frac{31\,\omega^6}{635040} - \frac{127\,\omega^8}{18144000} - \frac{511\,\omega^{10}}{316141056}\right). \qquad (25)$$

If a be the radius of each sphere, we have

$$\lambda^2/a^2 = x/a + x^2/(2a)^2 \quad \text{and} \quad \omega = 2\log\left(\lambda/a + \sqrt{1 + \lambda^2/a^2}\right).$$

Formulæ (22)—(25) were used in computing the following tables :—

x/a.	$k_{1,1}/a$.		$-k_{1,2}/a$.	
	Formula (22).	Kelvin.	Formula (23).	Kelvin.
0·5	1·25302*	1·25324	0·52537	0·52537
0·45	1·27420		0·54682	
0·4	1·29316	1·29316	0·57202	0·57202
0·35	1·31830		0·60049	
0·3	1·34828	1·34827	0·63384	0·63395
0·25	1·38491		0·67321	
0·2	1·43131	1·43131	0·72378	0·72378
0·15	1·49328		0·78927	
0·1	1·58396	1·58396	0·88352*	0·88175

* Computed also by Kirchhoff (*loc. cit.*).

x/a.	$k_{1,1}/a$.	$-k_{1,2}/a$.	x/a.	$k_{1,1}/a$.	$-k_{1,2}/a$.
0·09	1·60805	0·90833	0·02	1·96643	1·27181
0·08	1·63519	0·93619	0·01	2·13667	1·44246
0·07	1·66622	0·96795	$0\cdot0_21$	2·70920	2·01598
0·06	1·70234	1·00480	$0\cdot0_31$	3·28439	2·59124
0·05	1·74543	1·04862	$0\cdot0_41$	3·85998	3·16684
0·04	1·79864	1·10256	$0\cdot0_51$	4·43563	3·74249
0·03	1·86788	1·17253	$0\cdot0_61$	5·01124	4·31810

x/a.	$-\frac{1}{2}(\partial k_{1,1}/\partial x)$.		$\frac{1}{2}(\partial k_{1,2}/\partial x)$.	
	Formula (24).	Kelvin.	Formula (25).	Kelvin.
0·5	0·17424*	0·17432	0·20630	0·20630
0·45	0·20105		0·23107	
0·4	0·23158	0·23159	0·26464	0·26464
0·35	0·27319		0·30673	
0·3	0·32924	0·32917†	0·36325	0·36357
0·25	0·40848		0·44299	
0·2	0·52853	0·52852	0·56352	0·56350
0·15	0·73056		0·76603	
0·1	1·13844	1·13844	1·17439	1·17439

* Given also by Kirchhoff (*loc. cit.*).

† Kelvin had calculated the value previously by himself and made it 0·32926 (see Reprint, p. 23).

x/a.	$-\frac{1}{2}(\partial k_{1,1}/\partial x)$.	$\frac{1}{2}(\partial k_{1,2}/\partial x)$.	x/a.	$-\frac{1}{2}(\partial k_{1,1}/\partial x)$.	$\frac{1}{2}(\partial k_{1,2}/\partial x)$.
0·09	1·2751	1·3111	0·02	6·1043	6·1410
0·08	1·4461	1·4823	0·01	12·340	12·377
0·07	1·6665	1·7028	$0{\cdot}0_21$	124·82	124·86
0·06	1·9608	1·9972	$0{\cdot}0_31$	1249·8	1249·8
0·05	2·3736	2·4101	$0{\cdot}0_41$	12500	12500
0·04	2·9938	3·0304	$0{\cdot}0_51$	125000	125000
0·03	4·0294	4·0661	$0{\cdot}0_61$	1250000	1250000

Simplified Formulæ for Equal Spheres.

When x/a is small, we find that

$$k_{1,1} = \frac{a}{2}\left(1+\frac{x}{6a}-\frac{x^2}{180\,a^2}\right)\left(1{\cdot}96351+\tfrac{1}{2}\log_e\frac{a}{x}+\frac{x}{72\,a}-\frac{213\,x^2}{43200\,a^2}\right), \tag{26}$$

$$-k_{1,2} = \frac{a}{2}\left(1+\frac{x}{6a}-\frac{x^2}{180\,a^2}\right)\left(0{\cdot}577216+\tfrac{1}{2}\log_e\frac{a}{x}+\frac{7\,x}{72\,a}-\frac{251\,x^2}{43200\,a^2}\right). \tag{27}$$

Putting $x/a = 1/10$ in these formulæ, we find that $k_{1,1} = 1{\cdot}58397$ and $-k_{1,2} = 0{\cdot}88355$. The true values are 1·58395 and 0·88352. Hence (26) and (27) can be used in all practical calculations when x/a is not greater than a tenth.

When an accuracy of the hundredth part of 1 per cent. suffices and x/a is not greater than 1/10, we may use the formulæ

$$\tfrac{1}{2}(k_{1,1}-k_{1,2}) = \frac{a}{2}\left(1+\frac{x}{6a}\right)\left(1{\cdot}2704+\tfrac{1}{2}\log_e\frac{a}{x}+\frac{x}{18\,a}\right), \tag{28}$$

and

$$2(k_{1,1}+k_{1,2}) = a\left(1+\frac{x}{6a}\right)\left(1{\cdot}3863-\frac{x}{12\,a}\right) \tag{29}$$

for the capacity between the two spheres and their joint capacity respectively.

It will be seen that this latter capacity is very little greater than $2a \log_e 2$, the value which it has when they touch.

Similarly, when the squares and higher powers of x/a are negligibly small compared with unity, the electrostatic force F between them is given by

$$F = -\frac{a(v_1-v_2)^2}{8x}\left\{1-\frac{12\gamma+1}{36}\frac{x}{a}-\frac{1}{6}\frac{x}{a}\log\frac{a}{x}+\frac{x^2}{90a^2}\log\frac{a}{x}\right\}$$

$$+\frac{\log 2-\frac{1}{4}}{12}(v_1^2+v_2^2). \qquad (30)$$

When x/a is small, and $(v_1-v_2)^2$ is not very small compared with $v_1^2+v_2^2$, the magnitude of the attractive force is approximately

$$\frac{a(v_1-v_2)^2}{8x}. \qquad (31)$$

We see, therefore, that when the spheres are very close together the attractive force varies as the diameter of either sphere, as the square of their difference of potential, and inversely as the least distance between them.

When the spheres are at the same potential, the formula given by (30) agrees with that found by Kelvin for the repulsive force between two spheres in contact. As he pointed out, this force is independent of the size of the spheres.

Finally, if the charges on the spheres be $+q$ and $-q$ respectively, we find by (28) and (31) that the attraction between them is given by

$$F = q^2/2ax\,\{1{\cdot}2704+\tfrac{1}{2}\log_e(a/x)\}^2,$$

which when 2·54 can be neglected compared with $\log_e(a/x)$ may be written

$$F = 2q^2/ax\,\{\log_e(a/x)\}^2.$$