The Cohomology of Lattices in $\mathbf{SL}(2, \mathbb{C})$

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CONTENTS

This paper contains both theoretical results and experimental data on the behavior of the dimensions of the cohomology spaces $H^1(\Gamma, E_n)$, where Γ is a lattice in $\mathrm{SL}(2,\mathbb{C})$ and $E_n = \mathrm{Sym}^n \otimes \overline{\mathrm{Sym}}^n$, $n \in \mathbb{N} \cup \{0\}$, is one of the standard self-dual modules. In the case $\Gamma = \mathrm{SL}(2, \mathcal{O})$ for the ring of integers $\mathcal{O}$ in an imaginary quadratic number field, we make the theory of lifting explicit and obtain lower bounds linear in n. We present a large amount of experimental data for this case, as well as for some geometrically constructed and mostly nonarithmetic groups. The computations for $\mathrm{SL}(2, \mathcal{O})$ lead us to discover two instances with nonlifted classes in the cohomology. We also derive an upper bound of size $O(n^2/\log n)$ for any fixed lattice Γ in the general case. We discuss a number of new questions and conjectures suggested by our results and our experimental data.

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1. INTRODUCTION

For a semisimple Lie group G and a lattice Γ in G (i.e., a discrete subgroup of finite covolume), it is natural to consider the cohomology groups $H^*(\Gamma, E)$ of Γ with coefficients in finite-dimensional representation spaces E of G. If K is a maximal compact subgroup of G and $X = G/K$ the associated Riemannian symmetric space, these cohomology groups are canonically isomorphic to the cohomology groups of the quotient orbifold $\Gamma \setminus X$ with coefficients in the local system associated to E.

By the main result of [Franke and Schwermer 98], at least for arithmetic Γ the cohomology of Γ can be described by automorphic forms. For the contribution of the cuspidal spectrum one has (for general Γ) a generalized Matsushima formula describing the so-called cuspidal cohomology in terms of the multiplicities of cohomological unitary representations of G in $L^2_{\mathrm{cusp}}(\Gamma \setminus G)$, a well-known result of [Borel 81] predating [Franke and Schwermer 98]. The closer study of the noncuspidal part is the object of the theory of Eisenstein cohomology initiated by G. Harder (see, for example, the works cited above and [Harder 87]), and there are fairly complete results in many cases.

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While there is therefore a complete correspondence between cohomology and representation theory for the cuspidal part, generalizing the classical Eichler–Shimura homomorphism for the case $G = \mathrm{SL}(2,\mathbb{R})$ to all G, the behavior of the dimensions of the cohomology spaces is understood only under the hypothesis that G has a compact Cartan subgroup.

In this case, one can compute these dimensions using Euler–Poincaré characteristics [Serre 71] and the trace formula [Arthur 89]. If the highest weight of the representation E is regular, Arthur obtained an explicit formula for the dimension of the cohomology, which in this case is accounted for by the packet of discrete series representations with the same infinitesimal character as the dual of E. In particular, the leading term in his formula is a constant multiple of the dimension of E. The question of computing the cohomology for nonregular highest weights and of separating the individual representations in the packet is connected to the stabilization of the trace formula and to Arthur's conjectures.

In the easiest case, namely for lattices Γ in $G = \mathrm{SL}(2,\mathbb{R})$, it is well known that the dimension of the cohomology with coefficients in the symmetric power representations of G can be explicitly computed in terms of the basic invariants of Γ (i.e., the covolume and the orders of the elliptic elements), and in fact, such a dimension formula follows without difficulty from the description of the group-theoretic structure of Γ or from the Riemann–Roch theorem (see (1–4) below).

The situation is different if G has no compact Cartan subgroup and there are therefore no discrete series representations. No explicit dimension formulas are known in this case. We consider the simplest case of this type, namely lattices in the Lie group $G = \mathrm{SL}(2,\mathbb{C})$. Although the structure of this Lie group is very simple, the study of the cohomology of lattices in G presents a number of deep problems. The irreducible finite-dimensional representations of G are given by the tensor products

$$E_{n,m} = \mathrm{Sym}^n \otimes \overline{\mathrm{Sym}}^m \quad (n,m \in \mathbb{Z},\ n,m \geq 0). \quad (1\text{–}1)$$

Here Sym^n stands for the nth symmetric power of the standard two-dimensional representation of G and $\overline{\mathrm{Sym}}^m$ for its complex conjugate.

For a lattice Γ in $G = \mathrm{SL}(2,\mathbb{C})$ we consider therefore the finite-dimensional cohomology spaces

$$H^i(\Gamma, E_{n,m}).$$

The main problem studied in this paper is the behavior of the dimension of these spaces as a function of n and m for a fixed Γ. Another problem that we consider is the behavior of the dimensions as Γ ranges over the subgroups of finite index in a lattice Γ_0. Since the virtual cohomological dimension of Γ is 3 in the cocompact case, and 2 otherwise, the only dimensions with interesting cohomology are $i = 1$ and $i = 2$.

We now define the subspaces of cuspidal cohomology classes and give their description in terms of automorphic forms. As a consequence, it will turn out that we have only to consider the case $n = m$ and may in addition restrict to the first cohomology.

Consider the set of all proper parabolic subgroups $P = MU$ of $\mathrm{SL}(2,\mathbb{C})$ with the property that $\Gamma \cap U$ is a lattice in U. Here U is the unipotent radical of P, and M is a Levi subgroup of P. Let $\mathcal{C}$ be a system of representatives for the finitely many classes of such parabolics under Γ-conjugation and consider the direct sum of restriction maps

$$H^i(\Gamma, E_{n,m}) \longrightarrow U^i(\Gamma, E_{n,m}) = \bigoplus_{P \in \mathcal{C}} H^i(\Gamma \cap P, E_{n,m}). \quad (1\text{–}2)$$

The kernel of this map is called the cuspidal cohomology of Γ and denoted by

$$H^i_{\mathrm{cusp}}(\Gamma, E_{n,m}) \subseteq H^i(\Gamma, E_{n,m}).$$

If Γ is cocompact, the set $\mathcal{C}$ is empty and we have $H^i_{\mathrm{cusp}}(\Gamma, E_{n,m}) = H^i(\Gamma, E_{n,m})$.

We can also describe this construction geometrically. The group of orientation-preserving isometries of three-dimensional hyperbolic space $X = \mathbb{H}^3$ can be identified with $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}$, and every lattice Γ of $\mathrm{SL}(2,\mathbb{C})$ gives rise to a quotient orbifold $\Gamma \setminus X$. If this orbifold is not compact, it can be compactified by adding a boundary $\partial(\Gamma \setminus \mathbb{H}^3)$, which consists of finitely many disjoint two-dimensional tori or spheres. The inclusion

$$\Gamma \setminus \mathbb{H}^3 \hookrightarrow \widehat{\Gamma \setminus \mathbb{H}^3} := \Gamma \setminus \mathbb{H}^3 \cup \partial(\Gamma \setminus \mathbb{H}^3)$$

is a homotopy equivalence. The cohomology of Γ with coefficients in $E_{n,m}$ can be computed as the cohomology of a sheaf $\hat{E}_{n,m}$ on the compactified orbifold $\widehat{\Gamma \setminus \mathbb{H}^3}$.

The restriction map

$$H^i(\Gamma, E_{n,m}) \cong H^i(\widehat{\Gamma \setminus \mathbb{H}^3}, \hat{E}_{n,m}) \to H^i(\partial(\Gamma \setminus \mathbb{H}^3), \hat{E}_{n,m})$$

coincides with the restriction map (1–2) (see [Grunewald and Singhof 08] for a more detailed account). The spaces H^1_{cusp} and H^2_{cusp} are then dual to each other under Poincaré duality (cf. [Borel and Wallach 80, Section I.7], [Grunewald and Singhof 08]), and we will therefore restrict to the case $i = 1$ in the following.

Furthermore, by a result of Serre [Serre 70, Théorème 8], for $i = 1$ the dimension of the image of the map in (1–2) is one-half the dimension of the target space $U^1(\Gamma, E_{n,m})$. It is not difficult to calculate the latter dimension explicitly. For example, if $\Gamma \cap P \subseteq \pm U$ for all $P \in \mathcal{C}$ and $n + m$ is even, the dimension of the image is equal to the number of cusps (the number of elements of $\mathcal{C}$) by [Serre 70, Corollaire 1]. For $i = 2$ one sees immediately from the long exact cohomology sequence for the pair $(\widehat{\Gamma \setminus \mathbb{H}^3}, \partial(\Gamma \setminus \mathbb{H}^3))$ that the image of (1–2) is the entire target space, except in the case $n = m = 0$, where it has codimension one. It therefore remains to study the space H^1_{cusp}.

The theorem of Borel mentioned above yields an isomorphism

$$H^1_{\mathrm{cusp}}(\Gamma, E_{n,m}) \simeq H^1(\mathfrak{g}, K; L^2_{\mathrm{cusp}}(\Gamma \setminus G)^\infty \otimes E_{n,m}),$$

where the superscript ∞ denotes the subspace of smooth vectors. Since the space of cuspidal functions decomposes discretely as a representation of G, we can also write

$$\begin{aligned} &H^1_{\mathrm{cusp}}(\Gamma, E_{n,m}) \qquad\qquad (1\text{–}3)\\ &\quad\simeq \bigoplus_{\pi \in \hat{G}} \mathrm{Hom}(\pi, L^2_{\mathrm{cusp}}(\Gamma \setminus G)) \otimes H^1(\mathfrak{g}, K; H^\infty_\pi \otimes E_{n,m}), \end{aligned}$$

where $\hat{G}$ denotes the unitary dual of G.

From the computation of the $(\mathfrak{g}, K)$-cohomology of admissible irreducible representations of G [Borel and Wallach 80, Chapter II], we can deduce the following vanishing theorem:

$$H^1_{\mathrm{cusp}}(\Gamma, E_{n,m}) = \{0\}, \quad \text{for } n \neq m.$$

In the case $n = m$ the module

$$E_n := E_{n,n} = \mathrm{Sym}^n \otimes \overline{\mathrm{Sym}}^n$$

is self-dual in the terminology of [Borel and Wallach 80], and the dimension of $H^1_{\mathrm{cusp}}(\Gamma, E_n)$ is equal to the multiplicity of the principal series representation $\pi_{2n+2,0}$ (the representation unitarily induced from the character $z \mapsto (z/|z|)^{2n+2}$ of the maximal torus $T \simeq \mathbb{C}^\times$) in the space $L^2_{\mathrm{cusp}}(\Gamma \setminus G)$.

We will use this connection extensively in the following. We would like to stress that it does not yield an explicit dimension formula.

In the following we study the behavior of the dimension of $H^1(\Gamma, E_n)$ both theoretically and numerically. We focus primarily on the following problems:

A: How does the dimension of $H^1(\Gamma, E_n)$ behave when Γ is fixed and n grows?

B: Are there formulas for the dimension of $H^1(\Gamma, E_n)$ in terms of n at least for some groups Γ? Are there formulas for the dimension valid for all $n \geq n_0$, where n_0 is allowed to depend on Γ?

C: Are there lattices Γ such that $H^1_{\mathrm{cusp}}(\Gamma, E_n) = 0$ or $H^1(\Gamma, E_n) = 0$ for all n (necessarily cocompact in the latter case)?

D: How do the dimensions of $H^1(\Gamma, E_n)$ and $H^1_{\mathrm{cusp}}(\Gamma, E_n)$ behave when n is fixed and Γ ranges over the subgroups of finite index of a fixed lattice in G?

E: How does the asymptotic behavior of $H^1(\Gamma, E_n)$ and that of $H^1_{\mathrm{cusp}}(\Gamma, E_n)$ for $n \to \infty$ change as Γ ranges over the subgroups of finite index of a fixed lattice in G?

While we do not know of previous work on problems A, B, C, and E, problem D has been studied qualitatively in the context of limit multiplicities [Calegari and Emerton 09, de George and Wallach 78, Lott and Lück 95, Savin 89] and in that of the conjecture of Waldhausen and Thurston (cf. [Dunfield and Thurston 03]) in three-manifold topology.

The results and computations described below are of a very preliminary nature, but we hope to provide at least some evidence about what might be true. In Section 1.1 we summarize our theoretical results on upper and lower bounds for the dimension of $H^1(\Gamma, E_n)$. In Section 1.2 we shall formulate more specific questions about the behavior of the these dimensions and discuss the numerical evidence accumulated in the later sections.

1.1 Theoretical Results

We now describe our theoretical results. Before we proceed, let us briefly comment on the situation for Fuchsian groups, i.e., discrete subgroups Γ of $\mathrm{SL}(2,\mathbb{R})$ of finite covolume. Let g be the genus of Γ, k the number of cusps, and $r_1, \ldots, r_s$ the orders of the elliptic elements in the image of Γ in $\mathrm{PSL}(2,\mathbb{R})$, considered up to conjugacy. For an integer n and a positive integer r let μ be the remainder of n after division by $2r$ and set

$$d(n,r) = \begin{cases} 1 - \frac{\mu+1}{r}, & \mu \text{ even}, \\ -\frac{\mu+1}{r}, & 0 \leq \mu < r \text{ odd}, \\ 2 - \frac{\mu+1}{r}, & r \leq \mu < 2r \text{ odd}. \end{cases}$$

Then a consideration of the group-theoretic structure of Γ shows that

$$\dim H^1(\Gamma, \mathrm{Sym}^n) = \Big(2g - 2 + k + \sum_{i=1}^{s} \Big(1 - \frac{1}{r_i}\Big)\Big)(n+1) - \sum_{i=1}^{s} d(n, r_i) \qquad (1\text{–}4)$$

for all $n > 0$, where for $-1 \in \Gamma$ one has in addition to assume n to be even (the cohomology spaces vanish for odd n in this case). Thus the dimension of the cohomology is given by simple linear functions on congruence classes. Note also that the coefficient of $n+1$ in (1–4) is equal to $\mathrm{vol}(\Gamma \setminus \mathbb{H}^2)/2\pi$.

Our first theoretical result on lattices in $\mathrm{SL}(2, \mathbb{C})$ concerns a general upper bound for the dimension of the cohomology.

Theorem 1.1. *Let $\Gamma \subseteq \mathrm{SL}(2, \mathbb{C})$ be a discrete subgroup of finite covolume. Then*

$$\dim\ H^1(\Gamma, E_n) = O\left(\frac{n^2}{\log n}\right)$$

as $n \to \infty$.

This result is obtained by an application of the Selberg trace formula in Section 5. Note that the module E_n has dimension $(n+1)^2$. Since a group Γ as above is finitely presented, $\dim\ H^1(\Gamma, E_n) = O(n^2)$ is the trivial upper bound (cf. Lemma 3.1 below). Nontrivial lower bounds are not known for general lattices $\Gamma \subseteq \mathrm{SL}(2, \mathbb{C})$. In fact, our examples in Sections 7.2 and 7.3 indicate that there are probably none.

Lattices in $\mathrm{SL}(2, \mathbb{C})$ can be classified into arithmetic and nonarithmetic ones. The arithmetic lattices arise from quaternion algebras over number fields with precisely one complex place, and are intimately connected to number theory. They are distinguished by the existence of a large algebra of correspondences acting on their cohomology, the Hecke algebra (cf. Section 3.2 below). The primary examples are the Bianchi groups $\mathrm{SL}(2, \mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K. It is well known that every noncocompact arithmetic lattice in $\mathrm{SL}(2, \mathbb{C})$ is commensurable with a Bianchi group.

For congruence subgroups of arithmetic groups one can use Langlands functoriality (base change, automorphic induction, and the Jacquet–Langlands correspondence) to obtain lower bounds on the cohomology in certain cases (cf. [Clozel 87, Labesse and Schwermer 86, Rajan 04]). Unfortunately, because base change is available only for solvable extensions of number fields, the results are not complete (cf. [Lackenby et al. 08, Section 6]). Also, the Langlands conjectures relate the cohomology of these groups to important number-theoretic objects, namely ℓ-adic Galois representations and motives.

In the case of congruence subgroups of the Bianchi groups, [Harris et al. 93, Taylor 94] have indeed associated (under a restriction on the central character) ℓ-adic Galois representations of the absolute Galois group of the corresponding imaginary quadratic field K to eigenclasses of the Hecke algebra. More recently, these authors' work has been completed in [Berger and Harcos 07]. In the special case of trivial coefficients, one expects that Hecke eigenclasses correspond to elliptic curves and abelian varieties over the field K (cf. [Cremona 84, Grunewald et al.$\tilde{7}$8, Grunewald and Mennicke 78]).

The structure of the nonarithmetic lattices is even less well understood. The explicit examples considered in this paper are on the one hand Bianchi groups for certain K of small discriminant, and on the other hand certain geometrically constructed lattices and series of lattices, almost all of which are nonarithmetic.

A related (but in general not equivalent) method for obtaining lower bounds is based on studying the action of the complex conjugation automorphism c of $\mathrm{SL}(2, \mathbb{C})$ on the cohomology if the lattice Γ is invariant under c (as the Bianchi groups are, for example). In this case, one can use the Lefschetz fixed-point formula to compute the trace of this involution acting on $H^1_{\mathrm{cusp}}(\Gamma, E_n)$, and thereby obtain a lower bound for the dimension of this space.

For the case of the Bianchi groups with trivial coefficients this approach was carried out in [Krämer 85, Rohlfs 85]. Here the results turn out to be equivalent to those given by the theory of base change. While this method is not restricted to arithmetic groups, on the other hand, it does not cover all lower bounds obtainable by Langlands functoriality for arithmetic groups.

We work out the consequences of base change and automorphic induction (CM automorphic forms) for the cohomology of the Bianchi groups in Section 4 below. The base change construction detailed there associates to holomorphic automorphic forms for certain congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$ elements of $H^1(\mathrm{SL}(2, \mathcal{O}_K), E_n)$. Let us write

$$H^1_{\mathrm{bc}}(\mathrm{SL}(2, \mathcal{O}_K), E_n) \subseteq H^1_{\mathrm{cusp}}(\mathrm{SL}(2, \mathcal{O}_K), E_n)$$

for the corresponding subspace. Our main result here is a precise formula for the dimension of $H^1_{\mathrm{bc}}(\mathrm{SL}(2, \mathcal{O}_K), E_n)$

in terms of the prime factorization of the discriminant of K. To state it, we need to introduce functions ε_k and μ_k defined for all integers k that depend only on the residue classes of k modulo 4 and 3, respectively, and a function $\nu_{K,k}$ depending on K and at most on k modulo 2 or 3 (see Sections 4.1 and 4.3 below for the precise definitions).

Theorem 1.2. *Let K be an imaginary quadratic extension of $\mathbb{Q}$, $\mathcal{R}$ the set of primes ramified in K (the prime divisors of the discriminant of K), and for each $p \in \mathcal{R}$ let ν_p be the exact power of p dividing the discriminant. Then there are nonnegative constants c_2, c_3, and c_4 (depending on K) such that*

$$\begin{aligned}\dim H^1_{\mathrm{bc}}&(\mathrm{SL}(2,\mathcal{O}_K),E_n)\\ &= \Big(\frac{1}{24}\prod_{p\in\mathcal{R}}(p^{\nu_p}+1)+c_2(-1)^{n+1}\Big)(n+1)\\ &\quad -\nu_{K,n}\frac{h_K}{2}-2^{|\mathcal{R}|-2}+c_4\varepsilon_{n+2}+c_3\mu_{n+2}+\delta_{n,0}\end{aligned}$$

for all $n \geq 0$, where h_K is the class number of K and $\delta_{n,0}$ denotes the Kronecker delta.

Note that for every K the dimension of H^1_{bc} is given by linear functions of n spread out over the congruence classes modulo 12. If one takes the precise value of c_2 given in Section 4.3 into account, one sees that the coefficient of n in these linear functions is always positive, and that therefore the dimension of $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$ grows at least linearly with n. Also, for a fixed n the dimension grows linearly in the (absolute value of the) discriminant. This implies nonvanishing results for the cuspidal cohomology. The first results of this nature (in a much more limited situation) were obtained in [Grunewald and Schwermer 82, Grunewald and Schwermer 81]. Let us describe the special cases $K=\mathbb{Q}(\sqrt{d})$, $d=-2$, -7, -11, more explicitly.

Proposition 1.3. *For all $n \geq 1$ we have*

$$\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_{-2}),E_n)=\begin{cases}(n-1)/2, & n\equiv 1 \pmod 2,\\ (n-2)/4, & n\equiv 2 \pmod 4,\\ (n-4)/4, & n\equiv 0 \pmod 4,\end{cases}$$

$$\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_{-7}),E_n)=\begin{cases}(n-3)/3, & n\equiv 0 \pmod 3,\\ (n-1)/3, & n\equiv 1 \pmod 3,\\ (n-2)/3, & n\equiv 2 \pmod 3,\end{cases}$$

$$\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_{-11}),E_n)=\begin{cases}(n-1)/2, & n\equiv 1 \pmod 2,\\ n/2, & n\equiv 2 \pmod 4,\\ (n-2)/2, & n\equiv 0 \pmod 4.\end{cases}$$

A second construction of cohomology classes is via automorphic induction from Hecke characters of quadratic extensions of K (in fact necessarily biquadratic extensions of $\mathbb{Q}$ unramified over K). In Section 4.2 we describe the corresponding contribution H^1_{CM} to the cuspidal cohomology. In many cases, it is already contained in H^1_{bc}. The precise criterion for an additional CM contribution to the cohomology is as follows.

Proposition 1.4. *Let K be an imaginary quadratic field. There is a CM contribution to a space $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$, $n \geq 0$, which is not contained in $H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$, if and only if for some real quadratic field L' such that KL'/K is unramified, the narrow class number $h^+_{L'}$ is strictly greater than the corresponding number of genera $g^+_{L'}=2^{|\mathcal{R}(L')|-1}$, where $\mathcal{R}(L')$ denotes the set of primes ramified in L'.*

See Section 4.2 for the smallest examples of real quadratic fields L' with this property. In any case, the additional CM contribution is always constant on residue classes modulo 12, and the total contribution from base change and automorphic induction is therefore again given by linear functions on residue classes modulo 12.

