THE COHOMOLOGY OF A COXETER GROUP WITH GROUP RING COEFFICIENTS

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Introduction.

Let (W, S) be a Coxeter system with S finite (that is, W is a Coxeter group and S is a distinguished set of involutions which generate W, as in [B, p. 11.]). Associated to (W, S) there is a certain contractible simplicial complex Σ, defined below, on which W acts properly and cocompactly. In this paper we compute the cohomology with compact supports of Σ (that is, we compute the "cohomology at infinity" of Σ). As consequences, given a torsion-free subgroup Γ of finite index in W, we get a formula for the cohomology of Γ with group ring coefficients, as well as, a simple necessary and sufficient condition for Γ to be a Poincaré duality group.

Given a subset T of S denote by W_T the subgroup generated by T. (If T is the empty set, then W_T is the trivial subgroup.) Denote by $\mathcal{S}^f$ the set of those subsets T of S such that W_T is finite; $\mathcal{S}^f$ is partially ordered by inclusion. Let $W\mathcal{S}^f$ denote the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. $W\mathcal{S}^f$ is also partially ordered by inclusion.

The simplicial complex Σ is defined to be the geometric realization of the poset $W\mathcal{S}^f$. The geometric realization of the poset $\mathcal{S}^f$ will be denoted by K.

For each s in S, let $\mathcal{S}^f_{\geq\{s\}}$ be the subposet consisting of those $T \in \mathcal{S}^f$ such that $s \in T$ and let K_s be its geometric realization. So, K_s is a subcomplex of K. (K is called a *chamber* of Σ and K_s is a *mirror* of K.) For any nonempty subset T of S, set

$$K^T = \bigcup_{s \in T} K_s.$$

K is a contractible finite complex; it is homeomorphic to the cone on K^S.

For each $w \in W$, set

$$\begin{aligned} S(w) &= \{s \in S | \ell(ws) < \ell(w)\} \\ T(w) &= S - S(w), \end{aligned}$$

where $\ell(w)$ is the minimum length of word in S which represents w. Thus, $S(w)$ is the set of elements of S in which a word of minimum length for w can end.

For any locally finite simplicial complex Y, let $C^*_c(Y)$ denote the cochain complex of compactly supported simplicial cochains on Y. Its dual chain complex, $C^{lf}_*(Y)$, of *locally finite chains*, is the chain complex of, possibly infinite, linear combinations of oriented simplices in Y.

We will use $\prod$ to denote a direct product and $\oplus$ for a direct sum.

Theorem A. $H^*_c(\Sigma) \cong \underset{w \in W}{\oplus} H^*(K, K^{T(w)})$.

The corresponding result for homology is

$$H^{lf}_*(\Sigma) \cong \prod_{w \in W} H_*(K, K^{T(w)}).$$

Corollary. *For any subgroup Γ of finite index in W,*

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \bigoplus_{w \in W} H^*(K, K^{T(w)}).$$

Using this result one can determine when W is a virtual Poincaré duality group (that is, when the classifying space $B\Gamma$ satisfies Poincaré duality for any torsion-free subgroup Γ of finite index in W).

The poset $\mathcal{S}^f_{>\emptyset}$, consisting of those T in $\mathcal{S}^f$ other than the empty set, is isomorphic to the poset of simplices in a simplicial complex. We denote this simplicial complex by L (or $L(W,S)$). Thus, the vertex set of L is S and a subset of T of S spans a simplex if and only if $T \in \mathcal{S}^f_{>\emptyset}$.

The space Σ can be cellulated so that the link of each vertex is L (e.g., see [D3], [M] or Section 6 of [CD]). Thus, Σ is a polyhedral homology n-manifold if and only if L is a "generalized homology $(n-1)$-sphere" in the sense that it is a homology $(n-1)$-manifold with the same homology as S^{n-1}. Moreover, it is proved in [D1] that if L is a generalized homology $(n-1)$-sphere, then the singularities of Σ can be resolved to get a contractible manifold X with a proper cocompact W-action. Hence, if this condition holds, then, for any torsion-free subgroup Γ of finite index in W, $B\Gamma$ is homotopy equivalent to the closed manifold X/Γ and consequently W is a virtual Poincaré duality group.

The next result states that this is essentially the only way in which W can be a virtual Poincaré duality group.

Theorem B. *Suppose W is a virtual Poincaré duality group of dimension n. Then W decomposes as a direct product $W = W_{T_0} \times W_{T_1}$ where W_{T_1} is finite and where $L(W_{T_0}, T_0)$ is a generalized homology $(n-1)$-sphere.*

Theorem A is proved in §4 and Theorem B in §5. In §6 we show how to generalize Theorems A and B to groups constructed by the "reflection group trick" (in Theorems 6.5 and 6.10, respectively).

In [BB], Bestvina and Brady construct the first known examples of groups which are type FP but not finitely presented. In Example 6.7, we use the reflection group trick to show how the Bestvina-Brady examples can be promoted to examples of Poincaré duality groups. This gives the following result.

Theorem C. *In each dimension ≥ 4, there are examples of Poincaré duality groups which are not finitely presented.*

The classifying space of such a Poincaré duality group cannot be homotopy equivalent to a closed manifold (since the fundamental group of a closed manifold is finitely presented).

There is a geometric context in which to view these results. G. Moussong proved in [M] that a natural polyhedral metric on Σ is $CAT(0)$. (This means that, in addition to being contractible, Σ is nonpositively curved.) It follows, as in [Dr], that

Σ can be compactified by adding its ideal boundary $\Sigma(\infty)$, the space of asymptoty classes of geodesic rays. Moreover, as explained in [Be2], $\Sigma(\infty)$ is a Z-set in $\Sigma \cup \Sigma(\infty)$. It follows that $H_c^*(\Sigma) \cong \check{\mathrm{H}}^{*-1}\big(\Sigma(\infty)\big)$ and $H_*^{lf}(\Sigma) \cong H_{*-1}^{st}\big(\Sigma(\infty)\big)$ where $\check{\mathrm{H}}^*$ and H_*^{st} denote, respectively, Čech cohomology and Steenrod homology (as explained in [F]). Thus, W is a virtual Poincaré duality group of dimension n if and only if $\Sigma(\infty)$ has the homology of an $(n-1)$-sphere. "Nonresolvable" ANR homology manifolds which are homotopy equivalent to S^{n-1} are constructed in [BFMW]. A. D. Dranishnikov has pointed out that Theorem B implies $\sum(\infty)$ can never be such a space, i.e., nonresolvable homotopy spheres do not occur as boundaries of Coxeter groups.

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§1. **Preliminaries on Coxeter groups.**

The basic reference for this material is Chapter IV of [B].

(W, S) is a Coxeter system and $S(w), T(w)$ and $\mathcal{S}^f$ are as in the Introduction. For each subset T of S define the following subsets of W:

$$
\begin{aligned}
A_T &= \{w \in W | \ell(wt) > \ell(w), \text{ for all } t \in T\},\\
B_T &= \{w \in W | \ell(tw) > \ell(w), \text{ for all } t \in T\},\\
W^T &= \{w \in W | S(w) = T\}.
\end{aligned}
$$

An element w is in A_T (resp. B_T) if and only if any reduced word which represents it doesn't end (resp. doesn't begin) with an element of T.

