The cognitive basis of arithmetic

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1 Introduction

Arithmetic is the theory of the natural numbers and one of the oldest areas of mathematics. Since almost all other mathematical theories make use of numbers in some way or other, arithmetic is also one of the most fundamental theories of mathematics. But numbers are not just abstract entities that are subject to mathematical ruminations—they are represented, used, embodied, and manipulated in order to achieve many different goals, e.g., to count or denote the size of a collection of objects, to trade goods, to balance bank accounts, or to play the lottery. Consequently, numbers are both abstract and intimately connected to language and to our interactions with the world.

In the present paper we provide an overview of research that has addressed the question of how animals and humans learn, represent, and process numbers. The interrelations among mathematics, the world, and the cognitive capacities that are frequently discussed in terms of mind and brain have been the subject of many theories and much speculation. Figure 1a shows that the four basic concepts that anchor this discussion (mathematics, world, mind, brain) enable six possible binary relationships (four edges and two diagonals), each of which raises fundamental philosophical questions. Traditionally, philosophy of mathematics focuses on the triangle between mind, mathematics, and the world (Figure 1b, Φ), asks how mushy minds can grasp abstract numerical concepts, wonders about the nature of mathematical truth, and is puzzled by "the uncanny usefulness of mathematical concepts" (Wigner, 1960). In contrast, psychologists and their colleagues from cognitive science and neuroscience investigate the relationship between mind and brain and its relation to the world, that is further sub-divided into

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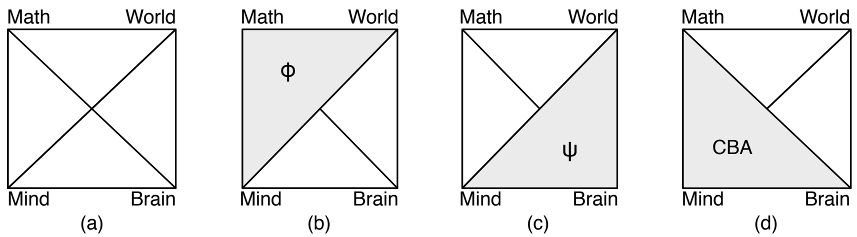

FIGURE 1. Schematic views of possible perspectives on the cognitive basis of arithmetic.

physical and social environments (Figure 1c, Ψ). From this perspective, the ability to understand and solve mathematical problems is just one accidental topic among many others, and is typically overshadowed by more prominent issues of perception, categorization, language, memory, judgment, and decision making.

As a departure from both of these traditional perspectives, this paper on the cognitive basis of arithmetic focuses on the manifold relations between mathematics, mind, and brain (Figure 1d, CBA). To illuminate this triangle, we shall cross many disciplinary boundaries and collect past and present insights from philosophy, animal learning, developmental psychology, cultural anthropology, cognitive science, and neuroscience. Although mathematics consists of far more than arithmetic and certainly involves cognitive faculties that extend beyond the ones discussed here (like reasoning with diagrams and infinite objects), we restrict ourselves to the cognitive basis of arithmetic. Such a foundation will provide the groundwork for a more comprehensive understanding of mathematics from a cognitive perspective.

Some basic terminological distinctions. By *numbers* we mean the abstract entities that are denoted by number words like 'seventeen' or numerals like '42.' The properties of these objects are studied by mathematicians. In contrast, what we encounter with our senses are *collections* of things, which are also called *numerosities*.[1] These are discrete, concrete numerical quantities of objects, like a pile of peas or the musicians in a band called 'The Beatles.' The magnitude or size of such collections are cardinal numbers,

[1]In ordinary parlance one also speaks of *sets* of objects, but we shall try to avoid this term, because it should not be confused with the mathematical notion of set. Some philosophers play down this difference, in order to account for our epistemic access to sets (Maddy, 1992, pp. 59–61).

	one	two	three	four	five	six
(a) Tally system:	𝍷	𝍷𝍷	𝍷𝍷𝍷	𝍷𝍷𝍷𝍷	𝍸	𝍸𝍷
(b) Roman numerals:	I	II	III	IIII	V	VI
(c) Greek alphabetic:	α	β	γ	δ	ε	ϝ
(d) Arabic decimal:	1	2	3	4	5	6
(e) Binary digits:	1	10	11	100	101	110

	seven	eight	nine	ten	eleven	twelve
(a) Tally system:	𝍸𝍷𝍷	𝍸𝍷𝍷𝍷	𝍸𝍷𝍷𝍷𝍷	𝍸 𝍸	𝍸 𝍸𝍷	𝍸 𝍸𝍷𝍷
(b) Roman numerals:	VII	VIII	VIIII	X	XI	XII
(c) Greek alphabetic:	ζ	η	ϑ	ι	ια	ιβ
(d) Arabic decimal:	7	8	9	10	11	12
(e) Binary digits:	111	1000	1001	1010	1011	1100

TABLE 1. Different representational systems to represent numbers. Note that the letter for the Greek alphabetic numeral for 6 is the now obsolete Greek letter *digamma*, and that we represent the Roman numerals without the 'subtractive' notation (i.e., representing 4 as IV, instead of IIII) that was introduced in the Middle Ages.

while ordinal numbers indicate positions in an ordered sequence (e.g., first, second, third, etc.).[2]

Numbers can be represented in multiple ways, and it is important to distinguish between *external* and *internal* representations. The two main external representations are *numerical* and *lexical notation systems* (Chrisomalis, 2004). The latter are sequences of numeral words in a language, either written or spoken, with a distinctive phonetic component, while the former are sequences of *numeral phrases* or simply *numerals*, themselves consisting of a group of elementary *numeral signs* or *symbols*. In our familiar decimal place-value numeral system, these are also called *digits*. Clearly, different numerals can denote the same number: Each column in Table 1 denotes the same numerical quantity. In ordinary parlance a numerical notation system is also referred to as a 'number system' and authors frequently use terms like 'Roman numbers' to refer to Roman numerals. We might also slip into this habit, if the context provides sufficient information to prevent ambiguity. Internal representations of numbers are how numbers are represented 'in the head', which can refer to the level of neurons in the brain, but also to a higher, more abstract conceptual or representational level. Note that in the psychological literature a *concept* is typically understood to be a mental entity, which is not necessarily so in the philosophical literature. Indeed, in the tradition of analytic philosophy concepts are expressly

[2]The related, but somewhat different, technical notions of cardinal and ordinal numbers in set theory are beyond the scope of this paper.

not considered to be mental entities.[3] Thus, cognitive scientists often refer to mental *representations* as ways in which a particular system of numerals is mapped to internal concepts and also mention mental *processes* as operations that translate between external and internal formats and that manipulate internal representations.

During the following discussion it is useful to keep in mind the distinction between different levels of *numerical competence*. The most basic level consists in the ability to recognize and distinguish small numerosities. The second level, that of *counting*, involves mastering at least an initial segment of a numerical or lexical notation system and the ability to systematically map numerosities to that system in a one-to-one fashion. The realization that there is no greatest number and that the numeral systems are potentially infinite, i.e., that the process of counting can be continued indefinitely, can be considered a further step in the acquisition of knowledge about numbers. *Arithmetic competence* begins with the ability to perform basic computations, i.e., to correctly apply the operations of addition, subtraction, multiplication, and division.[4] In analogy to the distinction between internal and external notation systems, we can also distinguish between internal and external arithmetic. As we shall see in Section 4.3, however, there is evidence that these systems are closely related. Finally, *logical reasoning* about numbers and the ability to *prove* and understand arithmetic theorems, e.g., that there are infinitely many prime numbers, constitutes the most advanced level of numerical competence.

Mathematical, philosophical, and psychological perspectives on arithmetic. Our above characterization of arithmetic as the theory of the natural numbers is one that a mathematician would provide. From this point view, practicing arithmetic mainly involves establishing properties of numbers by means of proofs, resulting in discoveries like Euclid's theorem about the infinity of prime numbers and conjectures like Fermat's Last Theorem. In the late nineteenth century arithmetic was even considered to be the most fundamental mathematical theory to which all others should be reduced (see Klein, 1895). In modern mathematics, however, there are no restrictions on the methods used for studying numbers; and while mathematicians operate with numerical representations, they usually do not worry too much about them, since they are interested in establishing relationships between numbers at an abstract level.

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[3] The analytic tradition follows Frege's distinction between concepts as logical entities and ideas as psychological entities (Frege, 1884, pp. xxi–xxii). We return to this issue in Section 5.

[4] The difficulties involved in learning the basic arithmetical operations and how these are related to particular systems of numerals are discussed in (Lengnink and Schlimm, 2010, this volume).

The metaphysical nature of numbers has been a topic of philosophical discussion since the ancient Greeks, most famously by Pythagoras, Plato, and Aristotle. Surprising and counterintuitive results in the mathematics of the nineteenth century kindled an interest in foundational questions.[5] In order to secure these foundations, Frege (1884) attempted to reduce the concept of number to purely logical notions while Dedekind (1888) and Peano (1889) provided axiomatizations of the natural number structure. These developments greatly influenced the practice and understanding of mathematics. Indeed, many contemporary philosophers characterize mathematics as the science of *structures* (Shapiro, 2000). According to this view it makes no sense to regard individual numbers in isolation; instead, they must be regarded as positions within a natural number structure. This raises the important question of how we can have access to and knowledge of such an abstract and infinite structure. Traditionally, philosophers also have paid very little attention to the representations of numbers, except to motivate their accounts of our epistemic access to them. In general, they seem to be content to mention *ad hoc* accounts based on anecdotal evidence and to outsource these investigations to cognitive scientists.[6]

Since mathematical reasoning is often considered to be a fundamental human ability, the relatively young discipline of cognitive science has shown great interest in exploring it, with its main concern being how the brain or the mind processes numerical reasoning (see, e.g., Dehaene, 1997; Butterworth, 1999; Lakoff and Númez, 2000).

One of the basic issues concerns the relation between basic numerical processing and the use of language. In particular, cognitive scientists investigate how the levels of arithmetic competence are related to various internal and external representations. Theory formation in cognitive science often goes hand in hand with the gathering of empirical data on arithmetical abilities of animals and infants, as well as on the use of number words in different cultures (see also François and Van Kerkhove, 2010, this volume). Thus, cognitive scientists are primarily concerned with the lower levels of numerical competence and with internal representations and rarely discuss how their empirical findings relate to higher-level mathematical abilities.

Overview. In the following sections we discuss a wide range of empirical findings on the cognitive foundations of arithmetic from a variety of scientific disciplines and perspectives. Of particular interest with regard to the phylogenetic and ontogenetic developments of numerical abstractions are the mathematical abilities of animals as well as those of infants (Sections 2.2

[5]See (Buldt and Schlimm, 2010, this volume), for a general discussion of these developments.