The techniques of Langlands functoriality have been previously used to prove the conjecture of Waldhausen and Thurston for some arithmetic groups Γ, i.e., to establish the existence of a finite-index subgroup Δ of Γ with $H^1(\Delta,\mathbb{C} \neq 0$. In the special case of arithmetic groups Γ associated to quaternion algebras defined over fields L such that the extension L/L^{tr}, where L^{tr} is the maximal totally real subfield of L, is quadratic, the methods of [Labesse and Schwermer 86] imply the following result on problem E.

Proposition 1.5. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{C})$ such that the field of definition L of the corresponding quaternion algebra is a quadratic extension of its maximal totally real subfield L^{tr}. Then for every $c>0$ there exists a finite-index subgroup Δ of Γ such that*

$$\dim H^1(\Delta,E_n)>cn$$

for all $n \geq 0$.

For solvable extensions L/L^{tr} one obtains a corresponding result for certain twisted variants of the representations E_n (cf. [Rajan 04]). Such a result is expected to be true for all arithmetic lattices Γ.

We now turn to a result that gives an upper bound for special cases of problem D. We consider finite-index subgroups of the Bianchi groups $\mathrm{SL}(2,\mathcal{O}_K)$. For any nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ we have the classical congruence subgroup

$$\Gamma_0(\mathfrak{a}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathcal{O}_K) \,\middle|\, c \in \mathfrak{a} \right\}.$$

The index of $\Gamma_0(\mathfrak{a})$ in $\mathrm{SL}(2,\mathcal{O}_K)$ is given by the multiplicative function

$$\iota(\mathfrak{a}) = \mathrm{N}(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 + \frac{1}{\mathrm{N}(\mathfrak{p})}\right).$$

The following theorem gives a bound on the dimension of $H^1(\Gamma \cap \Gamma_0(\mathfrak{a}), E_n)$ for each subgroup Γ of finite index in $\mathrm{SL}_2(\mathcal{O}_K)$, which improves the trivial bound $O(\iota(\mathfrak{a}))$ by a logarithm.

Theorem 1.6. *Let Γ be a subgroup of finite index in $\mathrm{SL}(2,\mathcal{O}_K)$, K an imaginary quadratic field. Then for any fixed $n \geq 0$ we have*

$$\dim H^1(\Gamma \cap \Gamma_0(\mathfrak{a}), E_n) = O\left(\frac{\iota(\mathfrak{a})}{\log \mathrm{N}(\mathfrak{a})}\right), \quad \mathrm{N}(\mathfrak{a}) \to \infty. \tag{1–5}$$

This theorem, which again results from an application of the trace formula, should be compared to the limit multiplicity results of [de George and Wallach 78, Lott and Lück 95, Savin 89], which imply

$$\lim_{i\to\infty} \frac{\dim H^1(\Gamma_i, E_n)}{[\Gamma : \Gamma_i]} = 0$$

for fixed n and towers of normal subgroups Γ_i of a fixed lattice Γ such that $\bigcap_i \Gamma_i = \{1\}$. That the trace formula implies a bound of this form has been known for some time in the case of Maass forms of eigenvalue $\frac{1}{4}$ for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ [Iwaniec 84, p. 173, (3.5)] (and similarly for modular forms of weight one for such groups).

Recently, [Calegari and Emerton 09] obtained by p-adic methods a much better bound in the special case of the principal congruence subgroups $\Gamma(\mathfrak{p}^k)$ of $\mathrm{SL}(2,\mathcal{O}_K)$ whose level is a power of a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$:

$$\dim H^1(\Gamma\left(\mathfrak{p}^k\right), E_n) = O\left(\mathrm{N}(\mathfrak{p})^{(3-[K_\mathfrak{p}:\mathbb{Q}_p]^{-1})k}\right), \quad k \to \infty,$$

if n is fixed, whereas

$$[\mathrm{SL}(2,\mathcal{O}_K) : \Gamma(\mathfrak{p}^k)] = \mathrm{N}(\mathfrak{p})^{3k}(1 - \mathrm{N}(\mathfrak{p})^{-2}).$$

Here, $K_\mathfrak{p}$ denotes the completion of K at $\mathfrak{p}$. The results of [Calegari and Emerton 09] cover all arithmetic lattices in $\mathrm{SL}(2,\mathbb{C})$.

Note also that for $\mathfrak{a} = a\mathcal{O}_K$, a a positive integer, we can get by base-change arguments a lower bound of the form $Ca = C\mathrm{N}(\mathfrak{a})^{1/2}$. If $\mathfrak{a}$ and its conjugate are relatively prime, there is no nontrivial lower bound known. In fact, in the papers [Boston and Ellenberg 06, Calegari and Dunfield 06], examples of cocompact (arithmetic and nonarithmetic) lattices Γ in $\mathrm{SL}(2,\mathbb{C})$ and infinite towers of congruence subgroups Γ_i of Γ (of $\mathfrak{p}$-power level for a suitable prime ideal $\mathfrak{p}$ of a number field K associated to Γ) are given that satisfy $H^1(\Gamma_i,\mathbb{C}) = 0$ for all i. Using certain specific link complements and cyclic covers, it is possible to construct nested sequences of subgroups Γ_i ($i \in \mathbb{N}$) of certain lattices Γ with $H^1_{\mathrm{cusp}}(\Gamma_i,\mathbb{C}) = 0$ for all i (see [Grunewald and Hirsch 95]).

On the other hand, it is certainly not possible to improve the trivial bound $O([\mathrm{SL}(2,\mathcal{O}_K) : \Gamma])$ on the dimension of $H^1(\Gamma, E_n)$ for finite-index subgroups Γ of $\mathrm{SL}(2,\mathcal{O}_K)$ without making any assumption on Γ. This follows easily from the fact that the Bianchi groups (and more generally all noncocompact lattices in $\mathrm{SL}(2,\mathbb{C})$) are large, i.e., contain a finite-index subgroup surjecting onto a nonabelian free group. This property is in fact conjectured to be true for all lattices (cf. [Lackenby et al. 08]).

1.2 Experimental Results and Questions

Here we formulate more specific versions of problems A through E from above. We shall also discuss the numerical results accumulated in the later sections.

Let us begin by reporting on our numerical calculations. In Section 3.1 we describe how the cohomology space $H^1(\Gamma, E_n)$ can be effectively computed from a presentation of Γ together with explicit matrices for the generators. We have developed computer codes for this task. The results of the computations are documented in Sections 6 and 7.

Consider first the case of Bianchi groups explained in Section 6. We consider the fields $K = \mathbb{Q}(\sqrt{d})$ for $d = -1, -2, -3, -5, -6, -7, -10, -11, -14, -19$. From Proposition 1.4 it follows immediately that $H^1_{\mathrm{CM}} \subseteq H^1_{\mathrm{bc}}$ in all these cases. In fact, in our computations we had in all cases except two $H^1_{\mathrm{cusp}} = H^1_{\mathrm{bc}}$. The precise range of the computations can be found in Section 6. The two exceptions are given in the following proposition.

Proposition 1.7. *In the spaces*

$$H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-7}), E_{12})$$

and

$$H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-11}), E_{10})$$

the subspace of cohomology classes obtained from base change has codimension two.

In both cases there is a uniquely determined two-dimensional complement invariant under the action of the Hecke algebra. We document the eigenvalues of the first few Hecke operators on these subspaces in Section 6.2. As mentioned above, by [Berger and Harcos 07, Harris et al. 93, Taylor 94] there exist compatible systems of ℓ-adic Galois representations associated to these Hecke eigenclasses. The eigenvalues satisfy the Ramanujan conjecture within the range of computation, which supports the conjecture that the associated Galois representations are motivic (note that this does not follow from their construction).

The computations suggest the following questions.

Question 1.8. For a given imaginary quadratic field K, is it true that

$$H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_K),E_n) = H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_K),E_n) + H^1_{\mathrm{CM}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$$

for all but finitely many n?

Question 1.9. Is it the case that

$$H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_K),E_n) = H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$$

for all n for some K, for example $K = \mathbb{Q}(\sqrt{d})$, $d = -1,-2,-3,-5,-6,-10,-14,-19$?

Question 1.8 is related to the conjectures and (mostly) conditional results of Calegari and Mazur on p-adic deformations of Galois representations over non-totally-real base fields [Calegari and Mazur 09]. In fact, an affirmative answer would imply special cases of a suitable automorphic analogue of [Calegari and Mazur 09, Conjecture 1.3]. The nonlifted cohomology classes in $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-7}),E_{12})$ and $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-11}),E_{10})$ can be used to give further examples of the phenomenon exhibited in [Calegari and Mazur 09, Theorem 1.1].

Regarding Question 1.9, recent experimental results in [Ash and Pollack 08] suggest that the cohomology of the group $\mathrm{SL}(3,\mathbb{Z})$ behaves similarly to the cases $d = -1,-2,-3,-5,-6,-10,-14,-19$ above. Here all the cuspidal cohomology seems to be in the image of the symmetric square lift from classical modular forms for $\mathrm{SL}(2,\mathbb{Z})$.

In Section 7 we consider some examples of (mostly) nonarithmetic lattices. All examples are compatible with an affirmative answer to the following question.

Question 1.10. For a given lattice Γ in $\mathrm{SL}(2,\mathbb{C})$, do there exist integers $n_0 \geq 0$, $N > 0$, depending on Γ such that for each $n \geq n_0$ and n in a fixed residue class modulo N, the dimension $\dim H^1(\Gamma,E_n)$ is given by a linear function in n?

As we have seen, an affirmative answer to Question 1.8 would imply that one might take $N = 12$ for the Bianchi groups. A weaker but still unresolved question is the following.

Question 1.11. Do we have $\dim H^1(\Gamma,E_n) = O(n)$ as $n \to \infty$ for every lattice Γ, or is it possible that these dimensions grow faster than linearly in n?

The computations in Sections 7.2 and 7.3 suggest an affirmative answer to the following question.

Question 1.12. Are there lattices Γ such that $\dim H^1(\Gamma,E_n)$ remains bounded as $n \to \infty$? Is it possible that $H^1_{\mathrm{cusp}}(\Gamma,E_n) = 0$ or even $H^1(\Gamma,E_n) = 0$ for all n (Γ being necessarily cocompact in the latter case)?

In Section 7.2, an infinite sequence of nonarithmetic groups with one cusp is considered, which provides candidates for lattices with $H^1_{\mathrm{cusp}}(\Gamma,E_n) = 0$ for all n. In Section 7.3 we consider a cocompact nonarithmetic lattice and its finite-index subgroups of low index and obtain many candidates for lattices with $H^1(\Gamma,E_n) = 0$ for all n. Concerning problem D, we pose the following variant of the conjecture of Waldhausen and Thurston as a question.

Question 1.13. Given a lattice Γ and $n \geq 0$, is there a subgroup Δ of finite index in Γ such that $H^1_{\mathrm{cusp}}(\Delta,E_n) \neq 0$? More strongly, is there a subgroup Δ such that $H^1_{\mathrm{cusp}}(\Delta,E_n) \neq 0$ for all n?

We are able to provide an affirmative answer for all examples that we computed (see Sections 7.2 and 7.3).

We have also made extensive computations of $\dim H^1(\Gamma_0(\mathfrak{p}),\mathbb{C})$ for the standard congruence subgroups $\Gamma_0(\mathfrak{p})$ of $\mathrm{SL}(2,\mathcal{O}_{-1})$ associated with degree-one prime ideals $\mathfrak{p}$ of $\mathcal{O}_{-1}$. The results are documented in Section 6.3.

The cohomology groups $H^1(\Gamma_0(\mathfrak{p}),\mathbb{C})$ are particularly interesting for number theory, since their nonvanishing is conjectured to be related to the existence of certain elliptic curves (or more generally abelian varieties) defined over $K = \mathbb{Q}(i)$ (cf. [Cremona 84, Grunewald et al. 78, Grunewald and Mennicke 78]). Also, the methods of Langlands functoriality do not provide any nontrivial lower bound for the dimension, and in fact, there are many examples of prime ideals $\mathfrak{p}$ with $H^1(\Gamma_0(\mathfrak{p}), \mathbb{C}) = 0$. In analogy with distribution questions for elliptic curves (cf. [Brumer and McGuinness 90]), we pose the following question.

Question 1.14. Is there a constant C such that the asymptotic relation

$$\sum_{\mathfrak{p},\, N(\mathfrak{p}) \le x} \dim H^1(\Gamma_0(\mathfrak{p}), \mathbb{C}) \sim C \frac{x^{5/6}}{\log x}$$

holds as x tends to infinity, where the sum is to be extended over all degree-one prime ideals $\mathfrak{p}$ of $\mathcal{O}_{-1}$ of norm at most x?

If the cohomology indeed corresponds to abelian varieties and the distribution heuristics of [Brumer and McGuinness 90] are valid, this question asks simply whether a positive proportion of the cohomology is accounted for by elliptic curves. Note that this is definitely not the case for the cohomology of the congruence subgroups $\Gamma_0(p)$ of $\mathrm{SL}(2, \mathbb{Z})$. The computational results in Section 6.3 are compatible with an affirmative answer, but the range of our computations seems to be too small to allow a more detailed analysis (cf. [Brumer and McGuinness 90]). The behavior of the dimensions $\dim H^1(\Gamma_0(\mathfrak{p}), E_n)$ seems to be quite different if $n \ge 1$ is fixed and $\mathfrak{p}$ varies (see Section 6.3).

Finally, we pose the following question regarding problem E. Here, our computations do not suggest a general answer.

Question 1.15. For a given lattice Γ, does there exist a subgroup Δ of finite index such that

$$\liminf_{n \to \infty} \frac{\dim H^1(\Delta, E_n)}{n} > 0?$$

The theoretical evidence summarized in Section 1.1 shows that this question has an affirmative answer for some arithmetic lattices Γ. Our computations for nonarithmetic groups are inconclusive. Namely, for the groups considered in Sections 7.2 and 7.3 we were not able to find such a finite-index subgroup Δ, but to search through all subgroups of a given index very quickly becomes prohibitive.

2. THE BIANCHI GROUPS

This section contains some notation and preliminary material concerning the Bianchi groups, as well as the explicit finite presentations on which our computer calculations are based. The first subsection fixes notation that we will use throughout this paper. The results needed from algebraic number theory are contained in [Lang 94]. We also follow this book in our notational conventions.

2.1 The Bianchi Groups and Their Congruence Subgroups

Let d be a square-free negative integer, $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{C}$ the corresponding imaginary quadratic number field, and $\mathcal{O} = \mathcal{O}_d = \mathcal{O}_K$ its ring of integers. The ring $\mathcal{O}_d$ has a $\mathbb{Z}$-basis consisting of 1 and ω_d, where

$$\omega = \omega_d = \begin{cases} \sqrt{d}, & \text{if } d \not\equiv 1 \pmod 4, \\ \dfrac{1+\sqrt{d}}{2}, & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

The discriminant of the field K is

$$D = D_d = \begin{cases} d, & \text{if } d \equiv 1 \pmod 4, \\ 4d, & \text{if } d \equiv 2,3 \pmod 4. \end{cases}$$

We set $\mathcal{R} = \mathcal{R}_d$ for the set of rational primes p ramified in K. The set $\mathcal{R}_d$ consists exactly of the prime divisors of D_d.

We also fix the following notation concerning subgroups of $\mathrm{SL}(2, \mathbb{C})$ commensurable with the Bianchi groups $\mathrm{SL}(2, \mathcal{O})$. Let $\mathfrak{a} \subseteq \mathcal{O}$ be a nonzero ideal. The subgroup

$$\Gamma(\mathfrak{a}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathcal{O}) \mid a-1, b, c, d-1 \in \mathfrak{a} \right\} \subseteq \mathrm{SL}(2, \mathcal{O})$$

is called the *full congruence subgroup* of level $\mathfrak{a}$. It clearly has finite index in $\mathrm{SL}(2, \mathcal{O})$. A subgroup $\Gamma \subseteq \mathrm{SL}(2, K)$ is called a *congruence subgroup* if $\Gamma \cap \mathrm{SL}(2, \mathcal{O})$ has finite index in both Γ and $\mathrm{SL}(2, \mathcal{O})$, and if Γ contains a full congruence subgroup $\Gamma(\mathfrak{a})$ for a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}$.

Let $\mathfrak{a} \subset K$ now be a fractional ideal of $\mathcal{O}$, that is, $\mathfrak{a}$ is a nonzero finitely generated $\mathcal{O}$-submodule of K. We define

$$\mathrm{SL}(2, \mathfrak{a}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, K) \mid a, d \in \mathcal{O},\ c \in \mathfrak{a},\ b \in \mathfrak{a}^{-1} \right\}.$$

Notice that $\mathrm{SL}(2, \mathfrak{a})$ is a congruence subgroup of $\mathrm{SL}(2, K)$. It is equal to the stabilizer in $\mathrm{SL}(2, K)$ of the $\mathcal{O}$-submodule $\mathcal{O} \oplus \mathfrak{a}$ of K^2. We write $\mathrm{PSL}(2, \mathcal{O})$ or $\mathrm{PSL}(2, \mathfrak{a})$ for the images of the corresponding subgroups of $\mathrm{SL}(2, K)$ in $\mathrm{PSL}(2, \mathbb{C})$.

We write $\mathbb{A}_K$ for the ring of adeles of K and $\mathbb{A}_{K,f}$ for the ring of finite adeles. We view $\mathbb{A}_{K,f}$ as the subring of $\mathbb{A}_K$ consisting of those elements that are 0 at the infinite place of K. The adele rings $\mathbb{A}_K$, $\mathbb{A}_{K,f}$ and their unit groups $\mathbb{A}_K^*$, $\mathbb{A}_{K,f}^*$ are equipped with their standard topologies (see [Lang 94]). We also consider the profinite completion of the ring $\mathcal{O}$, which we denote by $\hat{\mathcal{O}}$, to be embedded as a compact and open subring of $\mathbb{A}_{K,f}$ in the usual way.

Recall the standard description of the adelic coset space $\mathrm{GL}(2,K)\setminus\mathrm{GL}(2,\mathbb{A}_K)$ in terms of the coset spaces $\Gamma\setminus\mathrm{GL}(2,\mathbb{C})$ for congruence subgroups Γ of $\mathrm{GL}(2,K)$. By the strong approximation theorem, for any compact open subgroup $\mathcal{K}$ of $\mathrm{GL}(2,\mathbb{A}_{K,f})$, the determinant map identifies the space of connected components of

$$\mathrm{GL}(2,K)\setminus\mathrm{GL}(2,\mathbb{A}_K)/\mathcal{K}$$

with the finite set $\mathbb{A}_{K,f}^*/K^*\det(\mathcal{K})$, and for a set $S\subset\mathrm{GL}(2,\mathbb{A}_{K,f})$ with the property that $\det(S)$ forms a system of representatives for $\mathbb{A}_{K,f}^*/K^*\det(\mathcal{K})$ we have

$$\mathrm{GL}(2,K)\setminus\mathrm{GL}(2,\mathbb{A}_K)/\mathcal{K}=\bigcup_{s\in S}\Gamma_s\setminus\mathrm{GL}(2,\mathbb{C}), \tag{2–1}$$

where $\Gamma_s=\mathrm{GL}(2,K)\cap s\mathcal{K}s^{-1}$.

To obtain the special case of the groups $\mathrm{SL}(2,\mathfrak{a})$, let $\mathcal{K}_0=\mathrm{GL}(2,\hat{\mathcal{O}})$ be the standard maximal compact subgroup of $\mathrm{GL}(2,\mathbb{A}_{K,f})$, and for each finite-index subgroup Δ of $\hat{\mathcal{O}}^*$ set

$$\mathcal{K}(\Delta)=\{g\in\mathcal{K}_0\mid\det g\in\Delta\}. \tag{2–2}$$

If $\Delta\cap\mathcal{O}^*=\{1\}$, the groups Γ_s in (2–1) can be identified with the groups $\mathrm{SL}(2,\mathfrak{a})$, where $\mathfrak{a}$ runs over a system of representatives for the ideal classes of K and each group appears with multiplicity $[\hat{\mathcal{O}}^*:\Delta\mathcal{O}^*]$.

Denote by $X(\Delta)$ the set of all characters of $\mathbb{A}_{K,f}^*/\Delta K^*$. It evidently has cardinality $|X(\Delta)|=h_K[\hat{\mathcal{O}}^*:\Delta\mathcal{O}^*]$, where h_K is the class number of K.

2.2 Presentations

This subsection contains explicit finite presentations for some of the Bianchi groups. We include them here, because some of them have not yet appeared in print. The presentations are taken from [Flöge 83, Schneider 85, Swan 71]. We use the standard notation for presentations of groups: $G=\langle g_1,\dots,g_n\mid R_1,\dots,R_l\rangle$ means that the group G is generated by $g_1,\dots,g_n$ and presented by the words $R_1,\dots,R_l$.