In the first lemma we collect a few tautologies.

Lemma 1.1. *Let T be a subset of S and $w \in W$.*

(i) $W_T \subset B_{S-T}$

(ii) $W^T \subset A_{S-T}$. In particular, $w \in A_{T(w)}$.

(iii) $S(w) = \emptyset$ if and only if $w = 1$.

Lemma 1.2. (Lemma 7.12 in [D1]). *For each $w \in W$, $S(w) \in \mathcal{S}^f$ (i.e., $W_{S(w)}$ is a finite group).*

Lemma 1.3. (Exercise 3, p. 37 in [B]). *Given an element w in W and a subset T of S, there is a unique element $p'_T(w)$ (resp. $p_T(w)$) of shortest length in the coset wW_T (resp. in $W_T w$). Moreover, the following statements are equivalent:*

(i) $p'_T(w) = w$ (resp. $p_T(w) = w$),

(ii) $w \in A_T$ (resp. $w \in B_T$),

(iii) For each $u \in W_T$, $\ell(wu) = \ell(w) + \ell(u)$ (resp. $\ell(uw) = \ell(u) + \ell(w)$).

Lemma 1.4. (Exercise 22, p. 43 in [B]). *Suppose $T \in \mathcal{S}^f$. Then there is a unique element w_T in W_T of longest length. Moreover, the following statements are true.*

(i) w_T is an involution.

(ii) For any $x \in W_T$, $\quad x = w_T$ if and only if $S(x) = T$.

(iii) For any $x \in W_T$, $\quad \ell(w_T x) = \ell(w_T) - \ell(x)$.

Lemma 1.5. (Lemma 2 (iv) in [D2]). *Suppose $T \in \mathcal{S}^f$. For any $v \in W^T$ and $x \in W_T$, $\ell(vx) = \ell(v) - \ell(x)$.*

Lemma 1.6. *Suppose $T \in \mathcal{S}^f$ and $w \in W$. Then there is a unique element in wW_T of longest length (namely, the element $p'_T(w)w_T$). Moreover, the following statements are equivalent:*

(i) w is the element of longest length in wW_T,

(ii) $w = uw_T$, for some $u \in A_T$,

(iii) $T \subset S(w)$.

Proof. Let $u = p'_T(w)$ be the element of shortest length in wW_T. Then $u \in A_T$ (Lemma 1.3). The other elements in this coset have the form ux, with $x \in W_T$. By Lemma 1.3, $\ell(ux) = \ell(u) + \ell(x)$; hence, uw_T, where w_T is the element of longest length in W_T, is the unique element of longest length in wW_T. Thus, (i) is equivalent to (ii).

Suppose (ii) holds. Since for each $t \in T$, $\ell(uw_T t) = \ell(u) + \ell(w_T t) = \ell(u) + \ell(w_T) - 1 = \ell(uw_T) - 1$, we have that $T \subset S(w)$. So $(ii) \Rightarrow (iii)$. Conversely, suppose (iii) holds. Set $u = ww_T$. Since $w_T \in W_T \subset W_{S(w)}$, Lemma 1.5 implies that $\ell(u) = \ell(w) - \ell(w_T)$ and hence, u is the element of shortest length in wW_T. So, $(iii) \Rightarrow (ii)$. □

Lemma 1.7. (Théorème 2, p. 20 in [B]). *For any subsets T and T' of S, $W_T \cap W_{T'} = W_{T \cap T'}$.*

Lemma 1.8. (p. 18 in [B]). *Suppose $s, s' \in S$ and $w \in W$ are such that $w \in B_{\{s\}}$ and $ws' \notin B_{\{s\}}$. Then $sw = ws'$.*

For each pair of elements s, t in S, let $m(s,t)$ denote the order of st in W.

Lemma 1.9. *Suppose $T \in \mathcal{S}^f$ and $s \in S - T$. Then $sw_T = w_T s$ if and only if $m(s,t) = 2$ for all $t \in T$.*

Proof. If $m(s,t) = 2$, then s and t commute. Hence, if $m(s,t) = 2$ for all $t = T$, then s and w_T commute.

Conversely, suppose s and w_T commute, where $s \notin T$. Then $\ell(w_T s) = \ell(w_T) + 1$, so $s \in S(w_T s)$. Since $w_T s = sw_T$, $T \subset S(w_T s)$. Therefore, $S(w_T s) = T \cup \{s\}$. We want to show that $m(s,t) = 2$ for all $t \in T$. Suppose, to the contrary, that $m(s,t) > 2$, for some $t \in T$. Since $\{s,t\} \subset S(w_T s)$ and since $S(w_T s)$ generates a finite subgroup (Lemma 1.2), $m(s,t) \neq \infty$. Consider the dihedral group $W_{\{s,t\}}$

generated by $\{s,t\}$. By Lemma 1.5, for any $u \in W_{\{s,t\}}$, $\ell(w_T su) = \ell(w_T s) - \ell(u)$. In particular, consider the element $u = sts$. Since $m(s,t) > 2$, $\ell(sts) = 3$. Therefore, $\ell\big((w_T s)(sts)\big) = \ell(w_T s) - 3$. On the other hand, $\ell\big((w_T s)(sts)\big) = \ell(w_T ts) = \ell(w_T t) + 1 = \ell(w_T) = \ell(w_T s) - 1$, a contradiction. $\square$

The following lemma plays a key role in the proof of Theorem B in § 5.

Lemma 1.10. *Suppose that for some $T \in \mathcal{S}^f$, W^T is a singleton. Then W decomposes as a direct product: $W = W_{S-T} \times W_T$.*

Proof. Let w_T be the element of longest length in W_T. By Lemma 1.4 (ii), $S(w_T) = T$, and hence, since W^T is a singleton, $W^T = \{w_T\}$. Let $s \in S - T$. Clearly, $T \subset S(sw_T) \subset \{s\} \cup T$, and since $sw_T \notin W^T$, we must have $S(sw_T) = \{s\} \cup T$. Thus, $\ell\big(s(w_T s)\big) = \ell\big((sw_T)s\big) = \ell(sw_T) - 1 = \ell\big((sw_T)^{-1}\big) - 1 = \ell(w_T s) - 1$. In other words, $w_T s \notin B_{\{s\}}$. Since $w_T \in B_{\{s\}}$, Lemma 1.8 implies that $sw_T = w_T s$ and then Lemma 1.9 implies that $m(s,t) = 2$ for all t in T. Since this holds for each $s \in S - T$, the Coxeter system is reducible (if T is nonempty) and W decomposes as $W = W_{S-T} \times W_T$ (by Proposition 8, p. 22 in [B]).

§2. Preliminaries on simplicial complexes.

Let L be a simplicial complex. Denote by $V(L)$ the vertex set of L and by $P(L)$ the set of simplices in L together with the empty set. $P(L)$ is partially ordered by inclusion. Throughout this section, we shall identify any simplex in L with its vertex set. Thus, given a subset T of $V(L)$, $T \in P(L)$ if and only if T spans a simplex in L or $T = \emptyset$.