[6]See, e.g., (Maddy, 1992, pp. 50–74), (Resnik, 1997, Ch. 11), and (Shapiro, 2000, pp. 279–280).

and 2.3). The relation between these abilities and the use of number words is studied through investigations of the numerical abilities of cultures with only a limited repertoire of number words (Section 2.4). After discussing research on the localization of arithmetical processes in the brain (Section 3), we shall turn to questions about arithmetic notation (Section 4). In particular, we shall ask questions like: How do notations facilitate or constrain simple and complex arithmetic computations? What is the relationship between external notations and mental calculations? What is the impact of resources provided by the computational environment? To address these more theoretical issues we need to consider fundamental aspects of arithmetic notations. We shall provide some terminological distinctions and historical context for the comparative study of number systems. We then sketch a computational method that allows us to illustrate and quantify the trade-offs between specific numeration systems and the internal and external processes they require for performing calculations. Ultimately, we argue for a more nuanced view of the merits and faults of particular numeration systems and for a more careful analysis of the connections between internal and external representations in arithmetic reasoning. A comprehensive analysis of mathematical practice will have to study the complex interplay among representational systems, their biological and psychological bases, and their linguistic and cultural manifestations.

2 Developing arithmetic

2.1 Intuitive arithmetic

Up to the eighteenth century, philosophers of mathematics were primarily intrigued by the relationship between human cognition and the abstract objects that mathematical entities seem to be. Our apparent epistemic access to such objects needed an explanation. In the dialogue *Meno*, Plato proposed that our knowledge of geometry actually stems from recollecting (*ανάμνησις*) forms that we knew from before we were born. Descartes and Kant also thought that geometry derives from 'innate' knowledge—Kant's argument from geometry was an ambitious attempt to demonstrate that our cognitive capacities are reflected in Euclidean geometry.[7] Similar claims were made about numbers and arithmetic. Leibniz, for example, argued that mathematical knowledge must be innate, because it pertains to necessary truths rather than contingent facts. Nevertheless, he believed that this

[7] *Innateness* is a notion associated with a complex, shifting range of meanings. Today, under the influence of cognitive ethology, it has a distinctly biological meaning (as in genetically determined, or developmentally invariant) that it did not originally possess. As it is beyond the scope of this paper to present a detailed discussion of nativism in philosophy and psychology, suffice it to say that 'innate' for these authors was something akin to the notion of *a priori*.

innate knowledge needed to elicited through education: "The truths about numbers are in us; but still we learn them" (Leibniz, 1765, p. 85). Elsewhere he likens innate knowledge to veins in marble that outline a shape to be uncovered by a sculptor: our innate knowledge is uncovered through learning (Leibniz, 1765, p. 52). By contrast, Locke (1690) argued that numerical cognition can be traced back to perceptual knowledge. The number one, for him, is an idea, i.e., a mental representation due to perceptual input: "Amongst all the ideas we have, as there is none suggested to the mind by more ways, so there is none more simple, than that of unity, or one" (Locke, 1690, Book II, Ch. XVI). This "simplest and most universal idea" (Locke, 1690, Book II, Ch. XVI) can then be taken as a starting point to make other numbers; for example, by repeating the number one, we end up with larger natural numbers.

Since the late nineteenth century, philosophers of mathematics have turned away from examining the relationship between cognition and mathematics, focusing instead on formal properties and foundational ideas, such as how the natural numbers can be derived from set theory. Recently, however, philosophers of mathematics have taken a renewed interest in epistemic issues, primarily driven by the increased focus on mathematical practice, i.e., on mathematics as a human activity.

The emphasis on formal aspects of mathematics, such as proofs, is a recent phenomenon of Western culture that seems absent in other cultures with rich mathematical traditions, such as China, India, and the medieval Arabic world. Even Western mathematics up to the eighteenth century was result-driven, with proofs subservient to methods for solving specific mathematical problems. Today, intuitions have not disappeared in mathematical practice, as Thurston (1994) observes: mathematicians are born and enculturated in a rich fabric of pre-existing mathematical procedures and concepts. Some of these ideas are akin to living oral traditions in that they have never been published but yet are tacitly accepted by the mathematical community. Mathematicians have accorded a privileged role to intuition as a source of creativity. In their influential account of how mathematicians work, Davis and Hersh (1981, p. 399) go as far as to say: "[T]he study of mental objects with reproducible properties is called mathematics. Intuition is the faculty by which we can consider or examine these (internal, mental) objects."

Where does mathematical intuition come from? As we shall see in Section 2.3, some developmental psychologists argue for an innate basis of mathematical knowledge. A growing body of experimental literature indicates that infants can predict the outcomes of simple numerical operations. The study of numerical cognition in animals (see Section 2.2) predates this literature, again providing evidence of animals' successes in estimating cardinalities, comparing numbers of different magnitudes, and predicting the outcomes of arithmetic operations. Complementary to this is neuroscientific evidence (see Section 3), which shows that some areas of the human brain are consistently involved in arithmetical tasks, strengthening the case for evolved, numerical competence. Finally, in Section 4, we shall argue that humans also draw on their external environment to make mathematical problems more tractable. Thus, mathematical cognitive processes can be situated both internally (inside the head) and externally (in the world).

2.2 Animals' arithmetic

Examining the numeric competence of non-linguistic creatures presents a methodological challenge: In the absence of language the evidence for arithmetic abilities or their underlying representations has to be inferred from overt behavior. As failed attempts at meeting this challenge have lead to famous misattributions, popular accounts of animal arithmetic (e.g., Dehaene, 1997; Shettleworth, 1998) often begin with the cautionary tale of Clever Hans. Clever Hans was a horse that lived in the early 1900s and appeared to have astonishing arithmetic abilities. Among various verbal and calendar-related feats, Hans could add, subtract, multiply, divide, and even work out fractions, indicating the results by tapping his hoof. The skeptical inquiry of Oskar Pfungst (1907) revealed that Hans indeed was clever, but his abilities consisted in detecting the subtle cues that his questioners or audience inadvertently would provide. Even after debunking Hans' alleged abilities, Pfungst was unable to refrain from providing signals that the horse could use. Thus, the story of Clever Hans teaches an important lesson to comparative psychology: To prevent observer-expectancy effects, the number senses of animals and pre-verbal infants ought to be probed either without experimenter intervention or in double-blind designs in which neither the examined creature nor the experimenter is aware of the correct answer. More generally, we have to be cautious not to over-interpret the abilities of animals by anthropomorphizing them. Whenever animals—including humans—show surprising arithmetic abilities we need to distinguish between ingenious trickery, natural competence, and the results of extensive training.

After several decades of deep skepticism there has been a resurgence of research efforts to probe the arithmetic abilities of animals by behavioral means. In 1993 a prominent researcher concluded enthusiastically that "the common laboratory animals order, add, subtract, multiply, and divide representatives of numerosity [...]. Their ability to do so is not surprising if number is taken as a mental primitive [...] rather than something abstracted by the brain from sense data only with difficulty and long experience" (Gallistel, 1993, p. 222). We shall organize our discussion of animals' abilities according to different levels of numerical competence.

Numerosity discrimination. Using an operant conditioning paradigm that required rats to press n times on a lever A before obtaining a treat by pressing another lever B once, Mechner (1958) demonstrated that rats can associate rewards with a specific number of repeated actions. Interestingly, the rodents never performed fully error-free and the variance of their actual runs increased as n increased (from a minimum of 4 to a maximum of 16). As premature switches were punished (by not obtaining a reward at all) the rats' number of responses were skewed toward over-estimates, rather than under-estimates.

Because the number n of lever presses was confounded with the time t it took to perform these actions a rival explanation of the rats' alleged ability to discriminate between different numbers was that they could have used duration as a cue to estimate number. However, Mechner and Guevrekian (1962) ruled out this alternative account by depriving rats of water for different periods of times. Whereas thirsty animals pressed the lever much faster, their degree of deprivation had little effect on the number of responses. Meck and Church (1983) later showed that rats spontaneously attend to both the number and duration of a series of discrete events.

Counting. Capaldi and Miller (1988) provided evidence that rats count the number of rewarded trials. By randomly exposing them to sequences of trials RRRN and NRRRN (where R stands for a rewarded and N for a non-rewarded trial) rats learned that they could expect to be rewarded on three trials. A much slower speed on the last (N) trial of both sequences shows that rats no longer counted on being rewarded after having accumulated three rewards on earlier trials. Importantly, rats readily transfer their counts to other types of food and even integrate their counts across different types of food, suggesting that their internal counts are abstract rather than tied to concrete events. As counting the types and amount of food items obtained from a particular patch is a fundamental part of animal foraging such abilities may not come as a complete surprise (Shettleworth, 1998). But it is easily overlooked that systematically exploiting many food resources requires some basic—and possibly implicit—method for keeping track of both time and number.

Merely identifying and counting numerical quantities does not necessarily require an abstract concept of counting or number. As a possible mechanism Dehaene (1997) suggests the metaphor of an analog accumulator that gathers the amount of some continuous variable (like water) rather than discrete quantities (like pebbles). By incrementing and decrementing such an accumulator animals would possess an approximate representation of numerical quantities that would allow for basic comparisons, as well as elementary additions and subtractions. A fuzzy or noisy boundary of the elementary counting unit implies that larger quantities get increasingly im-precise. The resulting consequences that two numbers are more easily distinguished when they are further apart and that two numbers of a fixed distance are harder to discriminate as they get larger are known as the *distance* and *magnitude* effects in both animal and human experiments on number comparisons.

To model the identification of a number of objects from visual or auditory perception Dehaene and Changeux (1993) developed a neuronal network model that relies on number-detecting neurons. Despite its simplicity, this model can account for the detection and discrimination of numerosities in animals and pre-verbal infants without assuming any ability to count explicitly.

Abstract and symbolic representations. We just saw that comparing and counting numerosities does not yet imply the mastery of an abstract concept of number (see Shettleworth, 1998, p. 369). However, there is also evidence that rats can abstract from sensory modalities and add discrete events. Church and Meck (1984) trained rats to discriminate between two vs. four tones and two vs. four light flashes by teaching them the regularities $ll \rightarrow L$, $llll \rightarrow R$, $tt \rightarrow L$, and $tttt \rightarrow R$, where lowercase l and t stand for flashes of light or tones, and uppercase L vs. R correspond to pressing either a left or right lever, respectively. What happens if rats that have learned those contingencies are confronted with a stimulus configuration of $lltt$? Despite the double dose of stimuli that individually required a L response to "twoness" throughout the training phase, the rodents now pressed the right lever R, indicating that they instinctively added $2+2=4$. To emphasize the significance of this finding, Dehaene (1997) compares it to a fictitious experiment that trained rats to discriminate both between red and green objects and between square and circular shapes. Surely it would seem surprising if presenting a red square evoked the response for green and circular objects. Both our own intuition and the rats of (Church and Meck, 1984) suggest that discrete events are to be integrated in an additive fashion, rather than by a merging process that combines perceptual properties like color and shape.