The following three matrices are in the set of generators in almost all cases:

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad U=U_d=\begin{pmatrix}1&\omega_d\\0&1\end{pmatrix}.$$

First, we give the results for the cases $d=-1,-2,-3,-7,-11$, which are exactly the cases in which the ring of integers $\mathcal{O}_d$ is Euclidean:

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-1})\\&\quad=\big\langle A,B,U\mid B^2,(AB)^3,(BUBU^{-1})^3,AUA^{-1}U^{-1},\\&\qquad(BU^2BU^{-1})^2,(AUBAU^{-1}B)^2\big\rangle,\\&\mathrm{PSL}(2,\mathcal{O}_{-2})\\&\quad=\big\langle A,B,U\mid B^2,(AB)^3,AUA^{-1}U^{-1},(BU^{-1}BU)^2\big\rangle,\\&\mathrm{PSL}(2,\mathcal{O}_{-3})\\&\quad=\langle A,B,U\mid B^2,(AB)^3,AUA^{-1}U^{-1},(UBA^2U^{-2}B)^2,\\&\quad(UBAU^{-1}B)^3,AUBAU^{-1}BA^{-1}UBA^{-1}UBAU^{-1}B\rangle,\\&\mathrm{PSL}(2,\mathcal{O}_{-7})\\&\quad=\langle A,B,U\mid B^2,(BA)^3,AUA^{-1}U^{-1},(BAU^{-1}BU)^2\rangle,\\&\mathrm{PSL}(2,\mathcal{O}_{-11})\\&\quad=\langle A,B,U\mid B^2,(BA)^3,AUA^{-1}U^{-1},(BAU^{-1}BU)^3\rangle.\end{aligned} \tag{2–3}$$

Next we consider the case $d=-19$. In this case $\mathcal{O}_d$ is a non-Euclidean principal ideal ring. We have

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-19})\\&\quad=\big\langle A,B,U,C\mid B^2,(AB)^3,AUA^{-1}U^{-1},C^3,(CA^{-1})^3,\\&\qquad(BC)^2,\ (BA^{-1}UCU^{-1})^2\big\rangle\end{aligned}$$

with the matrix

$$C=\begin{pmatrix}1-\omega_{-19}&2\\2&\omega_{-19}\end{pmatrix}.$$

In the cases $d=-5,-6,-10$, the class number of $\mathcal{O}_d$ is equal to 2. We give presentations of both $\mathrm{PSL}(2,\mathcal{O}_d)$ and $\mathrm{PSL}(2,\mathfrak{a})$ for a nonprincipal ideal $\mathfrak{a}$. For $d=-5$ we have

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-5})\\&\quad=\big\langle A,B,U,C,D\mid B^2,(AB)^3,AUA^{-1}U^{-1},D^2,(BD)^2,\\&\qquad(BUDU^{-1})^2,AC^{-1}A^{-1}BCB,\\&\qquad AC^{-1}A^{-1}UDU^{-1}CD\big\rangle\end{aligned}$$

with matrices

$$C=\begin{pmatrix}-4-\omega_{-5}&-2\omega_{-5}\\2\omega_{-5}&-4+\omega_{-5}\end{pmatrix},\quad D=\begin{pmatrix}-\omega_{-5}&2\\2&\omega_{-5}\end{pmatrix},$$

and

$$\begin{aligned}&\mathrm{PSL}(2,\mathfrak{a}_{-5})\\&\quad=\langle A,V,C,D \mid AVA^{-1}V^{-1}, CDC^{-1}D^{-1}, (AC^{-1})^2,\\&\qquad (DV^{-1})^3, (CD^{-1}VA^{-1})^3\rangle\end{aligned}$$

with the ideal $\mathfrak{a}_{-5} = \langle 2, 1-\sqrt{-5}\rangle$ of $\mathcal{O}_{-5}$ and with the matrices

$$V = \begin{pmatrix} 1 & \frac{1+\sqrt{-5}}{2} \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 1 & 0 \\ 1-\sqrt{-5} & 1 \end{pmatrix}.$$

For $d = -6$, we have

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-6})\\&\quad=\langle A,B,U,C,D \mid B^2, (AB)^3, AUA^{-1}U^{-1}, D^2,\\&\qquad BCBC^{-1}, (BAUDU^{-1})^3,\\&\qquad A^{-1}CAUDU^{-1}C^{-1}D^{-1}, (BAD)^3\rangle\end{aligned}$$

with the matrices

$$C = \begin{pmatrix} 5 & -2\omega_{-6} \\ 2\omega_{-6} & 5 \end{pmatrix}, \quad D = \begin{pmatrix} -1-\omega_{-6} & 2-w_{-6} \\ 2 & 1+\omega_{-6} \end{pmatrix}$$

and

$$\begin{aligned}&\mathrm{PSL}(2,\mathfrak{a}_{-6})\\&\quad=\langle A,V,C,D,E \mid E^2, (CA^{-1})^2, (DV^{-1})^3, (DEV^{-1})^2,\\&\qquad (CEA^{-1})^2, CDC^{-1}D^{-1}, AVA^{-1}V^{-1},\\&\qquad (CDEV^{-1}A^{-1})^2\rangle\end{aligned}$$

with the ideal $\mathfrak{a}_{-6} = \langle 2, \sqrt{-6}\rangle$ of $\mathcal{O}_{-6}$ and with the matrices

$$V = \begin{pmatrix} 1 & \frac{\omega}{2} \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$
$$D = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & -1-\frac{\omega}{2} \\ 2-w & 2 \end{pmatrix}.$$

For $d = -10$, we have

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-10})\\&\quad=\langle A,B,U,C,D,E,F \mid B^2, (AB)^3, AUA^{-1}U^{-1}, C^2,\\&\qquad E^2, (BC)^2, (BE)^2, C^{-1}AD^{-1}BEBAD,\\&\qquad U^{-1}E^{-1}UFCF^{-1}, D^{-1}E^{-1}B^{-1}DU^{-1}DBCD^{-1}U,\\&\qquad D^{-1}B^{-1}ADC^{-1}U^{-1}EDA^{-1}BD^{-1}U,\\&\qquad U^{-1}DA^{-1}B^{-1}D^{-1}UFD^{-1}BADF^{-1}\rangle\end{aligned}$$

with the matrices

$$C = \begin{pmatrix} -\omega & 3 \\ 3 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} \omega-1 & -4 \\ 3 & \omega+1 \end{pmatrix},$$
$$E = \begin{pmatrix} \omega & 3 \\ 3 & -\omega \end{pmatrix}, \quad F = \begin{pmatrix} 11 & 5\omega \\ 2\omega & -9 \end{pmatrix}$$

and

$$\begin{aligned}&\mathrm{PSL}(2,\mathfrak{a}_{-10})\\&\quad=\langle A,V,C,D,E,F \mid E^2, (CA^{-1})^2, (FE)^2, (DEV^{-1})^2,\\&\qquad (DF^{-1}V^{-1})^3, CDC^{-1}D^{-1}, AVA^{-1}V^{-1},\\&\qquad (FC^{-1}EA)^2, F^3, (CF^{-1}A^{-1})^3, (CDF^{-1}A^{-1}V^{-1})^3\rangle\end{aligned}$$

with the ideal $\mathfrak{a}_{-10} = \langle 2, \sqrt{-10}\rangle$ of $\mathcal{O}_{-10}$ and with the matrices

$$C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -2 & -\frac{\omega}{2} \\ -\omega & 2 \end{pmatrix},$$
$$F = \begin{pmatrix} -3 & -1-\frac{\omega}{2} \\ 2-\omega & 2 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & \frac{\omega}{2} \\ 0 & 1 \end{pmatrix}.$$

The ideal class group of $\mathcal{O}_{-14}$ is cyclic of order 4. The ideal $\mathfrak{a}_{-14} = \langle 3, 1+\sqrt{-14}\rangle$ is not a square in the ideal class group:

$$\begin{aligned}&\mathrm{PSL}(2,\mathcal{O}_{-14})\\&\quad=\langle A,B,U,C,D,E,F \mid B^2, (AB)^3,\\&\qquad (A^{-1}C^{-1}BDBAD^{-1}C)^2, AUA^{-1}U^{-1},\\&\qquad (A^{-1}CD^{-1}ABDBC^{-1})^2, D^{-1}CE^{-1}A^{-3}DC^{-1}A^3E,\\&\qquad CB^{-1}C^{-1}FC^{-1}BCF^{-1},\\&\qquad C^{-1}DA^{-1}B^{-1}D^{-1}B^{-1}CA\\&\qquad - E^{-1}A^{-2}CBD^{-1}BA^{-1}DC^{-1}A^3E,\\&\qquad ACB^{-1}D^{-1}B^{-1}A^{-1}DC^{-1}\\&\qquad - AFA^{-1}C^{-1}BDBAD^{-1}CA^{-1}F^{-1}\rangle\end{aligned}$$

with the matrices

$$C = \begin{pmatrix} \omega & -5 \\ 3 & \omega \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 1+\omega \\ 1-w & 4 \end{pmatrix},$$
$$E = \begin{pmatrix} -5+4\omega & -23 \\ 4-\omega & 7+\omega \end{pmatrix}, \quad F = \begin{pmatrix} 13 & 6\omega \\ -2\omega & 13 \end{pmatrix},$$

and

$$\begin{aligned}&\mathrm{PSL}(2,\mathfrak{a}_{-14})\\&\quad=\langle A,U,C,D,E,F,G \mid G^2, CDC^{-1}D^{-1}, AUA^{-1}U^{-1},\\&\qquad (CA^{-1})^3, (DGU^{-1})^2, F^{-1}AE^{-1}A^{-1}UFEU^{-1},\\&\qquad (CGE^{-1}A^{-1}UGU^{-1}AEA^{-1})^3,\\&\qquad (AEU^{-1}DGE^{-1}A^{-1}UGD^{-1})^2,\\&\qquad DC^{-1}GU^{-1}AEGD^{-1}UE^{-1}F^{-1}\\&\qquad - CGE^{-1}A^{-1}UGU^{-1}AEA^{-1}F\rangle\end{aligned}$$

with the matrices

$$U = \begin{pmatrix} 1 & \frac{1-\omega}{3} \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 1+\omega & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} -3-\omega & -4 \\ 6 & 3-\omega \end{pmatrix}, \quad F = \begin{pmatrix} -3+\omega & -3 \\ 2+2\omega & -3+\omega \end{pmatrix},$$

$$G = \begin{pmatrix} -2 & \frac{\omega-1}{3} \\ 1+\omega & 2 \end{pmatrix}.$$

3. GROUP COHOMOLOGY

In this section we report basic definitions from the cohomology of groups. Section 3.1 reports a method to compute the first cohomology group for finitely presented groups. Our basic reference here is [Brown 82].

Let Γ be a group and M an $R\Gamma$-module for a commutative ring R. A derivation from Γ to M is a map $f : \Gamma \to M$ that satisfies

$$f(gh) = g \cdot f(h) + f(g)$$

for all $g, h \in \Gamma$. For $m \in M$ the map

$$f_m : \Gamma \to M, \quad f_m(g) = g \cdot m - m,$$

is a derivation and is called the *inner derivation* corresponding to m. We write $\mathrm{Der}(\Gamma, M)$ for the space of all derivations and $\mathrm{IDer}(\Gamma, M)$ for its subspace consisting of inner derivations. If $H^1(\Gamma, M)$ is the first cohomology group of Γ with coefficients in M, we have

$$H^1(\Gamma, M) = \mathrm{Der}(\Gamma, M)/\,\mathrm{IDer}(\Gamma, M). \tag{3–1}$$

Here we are interested in the case $\Gamma \subseteq \mathrm{SL}(2, L) \subseteq \mathrm{SL}(2, \mathbb{C})$, where $L \subset \mathbb{C}$ is a number field. The modules we consider are derived from the symmetric powers of the standard representation of $\mathrm{SL}(2, \mathbb{C})$. So let V be a two-dimensional L_1-vector space with basis x, y, where L_1 is a field between L and $\mathbb{C}$ invariant under complex conjugation. Let n be a nonnegative integer. The symmetric power $\mathrm{Sym}^n(L_1)$ has the L_1-basis $x^{n-i}y^i$, $0 \le i \le n$. The action of $g \in \mathrm{SL}(2, L_1)$ is given by

$$g \cdot x^{n-i}y^i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x^{n-i}y^i = (ax + cy)^{n-i}(bx + dy)^i,$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3–2}$$

The module $\overline{\mathrm{Sym}}^n(L_1)$ is equal to $\mathrm{Sym}^n(L_1)$ as an L_1-vector space, and the action is given by replacing g in (3–2) by its complex conjugate.

We often use the following simple facts from group cohomology without further notice. First of all, the spaces $H^1(\Gamma, \mathrm{Sym}^n(L_1) \otimes \overline{\mathrm{Sym}}^m(L_1)) \otimes \mathbb{C}$ and $H^1(\Gamma, \mathrm{Sym}^n(\mathbb{C}) \otimes \overline{\mathrm{Sym}}^m(\mathbb{C})))$ are isomorphic for all $n, m \ge 0$. Second, if $m + n$ is even, the action of Γ on $\mathrm{Sym}^n(L_1) \otimes \overline{\mathrm{Sym}}^m(L_1)$ factors through an action of the image $\tilde{\Gamma}$ of Γ in $\mathrm{PSL}(2, \mathbb{C})$, and $H^1(\Gamma, \mathrm{Sym}^n(L_1) \otimes \overline{\mathrm{Sym}}^m(L_1))$ is isomorphic to $H^1(\tilde{\Gamma}, \mathrm{Sym}^n(L_1) \otimes \overline{\mathrm{Sym}}^m(L_1))$.

3.1 $H^1(\Gamma, M)$ for Finitely Presented Groups

Here we explain how information about $H^1(\Gamma, M)$ can be computed from equation (3–1). We assume here that R is a Euclidean ring and M is a free R-module of finite rank in which a basis has been chosen. Let Γ be a finitely presented group given explicitly in the form

$$\Gamma = \langle g_1, \dots, g_s \mid R_1, \dots, R_t \rangle.$$

Here we consider the relations $R_1, \dots, R_t$ to be explicitly given words in the generators $g_1, \dots, g_s$ of Γ and their inverses. Assume also that the matrices for the action of $g_1, \dots, g_s$ on M are explicitly given.

Consider now the R-linear map

$$\Phi : \mathrm{Der}(\Gamma, M) \to M^s, \quad \Phi(f) = (f(g_1), \dots, f(g_s)).$$

The image of Φ lies in the kernel of the linear map $\Lambda : M^s \to M^t$, which is obtained by formally expanding the image of each of the relators $R_1, \dots, R_t$ under a derivation $f : \Gamma \to M$ in terms of the values $f(g_1), \dots, f(g_s)$. It is easily seen that Φ maps $\mathrm{Der}(\Gamma, M)$ isomorphically to the kernel $\ker(\Lambda)$ of Λ.

Since M^s is a free R-module, a basis for the free module $\ker(\Lambda)$ can be computed. Consider now the linear map

$$\mu : M \to \ker(\Lambda), \quad \mu(m) = ((g_1 - 1)m, \dots, (g_s - 1)m).$$

The image of μ may then be described as the linear span of the images of the basis elements of M. If we express these in terms of the previously computed basis of $\ker(\Lambda)$, we see that the effective version of the elementary divisor theorem can be used to compute the structure of

$$H^1(\Gamma, M) = \mathrm{Der}(\Gamma, M)/\,\mathrm{IDer}(\Gamma, M) = \ker(\Lambda)/\mathrm{Im}(\mu). \tag{3–3}$$

If R is a field, the dimension of $H^1(\Gamma, M)$ can be computed by this method.

Apart from being important for the computation of cohomology spaces, (3–3) leads to the following (trivial) estimate.

Lemma 3.1. *Let Γ be a group generated by s elements and M a finite-dimensional $R\Gamma$-module for some field R. Then* $\dim H^1(\Gamma, M) \le s \dim M$.

A typical problem encountered in our computations is that the module M can be a vector space of big dimension (up to around 50,000) over an algebraic number field, and that the direct computation of the dimension of $H^1(\Gamma, M)$ from (3–3) is infeasible.

All discrete subgroups $\Gamma \subseteq \mathrm{SL}(2,\mathbb{C})$ considered in this paper have the property that they are contained in $\mathrm{SL}(2,R)$ for a finitely generated ring R inside an algebraic number field. Let $\mathcal{O}_\Gamma$ be a ring containing R and its complex conjugate.

Suppose p is a prime and $\mathcal{O}_\Gamma \to \mathbb{F}_p$ is a surjective ring homomorphism. Then $E_n(\mathbb{F}_p)$ inherits the structure of a Γ-module. By the usual universal coefficient theorem we have

$$\dim_{\mathbb{F}_p} H^1(\Gamma, E_n(\mathbb{F}_p)) \geq \dim_{\mathbb{C}} H^1(\Gamma, E_n).$$

A standard argument using Čebotarev's density theorem shows that $\dim_{\mathbb{C}} H^1(\Gamma, E_n)$ is equal to the minimum of the dimensions $\dim_{\mathbb{F}_p} H^1(\Gamma, E_n(\mathbb{F}_p))$, where p ranges over all primes with the above compatibility property. For all real numbers x we define

$$\dim_{\leq x} H^1(\Gamma, E_n) = \inf_{p \leq x} \{ \dim_{\mathbb{F}_p} H^1(\Gamma, E_n(\mathbb{F}_p)) \},$$

where p ranges over all primes with $p \leq x$ that admit a surjective ring homomorphism $\mathcal{O}_\Gamma \to \mathbb{F}_p$. The numbers $\dim_{\leq x} H^1(\Gamma, E_n)$ are much cheaper to compute than the actual dimensions $\dim_{\mathbb{C}} H^1(\Gamma, E_n)$. Of course, in the computations below we hope to have chosen the bound x large enough to capture $\dim_{\mathbb{C}} H^1(\Gamma, E_n)$. Also, if a lower bound for this dimension is known beforehand, we can by this method verify that the actual dimension is equal to the bound.

3.2 Hecke Operators

In this section we introduce the Hecke operators on cohomology spaces in a way suitable for explicit computations. We chose a treatment similar to [Shimura 71, Section 8.5]; see also [Grunewald et al.78].

If H is a subgroup of a group Γ, and M is a Γ-module, the inclusion $H \hookrightarrow \Gamma$ induces a restriction map $\mathrm{res}^\Gamma_H : H^*(\Gamma, M) \longrightarrow H^*(H, M)$. When $[\Gamma : H] < \infty$, there is also a map $\mathrm{tr} : H^*(H, M) \longrightarrow H^*(\Gamma, M)$ in the opposite direction, called the *transfer map* (cf. [Brown 82]). The composition $\mathrm{tr} \circ \mathrm{res}^\Gamma_H$ is multiplication by $[\Gamma : H]$ on $H^*(\Gamma, M)$.

Now let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in an imaginary quadratic number field K, and let M be one of the $\mathrm{GL}(2,\mathbb{C})$-modules $E_{n,m}$. The groups Γ and $\delta\Gamma\delta^{-1}$ are easily seen to be commensurable for every $\delta \in \mathrm{GL}(2,K)$. Define the Hecke operator $T_\delta : H^1(\Gamma, M) \to H^1(\Gamma, M)$ by the diagram

$$\begin{array}{ccc} H^1(\Gamma, M) & \xrightarrow{T_\delta} & H^1(\Gamma, M) \\ {\scriptstyle \mathrm{res}}\downarrow & & \uparrow{\scriptstyle \mathrm{tr}} \\ H^1(\Gamma \cap \delta\Gamma\delta^{-1}, M) & \xrightarrow{\tilde{\delta}} & H^1(\delta^{-1}\Gamma\delta \cap \Gamma, M) \end{array}$$

where $\tilde{\delta}$ is the isomorphism in cohomology induced by conjugation with δ. For a nonzero element $a \in \mathcal{O}$ we define

$$T_a = T_{\delta_a} \quad \text{with } \delta_a = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}. \tag{3–4}$$

The following properties of the linear maps $T_\delta : H^1(\Gamma, M) \to H^1(\Gamma, M)$, $\delta \in \mathrm{GL}(2,K)$, are well known (cf. [Shimura 71, Section 8.5], [Grunewald et al.78]):

- Each T_δ is diagonalizable.
- The characteristic polynomial of T_δ has integral coefficients and its zeros are real numbers.
- T_δ depends only on the double coset $\Gamma\delta\Gamma$.
- If $\Gamma = \mathrm{SL}(2,\mathcal{O})$, then all operators T_δ commute with each other.

3.3 The Eichler–Shimura Isomorphism

In this subsection we briefly recall the generalized Eichler–Shimura isomorphism sketched already in the introduction, which will give us the possibility of using results from the theory of automorphic forms in our study of cohomology spaces. See also [Harder 87] and [Urban 95, Théorème 3.2] for the case of congruence subgroups of $\mathrm{GL}(2,K)$, K an imaginary quadratic field.

From [Borel and Wallach 80, Chapter II] we know that for any integer $n \geq 0$, and any unitary representation π of $G = \mathrm{SL}(2,\mathbb{C})$, the $(\mathfrak{g}, K)$-cohomology space $H^1(\mathfrak{g}, K; H_\pi^\infty \otimes E_n)$ is nontrivial if and only if π is the principal series representation $\pi_{2n+2,0}$ (the representation unitarily induced from the character $z \mapsto (z/|z|)^{2n+2}$ of the maximal torus $T \simeq \mathbb{C}^\times$; cf. Section 5), and one-dimensional in this case. Therefore, we can deduce from (1–3) the more explicit isomorphism

$$\mathrm{Hom}(\pi_{2n+2,0}, L^2_{\mathrm{cusp}}(\Gamma \setminus \mathrm{SL}(2,\mathbb{C})) \simeq H^1_{\mathrm{cusp}}(\Gamma, E_n)$$

for any lattice Γ of G.

For use in Section 4, we quickly rewrite this isomorphism in a form involving $\mathrm{GL}(2,\mathbb{C})$. Define a unitary character of $\mathbb{C}^*$ by $\chi_\infty(x) = x/|x|$, and for each integer $n \geq 0$ consider the principal series representation

$\rho_\infty^n = \mathrm{PS}(\chi_\infty^{n+1}, \chi_\infty^{-n-1})$ of $\mathrm{GL}(2,\mathbb{C})$. Let $Z_\infty \subset \mathrm{GL}(2,\mathbb{C})$ be the center of $\mathrm{GL}(2,\mathbb{C})$. Then we have an isomorphism

$$\mathrm{Hom}(\rho_n^\infty, L^2_{\mathrm{cusp}}(\Gamma \setminus \mathrm{GL}(2,\mathbb{C})/Z_\infty)) \simeq H^1_{\mathrm{cusp}}(\Gamma, E_n).$$

4. BASE CHANGE

This section contains our results on the construction of cohomology classes for the Bianchi groups by base change from classical modular forms for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and automorphic induction from Hecke characters of quadratic extensions. In particular, we derive explicit dimension formulas for the corresponding subspaces of the cohomology. For this we fix an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ and use the notation of Section 2.1.

4.1 General Results on the Base-Change Construction

Here we present the consequences of the theory of base change and automorphic induction for the cohomology of the groups $\mathrm{SL}(2,\mathfrak{a})$. We give a precise description of the base-change process and the relevant spaces of holomorphic elliptic modular forms. The notation and concepts from the theory of automorphic forms are taken from [Borel and Wallach 80]. For a quadratic extension L of $\mathbb{Q}$ denote by ω_L the associated quadratic character of $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*$.