Let P be any poset. Given $p \in P$, set $P_{\le p} = \{x \in P | x \le p\}$. The subposets $P_{\ge p}, P_{<p}$ and $P_{>p}$ are similarly defined. P is an *abstract simplicial complex* if it is isomorphic to $P(L)_{>\emptyset}$ for some simplicial complex L; L is called the *realization* of P.

For any $T \in P(L)$, the poset $P(L)_{>T}$ is an abstract simplicial complex; its realization is denoted $Lk(T,L)$ and called the *link* of T in L. (If $T = \emptyset$, then $Lk(T,L) = L$.) In fact, $Lk(T,L)$ can be identified with the subcomplex of L consisting of all simplices T' such that $T' \cap T = \emptyset$ and $T' \cup T$ spans a simplex in L (this simplex is called the *join* of T' and T).

The *derived complex* of a poset P, denoted by P', is the set of all finite chains in P, partially ordered by inclusion. It is an abstract simplicial complex. The *geometric realization* of P, denoted *geom*(P), is defined to be the realization of P'.

Given a simplicial complex L we define another simplicial complex K and a subcomplex ∂K by

$$K = geom\big(P(L)\big)$$
$$\partial K = geom\big(P(L)_{>\emptyset}\big)$$

Then ∂K is isomorphic to the barycentric subdivision of L and K is the cone on ∂K (the empty set provides the cone point). For any $T \in P(L)$, define subcomplexes K_T and ∂K_T of K by

$$K_T = geom\big(P(L)_{\geq T}\big)$$
$$\partial K_T = geom\big(P(L)_{>T}\big).$$

K_T is called the *dual face* to T; it is isomorphic to the cone on ∂K_T and ∂K_T is isomorphic to the barycentric subdivision of $Lk(T, L)$. If v is a vertex, write K_v instead of $K_{\{v\}}$. Thus, K_v is the closed star of v in the barycentric subdivision of L.

For any nonempty subset J of $V(L)$, set

$$K^J = \bigcup_{v \in J} K_v.$$

§3. **The simplicial complex Σ.**

(W, S) is a Coxeter system and $\mathcal{S}^f$ and $W\mathcal{S}^f$ are the posets defined in the Introduction. $\mathcal{S}^f_{>\emptyset}$ is an abstract simplicial complex; its realization is denoted L (so that $V(L) = S$ and $P(L) = \mathcal{S}^f$). Also,

$$K = geom(\mathcal{S}^f) \text{ and}$$
$$\Sigma = geom(W\mathcal{S}^f).$$

(So, K is the cone on the barycentric subdivision of L.) The group W acts on Σ via simplicial automorphisms.

The natural projection $W\mathcal{S}^f \to \mathcal{S}^f$ defined by $wW_T \to T$ induces a projection $\Sigma \to K$ which is constant on W-orbits and induces an identification $\Sigma/W \cong K$. The embedding $\mathcal{S}^f \to W\mathcal{S}^f$ defined by $T \to W_T$ induces an embedding $K \to \Sigma$ which we regard as an inclusion. A translate of K by an element w in W is denoted wK and called a *chamber* of Σ.

A simplex σ in K corresponds to chain $T_0 < T_1 \cdots < T_n$ in $\mathcal{S}^f$. Set $S(\sigma) = T_0$. (So, $K_{S(\sigma)}$ is the smallest dual face which contains σ.) The stabilizer of σ is $W_{S(\sigma)}$ and this group fixes $K_{S(\sigma)}$ pointwise.

For any subset X of W define a subcomplex $\Sigma(X)$ of Σ by

$$\Sigma(X) = \bigcup_{w \in X} wK.$$

We are particularly interested in the subcomplexes $\Sigma(B_T)$, where B_T is as defined in §1. We shall call such a subcomplex a *positive sector*. (It is "positive" because it contains the fundamental chamber K.) Similarly, for each $s \in S$, $\Sigma(B_{\{s\}})$ is a *positive half-space*. The involution s acts on Σ as a "reflection" interchanging the half-spaces $\Sigma(B_{\{s\}})$ and $\Sigma(sB_{\{s\}})$. For each subset T of S, $\Sigma(B_T)$ is the intersection of the positive half-spaces, $\Sigma(B_{\{s\}}), s \in T$.

The map $p_T : W \to B_T$, defined in Lemma 1.3, induces a projection $\pi_T : \Sigma \to \Sigma(B_T)$ which sends a simplex $w\sigma$ to $p_T(w)\sigma$. The map π_T is constant on W-orbits and induces an identification $\Sigma/W_T = \Sigma(B_T)$.

§4. The isomorphism ρ.

For any subset T of S, we have that $\Sigma(B_T - \{1\}) \cap K = K^{S-T}$. Hence, $C^*(\Sigma(B_T), \Sigma(B_T - \{1\}))$ can be identified with $C^*(K, K^{S-T})$. Let j_T denote the inclusion, $C^*(K, K^{S-T}) = C^*(\Sigma(B_T), \Sigma(B_T - \{1\})) \to C^*_c(\Sigma(B_T))$. If W_T is finite, then the projection map $\pi_T : \Sigma \to \Sigma(B_T)$ is finite-to-one; hence, it induces a cochain map $\pi^{\#}_T : C^*_c(\Sigma(B_T)) \to C^*_c(\Sigma)$. So, for each $T \in \mathcal{S}^f$, we have the cochain map $\rho_T = \pi^{\#}_T \circ j_T : C^*(K, K^{S-T}) \to C^*_c(\Sigma)$. For each $w \in W$ define $\rho_w : C^*(K, K^{T(w)}) \to C^*_c(\Sigma)$ by

$$\rho_w = w^{-1} \circ \rho_{S(w)},$$

where $w^{-1} : C^*_c(\Sigma) \to C^*_c(\Sigma)$ is the automorphism induced by translation by w^{-1}.

If $a \in C^k_c(\Sigma)$ and τ is an oriented k-simplex in Σ, then denote the value of a on τ by $\langle a, \tau\rangle$.

Lemma 4.1. *Suppose that $v, w \in W$, that $a \in C^k(K, K^{T(w)})$ and that σ is an oriented k-simplex in K. Then*

$$\langle \rho_w(a), v\sigma\rangle = \begin{cases} \langle a, \sigma\rangle & ; \textit{ if } v \in wW_{S(w)} \\ 0 & ; \textit{ otherwise} \end{cases}$$

Consequently,

(i) $\langle \rho_w(a), \sigma\rangle = \langle a, \sigma\rangle$, and

(ii) if $\ell(v) \geq \ell(w)$ and $v \neq w$, then $\langle \rho_w(a), v\sigma\rangle = 0$.

Proof. By definition,

$$\langle \rho_w(a), v\sigma\rangle = \begin{cases} \langle a, \sigma\rangle & ; \text{ if } p_{S(w)}(w^{-1}v) \in W_{S(\sigma)} \\ 0 & ; \text{ otherwise} \end{cases}$$

Suppose $\langle a, \sigma\rangle \neq 0$. Then $S(\sigma) \subset S(w)$. (If $S(\sigma)$ is not contained in $S(w)$, then σ is contained in $K^{T(w)}$ and consequently, $\langle a, \sigma\rangle = 0$.) Hence, $p_{S(w)}(w^{-1}v) \in W_{S(w)}$, i.e., $v \in wW_{S(w)}$. This proves the first formula in the lemma. Formula (i) follows immediately. By Lemma 1.6, w is the element of longest length in $wW_{S(w)}$. Hence, if $v \neq w$ and $\ell(v) \geq \ell(w)$, then $v \notin wW_{S(w)}$. Therefore, by the first formula, $\langle \rho_w(a), v\sigma\rangle = 0$. □

Define

$$\rho : \bigoplus_{w \in W} C^*(K, K^{T(w)}) \to C^*_c(\Sigma)$$

to be the sum of the ρ_w).