To address the question of whether animals can associate and manipulate numeric symbols we have to turn to parrots and chimpanzees. Pepperberg (1994) trained the African grey parrot Alex to vocally label collections of 2 to 6 simultaneously presented homogeneous objects and showed that he could then identify quantities of subsets in heterogeneous collections. For instance, Alex would be shown a collection of blue and red keys and cups and then identified the number of blue cups with an overall accuracy of over 80%. Boysen and Berntson (1989) demonstrated that their chimpanzee Sheba could assign Indo-Arabic[8] numerals to collections of objects and *vice versa*. In addition, Sheba could add up small numbers of oranges or numerals (up to a total of 4) when they were hidden in different locations.[9]

This section has shown that an assessment of the numeric competence of animals needs to strike a balance between two extremes: On the one hand, abundant credulity or naïve enthusiasm about animals' numeric feats would overlook that the numeric competence of non-human animals is fundamentally different from that demonstrated by humans. Almost always any abstract mastery of numeric symbols is the result of extensive training, is specific to a few numbers (i.e., difficult to generalize), and remains notoriously error-prone, particularly with numbers beyond 6 or 8. On the other hand, a refusal to acknowledge that lower animals can distinguish, count, and represent numerical quantities in some way would border on species chauvinism. There is no reason in principle why the perception of numerosity ought to be more complex than that of color, shape, or spatial orientation. And as detecting the amounts of prey, predators, or potential mates conveys a clear advantage for survival we should not be surprised that evolution has endowed non-human animals with at least some rudimentary number sense.

2.3 Infants' arithmetical skills

Prior to the late 1970s, developmental psychologists interested in the domain of numerical competence almost exclusively examined the development of explicit counting and exact positive integer representation during the preschool years. The early focus on explicit skills was partly due to methodological limitations (how to study cognition in infants) and partly due to firm conceptions about the cognitive foundations of arithmetical skills. Piaget's 1952 seminal work places the development of arithmetical skills late in cognitive development, between 5 and 12 years of age. Piaget thought that children must first master abstract reasoning skills, such as transitive reasoning or one-to-one correspondence. A problem with this framework is that it assumes that abstract reasoning skills are psychologically primitive for understanding number. The attraction of this view is that features such as one-to-one correspondence do play an important role in foundational work on mathematics, such as attempts to reduce arithmetic to set theory. However, it may be a category mistake to take that which is primitive in the development of formal arithmetic as psychologically primitive. As we shall see, infants and young children have some understanding of number which develops independently of other abstract reasoning skills. In 1978,

[8] See Section 4.2 for this terminology.

[9] See the review chapters by Boysen (1993) and Rumbaugh and Washburn (1993) for more details on the numeric competence of monkeys and chimpanzees.

Gelman and Gallistel published an influential monograph on arithmetical skills in preschoolers. From then on, the road was open for developmental psychologists to examine numerical capacity in infants. The early focus on explicit number representations has given way to the study of a broad domain of mathematical skills that are related to quantities, including the exact and immediate counting of small numerosities (subitizing), relative numerical judgments, and approximate systems of counting of larger sets (estimation). In this short review, we shall focus on arithmetic skills.

In a pioneering series of experiments, Karen Wynn (1992) tested the ability of five-month-olds to perform addition and subtraction on small quantities. To probe her subjects' capacities, she relied on the *looking time procedure* and the *violation of expectation paradigm.* The looking time procedure aims to probe cognitive abilities with a minimum of task demands. Clearly, infants cannot speak, so any test to probe infant knowledge is necessarily non-verbal (as was also the case with animals in Section 2.2). Moreover, human infants are motorically helpless (e.g., they are unable to release objects intentionally until 9 months of age), so one cannot rely on tasks that involve manual dexterity—this is importantly different from animal studies, which frequently require the subject to perform some particular action (e.g., pecking, pressing a lever). The violation of expectation paradigm exploits the propensity of humans and other animals to look longer at unexpected than at expected events. Our knowledge of the world enables us to make predictions of how objects will behave. For example, we expect coffee to remain in a stationary cup, but to flow out of a cup in which holes were drilled. When something happens that violates these predictions, we are surprised. Prior to the test trials, infants are exposed to habituation or familiarization trials to acquaint them with various aspects of the test events. With appropriate controls, evidence that infants look reliably longer at the unexpected than at the expected event is taken to indicate that they (1) possess the expectation under investigation, (2) detect the violation in the unexpected event, and (3) are surprised by this violation. The term 'surprise' is used here simply as a short-hand descriptor to denote a state of heightened attention or interest caused by an expectation violation.

In one of Wynn's experiments, a group of infants watched a $1+1$ operation: a Mickey Mouse doll was placed on a display stage, a screen rotated upwards to temporarily hide it from view, a hand entered the display stage with another identical looking doll, and placed it behind the screen. Then the screen was lowered to reveal either the possible outcome $1+1=2$, or the impossible outcome $1+1=1$. The infants looked significantly longer at the impossible outcome than at the possible one, suggesting to Wynn that they expected the outcome of $1+1$ to be 2. Similarly, they gazed longer at $2-1=2$ than at $2-1=1$. A methodological problem with the looking time procedure is that one cannot be sure what causes the longer looking times. Wynn (1992) favored an account in terms of fairly advanced conceptual cognition, namely that infants possess the ability to reason about number and perform arithmetical operations.

Wynn's interpretation is not the only possible way to account for these data. It is equally possible that the results are caused by lower-level cognitive capacities, such as a preference for visual stimuli that are familiar. For instance, Cohen and Marks (2002) proposed that the infants' longer looking time could be explained by a familiarity preference: they looked longer at one doll in the case of $1+1=1$ or 2, because during habituation, when the infants were familiarized with the setup, they saw one doll. Similarly, for the case of $2-1=2$ or 1, they looked longer at two dolls since that is what they saw during the familiarization trials. Importantly, developmental psychologists who probe innate knowledge do not exclude this possibility—indeed, they attempt to minimize familiarization effects by designing controls. Several subsequent experiments in independent labs (e.g., Kobayashi et al., 2004) have attempted to control for these alternative explanations, such as placing the puppets on rotating platforms, or familiarizing the infants equally with one, two, and three puppets. The results of these studies have supported Wynn's original experiment, and by controlling for lower-level cognitive accounts, have made the case for early developed numerical skills stronger.

Still, it is important to note that translating the experimental setup into mathematical notation can be misleading; it is not evident that Wynn's experiments show that infants are capable of operations that are equivalent to the mathematical notions of addition and subtraction. For instance, Uller et al. (1999) have argued that the experiments show that infants represent the objects that are being added and subtracted not as integers, but as object-files. According to this view, an object-file of two entities is represented as follows: there is an entity, and there is another entity numerical distinct from it, and each entity is an object, and there is no other object, i.e.,

$$(\exists x)(\exists y)\{(\text{object}[x] \;\&\; \text{object}[y]) \\ \&\; x \neq y \;\&\; \forall z(\text{object}[z] \rightarrow [z = x] \vee [z = y])\}.$$

This conception of numerosity is different from formal mathematical notions, but it is compatible with the empirical data on infants.

Later studies have probed whether infants can reason with larger quantities, and predict the outcomes of arithmetical operations that yield an absence of objects. McCrink and Wynn (2004), for example used a similar setup to Wynn's original experiment to investigate whether or not infants of 10 months of age can predict that $5+5=10$ and not 5, and that $10-5=5$

and not 10. This time, they used a computer-animated setup so that they could adequately control for total surface area (e.g., in the case of $5 + 5$, either ten objects of the same size as the original objects were shown, or five very large objects). The fact that infants could reliably predict these results strengthens the view that ten-month-olds can predict the outcomes of additions and subtractions of items in a visual display over larger numbers than previously studied, even when other factors such as the size of objects are controlled for.

The case of operations that yield an absence of objects reveals some limitations of this intuitive arithmetic. Wynn and Chiang (1998) used a looking-time experiment to show eight-month-olds subtraction events which had outcomes of no items (e.g., $1 - 1 = 0$). In contrast to the earlier experiments, the infants' looking time did not differ between the expected, correct result of $1 - 1 = 0$ and the incorrect, surprising outcome of $1 - 1 = 1$. This might suggest that infants have difficulties representing zero as a cardinal number. Unrelated experiments with chimpanzees yield similar results: although these animals can learn to distinguish between numbers up to 9 with good accuracy, they keep on confusing zero with very small natural numbers (1 and 2; Biro and Matsuzawa, 2001). These results are in tune with observations of mathematical practice in history and across cultures: most indigenous mathematical systems do not have a zero, neither as a placeholder symbol nor as a number.

Other limitations on infant arithmetic are related to working memory. In one study, Feigenson et al. (2002) presented infants of 10 and 12 months with the choice between two opaque buckets whose contents they were unable to see. In each of them, a number of crackers were dropped, one by one, so that at the end of each trial the buckets contained different numbers of crackers. After presentation, the subjects were allowed to crawl to the bucket of their choice to retrieve the crackers. Although the infants could successfully choose between 3 vs. 2 (i.e., they realized that $1+1 < 1+1+1$), they performed at chance level in the two versus four and three versus six conditions, despite the highly discriminable ratio between the quantities. Control tests ensured that this experiment cannot be explained by movement complexity. Possibly, working memory demands are a limiting factor: it is perhaps difficult to keep two collections with more than three objects each hidden from view in working memory. Indeed, as will be discussed in Section 3, humans rely on a host of complementary resources when doing arithmetic, including spatial representations, verbal labeling, finger representations, and imagined motion.

The results from studies with nonhuman animals and infants suggest that humans are furnished with an unlearned, early-developing capacity to perform simple arithmetical operations. However, one may wonder whether these studies can tell us anything relevant about arithmetic as a culturally elaborated skill. Some authors (e.g., Rips et al., 2008) remain deeply skeptical about the role of such evolved competencies in formal mathematical reasoning. It is indeed possible that similarities between infants' performance on some tasks and adult mathematical knowledge are superficial, and that there is no overlap between intuitive and formal mathematical concepts. Although we can *describe* Wynn's (1992) experiment in mathematical terms as $1 + 1 = 1$ or 2, does this mean that babies actually know that $1 + 1 \neq 1$? It may be problematic to use symbolic notation to describe such events, given that symbolic notations themselves influence mathematical cognition, a point that will be developed in more detail in Section 4. Non-numerical factors like language and verbal memory play an important role in elementary mathematics education, as is demonstrated by the memorizing of exact addition facts like $5 + 7 = 12$ or multiplication facts like $7 \times 9 = 63$. Young children also rely extensively on fingers and hands when they add and subtract.