Let $\mathcal{A}_K$ be the set of all cuspidal automorphic representations of $\mathrm{GL}(2,\mathbb{A}_K)$. See [Langlands 80] for information on the base-change map $\pi \mapsto \pi_K$ from $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ to $\mathrm{GL}(2,\mathbb{A}_K)$. We shall be interested in the following subset of $\mathcal{A}_K$.

Definition 4.1. The set $\mathcal{A}_K^{bc}$ of (twisted) base-change representations is the set of all $\Pi \in \mathcal{A}_K$ such that $\Pi \simeq \pi_K \otimes \chi$ for an automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ and an idele class character χ of K.

Recall from Section 3.3 the definition of the representations ρ_∞^n of $\mathrm{GL}(2,\mathbb{C})$. For an integer $n \geq 0$ and a finite-index subgroup Δ of $\hat{\mathcal{O}}^*$ consider

$$\mathcal{A}_K^1(n,\Delta) = \{\Pi \in \mathcal{A}_K \mid \Pi_\infty \simeq \rho_\infty^n, \Pi_f^{\mathcal{K}(\Delta)} \neq 0\},$$

(where $\mathcal{K}(\Delta)$ was defined in Section 2.1) and set

$$\mathcal{A}_K^1(n) = \bigcup_\Delta \mathcal{A}_K^1(n,\Delta).$$

Furthermore, let

$$\mathcal{A}_K^{1,bc}(n,\Delta) = \mathcal{A}_K^1(n,\Delta) \cap \mathcal{A}_K^{bc}$$

and

$$\mathcal{A}_K^{1,bc}(n) = \mathcal{A}_K^1(n) \cap \mathcal{A}_K^{bc}.$$

Recall that each representation in $\mathcal{A}_K$ occurs with multiplicity one in $L^2_{\mathrm{cusp}}(\mathrm{GL}(2,K)\backslash\mathrm{GL}(2,\mathbb{A}_K))$. Furthermore, $\Pi \in \mathcal{A}_K^1(n)$ is equivalent to the conditions that $\Pi_\infty \simeq \rho_\infty^n$ and that the local components $\Pi_\mathfrak{p}$ at the finite places $\mathfrak{p}$ be twists of unramified principal series representations by characters. Therefore, for $\Pi \in \mathcal{A}_K^1(n,\Delta)$ the space $\Pi_f^{\mathcal{K}(\Delta)}$ is actually one-dimensional.

If we take a subgroup Δ of $\hat{\mathcal{O}}^*$ with the property $\Delta \cap \mathcal{O}^* = \{1\}$, we have by (2–1) an isomorphism

$$\Big(\bigoplus_\mathfrak{a} \mathrm{Hom}(\rho_n^\infty, L^2_{\mathrm{cusp}}(SL(2,\mathfrak{a}) \setminus \mathrm{GL}(2,\mathbb{C})/Z_\infty))\Big)^{[\hat{\mathcal{O}}^*:\Delta\mathcal{O}^*]} \simeq \mathrm{Hom}(\rho_n^\infty, L^2_{\mathrm{cusp}}(\mathrm{GL}(2,K) \setminus \mathrm{GL}(2,\mathbb{A}_K)/Z_\infty\mathcal{K}(\Delta))),$$

where $\mathfrak{a}$ ranges over a system of representatives for the ideal classes of K. Combining the Eichler–Shimura isomorphism from Section 3.3 with multiplicity one and the fact that $\dim \Pi_f^{\mathcal{K}(\Delta)} = 1$ for $\Pi \in \mathcal{A}_K^1(n,\Delta)$, we obtain the relation

$$\sum_\mathfrak{a} \dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}), E_n) = \frac{|\mathcal{A}_K^1(n,\Delta)|}{[\hat{\mathcal{O}}^* : \Delta\mathcal{O}^*]}.$$

It is not difficult to obtain also a finer description that distinguishes the individual cohomology spaces $\dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}), E_n)$ for representatives $\mathfrak{a}$ of different ideal classes.

For this, consider the action of the abelian group $X(\Delta)$ (see Section 2.1) on the space

$$\mathrm{Hom}(\rho_n^\infty, L^2_{\mathrm{cusp}}(\mathrm{GL}(2,K) \setminus \mathrm{GL}(2,\mathbb{A}_K)/Z_\infty\mathcal{K}(\Delta)))$$

given by letting $\xi \in X(\Delta)$ act as multiplication of functions on

$$\mathrm{GL}(2,K) \setminus \mathrm{GL}(2,\mathbb{A}_K)/Z_\infty\mathcal{K}(\Delta)$$

by $\xi \circ \det$.

Considering a basis of

$$\mathrm{Hom}(\rho_n^\infty, L^2_{\mathrm{cusp}}(\mathrm{GL}(2,K) \setminus \mathrm{GL}(2,\mathbb{A}_K)/Z_\infty\mathcal{K}(\Delta)))$$

consisting of normalized (cf. [Urban 95, Section 5]) eigenfunctions for the Hecke algebra of $\mathcal{K}(\Delta)$ (which correspond to the representations in $\mathcal{A}_K^1(n,\Delta)$), one sees that the action of $X(\Delta)$ induces a permutation of this basis, and therefore the trace of the action of a nontrivial element $\xi \in X(\Delta)$ is equal to the number of elements $\Pi \in \mathcal{A}_K^1(n,\Delta)$ with $\Pi \otimes \xi \simeq \Pi$.

It is clear that this number can be nonzero only if ξ is quadratic, and indeed unramified quadratic, i.e., necessarily of the form $\omega_L \circ \mathrm{N}_{K/\mathbb{Q}}$ for an imaginary quadratic field $L \neq K$ such that LK/K is unramified (see Proposition 4.5 below).

Therefore, we get

$$\begin{aligned}\dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}), E_n) &= \frac{1}{|X(\Delta)|}\Big(\left|\mathcal{A}^1_K(n,\Delta)\right| \\ &\quad + \sum_{L\in\mathcal{L}(K)} \omega_L(\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})) \\ &\qquad \times \left|\{\Pi \in \mathcal{A}^1_K(n,\Delta) \mid \Pi \otimes \omega_L \circ \mathrm{N}_{K/\mathbb{Q}} \simeq \Pi\}\right|\Big),\end{aligned}$$

where $\mathcal{L}(K)$ denotes the set of all imaginary quadratic fields $L \neq K$ with LK/K unramified.

Furthermore, if $A \subseteq \mathcal{A}^1_K(n,\Delta)$ is any subset invariant under twisting by characters in $X(\Delta)$, we can consider inside the space $L^2_{\mathrm{cusp}}(\mathrm{GL}_2(K) \setminus \mathrm{GL}(2,\mathbb{A}_K)/Z_\infty \mathcal{K}(\Delta))$ the subspace spanned by representations in A and apply the same arguments to see that it splits as a direct sum of spaces of functions supported on a single connected component. This implies that it makes sense to speak of the contribution of representations in A to each space $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}), E_n)$ and that the dimension of the corresponding subspace is given by

$$\begin{aligned}\dim H^1_{\mathrm{cusp},A}(\mathrm{SL}(2,\mathfrak{a}), E_n) &= \frac{1}{|X(\Delta)|}\Big(|A| + \sum_{L\in\mathcal{L}(K)} \omega_L(\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})) \\ &\quad \times \left|\{\Pi \in A \mid \Pi \otimes \omega_L \circ \mathrm{N}_{K/\mathbb{Q}} \simeq \Pi\}\right|\Big).\end{aligned} \tag{4–1}$$

In particular, this dimension depends only on the genus of $\mathfrak{a}$ and it assumes its maximum on the principal genus. We are especially interested in evaluating the contribution of twisted base-change forms to the cohomology, i.e., in the case $A = \mathcal{A}^{1,bc}_K(n,\Delta)$.

Definition 4.2. For the set $A_{\mathrm{bc}} = \mathcal{A}^{1,bc}_K(n,\Delta)$ define

$$\begin{aligned}H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathfrak{a}), E_n) &:= H^1_{\mathrm{cusp},A_{\mathrm{bc}}}(\mathrm{SL}(2,\mathfrak{a}), E_n) \\ &\subseteq H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}), E_n).\end{aligned}$$

Note that this definition makes sense, since the right-hand side is indeed independent of the subgroup Δ with $\Delta \cap \mathcal{O}^* = \{1\}$.

Our first goal is to describe the set $\mathcal{A}^{1,bc}_K(n,\Delta)$ in terms of holomorphic automorphic forms for $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ satisfying explicit local conditions. We also need to distinguish the automorphic representations of CM type. Recall that for any quadratic extension E/F of number fields there is a canonical map from Hecke characters of E to automorphic representations of $\mathrm{GL}(2,\mathbb{A}_F)$ [Jacquet and Langlands 70], called *automorphic induction* (notation: $\mathrm{AI}_{E/F}$).

The map is characterized by $\mathrm{AI}_{E/F}(\theta\ \chi \circ \mathrm{N}_{E/F}) = \mathrm{AI}_{E/F}(\theta) \otimes \chi$ for Hecke characters χ of F and the L-function identity $L(s, \mathrm{AI}_{E/F}(\theta)) = L(s,\theta)$. The automorphically induced representation is cuspidal if and only if $\theta^\tau \neq \theta$, where τ is the automorphism of E/F, and the fibers of the automorphic induction map are precisely the orbits $\{\theta, \theta^\tau\}$ of τ. There are compatible local induction maps, which we also denote by AI.

In the following, we fix once and for all for each $p \in \mathcal{R}$ a character θ_p of K_p^* with θ_p/θ_p^c unramified quadratic. For characters α and β of $\mathbb{Q}_p^*$ let $\mathrm{PS}(\alpha,\beta)$ be the principal series representation of $\mathrm{GL}(2,\mathbb{Q}_p)$ unitarily induced from α and β (cf. [Bushnell and Henniart 06, Section 9.11]).

Definition 4.3. For each $n \geq 0$ let $\mathcal{A}^1_\mathbb{Q}(n)$ be the set of all cuspidal automorphic representations π of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$ such that π_∞ is the holomorphic discrete series representation of weight $n+2$ and π_p is unramified for $p \notin \mathcal{R}$ and is of one of the following three types for $p \in \mathcal{R}$:

1. unramified principal series,
2. $\mathrm{PS}(\alpha, \omega_{K,p}\beta)$ with α, β unramified characters of $\mathbb{Q}_p^*$,
3. $\mathrm{AI}_{K_p/\mathbb{Q}_p}(\theta_p) \otimes \gamma$ with an unramified character γ of $\mathbb{Q}_p^*$.

For any (necessarily imaginary) quadratic extension L of $\mathbb{Q}$ let $\mathcal{A}^1_\mathbb{Q}(n;L)$ be the subset of $\mathcal{A}^1_\mathbb{Q}(n)$ consisting of representations automorphically induced from L. Recall that $\pi \in \mathcal{A}^1_\mathbb{Q}(n;L)$ if and only if $\pi \otimes \omega_L \simeq \pi$ [Labesse and Langlands 79]. We will see that the set of possible extensions L is precisely $\mathcal{L}(K)$.

The basic classification statement is the following proposition. It shows that we obtain the representations in $\mathcal{A}^{1,bc}_K(n,\Delta)$ by base change and character twists from the elliptic modular forms satisfying the local conditions of Definition 4.3. Of course, the description depends on the choice of the local characters θ_p for $p \in \mathcal{R}$.

Proposition 4.4.

(1) *If $\Pi \in \mathcal{A}^{1,bc}_K(n)$, one can find $\pi \in \mathcal{A}^1_\mathbb{Q}(n) \setminus \mathcal{A}^1_\mathbb{Q}(n;K)$ such that $\Pi \simeq \pi_K \otimes \chi$ for some finite-order idele class character χ of K.*

(2) *If $\Delta \cap \mathcal{O}_K^\times = \{1\}$, then for any $\pi \in \mathcal{A}^1_{\mathbb{Q}}(n) \setminus \mathcal{A}^1_{\mathbb{Q}}(n;K)$ there exists a finite-order idele class character χ of K with $\pi_K \otimes \chi \in \mathcal{A}^1_K(n,\Delta)$. The set of all such characters χ is a principal homogeneous space for $X(\Delta)$.*

Proof: For an automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$ we have $\pi_K \in \mathcal{A}^1_K(n)$ if and only if π is not automorphically induced from K, π_∞ is up to a character twist the holomorphic or antiholomorphic discrete series representation of weight $n+2$, and each π_p satisfies up to a character twist the local condition of Definition 4.3. It is not difficult to deduce from this the assertions of the proposition. □

We can also classify the CM representations of interest to us as follows.

Proposition 4.5.

1. *If $\mathcal{A}^1_{\mathbb{Q}}(n;L)$ is nonempty, then L is an imaginary quadratic extension of $\mathbb{Q}$ such that for all primes p the character $\omega_{L,p}$ is either unramified or the product of $\omega_{K,p}$ and an unramified character.*

2. *If for $\Pi \in \mathcal{A}^1_K(n)$ there exists a character $\gamma \neq 1$ with $\Pi \otimes \gamma \simeq \Pi$, then the character γ is necessarily of the form $\omega_{KL/K} = \omega_L \circ \mathrm{N}_{K/\mathbb{Q}}$ for some quadratic extension $L/\mathbb{Q}$ as above.*

The set of all imaginary quadratic number fields different from K and satisfying the conditions of assertion (1) of Proposition 4.5 is precisely the set $\mathcal{L}(K)$ of imaginary quadratic fields different from K for which LK/K is unramified. Equivalently, it is the set of all imaginary quadratic fields L for which the discriminant d_L is a proper divisor of the discriminant of K and the two factors d_L and d_K/d_L are coprime.

Consider now for each $\pi \in \mathcal{A}^1_{\mathbb{Q}}(n) \setminus \mathcal{A}^1_{\mathbb{Q}}(n;K)$ the set

$$\mathcal{A}^1_K(n,\Delta;\pi) := \{\Pi = \pi_K \otimes \chi | \Pi \in \mathcal{A}^1_K(n,\Delta)\}.$$

Clearly, the sets $\mathcal{A}^1_K(n,\Delta;\pi)$ form a partition of $\mathcal{A}^{1,bc}_K(n,\Delta)$. Assuming $\Delta \cap \mathcal{O}_K^* = \{1\}$, the set $\mathcal{A}^1_K(n,\Delta;\pi)$ has cardinality $|X(\Delta)|$ if π is not automorphically induced from any quadratic field L, and $|X(\Delta)|/2$ otherwise. It remains to count for any $\pi \in \mathcal{A}^1_{\mathbb{Q}}(n)$ the number of π' with $\mathcal{A}^1_K(n,\Delta;\pi') = \mathcal{A}^1_K(n,\Delta;\pi)$. We first consider the non-CM representations.

Proposition 4.6. *For $\pi \in \mathcal{A}^1_{\mathbb{Q}}(n) \setminus \bigcup_{L\in\mathcal{L}(K)\cup\{K\}} \mathcal{A}^1_{\mathbb{Q}}(n;L)$ the set*

$$\{\pi' \in \mathcal{A}^1_{\mathbb{Q}}(n) | \mathcal{A}^1_K(n,\Delta;\pi') = \mathcal{A}^1_K(n,\Delta;\pi)\}$$

consists of the twists $\pi \otimes \gamma$ for all characters γ such that γ_p is unramified for all p where π_p is unramified, and γ_p is unramified or the product of $\omega_{K,p}$ and an unramified character at the primes p where π_p is ramified. In particular, it has cardinality $2^{|R(\pi)|}$, where $R(\pi) \subseteq \mathcal{R}$ denotes the set of all primes p where π_p is ramified.

Therefore, if we want to write the cardinality of $\mathcal{A}^{1,bc}_K(n,\Delta)$ as a sum over all representations

$$\pi \in \mathcal{A}^1_{\mathbb{Q}}(n) \setminus \mathcal{A}^1_{\mathbb{Q}}(n;K),$$

each non-CM representation

$$\pi \in \mathcal{A}^1_{\mathbb{Q}}(n) \setminus \bigcup_L \mathcal{A}^1_{\mathbb{Q}}(n;L)$$

has to be weighted by the factor $|X(\Delta)| 2^{-|R(\pi)|}$.

Example 4.7. Consider the case in which a single prime p is ramified in K. In this case, the set $\mathcal{A}^1_{\mathbb{Q}}(n)$ consists of the automorphic representations associated to classical modular forms of weight $n+2$ for $\mathrm{SL}(2,\mathbb{Z})$, for $\Gamma_0(p)$ with character ω_K, or of p-power level with $\pi_p \simeq \mathrm{AI}(\theta_p) \otimes \gamma_p$, γ_p unramified. The CM forms for K have to be omitted. In this case, there are no other fields L to be considered. To obtain the dimension of $H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}), E_n)$, the dimension of the corresponding spaces of modular forms has to be weighted by a factor $\frac{1}{2}$ except in the $\mathrm{SL}(2,\mathbb{Z})$ case.

In the count for the representations in $\mathcal{A}^{1,bc}_K(n,\Delta)$, the main term is therefore given by $|X(\Delta)| \sum_{\pi\in\mathcal{A}^1_{\mathbb{Q}}(n)} 2^{-|R(\pi)|}$. The contributions from CM representations have to be modified by omitting the representations automorphically induced from the field K and weighting the contribution of the representations induced from quadratic fields $L \in \mathcal{L}(K)$ by an additional factor $\frac{1}{2}$. The reason for this is that for these representations there are more equivalences $\mathcal{A}^1_K(n,\Delta;\pi') = \mathcal{A}^1_K(n,\Delta;\pi)$ than in the non-CM case.

To give some more details, we first explicate the local conditions on CM representations in $\mathcal{A}^1_{\mathbb{Q}}(n)$. Recall the definition of the local character at infinity χ_∞ in Section 3.3.

Lemma 4.8. *Let $L \in \mathcal{L}(K)$ be an imaginary quadratic field. Write $\mathrm{AI}_{K_p/\mathbb{Q}_p}(\theta_p) = \mathrm{AI}_{L_p/\mathbb{Q}_p}(\theta_{p,L_p})$ with a character θ_{p,L_p} of L_p^* for all $p \in \mathcal{R}$, where p is nonsplit in L (note that this is possible). For an idele class character ψ of L with $\psi_\infty = \chi_\infty^{-(n+1)}$ and unramified at primes not above primes in $\mathcal{R}$ we have $\mathrm{AI}_{L/\mathbb{Q}}(\psi) \in \mathcal{A}^1_{\mathbb{Q}}(n)$ if and only if the following local conditions are satisfied:*

(1) *If $p \in \mathcal{R}$ splits in L, ψ_p is either unramified or of the form $(\alpha_p, \omega_{K,p}\beta_p)$ or $(\omega_{K,p}\alpha_p, \beta_p)$ for unramified characters α_p and β_p of $\mathbb{Q}_p^*$.*

(2) *If $p \in \mathcal{R}$ is inert in L, ψ_p is either unramified or the product of θ_{p,L_p} or θ^c_{p,L_p} and an unramified character.*

(3) *If $p \in \mathcal{R}$ ramifies in L, ψ_p is either unramified or the product of θ_{p,L_p} and an unramified character.*

We can also explicate the equivalence relation $\mathcal{A}^1_K(n, \Delta; \pi') = \mathcal{A}^1_K(n, \Delta; \pi)$ for these representations.

Lemma 4.9. *If $\mathcal{A}^1_K(n, \Delta; \pi') = \mathcal{A}^1_K(n, \Delta; \pi)$ for π, $\pi' \in \mathcal{A}^1_{\mathbb{Q}}(n)$ that are automorphically induced from quadratic extensions, they are necessarily induced from the same quadratic extension L. Furthermore, for $\pi = \mathrm{AI}_{L/\mathbb{Q}}(\psi)$ and $\pi' = \mathrm{AI}_{L/\mathbb{Q}}(\psi')$ with ψ and ψ' as above, the equivalence $\mathcal{A}^1_K(n, \Delta; \pi') = \mathcal{A}^1_K(n, \Delta; \pi)$ is true if and only if either $\delta = \psi'/\psi = \gamma \circ \mathrm{N}_{L/\mathbb{Q}}$ for some idele class character γ of $\mathbb{Q}$ or $\delta = \psi'/\psi$ satisfies $\delta/\delta^c = \omega_K \circ \mathrm{N}_{L/\mathbb{Q}}$.*

With these descriptions in hand, one can obtain a preliminary formula for the cardinality of $\mathcal{A}^{1,bc}_K(n, \Delta)$, which will in a second step be refined to a completely explicit expression. Using (4–1) we can then compute the contribution to the cohomology of each individual group $\mathrm{SL}(2, \mathfrak{a})$.

To simplify the notation, we need the following definition: for an integer n and an imaginary quadratic field L define $\nu_{L,n} \in \{0, 1\}$ as follows:

1. If L is the field $\mathbb{Q}(\sqrt{-3})$, set
$$\nu_{L,n} = \begin{cases} 1, & \text{if } n \equiv 2 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$
2. If $L = \mathbb{Q}(i)$, set
$$\nu_{L,n} = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 2, \\ 0, & \text{otherwise.} \end{cases}$$
3. If L is not one of the two exceptional fields, we simply set $\nu_{L,n} = 1$ for all n.

Proposition 4.10.