Theorem A is an immediate consequence of the following result, after taking cohomology.

Theorem 4.2. *ρ is an isomorphism.*

Proof. The proof is similar to the argument on page 101 of [D2]. Order the elements of $W : w_1, w_2, \cdots$, so that $\ell(w_i) \le \ell(w_{i+1})$. Set

$$\Sigma_n = \Sigma(\{w_i | i > n\}),$$

i.e., Σ_n is the union of chambers $w_i K, i > n$. First observe that if $i \le n$, then, by part (ii) of Lemma 4.1, the image of ρ_{w_i} is contained in $C^*(\Sigma, \Sigma_n)$. We shall prove, by induction on n, that

$$\bigoplus_{i=1}^{i=n} \rho_{w_i} : \bigoplus_{i=1}^{i=n} C^*(K, K^{T(w_i)}) \to C^*(\Sigma, \Sigma_n) \tag{1}$$

is an isomorphism.

Since $C_c^*(\Sigma)$ is the direct limit of the $C^*(\Sigma, \Sigma_n)$ as $n \to \infty$, the theorem follows. Since $\Sigma_0 = \Sigma$, statement (1) holds trivially for $n = 0$. So, suppose $n \ge 1$ and that (1) holds for $n - 1$. Consider the triple $(\Sigma, \Sigma_{n-1}, \Sigma_n)$. We have a short exact sequence,

$$0 \to C^*(\Sigma, \Sigma_{n-1}) \to C^*(\Sigma, \Sigma_n) \to C^*(\Sigma_{n-1}, \Sigma_n) \to 0. \tag{2}$$

To simplify notation, put $w = w_n$. For any $m > n, w_m\, K \cap wK \subset wK^{T(w)}$. Hence, $C^*(\Sigma_{n-1}, \Sigma_n)$ can be identified with $C^*(wK, wK^{T(w)})$. Translation by w^{-1} gives an isomorphism,

$$C^*(wK, wK^{T(w)}) \cong C^*(K, K^{T(w)}).$$

Let λ denote the composition of the natural projection $C^*(\Sigma, \Sigma_n) \to C^*(\Sigma_{n-1}, \Sigma_n)$ with this isomorphism. So, (2) can be rewritten as

$$0 \to C^*(\Sigma, \Sigma_{n-1}) \to C^*(\Sigma, \Sigma_n) \xrightarrow{\lambda} C^*(K, K^{T(w)}) \to 0.$$

By part (i) of Lemma 4.1, the map $\rho_w : C^*(K, K^{T(w)}) \to C^*(\Sigma, \Sigma_n)$ splits λ. Therefore,

$$C^*(\Sigma, \Sigma_{n-1}) \oplus C^*(K, K^{T(w)}) \xrightarrow{\cong} C^*(\Sigma, \Sigma_n)$$

where the first factor is mapped by the inclusion and the second by ρ_w. Applying the inductive hypothesis, we see that (1) holds. $\square$

Remark 4.3. Let $\mathbb{Z}(W^T)$ denote the free abelian group W^T. Then the formula in Theorem A can be rewritten as

$$H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}^f} \mathbb{Z}(W^T) \otimes H^*(K, K^{S-T}).$$

Corollary 4.4.

(i) $H^*(W; \mathbb{Z}W) \cong \bigoplus_{w \in W} H^*(K, K^{T(w)})$

(ii) If Γ is a torsion-free subgroup of finite index in W, then

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \bigoplus_{w \in W} H^*(K, K^{T(w)}).$$

Proof. Statement (ii) follows from Proposition 7.5, p. 209 in [Br] and (i) follows from Exercise 4 on the same page.

§5. **Generalized homology spheres.**

In the first part of this section we return to the general situation of §2 : L is any simplicial complex and $K = geom\big(P(L)\big)$. Notation is as in §2, in particular, if $T \in P(L)$, then T is the vertex set of a simplex in L or $T = \emptyset$.

Let R be a commutative ring with unit. Then L is a *generalized homology m-sphere over R* if for each $T \in P(L)$ (including $T = \emptyset$) we have that

$$H_*\big(Lk(T, L); R\big) \cong H_*(S^{m-Card(T)}; R).$$

Thus, a generalized homology m-sphere is a polyhedral homology m-manifold with the same homology as S^m.

We say that $(K, \partial K)$ is a *generalized homology n-disk over R* if for each $T \in P(L)$,

$$H_*(K_T, \partial K_T; R) \cong H_*(D^{n-k}, S^{n-k-1}; R)$$

where $k = Card(T)$. (In particular, when $T = \emptyset$, this says that $(K, \partial K)$ has the same homology as (D^n, S^{n-1}).)

Since ∂K_T is the barycentric subdivision of $Lk(T, L)$, it is obvious that $(K, \partial K)$ is a generalized homology n-disk if and only if L is a generalized homology $(n-1)$-sphere.

Here is a third variation of this condition. Given an integer $n \geq 0$ and a commutative ring R, we say that $(K, \partial K)$ satisfies $hD(n; R)$ if the following condition holds:

$hD(n; R)$:

(a) for any $T \in P(L), T \neq \emptyset, H_i(K, K^{V(L)-T}; R) = 0,$ for all $i \geq 0$.

(b)
$$H_i(K, \partial K; R) = \begin{cases} 0 & , i \neq n \\ R & , i = n. \end{cases}$$

In other words, for each $T \in P(L)_{>\emptyset}$, $(\partial K, K^{V(L)-T})$ has the same homology as (S^{n-1}, D^{n-1}), while for $T = \emptyset$, ∂K has the same homology as S^{n-1}.

Lemma 5.1. *Let L be a simplicial complex and let $K = geom\big(P(L)\big)$. The following statements are equivalent:*

(i) L is a generalized homology $(n-1)$-sphere over R,

(ii) $(K, \partial K)$ is a generalized homology n-disk over R,

(iii) $(K, \partial K)$ satisfies $hD(n; R)$.

Proof. As was previously observed, (i) and (ii) are equivalent. We first show that $(iii) \Rightarrow (ii)$. So, suppose $(K, \partial K)$ satisfies $hD(n; R)$. We must show that $(K_T, \partial K_T)$ has the same homology as (D^{n-k}, S^{n-k-1}), where $k = Card(T)$. This holds for $T = \emptyset$ by part (b) of $hD(n; R)$.