Notwithstanding their sometimes problematic interpretation of the results, cognitive scientists offer the best hope of explaining our epistemic access to mathematical objects. Several lines of evidence indicate a causal connection between the early development of numerical skills and formal numerical competence. Halberda et al. (2008) found that children who are better at estimating numerical magnitudes (e.g., guessing the number of dots on a screen) also achieve better results in mathematics at school. Thus, approximate numerical skills are important for the development of more formalized ways of manipulating numbers such as symbolic arithmetic. Indeed, a study by Barth et al. (2006) found that both adults and preschoolers can perform additions and subtractions approximately, without the use of symbolic aids. In one of their experiments, inspired by Wynn's procedure, the preschoolers were shown a large number of blue dots. Then the blue dots were covered by a screen, and some more blue dots were shown to go hiding behind the screen. The children were then asked whether there were less or more blue dots compared to a set of visible red dots. The subjects answered well above chance level, indicating that approximate addition over large numbers develops prior to extensive training on arithmetical principles. Moreover, developmental dyscalculia, a disruption in the normal development of mathematical skills in some children, is correlated with an inability to grasp the concept of numerosity (Butterworth, 2005). Molko et al. (2003) studied the brain structure of subjects with developmental dyscalculia and found that their intraparietal sulci (which, as we shall see in Section 3, is implicated in numerical cognition) showed abnormal structural properties.

In sum, the evidence reviewed here strongly indicates that human infants possess elementary numerical skills. Combined with the evidence of numerical skills in animals, one could make a case for an evolutionary basis of numerical cognition. The importance of numerical knowledge in everyday decision making, such as foraging or forming groups, makes this evolutionary origin quite plausible.

2.4 Arithmetic in few-number cultures

As early as 1690, the philosopher John Locke (1690) mentioned possible effects of a limited numerical vocabulary on numerical cognition: "Some Americans I have spoken with (who were otherwise of quick and rational parts enough) could not, as we do, by any means count to 1000; nor had any distinct idea of that number." These Americans were the Tououpinambos, a culture from the Amazon forest in Brazil, who "had no names for numbers above 5." Although Locke thought that the absence of count words limited their ability to reason about large cardinalities, he mentioned that they could reckon well to twenty, by "showing their fingers, and the fingers of others who were present." He thus argues that count words are "conducive to our well-reckoning," but not strictly necessary for it (Locke, 1690, all citations from Book II, ch. XVI). By contrast, Alfred Russell Wallace, co-discoverer with Darwin of the principle of natural selection, believed that count words were essential for numerical cognition, in particular arithmetic: "if, now, we descend to those savage tribes who only count to three or five, and who find it impossible to comprehend the addition of two and three without having the objects actually before them, we feel that the chasm between them and the good mathematician is so vast, that a thousand to one will probably not fully express it" (Wallace, 1871, p. 339). The question of the role of language in arithmetic became the focus of recent experimental psychological studies in cultures with few number words, in particular the Pirahã and the Mundurukú, two cultures from the Amazon forest with an extremely limited number vocabulary.[10]

The Pirahã (Gordon, 2004) have only three words that consistently denote cardinality, 'hói', 'hoí' and 'baágiso'. These terms are not used as count words, but rather as approximations of perceived magnitude (not just cardinality). For example, the word 'hói' is used to denote single objects, but also as a synonym for small (as in a small child). 'Hoí' is used to denote a few items or a medium quantity, and 'baágiso' is used for large items or large quantities of items. One can ask 'I want only *hói* fish' to denote one fish, but one cannot use this phrase to ask for one very large fish, except as a joke (Everett, 2005). The imprecision of the Pirahã count words was recently demonstrated in a series of experiments (Frank et al., 2008a) in which

[10] The elaboration of mathematical ideas differs considerably between cultures. For an extensive discussion of ethnomathematics, see (François and Van Kerkhove, 2010), this volume.

Pirahã subjects were simply asked to say how many objects they saw. If the objects were presented in an *increasing* order, from 1 to 10 items, the subjects consistently said 'hói' for one item and 'hoí' for two items. For more than two items, some subjects said 'hoí' or 'baágiso'. By contrast, if the objects were presented in a *decreasing* order, the subjects said both 'hoí' or 'baágiso' for objects up to 7, and some claimed to see 'hói' starting at 6 items. Some years earlier, Gordon (2004) confronted Pirahã with a battery of experiments to test numeracy, such as probing the capacity to place objects into a one-to-one correspondence and memory for specific numbers of items. Their capacity to reason about exact magnitudes was severely compromised, especially for numerosities that are above the subitizing range ($n > 4$). An example of a matching task required that the subject draw as many lines as were presented to him or her by the experimenter. The accuracy dropped linearly as the target number of lines increased. After 7 items, none of the participants drew the correct number of lines. In one of the experiments that probed memory for numerosity, the participants witnessed a quantity of nuts being placed in a can, and then being withdrawn one by one. After each withdrawal, the subjects responded as to whether the can still contained nuts or was empty. This task proved extremely difficult, as the responses dropped to chance level between 4 and 5 items.

Authors who have studied Pirahã do not agree on the implications of these experiments on the role of external symbolic systems for numerical cognition. Gordon (2004, p. 498) claimed that his study "represents a rare and perhaps unique case for strong linguistic determinism." In contrast, Frank et al. (2008a) showed that Pirahã performed relatively well on tasks that did not involve memory, such as matching tasks (e.g., matching a number of objects to those that an experimenter showed them), by employing strategies that involve making one-to-one correspondences. These results suggest that count words do not create number concepts, but rather concur with Locke's view that they are "conducive to our well-reckoning." Similar results have been obtained with people from other non-numerate cultures, such as Australian aboriginal children who speak languages with few count words. In these studies, the children could even solve division problems if they could use one-to-one matching, such as dividing six or nine play-doh discs between three puppets. They simply dealt discs to each puppet one by one, until all discs were divided (Butterworth et al., 2008).

To better tease apart the role of language and other cultural factors, Frank et al. (2008b) conducted experiments with American college students that were very similar to those presented to the Pirahã. These tasks involved both one-to-one matching tasks and memory tasks. In the meantime, the participants performed a task that made it impossible for them to rely on subvocal counting. Apparently, the ability to perform one-to-one matchings was relatively unimpaired by the inability to count, but memory for numbers, as in the nuts-in-a-can task, was severely compromised (subjects answered correctly only 47 % of the time). In addition to language, other external tools may explain the limited numerical skills in the Pirahã. For example, Everett (2005) noted that they do not have individual names for fingers (e.g., ring finger, index), but collectively refer to their fingers as 'hand sticks'. In many cultures, finger counting plays a crucial role in the development of number concepts; the fact that words for 'one', 'four', and 'five' in many Indo-European languages are related to words for fingers (or digits) is indicative of this. If fingers are not differentiated, this might impair the formation of exact magnitude concepts, or vice versa.

The Mundurukú is another Amazonian culture with few number words (up to five), which are likewise used in an approximate fashion: pũg ('one'), xep-xep ('two'), e-ba-pũg (literally: 'your arms and one'), e-ba-dip-dip (literally, 'your arms and two', pũg-pog-bi (literally 'a handful' or 'a hand'). The approximate nature of these quantities is illustrated by the fact that the use of these terms is inconsistent when Mundurukú subjects have to denote three or more items. For example, when five dots are presented, the subjects respond pũg-pog-bi in only 28 % of the trials, and e-ba-dip-dip in 15 % of the trials. Above five, the Mundurukú do have words to denote numerosities, but these terms have very little consistency. Subjects refer to 10 items using the expressions ade ma ('really many'), adesũ ('not so many') and xep xep pog-bi ('two hands') (Pica et al., 2004). Pica et al. (2004) studied the effects of this limited vocabulary on arithmetic, revealing an interesting discrepancy between exact and approximate arithmetic. Mundurukú exact arithmetic proved to be highly compromised. For example, in one study, the subjects predicted how many objects would be left in a can after several had been removed. Although the results were small enough to be named with their number vocabulary (e.g., $6-4=2$), they were unable to predict them. In contrast, Mundurukú subjects did very well on approximate arithmetical tasks, where they were asked whether the addition of two large collections of dots (e.g., 16 and 16) in a can was smaller or bigger compared to given number of dots (e.g., 40). In this task, which involved quantities far above their count range, they did as well as French numerate adults.

Another aspect of numerical cognition that is clearly affected by external representations are questions regarding the distance between internal representations of different numbers, i.e., the shape of the 'mental number line'. Several studies have shown that young children (Siegler and Booth, 2004) and non-human animals (Nieder and Miller, 2003) represent numbers on a logarithmic, rather than a linear mental number line. In brief, a logarithmic mental number line is one where estimations of numerosities conform to the natural logarithms (ln) of these numbers. This typically leads to an overestimation of the distance between small numbers, such as 1 and 2, where the psychological distance is typically judged to be much larger than between larger numbers like 11 and 12. Young children make characteristic errors when plotting numbers on a scale. Siegler and Booth, for example, gave five- to seven-year-olds a number line with 0 at the left side and 100 at the right. Younger children typically place small numbers too far to the right. For example, they tend to place the number 10 in the middle of the scale, which is roughly in accordance with a logarithmic representation. As children become older, their number lines look more linear. From these results, Siegler and Booth (2004) conclude that our intuitive number representation is logarithmic, and that it becomes more linear when children learn to manipulate exact quantities. Dehaene et al. (2008) adapted this experiment in an elegant fashion to a study with Mundurukú participants, presenting them with a line with one dot to the left, and ten dots to the right. Then, the Mundurukú were given a specific numerical stimulus, either as a number of tones, or as a number word in Portuguese or Mundurukú. In all cases, the best fit of the responses was logarithmic, not linear. As the authors of this study acknowledge, language cannot be the sole factor responsible for linear numerical representations in Western people, as the Mundurukú responded logarithmically, regardless of the language or format in which the numbers were presented. Perhaps other external representations, such as rulers or the practice of measurement, can explain this change.