(1) *The cardinality of the set $\mathcal{A}^{1,bc}_K(n, \Delta)$ is given by*
$$\frac{\left|\mathcal{A}^{1,bc}_K(n, \Delta)\right|}{|X(\Delta)|} = \sum_{\pi \in \mathcal{A}^1_{\mathbb{Q}}(n)} 2^{-|R(\pi)|} - \nu_{K,n}\frac{h_K}{2} - \sum_{L \in \mathcal{L}(K)} \nu_{L,n} 2^{|\mathcal{R}|-|\mathcal{R}_L|-2} h_L.$$

(2) *For $L \in \mathcal{L}(K)$ we have*
$$\frac{\left|\{\Pi \in \mathcal{A}^{1,bc}_K(n, \Delta) | \Pi \otimes \omega_L \circ \mathrm{N}_{K/\mathbb{Q}} \simeq \Pi\}\right|}{|X(\Delta)|} = \nu_{L,n} 2^{|\mathcal{R}|-|\mathcal{R}_L|-2} h_L.$$

Combining this proposition with (4–1) and the fact that $\left|\mathcal{A}^1_{\mathbb{Q}}(n; K)\right| = \nu_{K,n} 2^{|\mathcal{R}|-1} h_K$, we can immediately deduce the following:

Proposition 4.11. *The dimension of the base-change part of the cohomology of the group $\mathrm{SL}(2, \mathfrak{a})$ is given by*
$$\begin{aligned}\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2, \mathfrak{a}), E_n) &= \sum_{\pi \in \mathcal{A}^1_{\mathbb{Q}}(n)} 2^{-|R(\pi)|} - \nu_{K,n}\frac{h_K}{2} \\ &\quad - \sum_{L \in \mathcal{L}(K), \omega_L(\mathrm{N}(\mathfrak{a}))=-1} \nu_{L,n} 2^{|\mathcal{R}|-|\mathcal{R}_L|-1} h_L.\end{aligned}$$

4.2 CM Classes

As a consequence we also obtain the following results on cohomology spaces associated to CM automorphic forms. We introduce the following notation.

Definition 4.12. If $A_{\mathrm{CM}} \subseteq \mathcal{A}^1_K(n, \Delta)$ is the subset of all automorphic representations automorphically induced from quadratic extensions of K, define
$$\begin{aligned}H^1_{\mathrm{CM}}(\mathrm{SL}(2, \mathfrak{a}), E_n) &:= H^1_{\mathrm{cusp}, A_{\mathrm{CM}}}(\mathrm{SL}(2, \mathfrak{a}), E_n) \\ &\subseteq H^1_{\mathrm{cusp}}(\mathrm{SL}(2, \mathfrak{a}), E_n).\end{aligned}$$

Note that the corresponding space is again independent of the choice of Δ with $\Delta \cap \mathcal{O}^* = \{1\}$.

First consider the intersection of this space with H^1_{bc}. The following proposition follows immediately from (4–1) and statement (2) of Proposition 4.10.

Proposition 4.13. *For $L \in \mathcal{L}(K)$, the representations in $\mathcal{A}_K^{1,bc}(n,\Delta)$ automorphically induced from KL contribute to $H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathfrak{a}),E_n)$ a space of dimension*

$$\begin{cases} \nu_{L,n} 2^{|\mathcal{R}|-|\mathcal{R}_L|-1} h_L, & \text{if } \omega_L(\mathrm{N}(\mathfrak{a}))=1, \\ 0, & \text{otherwise.} \end{cases}$$

We can also consider all representations in $\mathcal{A}_K^1(n,\Delta)$ automorphically induced from a fixed quadratic extension of K, necessarily of the form KL for an imaginary quadratic extension L as above. For this, let L' be the real quadratic subfield of LK and $h^+_{L'}$ its narrow ideal class number. The total number of such representations is then equal to

$$\frac{|X(\Delta)|}{2} \nu_{L,n} h_L h^+_{L'}. \tag{4–2}$$

Consequently, we obtain the following result.

Proposition 4.14. *For $L \in \mathcal{L}(K)$, the contribution of representations automorphically induced from KL to $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}),E_n)$ has dimension*

$$\begin{cases} \nu_{L,n} h^+_{L'} h_L, & \text{if } \omega_L(\mathrm{N}(\mathfrak{a}))=1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this is precisely the contribution of twisted base-change representations of the corresponding type times a factor of $h^+_{L'}/2^{|\mathcal{R}|-|\mathcal{R}_L|-1}$, which is the number of narrow ideal classes in a narrow genus of L'.

The following relation between the dimension of the cohomology spaces for $\mathrm{SL}(2,\mathfrak{a})$ and $\mathrm{SL}(2,\mathcal{O})$ follows immediately from (4–2) and (4–1), this time applied with $A = \mathcal{A}_K^1(n,\Delta)$.

Proposition 4.15. *For any fractional ideal $\mathfrak{a}$ of K we have*

$$\begin{aligned} &\dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}),E_n) \\ &\quad = \dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}),E_n) \\ &\quad - \sum_{\substack{L\in\mathcal{L}(K) \\ \omega_L(\mathrm{N}(\mathfrak{a}))=-1}} \nu_{L,n} h^+_{L'} h_L. \end{aligned}$$

Corollary 4.16. *For $L \in \mathcal{L}(K)$ there exist representations in $\mathcal{A}^1(n,\Delta)$ automorphically induced from KL that are not twisted base changes from $\mathbb{Q}$ if and only if the narrow ideal class number $h^+_{L'}$ of the real quadratic subfield L' of KL is greater than the corresponding number $g^+_{L'} = 2^{|\mathcal{R}(L')|-1}$ of genera. In this case, the contribution of these representations to the dimension of $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathfrak{a}),E_n)$ is independent of n if L is not one of the two exceptional fields $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(i)$ and in the two exceptional cases is constant on residue classes modulo 3 and 2, respectively,.*

The existence of such representations (i.e., the failure of the relation $H^1_{\mathrm{CM}} \subseteq H^1_{\mathrm{bc}}$ for the field K) is equivalent to the existence of a real quadratic field L' with $h^+_{L'} > g^+_{L'}$ and KL'/K unramified (equivalently, $d_{L'}$ divides the discriminant d_K, and $d_{L'}$ and $d_K/d_{L'}$ are coprime).

In the following table we give the real quadratic fields $L' = \mathbb{Q}(\sqrt{D})$ with the five smallest discriminants for which the criterion of Corollary 4.16 is satisfied:

$D = d_{L'}$	g^+	h^+
$136 = 8 \cdot 17$	2	4
$145 = 5 \cdot 29$	2	4
$205 = 5 \cdot 41$	2	4
$221 = 13 \cdot 17$	2	8
229	1	3

4.3 Dimension Formulas

We now deduce from the preliminary formula of Proposition 4.11 a completely explicit dimension formula for H^1_{bc}. For $p \in \mathcal{R}$ let ν_p be the exact power of p dividing the discriminant of K. We have $\nu_p = 1$ for $p \neq 2$ and $\nu_2 = 2$ or 3.

For any integer n set

$$\varepsilon_n = \begin{cases} \frac{(-1)^{n/2}}{4}, & \text{if } n \equiv 0 \pmod 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mu_n = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 3, \\ -\frac{1}{3}, & \text{if } n \equiv 2 \pmod 3, \\ \frac{1}{3}, & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

Theorem 4.17. *Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic number field with ring of integers $\mathcal{O}$ and let n be a nonnegative integer. We have*

$$\begin{aligned} &\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}),E_n) \\ &\quad = \left(\frac{1}{24} \prod_{p\in\mathcal{R}} (p^{\nu_p}+1) + c_2(-1)^{n+1} \right)(n+1) \\ &\qquad - \nu_{K,n}\frac{h_K}{2} - 2^{|\mathcal{R}|-2} + c_4\varepsilon_{n+2} + c_3\mu_{n+2} + \delta_{n,0}, \end{aligned}$$

where $\delta_{n,0}$ stands for the Kronecker delta. The constant c_2 is given by

$$c_2 = \begin{cases} 2^{|\mathcal{R}|-4}, & \text{if } p \equiv 1 \pmod 4 \text{ for all } p \in \mathcal{R},\ p \neq 2, \\ 0, & \text{otherwise.} \end{cases}$$

The constants c_4 and c_3 are given by

$$c_4 = \begin{cases} 2^{|\mathcal{R}|}, & \text{if } p \equiv 1 \text{ or } 3 \pmod 8 \text{ for all } p \in \mathcal{R}, \\ 2^{|\mathcal{R}|-1}, & \text{if } 2 \in \mathcal{R} \text{ and } p \equiv 1 \text{ or } 3 \pmod 8 \\ & \text{for all } p \in \mathcal{R},\ p \neq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c_3 = \begin{cases} 2^{|\mathcal{R}|-1}, & \text{if } p^{\nu_p} \equiv 1 \pmod 3 \text{ for all } p \in \mathcal{R}, \\ 2^{|\mathcal{R}|-2}, & \text{if } 3 \in \mathcal{R} \text{ and } p^{\nu_p} \equiv 1 \pmod 3 \\ & \text{for all } p \in \mathcal{R},\ p \neq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, for any fractional ideal $\mathfrak{a}$ of K we have

$$\begin{aligned} &\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathfrak{a}), E_n) \\ &\quad = \dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}), E_n) \\ &\qquad - \sum_{\substack{L \neq K \\ \omega_L(\mathrm{N}(\mathfrak{a}))=-1}} \nu_{L,k} 2^{|\mathcal{R}|-|\mathcal{R}_L|-1} h_L. \end{aligned}$$

It is interesting to compare the resulting lower bound for the dimension of the cohomology group $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}),\mathbb{C})$ with the lower bound obtained in [Rohlfs 85]. In [Krämer 85], a lower bound for the dimension of $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}),\mathbb{C})$ agreeing with the bound for $\dim H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}),\mathbb{C})$ given in the above theorem is derived by a different method.

Given Proposition 4.11, the proof of Theorem 4.17 rests on the dimension computations for spaces of holomorphic elliptic modular forms with fixed local components. We summarize the ingredients necessary to carry out this task in the remaining part of this subsection, while omitting some elementary computations. The possible local components are given in Definition 4.3. The dimension computation is based on the following proposition. Here, we denote by $S_k(\Gamma(N))$ the space of weight k elliptic modular forms for the principal congruence subgroup $\Gamma(N) \subseteq \mathrm{SL}(2,\mathbb{Z})$ of level N.

Proposition 4.18. *Let $N \geq 1$ and $k \geq 2$ be integers and σ a representation of $G_N = \mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ such that $\sigma(-\mathrm{Id}_2)$ is the scalar $(-1)^k$. Let $U_N \subseteq G_N$ be the subgroup of all upper triangular unipotent elements and S_3 and S_4 the images in G_N of elements of $\mathrm{SL}(2,\mathbb{Z})$ of order 3 and 4, respectively. Then*

$$\begin{aligned} &\dim \mathrm{Hom}_{G_N}(\sigma, S_k(\Gamma(N))) \\ &\quad = \frac{k-1}{12}\dim\sigma - \frac{1}{2}\dim\sigma^{U_N} + \varepsilon_k \operatorname{tr}\sigma(S_4) \\ &\qquad + \rho_k \operatorname{tr}\sigma(S_3) + \delta_{k,2}\dim\sigma^{G_N}. \end{aligned}$$

It is not difficult to prove this proposition using the description of $\mathrm{SL}(2,\mathbb{Z})$ as an amalgamated product of $\langle -S_3\rangle$ and $\langle S_4\rangle$ and the Eichler–Shimura isomorphism. Of course, it is also a consequence of the trace formula. By taking for σ a representation induced from the Borel subgroup, one recovers the classical dimension formulas for the group $\Gamma_0(N)$ with nebentype (cf. [Cohen and Oesterlé 77]).

It remains to make explicit the representations of $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$ corresponding to the local conditions of Definition 4.3 and to compute the terms appearing in Proposition 4.18. In fact, we will consider irreducible representations σ of $\mathrm{GL}(2,\mathbb{Z}/N\mathbb{Z})$ occurring in the automorphic representations π in question with multiplicity one and use their restrictions to $\mathrm{SL}(2,\mathbb{Z}/N\mathbb{Z})$.

The representations σ can be written as tensor products of representations σ_p of $\mathrm{GL}(2,\mathbb{Z}/p^{\nu_p}\mathbb{Z})$ for $p \in R(\pi)$. For the principal series representations of Definition 4.3 the necessary computation of dimensions and character values is standard, and we refer to [Cohen and Oesterlé 77]. For the convenience of the reader we repeat the results here.

The dimension of the corresponding representation σ_p is $p^{\nu_p-1}(p+1)$. The dimension of the space of $U_{p^{\nu_p}}$-invariants is 2. The character values are given by

$$\operatorname{tr}\sigma_p(S_3) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3, \\ 1, & \text{if } p = 3, \\ 2, & \text{if } p \equiv 1 \pmod 3, \end{cases}$$

and

$$\operatorname{tr}\sigma_p(S_4) = \begin{cases} 2(-1)^{(p-1)/4}, & \text{if } p \equiv 1 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

The parity of σ_p is equal to $\omega_{K,p}(-1)$.

For the supercuspidal components we can use the constructions of [Bushnell and Henniart 06].

Lemma 4.19. *Let $p \in \mathcal{R}$ and let $\mathbb{Q}_{p^2}$ be the unramified quadratic extension of $\mathbb{Q}_p$. We can write the representation $\pi_p = \mathrm{AI}_{K_p/\mathbb{Q}_p}(\theta_p)$ as $\mathrm{AI}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(\theta'_p)$ with a character θ'_p of $\mathbb{Q}^*_{p^2}$ satisfying $\theta'_p/(\theta'_p)^\tau = \omega_{K,p} \circ N_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}$. The minimal conductor of such a character is p^{ν_p}. Assume in the following that θ'_p has this minimal conductor. Then π_p contains (with multiplicity one) a unique representation of $\mathrm{GL}(2,\mathbb{Z}_p)$ that factors through $\mathrm{GL}(2,\mathbb{Z}/p^{\nu_p}\mathbb{Z})$. Let σ_p be the representation of $\mathrm{GL}(2,\mathbb{Z}/p^{\nu_p}\mathbb{Z})$ thus obtained. The dimension of σ_p is $p^{\nu_p-1}(p-1)$. If p is odd, σ_p is the cuspidal representation of $\mathrm{GL}(2,\mathbb{F}_p)$ associated to the character of $\mathbb{F}^*_{p^2}$ obtained by restricting θ'_p.*

Furthermore, we have the following values for the traces at the torsion elements of $\mathrm{SL}(2,\mathbb{Z})$*:*

$$\operatorname{tr}\sigma_p(S_3) = \begin{cases} 0, & \text{if } p \equiv 1 \pmod 3, \\ -2, & \text{if } p \equiv 2 \pmod 3,\ p > 2, \\ & \text{or } p = 2,\ \nu_p = 3, \\ -1, & \text{if } p = 3, \\ 2, & \text{if } p = 2, \nu_p = 2, \end{cases}$$

and

$$\operatorname{tr}\sigma_p(S_4) = \begin{cases} 2(-1)^{(p-3)/4}, & \text{if } p \equiv 3 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

The parity of σ_p *is* $-\omega_{K,p}(-1)$ *except in the case* $p = 2$, $\nu_p = 2$*, where it is* -1*, while* $\omega_{K,p}(-1) = -1$*.*

Proof: We briefly sketch the ingredients of the proof. Everything is based on the tame parametrization theorem of [Bushnell and Henniart 06, Theorem 20.2] and the explicit constructions in its proof. The case of odd p is covered by [Bushnell and Henniart 06, Proposition 19.1]. Dimensions and character values can then be read off from the standard description of cuspidal representations over finite fields in [Bushnell and Henniart 06, Theorem 6.4]. For $p = 2$ we need the constructions of [Bushnell and Henniart 06, Sections 19.3 and 19.4] together with [Bushnell and Henniart 06, Section 15.8] to describe the representations of $\mathrm{GL}(2,\mathbb{Z}_2)$ and to compute the character values. The dimension statement can be found in [Bushnell and Henniart 06, Lemma 27.6]. □

We can now finish the proof of Theorem 4.17. For any tensor product of local representations σ_p the dimensions and character values are obtained by multiplication. The space of U_N-invariants is nontrivial only if all local components are principal series representations. The last term in Proposition 4.18 appears only for representations of level one (i.e., for σ the trivial representation). It remains to compute for each n the sum of the contributions in Proposition 4.11 for all possible combinations of local components with total parity $(-1)^n$. This is a tedious but elementary computation, which we omit here.

4.4 Bounds for the Cohomology of Bianchi Groups

In this subsection we use the results of Section 4.3 to give some bounds for the dimension of the cohomology spaces $H^1(\mathrm{SL}(2,\mathcal{O}_d),E_n)$ as $|d|$ or n goes to infinity. The first result is a more or less obvious consequence of Theorem 4.17.

Corollary 4.20. *Let* K *be an imaginary quadratic number field with ring of integers* $\mathcal{O}_K$*. There is a bound* $C_1 > 0$ *such that*

$$\dim H^1(\mathrm{SL}(2,\mathcal{O}_K),E_n) \geq C_1 n \quad (\text{as } n \to \infty).$$

For the proof we have only to show that the coefficient of $n+1$ in the formula of Theorem 4.17 is nonnegative.

The second result we see as a complement to the following theorem, which is proved in [Belilopetsky et al. 10].

Theorem 4.21. *Let* G *be a simple Lie group with Haar measure* μ*. There is a constant* $C_2 > 0$ *such that* $d(\Gamma)$ *is at most* $C_2\mathrm{vol}(G/\Gamma)$ *for every lattice* Γ *in* G*, where* $d(\Gamma)$ *is the minimal number of generators of* Γ*.*

To use this theorem, note that

$$\mathrm{vol}\,(\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SL}(2,\mathcal{O}_d)) = \frac{|d|^{3/2}}{4\pi^2}\zeta_K(2),$$

where $\zeta_K(s)$ is the Dedekind zeta function of K; see [Elstrodt et al. 98, Section 7]. It is easy to see that $\zeta_K(2)$ is bounded between two positive real numbers for all imaginary quadratic fields K. In view of Lemma 3.1, we obtain the following corollary.

Corollary 4.22. *Let* n *be a (fixed) nonnegative integer. There is a constant* $C_3 > 0$ *such that*

$$\dim H^1(\mathrm{SL}(2,\mathcal{O}_d),E_n) \leq C_3|d|^{3/2} \quad \text{as } |d| \to \infty.$$

We remark that this result also follows from the trace-formula methods of Section 5 below.

Theorem 4.17 implies the following result.

Proposition 4.23. *Let* n *be a (fixed) nonnegative integer. There is a constant* $C_4 > 0$ *such that*

$$\dim H^1(\mathrm{SL}(2,\mathcal{O}_d),E_n) \geq C_4|d| \qquad \text{as } |d| \to \infty.$$

4.5 Base Change and Cocompact Arithmetic Groups

By [Labesse and Schwermer 86] and [Rajan 04], it is possible to use base change and the Jacquet–Langlands correspondence to study the cohomology of the cocompact arithmetic groups Γ associated with quaternion algebras defined over fields L such that the extension L/L^{tr}, where L^{tr} is the maximal totally real subfield of L, is solvable.

If L/L^{tr} is quadratic, one can obtain for all $n \geq 0$ a lower bound for $\dim H^1(\Gamma, E_n)$ by this method. The resulting bound is determined by the dimension of certain spaces of Hilbert modular forms of weight $(n+2, 2, \ldots, 2)$ for L^{tr}, and will again be linear on congruence classes. We do not go into the details here. Note that in contrast to the case of the Bianchi groups, for a particular group Γ the resulting bound will often be trivial. However, if we consider the collection of all congruence subgroups, we can easily obtain the following qualitative result concerning Question 1.15.

Proposition 4.24. *Let Γ be an arithmetic subgroup of $\mathrm{SL}(2,\mathbb{C})$ such that the field of definition L of the corresponding quaternion algebra is a quadratic extension of its maximal totally real subfield L^{tr}. Then for every $c > 0$ there exists a finite-index subgroup Δ of Γ such that*

$$\dim H^1(\Delta, E_n) > cn$$

for all $n \geq 0$.

Proof: By [Labesse and Schwermer 86], for a suitable Δ a lower bound for the dimension of the cohomology is given by the dimension of the space of Hilbert modular newforms of weight $(n+2, 2, \ldots, 2)$ for certain congruence subgroups of $\mathrm{GL}(2, L^{\mathrm{tr}})$. By adding additional local conditions, it is easily seen that it is possible to assume that the subgroups in question are torsion-free. Furthermore, their covolume can be made arbitrarily large by changing Δ. Shimizu's dimension formula [Shimizu 63] implies then that for any $c > 0$ we can find a subgroup Δ such that the dimension of the corresponding space of Hilbert modular forms is greater than or equal to $c(n+1)$ for all $n \geq 0$. □

Note that the conjecture of Waldhausen and Thurston has been verified in [Rajan 04] for all groups Γ with L/L^{tr} solvable. Under this assumption, his method also yields a generalization of Proposition 4.24, in which the representations E_n are replaced by certain twisted variants. If base change for $\mathrm{SL}(2)$ for arbitrary extensions of number fields were available, one could prove these statements for all arithmetic lattices.

5. UPPER BOUNDS FOR THE DIMENSION OF H^1

In this section we derive upper bounds for the dimension of the cohomology spaces $H^1(\Gamma, E_n)$ using the generalized Eichler–Shimura isomorphism to transform the problem into a question on multiplicities of representations in $L^2(\mathrm{SL}(2,\mathbb{C})/\Gamma)$ and then using the trace formula to obtain information on these multiplicities. We first set up the form of the trace formula we need by specializing the work of W. Hoffmann to our situation [Hoffmann 97, Hoffmann 99]. Then we consider the behavior of the dimension of $H^1(\Gamma, E_n)$ as a function of n and its behavior for fixed n as Γ varies over the standard congruence subgroups $\Gamma_0(\mathfrak{a})$ of a Bianchi group (our result is in fact slightly more general; cf. Theorem 5.5 below).

5.1 Review of the Invariant Trace Formula for $\mathrm{SL}(2,\mathbb{C})$

Let Γ be a general discrete subgroup of $G = \mathrm{SL}(2,\mathbb{C})$ of finite covolume and consider the discrete part of $L^2(G/\Gamma)$, which is a Hilbert space direct sum of irreducible unitary representations π of G, each one occurring with a finite multiplicity $m(\pi, \Gamma)$.