First consider the case where $T = \{v\}$, v a vertex in L. Set $\overline{L} = Lk(v, L)$ and $\overline{K} = geom\left(P(\overline{L})\right)$. Then K_v is isomorphic to $\overline{K}$ and for any $T \in P(\overline{L})$, $(\overline{K}, \overline{K}^{V(\overline{L})-T})$ is isomorphic to $(K_v, K_v \cap K^{V(L)-(\{v\}\cup T)})$. By excision, $H_*(K_v, K_v \cap K^{V(L)-(\{v\}\cup T)} \cong H_*(K^{V(L)-T}, K^{V(L)-(\{v\}\cup T)})$. By $hD(n;R)$, $K^{V(L)-(\{v\}\cup T)}$is acyclic and for $T \neq \emptyset$, $K^{V(L)-T}$ is also acyclic. Thus, for $T \neq \emptyset$, $H_i(\overline{K}, \overline{K}^{V(\overline{L})-T}) = 0$ for all i. When $T = \emptyset$, we have $H_i(\overline{K}, \partial\overline{K}) = H_i(K^{V(L)}, K^{V(L)-\{v\}})$. Since $K^{V(L)}(= \partial K)$ has the same homology as S^{n-1} and $K^{V(L)-\{v\}}$ is acyclic, $(\overline{K}, \partial\overline{K})$ has the same homology as (D^{n-1}, S^{n-2}). Thus, $(\overline{K}, \partial\overline{K})$ satisfies $hD(n-1;R)$.

Next suppose that T is the vertex set of an arbitrary simplex in L and that $Card(T) = k$. We may suppose by induction that $(K_{T'}, \partial K_{T'})$ satisfies $hD(n-k';R)$ for all $T' \in P(L)$ with $k' = Card(T')$ and $k' < k$. Let T' be the vertex set of a codimension-one face of the simplex spanned by T so that $T = \{v\} \cup T'$ (i.e., T is the join of v and T'). There is a natural identification $Lk(T, L) = Lk\big(v, Lk(T', L)\big)$. Then $hD(n-k-1;R)$ holds for $(K_{T'}, \partial K_{T'})$ by inductive hypothesis and hence, for $Lk(T, L)$ by the argument in the preceding paragraph. In particular, $(K_T, \partial K_T)$ has the same homology as (D^{n-k}, S^{n-k-1}) and therefore, $(K, \partial K)$ is a generalized homology n-disk.

Finally, we need to see that $(i) \Rightarrow (iii)$. Suppose that L is a generalized homology $(n-1)$-sphere and that $T \in P(L)$. If $T = \emptyset$, then $K^{V(L)} \quad (= \partial K)$ has the same homology as S^{n-1}. If $T \neq \emptyset$, then K^T is a regular neighborhood of the simplex T in the barycentric subdivision of L; hence, K^T is contractible. Since L is a polyhedral homology $(n-1)$-manifold, K^T is a homology $(n-1)$-manifold with boundary, its boundary being $K^T \cap K^{V(L)-T}$. By Poincaré duality, $(K^T, K^T \cap K^{V(L)-T})$ has the same homology as (D^{n-1}, S^{n-2}). Hence, $H_*(K^{V(L)}, K^{V(L)-T}) \cong H_*(K^T, K^T \cap K^{V(L)-T}) \cong H_*(D^{n-1}, S^{n-2})$ which implies condition $hD(n;R)$. $\square$

Definition 5.2. Let R be a commutative ring and Γ a torsion-free group. Then Γ is of *type FP over R* if R (regarded as a trivial $R\Gamma$-module) admits a finitely generated projective resolution of finite length. Γ is of *type FL over R* if R admits a finitely generated free resolution of finite length. The group Γ is an *n-dimensional Poincaré duality group over R* if it is of type FP over R and if

$$H^i(\Gamma; R\Gamma) = \begin{cases} 0 & ; \quad i \neq n \\ R & ; \quad i = n \end{cases}$$

where Γ acts on R via some homomorphism $w_1 : \Gamma \to \{\pm 1\}$. A group G is a *virtual Poincaré duality group over R* if it contains a torsion-free subgroup Γ of finite index such that Γ is a Poincaré duality group over R. (If we omit reference to R, then $R = \mathbb{Z}$.)

If Γ acts freely and cellularly on a CW-complex U, with U/Γ compact, and if U is acyclic over R, then Γ is of type FL over R. (The cellular chain complex, $C_*(U)$, provides the free resolution.) In particular, if $B\Gamma$ is homotopy equivalent to a finite complex, then Γ is of type FL (since the universal cover of $B\Gamma$ is contractible). Conversely, if Γ is finitely presented and of type FL, then $B\Gamma$is homotopy equivalent to a finite complex.

It can be shown that Γ is a Poincaré duality group if and only if $B\Gamma$ satisfies

Poincaré duality with respect to any local coefficient system (see Theorem 10.1, p. 222 in [Br]). So, if $B\Gamma$ has the homotopy type of a closed manifold (or even a homology manifold), then Γ is a Poincaré duality group. Similarly, if $B\Gamma$ has the homotopy type of a closed homology manifold over R, then Γ is a Poincaré duality group over R.

Remark 5.3. In [Fa] Farrell proved that, for R a field, if $H^i(\Gamma; R\Gamma) = 0$ for all $i < n$ and if $H^n(\Gamma; R\Gamma)$ is nonzero and finite dimensional over R, then Γ is a Poincaré duality group over R.

Example 5.4. In Lemma 11.3 of [D1] it is proved that given any finite simplicial complex X, there is a Coxeter system (W,S) with $L\big(=L(W,S)\big)$ equal to the barycentric subdivision of X. For example, we can find (W,S) with L a lens space $S^{2k-1}/\mathbb{Z}_m$ or the suspension of such a lens space. Such an L will be a generalized homology sphere over $\mathbb{Z}[\frac{1}{m}]$. Since Σ has a cell structure in which the link of each vertex is isomorphic to L, Σ is a homology manifold over $\mathbb{Z}[\frac{1}{m}]$ and consequently, W is a virtual Poincaré duality group over $\mathbb{Z}[\frac{1}{m}]$. On the other hand, for $m \neq 1$, $H^*(K, \partial K; \mathbb{Z}) \cong H^{*-1}(L; \mathbb{Z})$ has nontrivial m-torsion and hence, by Theorem A, so does $H^*_c(\Sigma; \mathbb{Z})$. Thus, W is not a virtual Poincaré duality group over $\mathbb{Z}$.

The following result is a more precise version of Theorem B in the Introduction.

Theorem 5.5. *A Coxeter group W is a virtual Poincaré duality group of dimension n over a principal ideal domain R if and only if W decomposes as a direct product $W = W_{T_0} \times W_{T_1}$ with $T_1 \in \mathcal{S}^f$, so that the simplicial complex L_0 associated to (W_{T_0}, T_0) is a generalized homology $(n-1)$-sphere over R.*

Proof. First suppose that W decomposes as a direct product as in the theorem and that L_0 is a generalized homology $(n-1)$-sphere over R. Then $\Sigma_0 \quad \big(=\Sigma(W_{T_0}, T_0)\big)$ is a homology n-manifold over R. Hence, W_{T_0} is a virtual Poincaré duality group over R and, since W_{T_1} is finite, so is W.