Taken together, these results suggest that approximate arithmetic relies less on external tools such as language than exact arithmetic. The animal, neuroimaging, and infant studies demonstrate that our intuitive numerical competence allows for approximate arithmetical tasks. External representational systems, such as fingers, count words, and numerical notation systems, serve to enhance exact numerical cognition that ventures beyond the range of our intuitive capabilities.

3 Arithmetic and the brain

3.1 Lesion studies

Neuropsychological studies offer the opportunity to study the neural correlates that underlie our capacity to perform arithmetical operations. What neural structures enable us to comprehend and compute with numbers? Are there differences between approximate arithmetic and exact arithmetic? How are external media, such as symbolic notation systems, reflected in the brain? The oldest method to study the neural basis of arithmetic relies on an examination of the effects of brain lesions on various cognitive tasks. This methodology was developed in the later decades of the 19th century when physicians like Broca and Wernicke noticed that specific lesions, i.e., patterns of brain damage led to an inability to speak. Such lesions can be the result of an external injury or a stroke (a blood-clot which momentarily deprives part of the brain of oxygen and nutrients), leading to specific patterns of cognitive impairment. Indirectly, one can infer from the correlation between damage to a given brain area X and loss of a certain cognitive function a, that X and a are functionally correlated.

Early studies by Gerstmann (1940) showed that patients with damage to the left inferior parietal lobule (a subsection of the parietal lobe) often had marked impairments in mathematical cognition. Lesions in this area often leave a patient unable to perform very simple arithmetical operations such as $3-1$ or 8×9. However, these lesions usually also affect other domains of cognition. This is exemplified in Gerstmann's syndrome (Gerstmann, 1940), a neurological condition that is associated with damage to the parietal lobe, and that is characterized by an inability to perform arithmetic, count, and do other numerical tasks, as well as by difficulties in writing (agraphia), the inability to recognize one's own fingers (finger agnosia), and left-right confusion (Chochon et al., 1999). The fact that loss of mathematical function is often accompanied by finger agnosia, agraphia, and left-right confusion might be due to the fact that lesions usually damage several adjacent functionally specialized brain areas. In that case, the cognitive functions are not really related, but their damage coincides because the areas correlated with them are in close anatomical proximity. Alternatively, one could take these findings as support for the view that finger counting, writing, and spatial skills play an important role in numerical processing. Evidence for this latter interpretation comes from several modern studies that impair finger cognition in an experimentally controlled and reversible way. In these repetitive transcranial magnetic stimulation (rTMS) experiments, brain activity was briefly disrupted in areas important for finger cognition, including the left intraparietal lobule (Sandrini et al., 2004) and the right angular gyrus (Rusconi et al., 2005). In both studies, disrupting finger cognition led to a marked increase in reaction time when subjects solved arithmetical operations. This suggests that finger recognition remains an important part of adult numerical cognition, even when we no longer count on our fingers.

Lesion studies have also examined whether or not language is essential for mathematical tasks. This has given rise to a nuanced picture. First, it seems that language, especially verbal memory, is more important for multiplication and addition than for division and subtraction. Lemer et al. (2003) assessed the differential contributions of brain areas specialized in language and number for diverse arithmetical operations. In their study, they examined a patient with a verbal deficit (caused by lesions in the left temporal lobe), and another patient with a numerical deficit (with a focal lesion in the left parietal lobe), but intact verbal skills. The authors hypothesized that language would play an especially important role in arithmetical tasks in which verbal memory is important, such as multiplication (due to memorization of multiplication facts like $5 \times 7 = 35$) and addition. By contrast, since we do not store subtraction facts in verbal memory, this capacity should be less affected by the loss of language. As predicted, Lemer et al. (2003) found that the patient with the language impairment performed worse on multiplications than on subtraction, whereas the patient with numerical impairments exhibited the reverse pattern. Thus it seems that verbal memory can play an important role in the performance of arithmetical tasks in the adult human brain.

Another study (Varley et al., 2005) probed whether language may be important for numerical cognition on a more deep, structural level. Parallels between recursive structures in mathematics and grammar have suggested to some authors that the generative power of grammar may provide a general cognitive template and a specific constituting mechanism for 'syntactic' mathematical operations involving recursiveness and structure dependency, such as the computation of arithmetical operations involving brackets, e.g., $50 - ((4 + 7) \times 4)$. Indeed, Hauser et al. (2002) argue that a domain-general and uniquely human capacity for recursion underlies our capacity for mathematics.[11] More specifically, they state that "Humans may be unique [...] in the ability to show open-ended, precise quantificational skills with large numbers, including the integer count list. In parallel with the faculty of language, our capacity for number relies on a recursive computation" (p. 1576). To test this relationship between language and numerical cognition, Varley et al. (2005) examined three severely agrammatic patients (i.e., people with an inability to comprehend and make grammatical sentences) on several numerical tasks, including multiplication tables and bracket operations. Despite their lack of grammar, all three men performed excellently on these tasks, solving problems like $80 - ((6 + 14) \times 2)$ accurately. One of the problems specifically examined the preservation of recursive capacities in the absence of grammar: it required the patients to come up with numbers smaller than 2, but larger than 1. Although none of the patients was capable of generating recursive linguistic expressions, they could solve these problems, coming up with numbers like $1, 1.9, 1.99, 1.999, \ldots$. From this, the authors conclude that, at least in the mature adult brain, the non-linguistic neural circuits that deal with recursive structure in mathematics are functionally independent of language. However, it does not follow that language is unimportant for the *development* of mathematical competence. For instance, Donlan et al. (2007) showed that eight-year-old children with specific language impairments (i.e., children with language impairments but

[11] A recursion consists of a few simple cases or objects, and rules to break down complex cases into simpler ones, e.g., my (full) brother is my blood relation (base case), anyone who is a blood relation of this brother is also a blood relation (recursion).

overall normal intelligence in other domains) are developmentally delayed for several numerical tasks compared to children without language impairments: as many as 40 % failed to count to twenty, and they showed problems in understanding the place-value system. By contrast, these children had no problems understanding high-level principles of arithmetic, such as commutativity.

3.2 Neuroimaging studies and EEG experiments

A more direct way to study which regions of the brain are involved in performing specific tasks is provided by functional neuroimaging techniques. All neuroimaging techniques exploit the fact that although the whole brain is always active, not every part is equally active. Regions that are more active require more energy (glucose) and oxygen. Neuroimaging techniques measure differential brain-activation after presentation of a relevant stimulus, and compare these activations to a carefully chosen control stimulus. If this effect is constant across subjects and if it is reproducible, the cerebral parts that are more active after presentation of the test stimulus compared to a control stimulus are taken as neural correlates for the task that the stimulus probes. The most frequently used neuroimaging technique for probing numerical competence is functional Magnetic Resonance Imaging (fMRI), which relies on strong magnetic fields to measure differences in oxygen-levels in cerebral blood flow. A problem with most neuroimaging techniques is that while they have a relatively good spatial resolution (i.e., they give a relatively accurate map of differential brain activity), they have a relatively poor temporal resolution (i.e., they are slow and may not pick up transient patterns of brain activity). By contrast, electroencephalography (EEG) scans, which measure electric activity in the brain through electrodes on the scalp, can pick up subtle and quick changes in brain activity, but have poor spatial resolution, as only areas at the surface of the brain can be accurately measured. EEG scans can be used to measure the specific response of the brain for a given task; these task-related patterns of electric brain activity are termed Event Related Potentials (ERPs).

Dehaene et al. (1999) investigated the relative importance of language and non-linguistic approximate representations of number in two brain-imaging studies: one with high temporal resolution (ERPs) and one with high spatial resolution (fMRI). First, they conducted a behavioral experiment with Russian-English bilinguals. The subjects were taught a series of exact or approximate sums of two-digit numbers in one of their languages, either Russian or English. The test condition consisted of a set of new additions. This was either an exact condition, in which they had to choose the correct sum from two numerically close numbers, or an approximate condition, in which they had to estimate the result and select

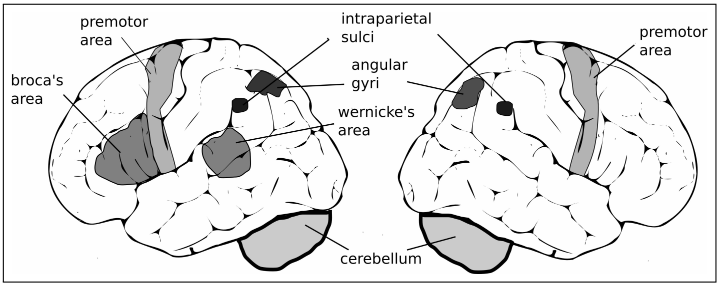

FIGURE 2. Regions of interest mentioned in the text, left hemisphere shown on the left.

the closest number. After training, response time and accuracy improved in both types of tasks. However, when tested in the exact condition, subjects performed much faster in the teaching language than in the untaught language. In contrast, for the approximate condition, there was no cost in response time when switching between languages. To the authors, this suggested that exact arithmetical facts are stored in a language-specific format; each new addition is separately stored from neighboring magnitudes, e.g., $9 + 1$ would be stored differently from $9 + 2$. Because there was no cost in the approximate condition when switching between languages, the authors assumed that number is also stored in a language-independent format (Dehaene et al., 1999, p. 971). The authors examined whether this apparent behavioral dissociation is the result of distinct cerebral circuits. In fMRI, the bilateral parietal lobes showed greater activation for the approximate task than for exact calculations. In the approximate task, the most active areas were the bilateral horizontal banks of the intraparietal sulci (IPS) (see Figure 2). Additional activation was found in the left dorsolateral prefrontal cortex and in the left superior prefrontal gyrus, as well as in the left cerebellum, the left and right thalami, and the left and right precentral sulci. Most of these areas fall outside of the areas associated with language. Exact calculations elicited a distinctly different pattern of brain activation, which was strictly left-lateralized in the inferior frontal lobe. Smaller activations were also noted in the left and right angular gyri. Previous studies have shown that the left inferior frontal lobe plays a critical role in verbal association tasks. Together with the left angular gyrus, this region may constitute a network involved in the language-dependent coding of exact addition facts (Dehaene et al., 1999).

Several studies since then have confirmed that the intraparietal sulci of both hemispheres, but predominantly of the left, are active during arithmetic and other numerical tasks. This is the case even in the absence of explicit tasks, for example, when subjects are only required to look passively at Indo-Arabic digits, or to listen to number words spoken out loud (Eger et al., 2003). The IPS seems to be an important neural correlate for numerical cognition, regardless of the format in which it is presented. This finding is confirmed by several studies that measure the firing rate (i.e., electric activity) of single neurons in monkeys. Tudusciuc and Nieder (2007) found that neurons in the intraparietal sulci of monkeys were sensitive to differences in numerosity, line length or both. The neurons were optimally tuned to a specific quantity (e.g., two items) and gradually showed less activity as the presented numerosity deviated from this preferred quantity. By focusing on which neural correlates are constant across numerical tasks, we have left open the question of whether the use of symbolic notations and other external tools affect numerical cognition at the neural level.