The irreducible unitary representations of G most important to us are the principal series representations $\pi_{m,i\nu}$ for integers m and real parameters ν, which are obtained by unitary induction from the characters

$$\sigma_{m,i\nu}(e^{u+i\theta}) = e^{i(\nu u + m\theta)}$$

of the maximal torus $T \simeq \mathbb{C}^\times$ of G. The representations $\pi_{m,i\nu}$ and $\pi_{-m,-i\nu}$ are equivalent.

We are interested in bounding the multiplicities $m(\pi_{m,0})$ from above. As explicated above, the dimension of $H^1_{\mathrm{cusp}}(\Gamma, E_n)$ is the same as the multiplicity $m(\pi_{2n+2,0})$.

We first recall the trace formula for $L^2(G/\Gamma)$ in the form in which it has been explicitly worked out by Hoffmann for lattices of rank one [Hoffmann 99]. We specialize his results to the simpler case of $G = \mathrm{SL}(2,\mathbb{C})$ and the trivial Hecke operator. As preparation, we need to recall the basic relations between orbital integrals and principal series characters and the explicit form of the Plancherel formula, for which we use [Knapp 86, Chapter XI] as a reference. We normalize measures as in that work, i.e., we use the Haar measure on G given by the product measure $dk\,dn\,da$ associated with the Iwasawa decomposition $G = KNA$, where dk gives $K = \mathrm{SU}(2)$ total measure one; dn is the standard measure on $N \simeq \mathbb{C}$, the upper triangular unipotent subgroup; and the measure da on the subgroup $A \simeq \mathbb{R}^{>0}$ of positive real diagonal matrices in G is du in the parameterization $u \mapsto \mathrm{diag}(e^u, e^{-u})$. We consider compactly supported functions $f \in C_c^\infty(G)$. To such a function is associated the function $F_f^T \in C_c^\infty(T)$

defined by

$$F_f^T(t) = e^{2u} \int_{K\times N} f(ktnk^{-1})\, dk\, dn.$$

For $g \in G$ set

$$D_G(g) = \det_{\mathfrak{g}/\mathfrak{g}_g}(1 - \mathrm{ad}(g)).$$

Then $|D_G(g)|^{1/2} = |t - t^{-1}|^2$ for a regular semisimple element g with eigenvalues t and t^{-1}, and $D_G(g) = 1$ otherwise. For $g \in G$ define the orbital integral

$$J_G(g, f) = |D_G(g)|^{1/2} \int_{G/G_g} f(xgx^{-1})dx.$$

Then $J_G(t, f) = F_f^T(t)$ for all regular elements $t \in T$ [Knapp 86, (11.13), (11.14)]. From this, one sees immediately that $F_f^T(t) = F_f^T(t^{-1})$. It is also easy to see that $F_f^T(\pm 1) = 8\pi J_G(\pm n_1, f)$ for $n_1 = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

The Fourier transform of F_f^T yields the characters of the principal series representations:

$$\Theta_{m,i\nu}(f) = \frac{1}{2\pi}\int_T F_f^T(e^{u+i\theta})e^{i(\nu u+m\theta)}\, du\, d\theta$$

and

$$F_f^T(e^{u+i\theta}) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\Theta_{m,i\nu}(f)e^{-i(\nu u+m\theta)}\, d\nu.$$

The Plancherel formula for G is given by [Knapp 86, Theorem 11.2] (up to a minor correction):

$$f(1) = \frac{1}{16\pi^2}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\Theta_{m,i\nu}(f)(m^2+\nu^2)\, d\nu.$$

For a discrete subgroup Γ of G of finite covolume let $\mathcal{C}$ be the set of all cuspidal parabolic subgroups of G, i.e., of all parabolic subgroups fixing a cusp of Γ. Let $\Gamma(*)$ be the set of all semisimple elements of Γ that do not fix a cusp together with the elements of $\Gamma \cap \{\pm 1\}$, and on the other hand, let Γ_{ce} be the set of semisimple elements of Γ different from ± 1 and stabilizing a cusp. For $\xi \in \Gamma_{ce}$ let $A(\xi)$ be the unique conjugate of the subgroup A contained in the centralizer of ξ. For the definition of the weight factor v_ξ for $\xi \in \Gamma_{ce}$ we refer to [Hoffmann 99, p. 105]. For each $P \in \mathcal{C}$ let $\Gamma_M(P)$ be the set of projections to a Levi component L of P of the elements of $\Gamma \cap P$. For $\eta \in \Gamma_M(P)$ we define constants $C(P, \eta, \Gamma)$ (called $C_P(\eta n_1, \chi_\Gamma)$ in [Hoffmann 99, p. 106]) in terms of Epstein zeta functions associated to $\eta\Gamma \cap N$. Namely, $C(P, \eta, \Gamma)$ is the constant term in the Laurent expansion at $z = 1$ of the meromorphic function

$$C(P, \eta, \Gamma; z) = \frac{2\mathrm{vol}(N/\Gamma\cap N)}{|\Gamma_M(P)|}\sum_{\xi\in\eta\Gamma\cap N, \xi\neq 1}\frac{1}{|u(\xi)|^{2z}},$$

where we choose $k_P \in K$ such that $k_P^{-1}Pk_P$ is the standard upper triangular Borel subgroup P_0 and write

$$\xi = k_P\begin{pmatrix}1 & u(\xi)\\ 0 & 1\end{pmatrix}k_P^{-1}, \quad \xi \in N.$$

The absolute value of $u(\xi)$ does not depend on the choice of k_P. For $\eta = 1$ we can write $C(P, 1, \Gamma) = 2\pi\kappa_{\Lambda(P)}/|\Gamma_M(P)|$ for the lattice $\Lambda(P) = u(\Gamma \cap N)$ in $\mathbb{C}$, where κ_Λ denotes the constant term in the expansion of $\frac{|\Lambda|}{\pi}\sum_{\lambda\in\Lambda\setminus\{0\}}|\lambda|^{-2z}$ at $z = 1$ (this notation agrees with [Hoffmann 99, Lemma 6.5.2]). We also need distributions $I_L(\eta)$, which are Arthur's invariant modifications of weighted orbital integrals (cf. [Hoffmann 99, Section 5]). Finally, let $\Phi(\sigma_{m,s})$ be the scattering matrix of Γ defined in [Elstrodt et al. 98, p. 122] (and denoted by $S(\chi_\Gamma, \tilde{w}, \sigma_\Lambda)$ there) and $\phi = \det\Phi$ its determinant (with respect to a suitable identification of the vector spaces in question, the choice of which is unimportant for our purposes). We can now quote [Hoffmann 99, Theorem 6.4], specialized to our situation.

Theorem 5.1. *For $f \in C_c^\infty(G)$ the trace of the corresponding convolution operator on the discrete part of $L^2(G/\Gamma)$ is given by*

$$\begin{aligned}
\mathrm{tr}\,\pi_\Gamma^{\mathrm{disc}}(f) &= \sum_{\{\xi\}_\Gamma\subset\Gamma(*)}\mathrm{vol}(G_\xi/\Gamma_\xi)|D_G(\xi)|^{-1/2}J_G(\xi, f)\\
&+\sum_{\{\xi\}_\Gamma\subset\Gamma_{ce}}\mathrm{vol}(G_\xi/\Gamma_\xi A(\xi))v_\xi|D_G(\xi)|^{-1/2}J_G(\xi, f)\\
&+\sum_{P\in\mathcal{C},\eta\in\Gamma_M(P)\cap\{\pm1\}}C(P,\eta,\Gamma)J_G\left(\eta\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), f\right)\\
&+\frac{1}{2}\sum_{P\in\mathcal{C},\eta\in\Gamma_M(P)}|\Gamma_M(P)|^{-1}I_L(\eta, f)\\
&+\frac{1}{4\pi}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{\phi'(\sigma_{m,i\nu})}{\phi(\sigma_{m,i\nu})}\Theta_{m,i\nu}(f)\, d\nu\\
&-\frac{1}{4}\,\mathrm{tr}\,\Phi(\sigma_{0,0})\Theta_{0,0}(f).
\end{aligned}$$

The distributions $I_L(\eta)$ can be explicitly described in terms of the character values $\Theta_{m,i\nu}(f)$. We need here only the limiting case $\eta = 1$. Denote by $\psi(s) = \Gamma'(s)/\Gamma(s)$ the logarithmic derivative of the gamma function. The following proposition follows easily from Hoffmann's work in [Hoffmann 97].

Proposition 5.2. *For the trivial element of G the distribution $I_L(1)$ is given by*

$$I_L(1,f) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\Omega_L(1,\sigma_{m,i\nu})\Theta_{m,i\nu}(f)\,d\nu + \frac{1}{2}\Theta_{0,0}(f)$$

with the function

$$\Omega_L(1,\sigma_{m,i\nu}) = \psi(1) - \operatorname{Re}\psi\left(\frac{m+i\nu}{2}\right).$$

Note that although the function ψ has a simple pole at $s=0$, the real part of $\psi(i\nu/2)$ is continuous at $\nu=0$, and in fact $\operatorname{Re}\psi(i\nu/2) = \operatorname{Re}\psi(1+i\nu/2)$.

Proof: Hoffmann considers invariant distributions I_P closely related to I_L; cf. [Hoffmann 97, p. 58 bottom] for their precise relation. The normalization factor $r_{\bar{P}P}(\sigma_{m,i\nu})$ there is up to a constant equal to $1/(|m|+i\nu)$ (cf. [Knapp and Stein 71]). The distributions $I_P(1)$ are explicitly given by [Hoffmann 97, Corollary, p. 96]. Note that we have only two roots α and $\bar\alpha$ and have to insert $\lambda(H_\alpha) = (m+i\nu)/2$ and $\lambda(H_{\bar\alpha}) = (-m+i\nu)/2$ into the expression given there. Putting everything together and using the well-known relation $\Gamma(s)\Gamma(1-s) = \pi/\sin\pi s$, one obtains the formula above. □

The reader may compare the resulting explicit trace formula, which involves only the function F_f^T on T and its Fourier transform, with the trace formula for K-bi-invariant functions f given in [Elstrodt et al. 98, Theorem 6.5.1].

5.2 The Dimension of H^1: Behavior with n

We now turn to the behavior of the multiplicities $m(\pi_{m,0},\Gamma)$ as $m\to\infty$ for a fixed group Γ. The method extends to cohomological representations of groups of real rank one without discrete series. It is an adaption of the method of [Duistermaat et al. 79, Section 9] for bounding the remainder term in Weyl's law. Our result is the following.

Theorem 5.3. *For any discrete subgroup $\Gamma\subseteq G$ of finite covolume one has*

$$m(\pi_{m,0}) = O(m^2/\log m), \quad m\to\infty.$$

As an immediate consequence we have the following corollary.

Corollary 5.4. *For any discrete subgroup $\Gamma\subseteq G$ of finite covolume one has*

$$\dim H^1(\Gamma, E_n) = O(n^2/\log n), \quad n\to\infty.$$

Proof of Theorem 5.3: By passing to a finite-index subgroup, we can assume that Γ is torsion-free and $\Gamma_M(P) = \{1\}$ for all P.

Let $m\geq 1$ and let g_0 be an even C^∞ function with support contained in $[-1,1]$, nonnegative Fourier transform h_0, and $h_0(0)>0$. Consider the functions

$$g(e^{u+i\theta}) = 2\varepsilon g_0(\varepsilon u)\cos m\theta$$

on T, with $\varepsilon>0$ being specified later. For $f\in C_c^\infty(G)$ with $F_f^T = g$ we have

$$\Theta_{\pm m,i\nu}(f) = h_0(\varepsilon^{-1}\nu),$$

and $\Theta_{n,i\nu}(f) = 0$ for $|n|\neq m$. Insert f into the trace formula of Theorem 5.1 and note that because of our assumption on Γ, the sum in the second line is empty, while the sums in the third and fourth lines involve only $\eta=1$. Also, the expression in the last line vanishes. Moving the integral involving the scattering matrix to the other side, we obtain an expression for

$$2\sum_\nu m(\pi_{m,i\nu})h_0(\varepsilon^{-1}\nu) - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\phi'(\sigma_{m,i\nu})}{\phi(\sigma_{m,i\nu})}h_0(\varepsilon^{-1}\nu)\,d\nu \tag{5–1}$$

as the sum of the remaining terms on the right-hand side (i.e., the sum of the first, third, and fourth lines). As usual, we split the sum in the first line as

$$\operatorname{vol}(G/\Gamma)f(1) + \sum_{\{\xi\}_\Gamma\subset\Gamma(*),\xi\neq1}\operatorname{vol}(G_\xi/\Gamma_\xi)|D_G(\xi)|^{-1/2}J_G(\xi,f).$$

By the Plancherel formula, we can express the first term as

$$\frac{\operatorname{vol}(G/\Gamma)}{8\pi^2}\int_{-\infty}^{\infty}h_0(\varepsilon^{-1}\nu)(m^2+\nu^2)d\nu = C_1\varepsilon m^2 + C_2\varepsilon^3$$

with constants C_1 and C_2 depending only on Γ and h_0.

By [Duistermaat et al. 79, pp. 90–91], since the absolute values of the orbital integrals $J_G(\xi,f)$ are bounded independently of m, the second term can be estimated by $C_3e^{C_4/\varepsilon}$. The third line of the trace formula is a constant multiple of $F_f^T(1)$, and therefore of the form $C_5\varepsilon$. To estimate the weighted orbital integrals, we use the standard approximation $\psi(s) = \log s + O(1)$ for $\mathrm{Re}\, s \geq \delta > 0$ to get

$$|\Omega_L(1,\sigma_{m,i\nu})| \leq \frac{1}{2}\log(m^2+\nu^2)+C_6 \leq \log m+\frac{\nu^2}{2m^2}+C_6.$$

From this one obtains

$$|I_L(1,f)| \leq (C_7 + C_8 \log m)\varepsilon + C_9\frac{\varepsilon^3}{m^2}.$$

Taking $\varepsilon = c/\log m$ with a suitable constant c, one may conclude that with this choice, the expression (5–1) is $O(m^2/\log m)$ as $m \to \infty$. Now for each m the determinant of the scattering matrix may be written as a Hadamard product

$$\phi(\sigma_{m,s}) = \phi(\sigma_{m,0})q_m^s \prod_{\eta \in P_m} \frac{s+\bar{\eta}}{s-\eta}$$

with positive real constants q_m bounded from above, where the product runs over the set P_m of poles of $\phi(\sigma_m, s)$, which all have negative real part. Note that for $m \geq 1$ the Eisenstein series and the scattering determinant cannot have a pole on the real axis, since the corresponding induced representations do not have any unitarizable subquotients. Taking logarithmic derivatives, one sees that $\log q_m - \phi'(\sigma_{m,i\nu})/\phi(\sigma_{m,i\nu})$ is positive real for all ν. Since h_0 was assumed to be nonnegative, (5–1) is therefore up to a term going to zero with m an upper bound for $m(\pi_{m,0})h_0(0)$. The theorem follows. □

We remark that for congruence subgroups of the Bianchi groups $\mathrm{SL}(2,\mathcal{O}_K)$, K imaginary quadratic, standard estimates for the logarithmic derivatives of Hecke L-functions imply that the contribution from the continuous spectrum in (5–1) is $O(\varepsilon \log m)$ as ε goes to zero for $m \to \infty$, and that it is therefore bounded with our choice of ε.

5.3 The Dimension of H^1: Congruence Subgroups of Bianchi Groups

We now consider finite-index subgroups of the Bianchi groups $\mathrm{SL}(2,\mathcal{O}_K)$. For any nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ we have the classical congruence subgroup

$$\Gamma_0(\mathfrak{a}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathcal{O}_K) \mid c \in \mathfrak{a} \right\}.$$

The index of $\Gamma_0(\mathfrak{a})$ in $\mathrm{SL}(2,\mathcal{O}_K)$ is given by the multiplicative function

$$\iota(\mathfrak{a}) = \mathrm{N}(\mathfrak{a}) \prod_{\mathfrak{p}|\mathfrak{a}} \left(1+\frac{1}{\mathrm{N}(\mathfrak{p})}\right).$$

We also need the following subgroups closely related to the principal congruence subgroups:

$$\tilde{\Gamma}(\mathfrak{a}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathcal{O}_K) \mid a \equiv d \bmod \mathfrak{a},\ b,c \in \mathfrak{a} \right\}.$$

The following theorem gives a bound for the multiplicity of a representation $\pi_{m,0}$ in $L^2(G/\Gamma\cap\Gamma_0(\mathfrak{a}))$, Γ a subgroup of finite index in $\mathrm{SL}(2,\mathcal{O}_K)$, which improves the trivial bound $O(\iota(\mathfrak{a}))$ by a logarithm.

Theorem 5.5. *Let Γ be a subgroup of finite index in $\mathrm{SL}(2,\mathcal{O}_K)$, K imaginary quadratic. Then for any fixed $m \geq 1$ we have*

$$m(\pi_{m,0}, \Gamma \cap \Gamma_0(\mathfrak{a})) = O\left(\frac{\iota(\mathfrak{a})}{\log \mathrm{N}(\mathfrak{a})}\right), \quad \mathrm{N}(\mathfrak{a}) \to \infty.$$

This theorem can be regarded as a quantitative variant of the limit multiplicity results of [de George and Wallach 78, Lott and Lück 95, Savin 89] (which, however, concern towers of normal subgroups). There is an analogous bound for the dimension of the space of Maass forms of eigenvalue $\frac{1}{4}$ for a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ [Iwaniec 84, p. 173, (3.5)]. See also [Calegari and Emerton 09] for quite strong bounds obtained by p-adic methods in restricted cases. Note also that for $\mathfrak{a} = a\mathcal{O}_K$, a a positive integer, we can get by base-change arguments a lower bound of the form $Ca = C\mathrm{N}(\mathfrak{a})^{1/2}$. If $\mathfrak{a}$ and its conjugate are relatively prime, there is no nontrivial lower bound known, and in general one does not expect one to exist (see Section 6.3 below and [Boston and Ellenberg 06, Calegari and Dunfield 06]).

Corollary 5.6. *Let Γ be a subgroup of finite index in $\mathrm{SL}(2,\mathcal{O}_K)$. Then for any fixed $n \geq 0$ we have*

$$\dim H^1(\Gamma \cap \Gamma_0(\mathfrak{a}), E_n) = O\left(\frac{\iota(\mathfrak{a})}{\log \mathrm{N}(\mathfrak{a})}\right), \quad \mathrm{N}(\mathfrak{a}) \to \infty.$$

Proof: The corresponding assertion for the cuspidal part is an immediate consequence of Theorem 5.5. To bound the dimension of the noncuspidal part use Lemma 5.7 below. □

The proof of Theorem 5.5 is again based on the trace formula. By passing to a finite-index subgroup, we can assume that Γ is torsion-free and $\Gamma_M(P) = \{1\}$ for all P. Let $\Delta \subseteq \Gamma$ be a subgroup of finite index. We may then write (cf. [Corwin 77]) for every $f \in C_c^\infty(G)$ the spectral side

$$\operatorname{tr}\pi_\Delta^{\mathrm{disc}}(f) - \frac{1}{4\pi}\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{\phi'_\Delta(\sigma_{m,i\nu})}{\phi_\Delta(\sigma_{m,i\nu})}\Theta_{m,i\nu}(f)\,d\nu + \frac{1}{4}\operatorname{tr}\Phi_\Delta(\sigma_{0,0})\Theta_{0,0}(f)$$

as the sum

$$[\Gamma:\Delta]\mathrm{vol}(G/\Gamma)f(1) + \sum_{\substack{\{\xi\}_\Gamma\subset\Gamma(*)\\ \xi\neq 1}} c_\Delta(\xi)\mathrm{vol}(G_\xi/\Gamma_\xi)|D_G(\xi)|^{-1/2}J_G(\xi,f) + \sum_{P\in\mathcal{C}_\Delta} C(P,1,\Delta)J_G(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),f) + \frac{1}{2}|\mathcal{C}_\Delta|I_L(1,f),$$

where we set

$$c_\Delta(\xi) = \left|\{\gamma\in\Delta\setminus\Gamma|\gamma\xi\gamma^{-1}\in\Delta\}\right|.$$

We now need a sequence of elementary lemmas to deal with the parabolic and hyperbolic contributions. The following well-known lemma is used to bound the parabolic contribution.

Lemma 5.7. *Let κ be the multiplicative function defined by*

$$\kappa(\mathfrak{p}^k) = \begin{cases}\mathrm{N}(\mathfrak{p})^{k/2}+\mathrm{N}(\mathfrak{p})^{k/2-1}, & k\equiv 0 \pmod 2,\\ 2\mathrm{N}(\mathfrak{p})^{(k-1)/2}, & k\equiv 1 \pmod 2.\end{cases}$$

Then we have

$$|\mathcal{C}_{\Gamma\cap\Gamma_0(\mathfrak{a})}| \le \kappa(\mathfrak{a})|\mathcal{C}_\Gamma| \le \frac{\iota(\mathfrak{a})}{\sqrt{\mathrm{N}(\mathfrak{a})}}|\mathcal{C}_\Gamma|,$$

and the first inequality is an equality for $\Gamma = \mathrm{SL}(2,\mathcal{O}_K)$.

To deal with the hyperbolic contribution, we need to consider first the numbers $c_{\Gamma\cap\Gamma_0(\mathfrak{a})}(\xi)$. The following lemma follows easily from the definitions.

Lemma 5.8. *Let $\xi\in\Gamma$ and let $\mathfrak{b}$ be the largest divisor of $\mathfrak{a}$ such that $\xi\in\tilde\Gamma(\mathfrak{b})$. Then*

$$c_{\Gamma\cap\Gamma_0(\mathfrak{a})}(\xi) \le c(\mathfrak{a},\mathfrak{b}) \le 2^{\nu(\mathfrak{a})}\mathrm{N}(\mathfrak{b}),$$

where $\nu(\mathfrak{a})$ denotes the number of prime divisors of $\mathfrak{a}$, and c is defined by extending multiplicatively

$$c(\mathfrak{p}^k,\mathfrak{p}^r) = \begin{cases}\mathrm{N}(\mathfrak{p})^r, & r<k,\\ \mathrm{N}(\mathfrak{p})^k+\mathrm{N}(\mathfrak{p})^{k-1}, & r\ge k.\end{cases}$$

For any semisimple element $\gamma\in G$ let its norm $\mathrm{N}(\gamma)\ge 1$ be the maximum value of $|t|^2$ for the two eigenvalues t of γ. We need to estimate the number of Γ-conjugacy classes of bounded norm that are contained in $\tilde\Gamma(\mathfrak{b})$. Such an estimate can be deduced from the following well-known lemma.