Conversely, suppose that W is a virtual Poincaré duality group over R of dimension n. Then

$$H^i_c(\Sigma; R) = \begin{cases} 0 & , \quad i \neq n \\ R & , \quad i = n \end{cases}$$

By Theorem 4.2,

$$H^*_c(\Sigma; R) \cong \bigoplus_{w \in W} H^*(K, K^{T(w)}; R).$$

Since R is a principal ideal domain only one summand on the right hand side can be nonzero. Therefore, there is a $T_1 \in \mathcal{S}^f$ such that

(a) if $T \in \mathcal{S}^f$ and $T \neq T_1$, then $H^i(K, K^{S-T}; R) = 0$ for all i,

(b) $H^i(K, K^{S-T_1}; R) = \begin{cases} 0 & , \quad i \neq n \\ R & , \quad i = n \end{cases}$

(c) W^{T_1} is a singleton.

According to Lemma 1.10, (c) implies that W decomposes as a direct product $W = W_{T_0} \times W_{T_1}$. It follows that

$$\begin{aligned} K &= K_0 \times K_1 \\ \Sigma &= \Sigma_0 \times \Sigma_1 \\ L &= L_0 * L_1 (\text{ the join of } L_0 \text{ and } L_1) \end{aligned}$$

where K_i, Σ_i, L_i are the complexes associated to (W_{T_i}, T_i). Moreover, L_1 is a simplex, while K_1 and Σ_1 are both cells. Thus,

(a)′ if $T \in \mathcal{S}^f(W_{T_0}, T_0)$ and $T \neq \emptyset$, then $H^i(K_0, K_0^{T_0 - T}; R) = 0$ for all i, and

(b)′ $H^i(K_0, \partial K_0; R) = \begin{cases} 0 & ; \quad i \neq n \\ R & ; \quad i = n \end{cases}$.

That is to say, K_0 satisfies $hD(n; R)$. By Lemma 5.1, L_0 is a generalized homology $(n-1)$-sphere. □

Suppose $W = W_{T_0} \times W_{T_1}$ is a virtual Poincaré duality group of dimension n over $\mathbb{Z}/2$. Then L_0 is a generalized homology $(n-1)$-sphere over $\mathbb{Z}/2$ and the fixed point set of each $s \in T_0$ on $\Sigma(W_{T_0}, T_0)$ is a contractible $\mathbb{Z}/2$-homology $(n-1)$- manifold. Since $C_W(s)$, the centralizer of s in W, acts properly and cocompactly on this fixed set, it follows that $C_W(s)$ is a virtual Poincaré duality group of dimension $(n-1)$ over $\mathbb{Z}/2$. (This observation is due to S. Prassidis.) If $s \in T_1$, then $C_W(s)$ is of finite index in W and hence, is a virtual Poincaré duality group of dimension n over $\mathbb{Z}/2$. These observations lead to the following corollary of Theorem 5.5.

Corollary 5.6. *Suppose that a Coxeter group W acts effectively, properly and cocompactly on a $\mathbb{Z}/2$-acyclic n-manifold M. Then W acts as a group generated by reflections in the following sense. For each $s \in S$, let M_s denote the fixed point set of s on M. Then M_s is a $\mathbb{Z}/2$-acyclic, $\mathbb{Z}/2$-homology $(n-1)$-manifold which separates M and s interchanges the two components of $M - M_s$.*

Proof. By Smith theory, the fixed point set of any involution in W is $\mathbb{Z}/2$-acyclic and a $\mathbb{Z}/2$-homology manifold. Since M is a $\mathbb{Z}/2$-acyclic manifold and W acts properly, effectivley and cocompactly, W is a virtual Poincaré duality group of dimension n over $\mathbb{Z}/2$. Hence, $W = W_{T_0} \times W_{T_1}$, as in Theorem 5.5. For any $s \in S$, since $C_W(s)$ acts properly and cocompactly on M_s and since M_s is $\mathbb{Z}/2$-acyclic, we see that the dimension of M_s must equal the virtual cohomological dimension of $C_W(s)$ over $\mathbb{Z}/2$. If $s \in T_1$, this virtual cohomological dimension is n; hence, $M_s = M$. Since the action is supposed to be effective, this can only happen if $T_1 = \emptyset$. So, $W = W_{T_0}$. If $s \in T_0$, then $\dim M_s = n - 1$. Then, by Alexander duality, $M - M_s$ has two components and these must be interchanged by s. □

Corollary 5.7. *Suppose that M is a symmetric space of noncompact type and that a Coxeter group W is a discrete, cocompact subgroup of the group of isometries of M. Then M must be a product of a Euclidean space and (real) hyperbolic spaces.*

Proof. Since W acts on M by isometries, the fixed point set of each s in S must be totally geodesic submanifold of M. By the previous corollary, this submanifold must be of codimension one. But symmetric spaces of noncompact type do not contain totally geodesic submanifolds of codimenion one unless each irreducible factor is $\mathbb{R}^1$ or a real hyperbolic space. $\square$

Remark 5.8. An m-dimensional simplicial complex L is a *Cohen-Macaulay complex* if a) every simplex is contained in an m-simplex, b) the reduced homology $\overline{H}_i(L)$ vanishes for $i \neq m$, and c) for each simplex T in L, $\overline{H}_i\big(Lk(T,L)\big)$ vanishes for $i \neq m - Card(T)$. A torsion-free group Γ of type FP is a *duality group* of dimension n, if $H^i(\Gamma;\mathbb{Z}\Gamma)$ vanishes for $i \neq n$. It then follows that $H^i(\Gamma;M) \cong H_{n-i}(\Gamma;D\otimes M)$ for any Γ-module M, where $D = H^n(\Gamma;\mathbb{Z}\Gamma)$. (See Theorem 10.1, p. 220, in [Br].) It follows easily from Theorem A that if $L(W,S)$ is a Cohen-Macaulay complex of dimension $(n-1)$, then W is a virtual duality group of dimension n.

§6. **The reflection group trick.**

Suppose we are given the following data:

1) a CW-complex X, a group π and an epimorphism $\varphi : \pi_1(X) \to \pi$,

2) a Coxeter system (W,S) with associated simplicial complex L, and

3) a continuous map $f : L \to X$.

Replacing f by a cellular approximation and X by the mapping cylinder of f, we can assume that L is a subcomplex of X and that f is the inclusion.

One can construct a W-space Ω from these data in exactly the same manner as Σ is constructed from K. Thus, $\Omega = (W \times X)/\sim$ where the equivalence relation $\sim$ is defined as follows: for each $s \in S$, let X_s denote the closed star of s in the barycentric subdivision of L, for each $x \in X$ let $S(x)$ be the set of s such that x belongs to X_s, then $(w,x) \sim (w',x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.

Let $p : \tilde{X} \to X$ be the covering space associated to $\varphi : \pi_1(X) \to \pi$. Thus, π acts on $\widetilde{X}$ as the group of deck transformations. Let $\widetilde{L}$ denote the inverse image of L in $\widetilde{X}$. The vertex set of $\widetilde{L}$ (i.e., $p^{-1}(S)$) is denoted $\widetilde{S}$. We define a Coxeter matrix on $\widetilde{S}$ as follows. Suppose $\widetilde{s}, \widetilde{t}$ are elements of $\widetilde{S}$ lying over s and t in S, respectively. Define $m(\widetilde{s},\widetilde{t})$ to be $m(s,t)$ (the order of st in W) if $\widetilde{s} = \widetilde{t}$ or if $\widetilde{s}$ and $\widetilde{t}$ are connected by an edge in $\widetilde{L}$ and to be ∞ otherwise. Denote the resulting Coxeter system by $(\widetilde{W},\widetilde{S})$; its associated simplicial complex is clearly $\widetilde{L}$. The fundamental group π of X acts on $\widetilde{S}$ (via deck transformations) and hence on $\widetilde{W}$. The group G is defined to be the semidirect product: $G = \widetilde{W} \rtimes \pi$.