An intriguing fMRI study by Tang et al. (2006) provides indirect support for the role of symbolic representation in numerical cognition.[12] In this study, both native English speakers and native Chinese speakers solved arithmetical operations. Although the IPS were active in both groups, they exhibited marked differences in other brain areas. Whereas the English speakers had a stronger activation in perisylvian, language-related areas (such as Broca and Wernicke's areas), the Chinese speakers showed an enhanced response in premotor areas, involved in the planning of motor actions. The authors offered a possible reason for this: whereas English speakers learn arithmetical facts in verbal memory (e.g., when they learn multiplication tables), Chinese speakers rely on the abacus in their schooling. These differences in schooling might still be reflected in arithmetical practice, with English speakers mentally relying on language-based strategies, and Chinese speakers on motor-based strategies.

Taken together, neuropsychological studies indicate that numerical cognition relies on an interplay of cognitive skills that are specific to number (primarily located in the IPS) and cognitive skills from other domains, including language, finger cognition, and motor skills. Such findings indicate that numerical cognition is a complex skill, which involves a variety of capacities that are coordinated in very specific ways.

4 The role of notation in arithmetic

In Sections 2.4 and 3 we have described some connections between intuitive arithmetic notions and language, and have seen how the use of a lexical numeral system greatly affects people's basic arithmetic abilities. We now turn to the role of notation and its relation to computations. We shall ar-

[12] For a more detailed discussion on the relationship between extended mind and mathematics, we refer to (Johansen, 2010), this volume.

4	9	2
3	5	7
8	1	6

FIGURE 3. Arranging the digits 1 to 9 into cells of a 3-by-3 magic square reveals that the game of number scrabble (see text for details) is isomorphic to tic-tac-toe.

gue for the thesis that mental arithmetic is an interplay between internal and external representations. The properties of external media and characteristics of the representational format profoundly affect the process of manipulating and computing the solutions to arithmetic problems.

4.1 Representational effects

It is hard to overestimate the relevance of representations for problem solving. As all deductive inferences (e.g., in mathematics) are essentially changes of representation, an extreme argument for the crucial role of representations is that solving a problem is nothing but a change in representation, or "solving a problem simply means representing it so as to make the solution transparent" (Simon, 1996, p. 132). Simon illustrates his claim by the game of number scrabble: Two players alternate in choosing a unique number from 1 to 9. The player who first manages to select a triple of numbers that sum to 15 wins the game.

Most people find this game rather abstract and have difficulties in choosing numbers strategically. However, when the game is represented as in Figure 3, it becomes apparent that number scrabble is structurally identical to the game of tic-tac-toe, in which players alternatively pick a cell of a 3-by-3 grid and win by first occupying three cells on a straight line. The spatial re-representation makes it easier to 'see' that some numbers (e.g., 5) are more valuable for winning than others, as they are part of more potential solutions and allow for the obstruction of more opponent moves.

The phenomenon that different representations of a problem can greatly change their level of difficulty is referred to as a *representational effect* and problems that are identical except for the their surface representation are called *isomorphs* (e.g., Kotovsky et al., 1985; Kotovsky and Simon, 1990). Our introductory distinction between numbers and numerical notation systems (see Table 1 on page 61) illustrated that many alternative representational systems can represent the same entities.

From a cognitive standpoint the distinction between computational and informational equivalence is important for characterizing alternative representations (Simon, 1978; Larkin and Simon, 1987). Two representations are *informationally equivalent* if they allow the same information to be represented, but they are *computationally equivalent* if in addition any information that can be inferred 'easily and quickly' from one representation can also be inferred 'easily and quickly' from the other. Larkin and Simon (1987) admit that these definitions are inherently vague, but they are still useful in describing the effects of different numeric representations on the ease or difficulty of arithmetic calculations. Two systems are informationally equivalent with regard to a set of tasks if they both allow the same tasks to be performed. They are additionally computationally equivalent if the relative difficulty of tasks is the same no matter which representation is used (Bibby and Payne, 1996; Payne, 2003).

The distinction between the two different types of equivalence can readily be applied to numeration systems. The representational systems illustrated in Table 1 are all informationally equivalent, as every natural number can be unambiguously expressed in each system. Nonetheless, the different systems differ in their computational properties. Whereas the tally system makes the summation of two numbers a simple matter of combining their respective number of elementary strokes, the Indo-Arabic decimal system requires the recognition of different symbol shapes and the retrieval of arithmetic facts from memory.

Particular representational systems can be exploited in different ways to reveal properties of the represented entities. For instance, judging whether a particular number is even or odd requires a cumbersome counting process when it is represented in tallies (e.g., |||||||||), but only requires looking at the last digit when the same number is represented in binary or decimal notation (e.g., 1001 or 9, respectively). As another example, a natural number represented in decimal notation is divisible by nine when the sum of its digits is divisible by nine. Importantly, these computational shortcuts are not only dependent upon particular representations, but they are only available when both the meaning of the elementary symbols and their arithmetic properties are *known* to and actively *used* by the problem solver.

The terminology introduced above allows us to state that different number systems are isomorphs for the mathematical theory of arithmetic. Differences between their usefulness for solving particular problems are representational effects, i.e., phenomena in which representational systems that are informationally equivalent lack computational equivalence. Importantly, any judgment about task difficulty needs to consider the triad of the specific arithmetic task at hand, the representational system used, and the problem solver's mental (representational and computational) capacities.

The crucial question to be addressed in the following sections is: What causes or explains representational effects? As the definition of computational equivalence included a reference to the ease of operations that can be internal or external we need to consider the internal and external mechanisms involved in solving a problem to explain the genesis of representational effects. For instance, the ease of tic-tac-toe when compared to number scrabble can possibly be explained by the claim that visuo-spatial representations are more compatible with the cognitive mechanisms of ordinary human beings than the arithmetic properties of small numbers. But the relative ease or difficulty of operations also depend on the task to be performed. Note that a game of number scrabble could be communicated much more concisely (e.g., during a telephone call) than the identical game of tic-tac-toe. For an analogous argument about the usefulness of number systems we need to investigate their properties and computational demands on cognition for particular tasks.

4.2 Computations with different notation systems

Traditional accounts of number systems (e.g., Dantzig, 1954; Ifrah, 1985) draw a basic distinction between additive and place-value systems. Moreover, it is usually claimed that the positional system is superior, because it allows for more complex calculations, like the application of multiplication algorithms.[13] However, assessing the strengths and weaknesses of different notational systems is not as simple as this suggests and demands a much more nuanced analysis.

Cipherization. By contrasting several numerical notation systems, in particular the Greek alphabetic system of numerals (see Table 1c), which is *not* positional, and the Babylonian sexagesimal place-value system, Boyer (1944) argued that it is not so much the use of the principle of place-value that guarantees the ease of computation, but 'independent representation' or 'cipherization.' Characteristic for the latter is that individual symbols, that are brief and easy to write and read, are introduced to represent numbers in a concise way, avoiding the frequent repetition of basic symbols. What makes the Babylonian system cumbersome to use, despite the fact that it is a place-value system, is that is uses an additive system to represent numbers less than 60. Thus, for example, 57 is essentially represented by $\langle\langle\langle\langle\langle\ |||||||$. By contrast, the Greek alphabetic system (see Table 2) represents numbers up to 999 by combining 27 different elementary symbols and uses at most three different symbols per number. Since the Greek system is not positional, it does not need a symbol to mark an empty position (zero), but—in contrast to the system of Roman numerals—its basic symbols never occur repeatedly in a numeral. As a consequence, numbers are represented by even fewer symbols on average than in our familiar decimal place-value notation. For example, the number 208 is written as 'ση', and 400 simply as 'υ'. As in the Roman system to be discussed below, larger numbers require new symbols or a systematic scheme for modifications (see Boyer, 1944).

[13]See, e.g., (Menninger, 1969, p. 294), (Ifrah, 1985, p. 431), and (Dehaene, 1997, p. 98).

	1	2	3	4	5	6	7	8	9
×1	α	β	γ	δ	ε	ϝ	ζ	η	ϑ
×10	ι	κ	λ	μ	ν	ξ	ο	π	ϟ
×100	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ

Table 2. The Greek alphabetic system of numerals. The numerals for 6, 90, and 900 are the now obsolete symbols digamma (ϝ), koppa (ϟ), and san (ϡ) (see Ifrah, 1985).

Typology for numerical notation. Boyer's observations have been taken up by Chrisomalis (2004), who developed a two-dimensional typology for numerical notation, with intraexponential (e.g., cumulative or ciphered) and interexponential structuring (e.g., additive or positional) as the two independent dimensions. In this terminology the Greek alphabetic and the Roman system differ with regard to their intraexponential representations: The former is ciphered, while the latter is cumulative. Using this classification he was able to discern some patterns in the evolutionary change of number systems that go beyond the traditional view that takes the development of numeral systems to be linear, always from additive systems to positional ones.

Trade-offs. We already mentioned that the succinctness of the Greek alphabetic system is achieved through the use of 27 elementary symbols to represent the numbers from 1 to 999. Thus, the Greek system requires more symbols to be memorized, and more single addition and multiplication facts to be learned than our familiar decimal system. We see here clearly a trade-off between the complexity of an external representation system and the cognitive capacities (in this case recognition memory) that are required to master it. Arguably the best way to evaluate such differences would be to familiarize oneself equally thoroughly with multiple systems and then conduct the same computations in different systems. Unfortunately, such comparisons have rarely been done—presumably due to considerable practical limitations. However, the historian of mathematics Paul Tannery reported in 1882 that he practiced the Greek alphabetic system and "found that this notation has practical advantages which he had hardly suspected before, and that the operations took little longer with Greek than with modern numerals" (Boyer, 1944, pp. 160 f.). Unfortunately, Tannery did not provide more details about his experiences.

In the following, two different numerical notation systems are presented in detail, and some trade-offs with regard to the ease of computations and the cognitive abilities needed to master each system are discussed.