Lemma 5.9. *There is a constant B depending only on Γ such that every semisimple conjugacy class $\{\gamma\}_\Gamma$ in Γ with $\mathrm{N}(\gamma)\le T$ contains a representative $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with $|a|^2,|b|^2,|c|^2,|d|^2\le BT$.*

The crude estimate of the next lemma is an easy consequence. The reader may verify that a better estimate would not change the final result (apart from the constant implicit in the O).

Lemma 5.10. *For every $\delta>0$ there is a constant C depending on Γ and δ such that for all nonzero ideals $\mathfrak{b}$ of $\mathcal{O}_K$, the number of Γ-conjugacy classes in $\Gamma(*)$ with norm $\le T$ that are contained in the normal subgroup $\Gamma\cap\tilde\Gamma(\mathfrak{b})$ of Γ is bounded by $CT^{2+\delta}\mathrm{N}(\mathfrak{b})^{-2}$.*

Proof: Apply Lemma 5.9 to see that each such conjugacy class has a representative $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with $|a|^2,|b|^2,|c|^2,|d|^2\le BT$. Furthermore, $bc\neq 0$, since the conjugacy class was assumed to lie in $\Gamma(*)$. The number of possible pairs (b,c) corresponding to elements of $\tilde\Gamma(\mathfrak{b})$ is therefore bounded by $C'T^2\mathrm{N}(\mathfrak{b})^{-2}$ with C' depending only on K. For each such pair the number of possible entries a and d with $ad = 1+bc\neq 0$ is clearly bounded by $O(T^\delta)$ with a constant depending only on δ. This proves the assertion. □

Proof of Theorem 5.5: We take a test function $f\in C_c^\infty(G)$ depending on $m\ge 1$ and ε as above. We fix m and assume at first only that ε is bounded. The identity contribution to the trace formula for $\Delta = \Gamma\cap\Gamma_0(\mathfrak{a})$ is bounded by $C_1\iota(\mathfrak{a})\varepsilon$. The lattices $\Lambda(P) = u(\Delta\cap N)$ appearing in the definition of $C(P,1,\Delta) = 2\pi\kappa_{\Lambda(P)}$ are all invariant under a fixed order of the field K, and belong therefore to finitely many classes up to multiplication by elements of K^*. Using that $\kappa_\Lambda + \log|\Lambda|$ is invariant under such homotheties, this implies that the constants $\kappa_{\Lambda(P)}$ are bounded by $C+\log\mathrm{N}(\mathfrak{a})$ for a constant C. By Lemma 5.7 the parabolic contribution is therefore bounded by $C_2\iota(\mathfrak{a})(\log\mathrm{N}(\mathfrak{a}))\mathrm{N}(\mathfrak{a})^{-1/2}\varepsilon$.

As for the contribution of classes in $\Gamma(*)$, the estimates of Lemma 5.8 and Lemma 5.10 show that it is bounded by

$$\Big(2^{\nu(a)}\sum_{\mathfrak{b}|\mathfrak{a}}\frac{1}{\mathrm{N}(\mathfrak{b})}\Big)C_3e^{C_4/\varepsilon}\leq C_5(\mu)\mathrm{N}(\mathfrak{a})^{\mu}e^{C_4/\varepsilon}$$

for any $\mu > 0$. Taking $\varepsilon = c/\log \mathrm{N}(\mathfrak{a})$ with a suitable constant c, we see that the geometric side of the trace formula is indeed $O(\iota(\mathfrak{a})/\log \mathrm{N}(\mathfrak{a}))$. A positivity argument as in the proof of Theorem 5.3 yields the result. □

6. COMPUTATIONAL RESULTS FOR BIANCHI GROUPS

This section contains computational results on the dimensions of the cohomology groups $H^1(\Gamma, E_n)$, where $\Gamma = \mathrm{SL}(2,\mathfrak{a})$ is one of the Bianchi groups of Section 2.2 and n a nonnegative integer. We also consider certain congruence subgroups of $\mathrm{SL}(2,\mathcal{O}_{-1})$. The method of computation is explained in Section 3.1.

Often the resulting systems of linear equations turned out to be far to big to do computations over the rational numbers. In these cases we were able to use the lower estimates of Section 4.3 to deduce the dimension of the solution space over the complex numbers from the dimension over various finite fields.

6.1 Dimensions of $H^1(\mathrm{SL}(2,\mathcal{O}), E_n)$

Let us start with a little table. In Table 1 we have listed the dimension of the cohomology spaces $H^1(\mathrm{SL}(2,\mathcal{O}_d), E_n)$ for $d = -1, -2, -3, -7, -11$ and $0 \leq n \leq 15$. To compare these values with the dimensions of the spaces of lifted forms given in Proposition 1.3 or more generally in Theorem 4.17, it is important to know the codimension of the cuspidal cohomology. Using [Serre 70, Théorème 8, Corollaire 1] (and further information contained there), it can be easily checked that

$$\dim H^1(\mathrm{SL}(2,\mathcal{O}_K), E_n) - \dim H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_K), E_n) = \nu_{K,n}h_K$$

for all imaginary quadratic fields K, using the notation introduced in Section 4.1.

We see that the cuspidal cohomology consists only of lifted forms except in the two cases marked in boldface. These two cases will be analyzed more closely below. As a result of some heavy computer calculations we can report the following results.

n	$d=-1$	-2	-3	-7	-11
0	0	1	0	1	1
1	1	1	0	1	1
2	0	1	1	1	2
3	1	2	0	1	2
4	0	1	0	2	2
5	2	3	1	2	3
6	0	2	1	2	4
7	3	4	1	3	4
8	0	2	1	3	4
9	3	5	1	3	5
10	1	3	2	4	**8**
11	4	6	2	4	6
12	0	3	1	**6**	6
13	5	7	2	5	7
14	1	4	3	5	8
15	5	8	2	5	8

TABLE 1. Dimensions of $H^1(\mathrm{SL}(2,\mathcal{O}_d), E_n)$.

Proposition 6.1. *For $d = -1, -2, -3, -7, -11$ and r_d equal respectively to* $104, 141, 116, 132, 153$*, we have*

$$H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_d), E_n) = H^1_{\mathrm{bc}}(\mathrm{SL}(2,\mathcal{O}_d), E_n)$$

in the range $0 \leq n \leq r_d$, except in the cases $d = -7$ and $n = 12$, $d = -11$ and $n = 10$, where H^1_{bc} has codimension two in H^1_{cusp}.

It remains to report the results of the computations in the non-Euclidean cases. We have found the following.

Proposition 6.2. *Let Γ be one of the groups $\mathrm{SL}(2,\mathcal{O}_d)$ with $d = -19, -5, -6, -10, -14$ or $\mathrm{SL}(2,\mathfrak{a}_{-5})$, $\mathrm{SL}(2,\mathfrak{a}_{-6})$, $\mathrm{SL}(2,\mathfrak{a}_{-10})$, $\mathrm{SL}(2,\mathfrak{a}_{-14})$, where the ideals $\mathfrak{a}$ are as in Section 2.2, and let the nonnegative integer n be in the range $0 \leq n \leq 60$. Then $H^1_{\mathrm{cusp}}(\Gamma, E_n) = H^1_{\mathrm{bc}}(\Gamma, E_n)$.*

6.2 Hecke Operators on Nonlifted Cohomology Classes

In this subsection we give the numerical values of some of the Hecke operators on the two spaces of nonlifted cohomology classes exhibited in Section 6.1.

6.2.1 Hecke Operators on $H^1(\mathrm{SL}(2,\mathcal{O}_{-7}), E_{12})$. We consider the prime element $\pi_{11} = 2+\sqrt{-7}$ of $\mathcal{O}_{-7}$, which has degree one and norm 11. By the methods described in Section 3.2, the characteristic polynomial of the corresponding Hecke operator

$$T_{\pi_{11}} : H^1(\mathrm{SL}(2,\mathcal{O}_{-7}), E_{12}) \to H^1(\mathrm{SL}(2,\mathcal{O}_{-7}), E_{12})$$

can easily be computed to be

$$\begin{aligned}P_{\pi_{11}}(X) &= (X - 9951764)\\ &\quad\times (X^2 + 1877432X - 54779120751344)\\ &\quad\times (X^3 - 2226532X^2 - 7410075237136X\\ &\qquad - 16787944740225591 68).\end{aligned}$$

We know that there is a unique two-dimensional complement of the space of base-change classes in the cohomology space $H^1_{\text{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-7}),E_{12})$ invariant under the Hecke operators. Identifying the Hecke operators on lifted classes (see Section 3.2), we infer that the kernel

$$\mathbf{NL}(-7,12)=\ker(T_{\pi_{11}}^2+1877432T_{\pi_{11}}-54779120751344)$$

is this space of nonlifted classes. We write L_π for the restriction of the Hecke operators T_π, π a prime element of $\mathcal{O}_{-7}$, to the space $\mathbf{NL}(-7,12)$.

The following properties of the linear maps L_π hold for all prime elements π of $\mathcal{O}_{-7}$:

$\mathrm{P}_{7.1}$ $L_{-\pi}=-L_\pi$.

$\mathrm{P}_{7.2}$ $L_{\bar\pi}=-L_\pi^{\mathrm{adj}}$.

$\mathrm{P}_{7.3}$ If $\pi=p$ is a prime of degree two, then L_π is an integer scalar denoted by λ_p.

$\mathrm{P}_{7.4}$ After a suitable choice of basis for $\mathbf{NL}(-7,12)$, the matrices giving the action of the Hecke operators L_π have integral entries.

$\mathrm{P}_{7.5}$ The simultaneous splitting field for the Hecke operators L_π is $\mathbb{Q}(\sqrt{7\cdot 239})$.

Here A^{adj} stands for the adjugate of a linear map A. If A is given by a 2×2 matrix, we have

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{\mathrm{adj}}=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}.$$

Property $\mathrm{P}_{7.2}$ follows by comparing the actions of the Hecke operator T_π on the two cohomology spaces

$$H^1\left(\mathrm{SL}(2,\mathcal{O}_{-7}),\mathrm{Sym}^{12}\otimes\overline{\mathrm{Sym}}^{12}\right)$$

and

$$H^1\left(\mathrm{SL}(2,\mathcal{O}_{-7}),\overline{\mathrm{Sym}}^{12}\otimes\mathrm{Sym}^{12}\right).$$

Property $\mathrm{P}_{7.1}$ is proved by computing the automorphism ϵ induced by a matrix $E\in\mathrm{GL}(2,\mathcal{O}_{-7})$ of determinant -1 on $H^1(\mathrm{PSL}(2,\mathcal{O}_{-7}),E_{12})$. This property implies that no nonzero class in $\mathbf{NL}(-7,12)$ is the restriction of a cohomology class in $H^1(\mathrm{GL}(2,\mathcal{O}_{-7}),E_{12})$. The rest of the above properties are clear.

Examples of the scalars λ_p for primes of degree two are contained in Table 2. Examples of the integral matrices corresponding to the linear maps L_π for primes π of degree one are contained in Tables 3, 4, and 5.

The Ramanujan conjecture for $\mathrm{SL}(2,\mathcal{O}_K)$ is the assertion that all eigenvalues λ_π of the Hecke operator T_π on $H^1_{\text{cusp}}(\mathrm{SL}(2,\mathcal{O}_K),E_n)$ satisfy the inequality

$$|\lambda_\pi|\le 2\mathrm{N}(\pi)^{\frac{n+1}{2}}.$$

p	λ_p
3	−1939626
5	−747491750
13	−252803502896086
17	4756247617499746
19	5094169624293878
31	−30279773153264109058
41	−948454707467278569518
47	−9168990821180522751074
59	123833654051598471764998
61	−105716258627702854298998
73	−707186203752039245531566
83	−5005894274852029376014346
89	−980936263375178621227022
97	84206314563458516168628866

TABLE 2. Scalars λ_p for L_p on $\mathbf{NL}(-7,12)$.

p	π	A_π
2	ω	$\begin{pmatrix} 0 & 1\\ 14432 & -50\end{pmatrix}$
23	$3+2\omega$	$\begin{pmatrix} -257854600 & 4457728\\ 64333930496 & -480741000\end{pmatrix}$
7	$-1+2\omega$	$\begin{pmatrix} 44800 & 1792\\ 25862144 & -44800\end{pmatrix}$
11	$1+2\omega$	$\begin{pmatrix} 581284 & 60800\\ 877465600 & -2458716\end{pmatrix}$

TABLE 3. Hecke operators L_π on $\mathbf{NL}(-7,12)$.

It is true for the subspace H^1_{bc} as a consequence of Deligne's proof of the corresponding statement for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. The truth of the Ramanujan conjecture is a necessary condition for the existence of a motive associated to a Hecke eigenclass. As expected, the eigenvalues of the matrices L_π satisfy the Ramanujan bound $|\lambda_\pi|\le 2\mathrm{N}(\pi)^{13/2}$ within the range of our computations.

p	π	A_π
29	$-1+4\omega$	$\begin{pmatrix} -114226222 & -627200\\ -9051750400 & -82866222\end{pmatrix}$
37	$1+4\omega$	$\begin{pmatrix} 8869653750 & -61610496\\ -889162678272 & 11950178550\end{pmatrix}$
43	$5+2\omega$	$\begin{pmatrix} 42167274700 & 293147008\\ 4230697619456 & 27509924300\end{pmatrix}$
53	$3+4\omega$	$\begin{pmatrix} -229421381350 & -843922944\\ -12179495927808 & -187225234150\end{pmatrix}$
67	$-1+6\omega$	$\begin{pmatrix} 914163852100 & -1805341824\\ -26054693203968 & 1004430943300\end{pmatrix}$
71	$7+2\omega$	$\begin{pmatrix} -600257601424 & -5497094400\\ -79334066380800 & -325402881424\end{pmatrix}$

TABLE 4. Hecke operators L_π on $\mathbf{NL}(-7,12)$.

p	π	A_π
79	$1+6\omega$	$\begin{pmatrix} -775382036248 & -11236492800 \\ -162165064089600 & -213557396248 \end{pmatrix}$
107	$9+2\omega$	$\begin{pmatrix} 11411424109300 & 66437695872 \\ 958828826824704 & 8089539315700 \end{pmatrix}$
109	$7+4\omega$	$\begin{pmatrix} 8058528373122 & 78253401600 \\ 1129353091891200 & 4145858293122 \end{pmatrix}$
113	$3-8\omega$	$\begin{pmatrix} 4624127056750 & 42202431488 \\ 609065491234816 & 2514005482350 \end{pmatrix}$
127	$5+6\omega$	$\begin{pmatrix} 4687450108000 & 367392485376 \\ 5302208348946432 & -13682174160800 \end{pmatrix}$
137	$1+8\omega$	$\begin{pmatrix} 79180120345450 & -113527447552 \\ -1638428123070464 & 84856492723050 \end{pmatrix}$
149	$9+4\omega$	$\begin{pmatrix} -71276735378522 & 562548416000 \\ 8118698739712000 & -99404156178522 \end{pmatrix}$

TABLE 5. Hecke operators L_π on $\mathbf{NL}(-7,12)$.

6.2.2 Hecke Operators on $H^1(\mathrm{SL}(2,\mathcal{O}_{-11}),E_{10})$. We consider the prime element $\pi_3=(1+\sqrt{-11})/2$ of $\mathcal{O}_{-11}$, which has degree one and norm 3. By the methods described in Section 3.2, the characteristic polynomial of the corresponding Hecke operator

$$T_{\pi_3}: H^1(\mathrm{SL}(2,\mathcal{O}_{-11}),E_{10}) \to H^1(\mathrm{SL}(2,\mathcal{O}_{-11}),E_{10})$$

can easily be computed to be

$$\begin{aligned} &P_{\pi_3}(X)\\ &\quad=(X-252)(X-67)(X^2+700X+40671)\\ &\qquad\times(X^4+403X^3-439713X^2-113276475X\\ &\qquad\quad+1097145000). \end{aligned}$$

We know that there is a unique two-dimensional complement of the space of base-change classes in the cohomology space $H^1_{\mathrm{cusp}}(\mathrm{SL}(2,\mathcal{O}_{-11}),E_{10})$ invariant under the Hecke operators. We infer that the kernel

$$\mathbf{NL}(-11,10)=\ker(T_{\pi_3}^2+700T_{\pi_3}+40671)$$

is equal to this space of nonlifted classes. We write L_π for the restriction of the Hecke operator T_π, π a prime element of $\mathcal{O}_{-11}$, to the space $\mathbf{NL}(-11,10)$. The following properties of the linear maps L_π hold for all prime elements π of $\mathcal{O}_{-11}$:

$P_{11.1}$ $L_{-\pi}=L_\pi$.

$P_{11.2}$ $L_{\bar\pi}=-L_\pi^{\mathrm{adj}}$.

$P_{11.3}$ If $\pi=p$ is a prime of degree two, then L_π is an integer scalar denoted by μ_p.

p	μ_p
2	−80
7	−818885550
13	1235127129530
17	45387811032610
19	−95158947964038
29	−8701360899198758
41	−429545462511285518
43	638559982027780650
61	10654154103002912922
73	34915634850910529970
79	−688424011186184859358
83	33668143605728046010
101	15523742571431406528202

TABLE 6. Scalars μ_p for L_p on $\mathbf{NL}(-11,10)$.

$P_{11.4}$ After a suitable choice of basis for $\mathbf{NL}(-11,10)$, the matrices giving the action of the Hecke operators L_π have integral entries.

$P_{11.5}$ The simultaneous splitting field for the Hecke operators L_π is $\mathbb{Q}(\sqrt{11\cdot 43\cdot 173})$.

$P_{11.6}$ The space $\mathbf{NL}(-11,10)$ is the restriction of a subspace of $H^1(\mathrm{GL}(2,\mathcal{O}_{-11}),E_{10})$.

The case $d=-11$, $n=10$, differs from the case $d=-7$, $n=12$, since we have $L_{-\pi}=L_\pi$ for $d=-11$ and $L_{-\pi}=-L_\pi$ for $d=-7$. In the case $d=-11$, this leads directly to property $P_{11.6}$.

Examples of the scalars μ_p for primes of degree two are contained in Table 6. Examples of the integral matrices corresponding to the linear maps L_π for primes π of degree one are contained in Tables 7, 8, 9. As expected, the eigenvalues of the matrices L_π satisfy the Ramanujan bound $|\lambda_\pi|\le 2\mathrm{N}(\pi)^{11/2}$ within the range of our computations.

p	π	A_π
3	ω	$\begin{pmatrix} 0 & 1 \\ -40671 & -700 \end{pmatrix}$
5	$-2+\omega$	$\begin{pmatrix} -14203 & -26 \\ 1057446 & 3997 \end{pmatrix}$
11	$-1+2\omega$	$\begin{pmatrix} -117612 & 0 \\ 0 & -117612 \end{pmatrix}$
23	$-5+\omega$	$\begin{pmatrix} 44565050 & 22561 \\ -917578431 & 28772350 \end{pmatrix}$
31	$-4+3\omega$	$\begin{pmatrix} -124944582 & -577125 \\ 23472250875 & 279042918 \end{pmatrix}$
37	$-5+3\omega$	$\begin{pmatrix} 351981325 & 819882 \\ -33345420822 & -221936075 \end{pmatrix}$

TABLE 7. Hecke operators L_π on $\mathbf{NL}(-11,10)$.

p	π	A_π
47	$-7+2\omega$	$\begin{pmatrix} 2959574800 & 5646848 \\ -229662955008 & -993218800 \end{pmatrix}$
53	$-5+4\omega$	$\begin{pmatrix} -3591316050 & -868224 \\ 35311538304 & -2983559250 \end{pmatrix}$
59	$-8+\omega$	$\begin{pmatrix} 2044859460 & 10611525 \\ -431581333275 & -5383208040 \end{pmatrix}$
67	$-8+3\omega$	$\begin{pmatrix} 5506303200 & -13041567 \\ 530413571457 & 14635400100 \end{pmatrix}$
71	$-4+5\omega$	$\begin{pmatrix} -20524885978 & -34309625 \\ 1395406758375 & 3491851522 \end{pmatrix}$
89	$-7+5\omega$	$\begin{pmatrix} 19167086435 & 60342700 \\ -2454197951700 & -23072803565 \end{pmatrix}$

TABLE 8. Hecke operators L_π on $\mathbf{NL}(-11, 10)$.

p	π	A_π
97	$-10+3\omega$	$\begin{pmatrix} 33903167375 & 94396752 \\ -3839210300592 & -32174559025 \end{pmatrix}$
103	$-5+6\omega$	$\begin{pmatrix} -127510128200 & -435253824 \\ 17702208275904 & 177167548600 \end{pmatrix}$
113	$-11+\omega$	$\begin{pmatrix} -257969686425 & -640536456 \\ 26051258201976 & 190405832775 \end{pmatrix}$
137	$-5+7\omega$	$\begin{pmatrix} 500270562475 & -968684668 \\ 39397374132228 & 1178349830075 \end{pmatrix}$
157	$-13+3\omega$	$\begin{pmatrix} -2300340926975 & -6625408122 \\ 269461973729862 & 2337444758425 \end{pmatrix}$
163	$-11+6\omega$	$\begin{pmatrix} -174488742500 & 315270144 \\ -12822352026624 & -395177843300 \end{pmatrix}$
179	$-13+5\omega$	$\begin{pmatrix} -933096107380 & -5594462825 \\ 227532397555575 & 2983027870120 \end{pmatrix}$

TABLE 9. Hecke operators L_π on $\mathbf{NL}(-11, 10)$.

6.3 Cohomology of Congruence Subgroups

In this subsection we give some computational results concerning the dimensions of the cohomology groups $H^1(\Gamma, E_n)$, where $\Gamma \subseteq \mathrm{SL}(2, \mathcal{O}_{-1})$ is a congruence subgroup.