We can construct a space $\widetilde{\Omega}$ from $\widetilde{W}$ and $\widetilde{X}$ exactly as before. For each $t \in \widetilde{S}$, let $\widetilde{X}_t$ denote the closed star of t in the barycentric subdivision of $\widetilde{L}$. Then $\widetilde{\Omega} = (\widetilde{W} \times \widetilde{X})/\sim$, where the equivalence relation is defined as before. The group π acts freely on $\widetilde{\Omega}$ if $\alpha \in \pi$ and $[\widetilde{w},\widetilde{x}] \in \widetilde{\Omega}$, then $\alpha \cdot [\widetilde{w},\widetilde{x}] = [\theta_\alpha(\widetilde{w}), \alpha\widetilde{x}]$, where θ_α is the automorphism of $\widetilde{W}$ induced by α. The orbit space $\widetilde{\Omega}/\pi$ is identified with Ω via the natural surjection $\widetilde{W} \times \widetilde{X} \to W \times X$. (So, $\widetilde{\Omega}$ is the covering space of Ω associated to the epimorphism $\varphi \circ r_* : \pi_1(\Omega) \to \pi$ where $r : \Omega \to X$ denotes natural retraction.) Furthermore, the actions of $\widetilde{W}$ and π generate an action of the semidirect product G on $\widetilde{\Omega}$ and $\widetilde{\Omega}/G \cong X$. (So, G is the group of homeomorphisms of $\widetilde{\Omega}$ consisting of all lifts of the W-action on Ω to $\widetilde{\Omega}$.)

Remark 6.1. Actually one can carry out the above construction under slightly weaker assumptions: the space L need not be the simplicial complex associated to a Coxeter system. All one need assume is that a) $\widetilde{L}$ is the simplicial complex associated to a Coxeter system $(\widetilde{W}, \widetilde{S})$, b) $\widetilde{L}/\pi = L$, and c) the Coxeter matrix $m(\widetilde{s}, \widetilde{t})$ is π-equivariant.

For the remainder of the paper we shall assume the following.

Hypotheses 6.2. *(i) The set S is finite (i.e., the Coxeter group W is finitely generated).*

(ii) X is a finite complex.

(iii) The covering space $\widetilde{X}$ is acyclic.

Remark 6.3. Hypotheses (ii) and (iii) are satisfied if X is a finite aspherical complex and $\varphi : \pi_1(X) \to \pi$ is the identity (so that $\widetilde{X}$ is the universal covering space).

Theorem 6.4. *Under Hypotheses 6.2 the following statements are true.*

(i) $\widetilde{\Omega}$ is acyclic.

(ii) G is virtually torsion-free.

(iii) If Γ is a torsion-free subgroup of finite index in G, then Γ is of type FL (and then, a fortiori, of type FP).

(iv) If X is aspherical and $\varphi : \pi_1(X) \to \pi$ is the identity and Γ is as above, then $B\Gamma$ is homotopy equivalent to a finite complex.

Proof. (i) That $\widetilde{\Omega}$ is acyclic follows from [D1] or [D2]. (Strictly speaking, Theorem 10 in [D1] is stated only for finitely generated Coxeter groups; however, as pointed out in [DL], the same argument works for $(\widetilde{W}, \widetilde{S})$, when $\widetilde{S}$ is infinite).

(ii) Since S is finite, W has a faithful representation into $GL(m; \mathbb{R})$ (where $m = \mathrm{card}(S)$). By Selberg's Lemma, this implies that W is virtually torsion-free. Let Γ' be a torsion-free subgroup of finite index in W, let $\widetilde{\Gamma}$ be its inverse image in $\widetilde{W}$ and Γ its inverse image in G. The natural surjection $\widetilde{W} \to W$ is injective when restricted to any finite subgroup, so $\widetilde{\Gamma}$ is torsion-free. Since π acts freely on the finite dimensional acyclic complex $\widetilde{X}$, it is torsion-free. It follows easily that $\Gamma = \widetilde{\Gamma} \rtimes \pi$ is also torsion-free.

(iii) $C_*(\widetilde{\Omega})$ provides the desired resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}\Gamma$ modules.

(iv) If X is aspherical and $\pi_1(X) = \pi$, then $\widetilde{\Omega}$ is contractible (by [D1]) and $B\Gamma = \widetilde{\Omega}/\Gamma$, which is a finite complex. □

The proofs of Theorem 4.2 and Corollary 4.4 go through to give the following.

Theorem 6.5. *Under Hypotheses 6.2,*

$$(i) \qquad H^*_c(\widetilde{\Omega}) \cong \bigoplus_{w \in \widetilde{W}} H^*_c(\widetilde{X}, \widetilde{X}^{T(w)})$$

(ii) For any subgroup Γ *of finite index in* G*,* $H^*(\Gamma; \mathbb{Z}\Gamma) = H^*_c(\widetilde{\Omega})$.

Remark 6.6. We call the above construction the "reflection group trick". It has been used in the following context. Start with a finite aspherical complex Y with fundamental group π. Then thicken Y to a compact manifold with boundary X (e.g., embed Y in Euclidean space and let X be its regular neighborhood). Take $\varphi : \pi_1(X) \to \pi$ to be the identity. Let L be a (sufficiently fine) triangulation of ∂X and let (W, S) be a Coxeter system with associated simplicial complex L. Finally, let Γ' be a torsion-free subgroup of finite index in W. Then Ω/Γ' is a closed manifold. It is aspherical since its universal cover $\widetilde{\Omega}$ is contractible (by Theorem 6.4 (i)). The natural retraction $\Omega \to X$ descends to a retraction $\Omega/\Gamma' \to X$. Thus, $\Gamma = \pi_1(X/\Gamma')$ retracts onto π. This reflection group trick was introduced by Thurston in the context of hyperbolic 3-manifolds. In the above generality it was explained in Remark 15.9 of [D1]. It can be used to constuct examples of closed aspherical manifolds with fundamental groups having various interesting properties. The idea is to start with a group π such that (a) $B\pi$ is homotopy equivalent to a finite complex Y and (b) π has some property which also holds for any group which retracts onto it. The reflection group trick then yields Ω/Γ', the fundamental group of which retract onto π. For example, this idea is used in [Me] to show that the fundamental group of an aspherical manifold need not be residually finite. A slight variation of this construction gives the following.