The decimal number system. It has been claimed that the numerical competence of the average teenager in today's Western societies exceeds that of an educated adult in antiquity by far (Nickerson, 1988, p. 198). Moreover, due to the nearly universal adoption of the decimal place-value system and our high degree of familiarity with it from early childhood it is hard for us to recognize it as a cultural convention. Despite its universality today, it originated in India and was only introduced to Europe in the 10th century C. E. by way of the Arabic world. We shall refer to it as the Indo-Arabic system.[14]

An important characteristic of the Indo-Arabic system is its economy of expression: Ten elementary symbols suffice to represent every conceivable natural number. This is possible because the basic values of a symbol and its position in the numeral sequence jointly determine its value. Hence, the *order* of digits in a numeral carries magnitude information and cannot be changed inconsequentially. Note how the value represented by the symbol '4' in the numeral 420 is 400, whereas the same symbol signifies 40 in the numeral 42. The fact that only numbers smaller than ten can be expressed within the numerical 'alphabet' of single symbols makes the Indo-Arabic system a *base-10* or *decimal* number system. This base, in turn, is responsible for the operational decade effects, which we discuss in Section 4.3. In general, in a place-value system with base p, the value of a numeral phrase containing $n+1$ symbols $a_n a_{n-1} \dots a_2 a_1 a_0$, where each a_i is an elementary symbol, is equal to $\sum_{i=0}^{n}(a_i \times p^i)$, i.e.,

$$(a_n \times p^n) + (a_{n-1} \times p^{n-1}) + \dots + (a_2 \times p^2) + (a_1 \times p) + a_0.$$

It becomes clear from this notation that the *position* of a symbol, which is denoted by the subscript, must also be taken into account when performing basic arithmetic operations. For example, *addition* of two numerals $a_n \dots a_0$ and $b_m \dots b_0$ in a place-value system, amounts to adding two polynomials, which results in

$$((a_n + b_n) \times p^n) + ((a_{n-1} + b_{n-1}) \times p^{n-1}) + \dots + ((a_1 + b_1) \times p) + (a_0 + b_0).$$

Note that only the as and bs that are at the same position, i.e., that have the same index, are added. Moreover, one has to take special care of the fact that every single digit of the result must be smaller than the base p. This is done by 'carrying' into the next position if $(a_i + b_i) \geq p$. Analogously, *multiplication* of two numerals in a place-value system amounts to

[14] Most commonly this system is known as the Hindu-Arabic system, but has also been referred to as 'Western,' since the digits have nothing to do with the Hindu religion and are different from Arabic scripts (Chrisomalis, 2004). A still better name might be the 'indic' system of numerals, as was suggested to us by Brendan Gillon. See (Menninger, 1969) and (Ifrah, 1985) for historic accounts of number system development.

multiplication of two polynomials, and the value of the result of multiplying the two numerals $a_n \dots a_0$ and $b_m \dots b_0$ is

$$\sum_{i=0}^{n} \sum_{j=0}^{m} (\, a_i \times b_j \times p^{i+j}\,).$$

From this form of representation two things become evident: that $m \times n$ operations (i.e., multiplications of two single digits) are necessary, and that one has to keep track of the *position* $(i + j)$ of the intermediate terms $a_i \times b_j$ in the resulting numeral. In the various multiplication algorithms for computations on paper this is achieved by the careful positioning of the intermediate terms (e.g., in columns).

The purpose of our presentation of the familiar Indo-Arabic system in these rather abstract terms is to bring to the fore its inner complexities, which are usually hidden from us due to our familiarity with it. Most educated adults will be able to instantly 'read off' the number denoted by the numeral 4711, without any deliberations, but the situation is very different for children or if we replaced the familiar symbols with unfamiliar ones. This shows that we have internalized the recognition and transformation of Indo-Arabic numerals in a very effective way and our familiarity with this system hides the underlying complexity of such processes. In turn, the fact that a numeral system looks unfamiliar to us should not play a role in its assessment.[15]

Place-value systems enable efficiency of representation in two important ways: Their use of only a finite number of basic symbols (in general, a base-n system requires n different symbols) and the relatively short length of their numerical phrases. The small number of symbols reduces the mental efforts needed to interpret and write the numerals as well as the number of basic facts that have been stored in long-term memory. (See Nickerson, 1988; Zhang and Norman, 1993, 1995, for a more detailed treatment of these issues.) On the other hand, the fact that the position of symbols has to be kept track of in computations increases the complexity of the algorithms that are needed. Dealing with columns and carries, which are both devices to keep track of positions, are the major stumbling blocks children face in learning to compute with Indo-Arabic numerals (see Lengnink and Schlimm, 2010).

One final issue about place-value numeral system concerns the choice of the base. This choice is essentially arbitrary, though it has been speculated that it may be due to the "anatomical accident" (Ifrah, 2000, p. 12) of the human body having ten fingers (see also Section 2.4, page 76). Ifrah also suggests that from a purely computational perspective a base of 11

[15] This was noted also by Anderson (1956), but see (Menninger, 1969, p. 294).

could in some circumstances be better (by virtue of being a prime number), and for trade purposes a base-12 system would yield benefits, as it would allow for more even divisors. A system with base 60 would fare even better in this last respect, and the ancient Babylonians did in fact use such a system. As another extreme, a base-2 (or binary) system would lead to the smallest number of basic symbols for a positional system, but also greatly increases the length of the numerals. Binary and hexadecimal (base-16) systems are in common usage in engineering and computer science, and their properties illustrate additional trade-offs between representational efficiency and implementation requirements. Thus, even the choice of the value of the base involves substantial trade-offs.

The system of Roman numerals. The system of Roman numerals is a purely additive system, in which each elementary symbol in a numeral phrase has a fixed value.[16] For example, the symbols I, V, X, L, C, D, M, stand for 1, 5, 10, 50, 100, 500, and 1000, respectively. The value of the numeral is then obtained by simply adding the values of its constituents. Thus, for a numeral $a_0a_1\dots a_n$, the value obtained is:

$$a_0 + a_1 + \cdots + a_n. \tag{1}$$

Comparing this with the process required for reading off the value of an Indo-Arabic numeral (in Equation 4.2 above) emphasizes the simplicity of additive systems. This internal simplicity comes at the cost of requiring a potentially infinite amount of elementary symbols if all natural numbers are to be represented. However, this theoretical limitation may be extenuated by the fact that only numbers up to a certain limit are used in practice.

A quantitative route for the evaluation of different notational systems in terms of their cognitive demands has been taken by Schlimm and Neth (2008). Using a computational cognitive modeling approach (along the lines of Payne et al., 1993), they analyzed algorithms for addition and multiplication with Roman and Indo-Arabic numerals in order to quantify the trade-offs between basic perceptual-motor operations and (short-term and long-term) memory requirements. For their comparisons, they modeled the common paper-and-pencil algorithms for addition and multiplication with the Indo-Arabic numerals, collecting information regarding the number of symbols used, single perceptual activities (e.g., reading a symbol), attentional shifts (to the next symbol in an array or to some absolute position on paper), memory usage (retrieval of addition and multiplication facts from long term memory, remembering of intermediate results, internal computations, etc.), and output activities (writing or deleting symbols). They also

[16] The ‘subtractive notation,’ according to which 4 is represented as IV and not as IIII in Roman numerals, was introduced only in the Middle Ages and is not discussed here.

devised paper-and-pencil algorithms for addition and multiplication with Roman numerals, and despite common prejudices, these algorithms were found not to be more difficult for humans to execute than those for the Indo-Arabic numerals (see Schlimm and Neth, 2008, for a description of the algorithms).

The analysis by Schlimm and Neth (2008) of the elementary information processes employed in the computations revealed that addition with Indo-Arabic numerals requires knowledge of many basic addition facts (like '$2 + 3 = 5$'), but that only few simplification rules (like 'IIIII $\rightarrow$ V') need to be mastered for additions with Roman numerals. The fact that a Roman numeral is on average longer than the Indo-Arabic numeral of the same value, has the effect that many more individual steps (perceptions, attention shifts, write operations) have to be carried out, but that these put little strain on working memory. The authors also noticed that the Indo-Arabic algorithms are highly optimized in order to reduce external computations and thus employ more internal ones, whereas the Roman algorithm they devised made heavy use of external representations. Thus, not only the different numeral systems themselves, but also the different computational strategies that they require, have considerable effects on the ease and speed with which both systems are used.

As mentioned above, one of the main disadvantages of the Roman numeral system in comparison with the Indo-Arabic one is the on average longer length of its numerals, which results in lengthier computations. However, it is important to keep in mind that this only holds on average (assuming that uniform ranges of natural numbers are used), since, for particular numbers, the Roman numerals can be shorter than the corresponding ones in the Indo-Arabic system. As an example, compare M with 1000. One could make a case that it is exactly these kinds of (round) numbers that are used most frequently in practice.

Computing with artifacts. We have argued above that notions of problem difficulty or computational equivalence between two problems require a reference to the machinery or mechanism involved in solving the problems (see Section 4.1, page 83). Our emphasis on the actual computational process also revealed the crucial role of external resources. For instance, the paper-and-pencil algorithms for arithmetic computations with Roman numerals discussed in Schlimm and Neth (2008) could also be carried out on an abacus, whereby the computations would be simplified considerably. This observation naturally leads to the consideration of the use of artifacts for computations. Examples of such artifacts are the digits on one's hands or toes, sand tablets, paper and pencil, the abacus, but also cash registers, pocket calculators, and modern computers. The interactions between such devices and arithmetical practice are manifold: On the one hand, each device operates with a particular representation of numbers, so that this representation affects the mechanical computations. For instance, one of the biggest challenges when constructing the first 'analytical engine' was to devise an automatic mechanism to carry the tens, which is needed even for simple additions such as $9 + 2 = 11$:

> "The most important part of the Analytical Engine was undoubtedly the mechanical method of carrying the tens. On this I laboured incessantly, each succeeding improvement advancing me a step or two. [...] At last I came to the conclusion that [...] nothing but teaching the Engine to foresee and then to act upon that foresight could ever lead me to the object I desired..." (Babbage, 1864, p. 114, Ch. VIII)

Babbage's conundrum has nothing to do with the mathematical features of addition, and everything to do with the arbitrary properties of the base-ten place-value notation system. Not only do the properties of notations constrain the design of calculators, but the availability of calculators can also influence our arithmetic abilities. For instance, the negative effects of the widespread availability and use of pocket calculators on students' mathematical abilities have been widely discussed.[17] While we cannot go into further detail about these interactions, they illustrate the subtle interplay between cultural (technological) and cognitive (mental) operations.

All the analyses we have described so far concern the ease with which external numeric representations are processed during computation. But arithmetic is often done 'in the head' suggesting that constraints imposed by external (perceptual or motor) processes should not apply. However, we shall argue that notation, in particular the Indo-Arabic decimal system, exerts some subtle effects on arithmetic performance, even when the tasks are performed primarily mentally.