Example 6.7. In [BB] Bestvina and Brady construct an example of a finite 2-complex Y, a group π and an epimorphism $\varphi : \pi_1(Y) \to \pi$ such that (i) the associated covering space $\widetilde{Y}$ is acyclic and (ii) the group π cannot be finitely presented. Such a group was the first example of a group of type FP which is not finitely presented. As in the previous remark, thicken Y to a compact n-manifold with boundary X (we can take X to be 4-dimensional) and apply the reflection group trick. Let $\Gamma = \widetilde{\Gamma} \rtimes \pi$ be a torsion-free subgroup of finite index in G as in the proof of Theorem 6.4 (ii). Since $\widetilde{\Omega}$ is an acyclic manifold,

$$H^i_c(\widetilde{\Omega}) = \begin{cases} \mathbb{Z} & ;\ i = n \\ 0 & ;\ i \neq n \end{cases},$$

so Γ is a Poincaré duality group of dimension n. Since π is a retract of Γ and π cannot be finitely presented, neither can Γ (see Lemma 1.3 in [W]). This example proves Theorem C.

We turn now to the question of finding necessary and sufficient conditions for G to be a virtual Poincaré duality group.

Definition 6.8. Suppose that A is a finite CW complex, that B is a subcomplex, that π is a group and that $\varphi : \pi_1(A) \to \pi$ is an epimorphism. Then any $\mathbb{Z}\pi$-module gives a local coefficient system on A. We say that (A, B) is a *Poincaré pair over*

π (of dimension n) if there is a class $\mu \in H_n(A, B; D)$ (where D denotes a local coefficient system on A defined via some homomorphism $w_1 : \pi_1(A) \to \{\pm 1\}$) such that

$$\cap\mu : H^i(A; M) \to H_{n-i}(A, B; D \otimes M)$$

is an isomorphism for all i and for any $\mathbb{Z}\pi$-module M. If $\pi = \pi_1(A)$ and φ is the identity, then (A, B) is simply a *Poincaré pair*.

For example, a compact manifold with boundary is a Poincaré pair.

Lemma 6.9. *Suppose that (A, B) is a pair of finite CW complexes, that $\varphi : \pi_1(A) \to \pi$ is an epimorphism, that $\widetilde{A}$ is the covering space of A defined by φ and that $\widetilde{B}$ is the inverse image of B in $\widetilde{A}$. Suppose further that $\widetilde{A}$ is acyclic. Then the following two statements are equivalent.*

(i) (A, B) is a Poincaré pair over π of dimension n.

(ii) $H^i_c(\widetilde{A}, \widetilde{B}) = \begin{cases} \mathbb{Z} & ; \ i = n \\ 0 & ; \ i \neq n \end{cases}.$

The proof in the absolute case (where $B = \emptyset$) can be found on pages 220-221 of [Br] and, in fact, the argument given there proves the lemma above.

Theorem 6.10. *Assume that Hypotheses 6.2 hold and that the group π is non-trivial. Then G is a virtual Poincaré duality group of dimension n if and only if*

a) L is an $(n-1)$-dimensional homology manifold, and

b) (X, L) is an n-dimensional Poincaré pair over π.

Proof. If conditions a) and b) hold then it is easy to see that Ω is a Poincaré space as is Ω/Γ for any torsion-free subgroup Γ of finite index in W and hence, that G is a virtual Poincaré duality group. (This was previously observed in [DH].) Conversely, suppose that G is a virtual Poincaré duality group. Then $H^i_c(\widetilde{\Omega}) = 0$ for $i \neq n$ and $H^n_c(\widetilde{\Omega}) = \mathbb{Z}$. As in the proof of Theorem 5.5, Theorem 6.5 implies that there is a subset $\widetilde{T}$ of $\widetilde{S}$ such that $\widetilde{W}^{\widetilde{T}}$ is a singleton. Since $\widetilde{T}$ generates a finite subgroup so does its image T in W. Suppose $T \neq \emptyset$. Then W splits as a nontrivial direct product and $L = L_0 * L_1$ where L_1 is a simplex. Since this implies that L is simply connected and since π is nontrivial, $\widetilde{L}$ has many components each of which would contribute to $H^n_c(\widetilde{\Omega})$, contradicting the assumption that it is $\mathbb{Z}$. Therefore, T and $\widetilde{T}$ are empty, and $H^n_c(\widetilde{\Omega}) = H^n_c(\widetilde{X}, \widetilde{L})$. So, Lemma 6.9 implies that (X, L) is an n-dimensional Poincaré pair. Moreover, the argument of Lemma 5.1 shows that $\widetilde{L}$ (and hence L) is an $(n-1)$-dimensional homology manifold. □

Suppose we want to use the reflection group trick to construct an example of a finitely presented Poincaré duality group which is not the fundamental group of a closed aspherical manifold. If we require that $\pi_1(X) = \pi$, so that X is aspherical, then Theorem 6.10 states that (X, L) must be a Poincaré pair and that L must be a homology manifold. So, essentially, L is a manifold. But the problem of finding such a pair (X, L) which is not homotopy equivalent rel L to a compact manifold with boundary is just the relative version of the original problem.

References

[Be1] M. Bestvina, *The virtual cohomological dimension of Coxeter groups*, Geometric Group Theory I, edited by G. Niblo and M. Roller, London Math. Soc. Lecture Notes 181, Cambridge University Press, Cambridge, 1993, 19-23.

[Be2] ———, *Local homology properties of boundaries of groups*, preprint, 1995.

[BB] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, preprint, 1995.

[B] N. Bourbaki, *Groupes et Algebrès de Lie*, Chapters IV-VI, Masson, Paris, 1981.

[Br] K. Brown, *Cohomology of groups*, Springer-Verlag, Berlin, Heidelberg, 1982.

[BFMW] J. Bryant, S. C. Ferry, W. Mio, S. Weinberger, *Topology of homology manifolds*, Ann. of Math. 143 (1996), 435-467.

[CD] R. Charney and M. W. Davis, *The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold*, Pac. J. Math. 171 (1995), 117-137.

[D1] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) 117 (1983), 293-325.

[D2] ———, *The homology of a space on which a reflection group acts*, Duke Math. J. 55 (1987), 97-104, Erratum: 56 (1988), 221.

[D3] ———, *Nopositive curvature and reflection groups*, Handbook of Geometric Topology, Elsevier, to appear.

[DH] M. W. Davis and J.-C. Hausmann, *Aspherical manifolds without smooth or PL structure*, Lecture Notes in Math. 1370, Springer-Verlag, Berlin, Heidelberg, 1989, 135-142.

[DL] W. Dicks and I. J. Leary, *On subgroups of Coxeter groups*, Proc. LMS Durham Symposium on Geometry and Cohomology in Group Theory, London Math. Soc. Lecture Notes, Cambridge Univ. Press, to appear.

[Dr] A. D. Dranishnikov, *The virtual cohomological dimension of Coxeter groups*, preprint, 1995.

[Fa] F. T. Farrell, *Poincaré duality and groups of type* (FP), Comment. Math. Helvetici 50 (1975), 187-195.

[F] S. Ferry, *Remarks on Steenrod homology*, In Novikov conjectures, index theorems and rigidity, edited by S. Ferry, A. Ranicki, J. Rosenberg, London Math. Soc. Lecture Notes 226, 227, Cambridge University Press, Cambridge, 1995.

[M] G. Moussong, *Hyperbolic Coxeter groups*, Ph. D. thesis, The Ohio State University, 1988.

[Me] G. Mess, *Examples of Poincaré duality groups*, Proc. Amer. Math. Soc. 110 (1990), 1145-1146.

[W] C. T. C. Wall, *Finiteness conditions for CW-complexes*, Ann. of Math 81 (1965), 56-69.