4.3 Notation and mental arithmetic

If a particular notation constrains the design of mechanical devices, what are the effects of adopting the Indo-Arabic decimal system on the 'machinery' of mental arithmetic? The very notion of 'mental' arithmetic might suggest that any effects are limited to the translation between input and output formats, as in the simple three-stage view of problem solving attributed to Craik (1943). He claims that, after some initial translation process, operations are carried out in some medium of thought (mentalese) before the final answer is returned. A more embodied and embedded view of cognition (as promoted in different ways by Clark, 1997; Wilson, 2002; Neth et al., 2007) would argue that this simple model needs to be elaborated with a

[17] See (Dehaene, 1997, pp. 134–136) and (Butterworth, 1999, p. 350), but more systematic studies are needed.

more interactive view of arithmetic that might allow the environment, including external representations, a more potent role in shaping mental representations and processes. However, most psychological studies of mental arithmetic have ignored notational effects and concentrate only on the representation of number and of number processing independently of specific numeral systems.

A notable exception is the existing research on *representational effects* in the psychological literature on mental arithmetic. One body of research on that topic analyzes the influence of particular representational systems on the ease with which mental operations can be carried out. For example, in comparing the Indo-Arabic number system with Roman numerals Nickerson (1988) and Zhang and Norman (1993, 1995) point out that each system selectively facilitates different subprocesses. Zhang and Norman analyze such differences in terms of which computational constraints are enforced by the notation, and which need to be maintained mentally. The thrust of this work is more theoretical than empirical. Some empirical work on the effects of different number systems has shown that notational effects can be measured on internal operations as well as on interactive read-write processes.

Gonzalez and Kolers (1982) showed that the reaction times of very simple addition tasks (with sums below 10) were influenced by notation. They used a verification/rejection task in which participants were presented with equations like IV+2 =VI, displayed in a variety of mixtures of Roman and Indo-Arabic numerals, and showed that the notation used affected the slope of problem-size effects. On this basis, they suggested that different mental operations were applied to Roman and Indo-Arabic numerals, so that the notation was exerting an effect even on internal transformation processes (Gonzalez and Kolers, 1987). Other examples of research directed at representational effects in arithmetic include investigations of linguistic number name effects (see, e.g., Miller, 1992), based on discrepancies in the regularities of number names in different languages.

More recently, Campbell and Fugelsang (2001) presented simple addition problems in a verification task in either Indo-Arabic digit or English number word format and monitored participants' adding strategies. As participants were less likely to retrieve results, but rather resorted to calculation when facing number word problems, and this difference increased with problem size, they concluded that presentation format does have an impact on central aspects of cognitive arithmetic. In a similar vein Nuerk et al. (2001) suggested on the basis of a number comparison paradigm that tens and units might not be represented on a single continuous mental number line.

The work of LeFevre et al. (1996) links two pervasive issues in experimental studies of arithmetic skills. First, even relatively simple arithmetic tasks involve a choice between alternative strategies. For instance, the problem $3+6$ can be solved by retrieving the answer 9 from memory or by using an incremental counting procedure to generate the answer. If some stepwise algorithm is used one might start with the number 3 and count up 6 units from it, or, more efficiently, one might reverse the left-right order of addends to start with the larger number 6 and count up 3 from that (see Siegler and Shrager, 1984; Siegler, 1987; Siegler and Lemaire, 1997; Shrager and Siegler, 1998; Siegler and Stern, 1997, for developmental studies of strategy selection).

The phenomenon that problems involving larger numbers (e.g., $4+5$) are generally solved more slowly than those with smaller numbers ($4+3$) is known as the *problem size effect* (Parkman and Groen, 1971; Groen and Parkman, 1972; Campbell, 1995; Zbrodoff, 1995; Geary, 1996). While explanations for the problem size effect remain controversial, LeFevre et al. (1996) have suggested that it may be related to the issue of strategy choice. If different participants resorted, at different points, to counting strategies (which show a linear relationship between addend size and count duration) then a problem size effect could be explained as a methodological consequence of averaging over an entire group of participants. (See also Siegler, 1987, for a similar point.)

Interestingly, a related analysis of addition tasks is sensitive to the properties of the Indo-Arabic decimal number system. LeFevre et al. (1996) report that when the sum of a pair of digits exceeds ten, an adder is more likely to use a counting rather than a fact retrieval strategy. According to this account, sums greater than ten (e.g., $6+7$) are sometimes *decomposed* into two stages: up to the decade ($6+4$), and beyond ($10+3$). Moreover, Geary (1996) reports evidence that even adults frequently use decomposition as a back-up strategy, particularly on larger-valued addition problems (with sums exceeding ten). If this strategy is indeed widespread, then it makes an interesting prediction, namely, that those additions that sum to ten are likely to be the most practiced of all additions. In this case, one would predict that sums to ten, or more generally, sums reaching decade boundaries will be even easier than smaller sums. Neth (2004) investigated this hypothesis by letting participants add up sequences of random single-digit numbers and measuring the time for each individual addition (see also Neth and Payne, 2001). The results show clear *decade effects* in mental addition. So-called complements (two addends adding up to a round sum, e.g., $16+4$) are added faster than sub-complements (e.g., $16+3$), which, in turn, are faster than super-complements (e.g., $16+5$). As the first of these results in particular cannot be explained by problem size effects, the duration of mental operations is influenced by arbitrary properties of a numeric notation. In line with this argument is the observation that post-complements (e.g., $20 + 6$) are computed faster than any other type of addition, presumably because such operations do not require any actual addition, but can be achieved by mere replacement of the unit digit 0 by the current addend 6. Thus, the neural gears of our minds seem to be affected by properties of our decimal notation just like Babbage's analytical engine required some special ingenuity to carry the tens.

In this section we highlighted various trade-offs between the richness of a representational system (in terms of its basic set of symbols), its demands on human cognition (e.g., the need to memorize symbol meanings and rules for symbol manipulation), and the resulting potential for algorithmic computations. This potential not only depends on the mental machinery of the human mind but is modulated by the availability of external tools (like paper and pencil, an abacus, or an electronic calculator). More generally, the difficulty of any arithmetic problem crucially depends on the relations among the specific task to be solved, the representational system used to grasp and frame it, and the internal and external capacities and resources that are available and required to solve it.

5 Conclusion

As was mentioned briefly in Sections 1 and 2.1, philosophers of mathematics in general, and analytic philosophers in particular, have shown great reservations toward taking seriously the work of psychologists on mathematical reasoning. This is possibly due to the influence of Frege's arguments against psychological accounts of mathematical objects, which he deemed either unsatisfactory or subjective. Since then anti-psychologistic tendencies have been popular in philosophy of mathematics, so that philosophers have shunned the idea that any psychological insights might be relevant to their enterprise. The distinction made between the contexts of discovery and justification (Reichenbach, 1938), together with the view that philosophy has nothing to say about the former, has further ingrained this attitude (see Schlimm, 2006, for a general discussion of these developments). Objections to these sentiments were raised mainly by mathematicians, who tried to get a better understanding of their practices, and who were fully aware that their activities had strong psychological components.[18]

Frege's emphasis on mathematical objects as logical entities led to a static view of mathematics and a focus on the ontological nature of mathematical objects, which dominated philosophy of mathematics for a long time. With the turn toward history and mathematical practice that was pushed by the work of Lakatos (1976), philosophy of mathematics began to show more interest in the cognitive foundations of mathematical reasoning. Nevertheless, the relationship between mathematics and cognition is still

[18] See, e.g., (Klein, 1926, p. 152) and (Hadamard, 1945).

tenuous, as can be seen by the reserved interest of philosophers in the first attempt to relate cognitive psychology to mathematical practice (Lakoff and Núñez, 2000).

One point that we have stressed in this presentation is that notation is relevant for understanding how we deal with numbers and that even mental arithmetic is best understood as an interplay between internal and external representations. This suggests that mathematical notation also plays an important role in our understanding of higher mathematics and mathematical practice in general. Indeed, if notation is an irreducible part of our mathematical cognition, this might have important consequences for philosophy of mathematics, in particular for our epistemic access to mathematical objects.

While philosophers can learn from psychologists about the cognitive underpinnings of mathematics, psychologists can also learn from philosophers, in particular when it comes to conceptual clarification. The basic terminological distinctions presented in Sections 1 and 4 are not always adhered to in psychology, where we frequently find the terms 'number,' 'number concept,' and sometimes even 'numeral' used interchangeably. We have also seen that what cognitive scientists mean by 'arithmetic' (simple computations with mostly natural numbers) is not necessarily what philosophers or mathematicians take it to be (e.g., the study of properties of prime numbers). And indeed, mathematics extends far beyond differentiating between heaps of discrete objects, and counting up to small numbers. Therefore, the question arises whether empirical results about the ways in which humans or lower animals deal with small numerosities tell us anything at all about high-level mathematics. Related to this are issues regarding the nature of infinity. Many mathematical objects—like the set of natural numbers, the continuum, the functions in analysis, and the lines in geometry—are essentially infinite. How human beings are able to reason about such structures is still largely a mystery from a cognitive perspective. This opens new opportunities for cognitive science, which hitherto has mainly dealt with elementary numerical cognition, to investigate the cognitive underpinnings of more complex mathematical thought.

At the beginning of this paper, we presented our aim as illuminating the relationship between mathematics, mind, and brain from various perspectives (recall Figure 1d on page 60). We now realize that even our interdisciplinary journey could not do full justice to all the nuances of our topic. Any comprehensive treatment of the cognitive basis of arithmetic will have to include specifications of the world in which mathematical problem solving is situated. Such references to the context must not only include descriptions of the physical environment (e.g., a characterization of the precise task to be solved, the shape of our bodies, and the availability of artifacts),

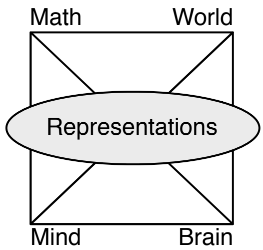

FIGURE 4. Representations shape and are shaped by the relations between mathematics, world, mind, and brain.

but also aspects of the social, linguistic, and historical context in which particular problems occurred. As we have seen, representations play a key role in the explanation of the various relationships between our four main reference points of mathematics, mind, brain, and world (Figure 4). Due to their mediating role representations not only shape our interactions with mathematical problems and constructs (e.g., by determining the difficulty of mathematical tasks), but they are themselves adapted and shaped by the need to solve specific mathematical problems in particular environments. The theory of arithmetic and the development of numerical notations as tools to solve mathematical tasks cannot meaningfully be studied in isolation from the physical, social, historical, psychological, and biological context in which theory and tools were conceived and applied. We hope that future work in this direction will further illuminate the cognitive basis of arithmetic and of mathematics as a whole.

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