6.3.1 The Case of Trivial Coefficients. Here we consider the congruence subgroups

$$\Gamma^0(\mathfrak{p}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathcal{O}_{-1}) \mid b \in \mathfrak{p} \right\}$$

of $\mathrm{SL}(2, \mathcal{O}_{-1})$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O} = \mathcal{O}_{-1}$ of degree one. Note that $\Gamma^0(\mathfrak{p})$ is conjugate in $\mathrm{SL}(2, \mathcal{O})$ to the congruence subgroup $\Gamma_0(\mathfrak{p})$ considered in Section 5.3. The norm of $\mathfrak{p}$ is either 2 or a rational prime p congruent to 1 modulo 4. The index of $\Gamma^0(\mathfrak{p})$ in $\mathrm{SL}(2, \mathcal{O}_{-1})$ is $p+1$. The cohomology groups $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$ are particularly interesting for number theory, since their nonvanishing is conjectured to be related to the existence of certain elliptic curves (or more generally abelian varieties) defined over $K = \mathbb{Q}(i)$ (cf. [Cremona 84, Grunewald et al.$\tilde{7}$8, Grunewald and Mennicke 78]).

We shall report here on extensive computations of the dimensions of the spaces $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$. Note that we have

$$H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = H^1_{\mathrm{cusp}}(\Gamma^0(\mathfrak{p}), \mathbb{C})$$

and

$$H^1(\mathrm{SL}(2, \mathcal{O}), \mathbb{C}) = 0,$$

and that $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$ consists therefore entirely of new classes (cf. Section 6.3.2).

The elements A^i, $0 \le i \le p-1$, and B (cf. Section 2.2) form a system of coset representatives for $\Gamma^0(\mathfrak{p})$ in $\mathrm{SL}(2, \mathcal{O}_{-1})$. From this we obtain the following generating system for $\Gamma^0(\mathfrak{p})$:

$$A^p,\ BAB,\ BUB,\ UA^\rho,\ A^{i'}BA^i,$$
$$1 \le i, i' \le p-1,\ ii' \equiv 1 \pmod{p},$$

where $\rho^2 + 1 = 0$ in the field $\mathbb{F}_p$.

From the presentation (2–3) we may compute a presentation of the finitely generated abelian group $\Gamma^0(\mathfrak{p})^{\mathrm{ab}}$ and in particular the dimension of $\Gamma^0(\mathfrak{p})^{\mathrm{ab}} \otimes \mathbb{C}$ (which is the same as the dimension of $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$). This computation may be speeded up in the following way, i.e., the presentation of $\Gamma^0(\mathfrak{p})$ obtained from the Reidemeister–Schreier method can be simplified considerably. We shall describe a result contained in [Grunewald et al.$\tilde{7}$8] that gives such a simplification.

Let R be a commutative ring and let $\mathbf{P}^1(R, p)$ be a $(p+1)$-dimensional free R-module with basis u_x indexed by the projective line $\mathbb{P}^1(\mathbb{F}_p)$. This module has rank $p+1$ and is a $\mathrm{PGL}(2, \mathbb{F}_p)$-module by the natural permutation action on the basis elements. Let $\mathbf{U}(R, p)$ be the submodule of $\mathbf{P}^1(R, p)$ generated by u_0 and the elements

$$u_x + u_{Bx}, \quad u_x + u_{Wx}, \quad u_x + u_{Sx} + u_{S^2x}, \quad u_x + u_{Yx} + u_{Y^2x},$$

$x \in \mathbb{P}^1(\mathbb{F}_p)$, with the matrices

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -\rho & 1 \\ 1 & 0 \end{pmatrix}.$$

Define $\Phi_{\mathfrak{p}} : \mathbf{P}^1(R, p) \to \Gamma^0(\mathfrak{p})^{\mathrm{ab}} \otimes R$ by setting $\Phi_{\mathfrak{p}}(u_i) = A^{i'}BA^i$ for $i \in \mathbb{F}_p$ with $i \neq 0$ and $\Phi_{\mathfrak{p}}(u_0) = \Phi_{\mathfrak{p}}(u_\infty) = 0$. The results of [Grunewald et al.$\tilde{7}$8, Section 3] imply that the map $\Phi_{\mathfrak{p}}$ is a surjective group homomorphism with kernel equal to $\mathbf{U}(R, p)$ if R is a field of characteristic 0.

137	233	257	277	509	569	733	977
1009	1013	1021	1049	1153	1277	1373	1489
1493	1753	1997	2053	2081	2377	2441	2521
2609	2729	2917	3109	3361	3929	4013	4177
4289	4421	4597	4621	4721	5021	5237	5741
5749	5801	6029	6361	6701	6781	6793	6857
6949	7001	7069	7121	7793	7937	8297	8377
8461	8513	8537	8753	9041	9413	10357	10369
10477	10657	10729	10861	10937	11701	11953	12253
12553	13381	13457	13633	15161	15497	15569	15629
15749	16097	16349	16649	16673	17209	17921	18289
18553	18701	18869	18913	19213	19417	19841	19997

TABLE 10. Norms of degree-one primes $\mathfrak{p}$ in $\mathcal{O}_{-1}$ with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = 1$.

To avoid heavy integer computations, we take $R = \mathbb{F}_q$ for various (small) primes q, compute the dimension of $\mathbf{P}^1(R, p)/\mathbf{U}(R, p)$, and set

$$\dim_{\leq x} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = \inf\{\dim_{\mathbb{F}_q} \mathbf{P}^1(\mathbb{F}_q, p)/\mathbf{U}(\mathbb{F}_q, p)\},$$

where q ranges over all primes below x. Of course, if this number is zero for some $x \geq 5$, then also $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = 0$, and if x is sufficiently large, then $\dim_{\leq x} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$ will be equal to the dimension of $H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$.

In Table 10 we give the norms of the degree-one primes $\mathfrak{p}$ in $\mathcal{O}_{-1}$ with $N(\mathfrak{p}) \leq 20000$ and $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C} = 1$. Table 11 covers the same range and gives the norms of the degree-one primes $\mathfrak{p}$ with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = 2$. The norms of the primes with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = 3$ are 941, 1777, 5113. Those with $\dim_{\leq 500} = 4$ are 8893, 17021. The values 5 and 6 are attained for 4517, 5309 respectively. There is no prime $\mathfrak{p}$ with $N(\mathfrak{p}) \leq 20000$ and $\dim_{\leq 500} \geq 7$.

In an even more extensive search we have gone through the degree-one primes $\mathfrak{p}$ in $\mathcal{O}_{-1}$ with $N(\mathfrak{p}) \leq 60000$ and have computed $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})$. There are altogether 3018 primes below 60,000 that are congruent to 1 modulo 4. We give the number $N(r, 60000)$ of such primes with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = r$ in Table 12.

The value 8 is attained for the prime 58313.

433	709	757	853	953	1321	1549	1901
1973	2657	2753	3313	3469	3529	3637	3877
5849	5857	6689	7577	8081	9349	9629	11437
12269	12953	13093	13477	15761	16921	17033	18757
19237	19937						

TABLE 11. Norms of degree-one primes $\mathfrak{p}$ in $\mathcal{O}_{-1}$ with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = 2$.

$\boldsymbol{r}$	0	1	2	3	4	5	6	7	8	≥ 9
$\boldsymbol{N(r, 60000)}$	2728	198	73	11	4	1	1	1	1	0

TABLE 12. The number $N(r, 60000)$ of primes below 60,000 congruent to 1 modulo 4 with $\dim_{\leq 500} H^1(\Gamma^0(\mathfrak{p}), \mathbb{C}) = r$.

Let us now define for real numbers x the function

$$S(x) = x^{\frac{1}{6}} \frac{\sum_{\mathfrak{p},\ N(\mathfrak{p}) \leq x} \dim H^1(\Gamma^0(\mathfrak{p}), \mathbb{C})}{|\{\mathfrak{p} \mid N(\mathfrak{p}) \leq x\}|},$$

where the sum is extended over all degree-one prime ideals of $\mathcal{O}_{-1}$. A positive answer to Question 1.14 of the introduction is clearly equivalent to $S(x)$ tending to a limit for $x \to \infty$. The function $S(x)$ can be tabulated in the range $x \leq 60000$ as follows:

$\boldsymbol{x/1000}$	6	12	18	24	30	36	42	48	54	60
$\boldsymbol{S(x)}$	3.21	3.39	3.62	3.99	4.15	4.18	4.24	4.31	4.37	4.52

6.3.2 The Case of Nontrivial Coefficients.

We now report on some computational results on the cohomology spaces $H^1(\Gamma^0(\mathfrak{p}), E_n)$, where $\mathfrak{p}$ is a prime of $\mathcal{O}_{-1}$ of degree one and $n \geq 1$.

Let π be a generator of $\mathfrak{p}$ and let $\delta_\pi \in GL(2, \mathbb{Q}(\sqrt{-1}))$ be defined as in (3–4). The two injective homomorphisms

$$\iota_1 : \Gamma^0(\mathfrak{p}) \to SL(2, \mathcal{O}_{-1}), \quad \iota_2 : \Gamma^0(\mathfrak{p}) \to SL(2, \mathcal{O}_{-1}),$$

where ι_1 is just the injection and ι_2 is induced by conjugation with the element δ_π, give rise to an injection

$$\iota : H^1(SL(2, \mathcal{O}_{-1}), E_n) \oplus H^1(SL(2, \mathcal{O}_{-1}), E_n) \hookrightarrow H^1(\Gamma^0(\mathfrak{p}), E_n).$$

The image of ι is traditionally called the *space of old classes*. It is invariant under the Hecke operators and has an invariant complement $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_n)$.

We have found the following:

- $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_1) = 0$ for all prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leq 1000$ except for the case $N(\mathfrak{p}) = 41$, where $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_1)$ has dimension 2.
- $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_2) = 0$ for all prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leq 600$.
- $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_n) = 0$ for all prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) \leq 90$ and $3 \leq n \leq 10$.

Table 13 contains some examples of the characteristic polynomials of the Hecke operators on $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_1)$ for $N(\mathfrak{p}) = 41$. The Hecke operators on $H^1_{\text{new}}(\Gamma^0(\mathfrak{p}), E_1)$ satisfy $T_{i\pi} = -T_\pi$ for all prime elements π of degree one. There is no apparent connection between T_π and $T_{\bar{\pi}}$. We thank Haluk Sengun for help with this computation.

p	π	A_π
2	$1+i$	x^2-x-10
5	$2+i$	$(x+4)^2$
5	$2-i$	$x^2+6x-32$
13	$3+2i$	$x^2+2x-40$
13	$3-2i$	$(x+10)^2$
17	$1+4i$	$x^2-22x+80$
17	$1-4i$	$x^2+24x-20$
29	$5+2i$	$x^2+48x-80$
29	$5-2i$	x^2-164
37	$1+6i$	$x^2+4x-160$
37	$1-6i$	$x^2+30x-800$
41	$5+4i$	$x^2+48x+412$
41	$5-4i$	$-$
61	$6+5i$	$x^2-108+292$
61	$6-5i$	$x^2-12x-620$
73	$8+3i$	$x^2-106x+2440$
73	$8-3i$	$x^2-2x-3320$

TABLE 13. Characteristic polynomials of Hecke operators on $H^1_{\mathrm{new}}(\Gamma^0(\mathfrak{p}_{41}), E_1)$.

7. COHOMOLOGY OF NONARITHMETIC GROUPS

This section contains computational results on the cohomology of various geometrically constructed and mostly nonarithmetic groups. The results are discussed in more detail in the introduction. See Section 3.1 for remarks on the method of computation and especially for the notation $\dim_{\leq x}$ used below.

7.1 Klimenko's Examples

The discrete subgroups $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{C})$ described here arose in an important attempt to classify simultaneous conjugacy classes of pairs of matrices generating discrete subgroups of $\mathrm{SL}(2,\mathbb{C})$ (see [Klimenko and Sakuma 98, Klimenko and Kopteva 02, Klimenko and Kopteva 05, Klimenko and Kopteva 07, Klimenko and Kopteva 06]). We follow the notation of [Klimenko and Kopteva 07, Klimenko and Kopteva 06]; see also [Grunewald et al. 10].

7.1.1 Groups of Finite Covolume. Let $k \geq 8$ be an even integer. We set

$$t = t_k = (\exp(\pi i/k) + \exp(-\pi i/k))^2 \qquad (7\text{–}1)$$
$$= \exp(2\pi i/k) + \exp(-2\pi i/k) + 2$$

and define the matrices

$$f = f_k = \begin{pmatrix} \exp(\pi i/k) & 0 \\ 0 & \exp(-\pi i/k) \end{pmatrix}$$

and

$$g = g_k = \begin{pmatrix} \frac{1}{2}\Big(\sqrt{\frac{t}{(t-3)(4-t)}} + \sqrt{\frac{3}{t-3}}\Big) & 1 \\ \frac{t-3}{4-t} & \frac{1}{2}\Big(\sqrt{\frac{t}{(t-3)(4-t)}} - \sqrt{\frac{3}{t-3}}\Big) \end{pmatrix}.$$

Let $\mathrm{GTet}_1[k,3,3]$ be the image in $\mathrm{PSL}(2,\mathbb{C})$ of the group generated by the matrices f and g. The following properties are known:

- $\mathrm{GTet}_1[k,3,3]$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ of finite covolume with one cusp [Klimenko and Kopteva 07, Klimenko and Kopteva 06].
- $\mathrm{GTet}_1[k,3,3]$ is commensurable with a reflection group [Klimenko and Kopteva 07, Klimenko and Kopteva 06].
- $\mathrm{GTet}_1[k,3,3]$ is nonarithmetic for all k ([Grunewald et al. 10]).
- We have [Klimenko and Kopteva 07]

$$\mathrm{GTet}_1[k,3,3] = \langle f, g \mid f^k, (gf^{k/2}zf^{k/2}g^{-1}z)^3, z^2, fzgf^{-1}g^{-1}z\rangle,$$

where $z = fgfg^{-1}f$.

Using this explicit presentation, we obtain

$$\dim_{\leq 1000} H^1(\mathrm{GTet}_1[k,3,3], E_n) = \begin{cases} 2(n-1)/3+1, & \text{if } n \equiv 1 \pmod 6,\\ 2(n-2)/3+2, & \text{if } n \equiv 2 \pmod 6,\\ 2(n-3)/3+2, & \text{if } n \equiv 3 \pmod 6,\\ 2(n-4)/3+3, & \text{if } n \equiv 4 \pmod 6,\\ 2(n-5)/3+4, & \text{if } n \equiv 5 \pmod 6,\\ 2(n-6)/3+4, & \text{if } n \equiv 6 \pmod 6, \end{cases}$$

in the range $8 \leq k \leq 100$ and $1 \leq n \leq 50$. Note that these groups Γ are invariant under the complex conjugation automorphism of $\mathrm{PSL}(2,\mathbb{C})$.

This opens up the possibility to compute the trace of this involution on $H^1_{\mathrm{cusp}}(\Gamma, E_n)$ and to obtain a lower bound for the dimension of this space, following the work of Rohlfs and Krämer [Krämer 85, Rohlfs 85] on the Bianchi groups. We hope to return to this question in the future.

7.1.2 Cocompact Groups. As in the case considered before, we take from [Klimenko and Kopteva 07] (see also [Grunewald et al. 10]) a series of explicit pairs of matrices generating a discrete subgroup $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{C})$. In this case the groups Γ act on three-dimensional hyperbolic space with a compact quotient.

Let $k \geq 8$ be an even integer. We define $t = t_k$ as in (7–1) and set

$$f = f_k = \begin{pmatrix} \exp(\pi i/k) & 0 \\ 0 & \exp(-\pi i/k) \end{pmatrix},$$

and

$$g = g_k = \begin{pmatrix} \frac{1}{2}\left(\sqrt{\frac{2(t-2)}{(t-3)(4-t)}}+\sqrt{\frac{2}{t-3}}\right) & 1 \\ \frac{t-3}{4-t} & \frac{1}{2}\left(\sqrt{\frac{2(t-2)}{(t-3)(4-t)}}-\sqrt{\frac{2}{t-3}}\right) \end{pmatrix}.$$

Define $\mathrm{GTet}_1[k,3,2] \subseteq \mathrm{PSL}(2,\mathbb{C})$ to be the image in $\mathrm{PSL}(2,\mathbb{C})$ of the group generated by f and g.

The following facts are known:

- $\mathrm{GTet}_1[k,3,2]$ is a discrete and cocompact subgroup of $\mathrm{PSL}(2,\mathbb{C})$ [Klimenko and Kopteva 07, Klimenko and Kopteva 06].
- $\mathrm{GTet}_1[k,3,2]$ is commensurable with a reflection group [Klimenko and Kopteva 07, Klimenko and Kopteva 06].
- $\mathrm{GTet}_1[k,3,2]$ is nonarithmetic for all $k \geq 14$ [Grunewald et al. 10].
- We have [Klimenko and Kopteva 07]

$$\mathrm{GTet}_1[k,3,2] = \langle f, g \mid f^k, (gf^{k/2}zf^{k/2}g^{-1}z)^2, z^2, fzgf^{-1}g^{-1}z\rangle,$$

where $z = fgfg^{-1}f$.

Again these results are sufficient to compute cohomology spaces. We obtain

$$\dim_{\leq 1000} H^1(\mathrm{GTet}_1[k,3,2], E_n) = \begin{cases} n/4, & \text{if } n \equiv 0 \pmod 4, \\ (n+1)/2, & \text{if } n \equiv 1 \pmod 2, \\ (n+2)/4, & \text{if } n \equiv 2 \pmod 4, \end{cases} \tag{7–2}$$

in the range $14 \leq k \leq 100$ and $1 \leq n \leq 30$. The groups $\mathrm{GTet}_1[8,3,2]$, $\mathrm{GTet}_1[10,3,2]$ and $\mathrm{GTet}_1[12,3,2]$ are arithmetic. Compared to (7–2), the dimensions of their cohomology groups show a similar but slightly more complicated behavior. In particular, the dimensions of the cohomology spaces $H^1(\mathrm{GTet}_1[10,3,2], E_n)$ are given by linear functions on the residue classes modulo 20 within the range of our computations. Again all these groups are invariant under the complex conjugation automorphism of $\mathrm{PSL}(2,\mathbb{C})$.

7.2 Helling's Examples

Here we report on a series of two-generator discrete subgroups of $\mathrm{SL}(2,\mathbb{C})$ described in [Helling 99]. We shall keep Helling's terminology. The phenomena seen here are new.

For a nonnegative integer k, let T_k and U_k be the standard Chebyshev polynomials [Erdélyi et al. 53]. They can be defined by the relation

$$\left(x + \sqrt{x^2-1}\right)^k = T_k(x) + U_{k-1}(x)\sqrt{x^2-1},$$

for example. For a nonnegative integer m we define polynomials

$$\tilde{p}_m(x) = \begin{cases} 2T_k\left(\frac{x}{2}\right), & \text{if } m = 2k, \\ U_k\left(\frac{x}{2}\right) - U_{k-1}\left(\frac{x}{2}\right), & \text{if } m = 2k+1, \end{cases}$$

and

$$f_m(x) = \begin{cases} \tilde{p}_{m+2}(x)^2 - x^2 + 4, & \text{if } m \text{ is even}, \\ \tilde{p}_{m+2}(x)^2 - x + 2, & \text{if } m \text{ is odd}. \end{cases}$$

Table 14 contains the first ten polynomials $f_m(x)$.

Helling shows that the polynomials f_m have only nonreal zeros. For a zero z of f_m define the matrices

$$A_m = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}, \quad B_m = \begin{pmatrix} 1 & 0 \\ \frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} & 1 \end{pmatrix}, \quad C_m = \begin{pmatrix} 1 & \frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} \\ 0 & 1 \end{pmatrix}.$$

An easy computation using properties of the Chebyshev polynomials confirms that these matrices satisfy the relations

$$A_m C_m A_m^{-1} = B_m^{-1}, \quad C_m B_m C_m^{-1} B_m^{-1} = A_m^m. \tag{7–3}$$

Define Θ_m to be the group generated by the above matrices:

$$\Theta_m = \langle A_m, B_m, C_m \rangle = \langle A_m, C_m \rangle \subseteq \mathrm{SL}(2,\mathbb{C}).$$

Helling shows that for every $m \in \mathbb{N}$ there is a zero $z \in \mathbb{C}$ of f_m such that the matrix group Θ_m satisfies the following:

m	$f_m(x)$
1	$x^2 - 3x + 3$
2	$x^4 - 5x^2 + 8$
3	$x^4 - 2x^3 - x^2 + x + 3$
4	$x^6 - 6x^4 + 8x^2 + 4$
5	$x^6 - 2x^5 - 3x^4 + 6x^3 + 2x^2 - 5x + 3$
6	$x^8 - 8x^6 + 20x^4 - 17x^2 + 8$
7	$x^8 - 2x^7 - 5x^6 + 10x^5 + 7x^4 - 14x^3 - 2x^2 + 3x + 3$
8	$x^{10} - 10x^8 + 35x^6 - 50x^4 + 24x^2 + 4$
9	$x^{10} - 2x^9 - 7x^8 + 14x^7 + 16x^6 - 32x^5 - 13x^4 + 26x^3$ $+3x^2 - 7x + 3$
10	$x^{12} - 12x^{10} + 54x^8 - 112x^6 + 105x^4 - 37x^2 + 8$

TABLE 14. The first ten polynomials $f_m(x)$.

- Θ_m is a discrete and torsion-free subgroup of $\mathrm{SL}(2,\mathbb{C})$.
- Θ_m has finite covolume and exactly one cusp.
- Θ_m is defined by the relations (7–3).
- The groups Θ_1, Θ_2 are arithmetic groups, but all the other Θ_m ($m \geq 3$) are nonarithmetic.

The zero z in question is specified (up to complex conjugation) by the condition

$$|z-2| < 4\sin^2\frac{\pi}{2m}$$

for $m \geq 3$ odd and

$$\mathrm{Re}(z) > 0, \quad |z^2-4| < 4\sin^2\frac{\pi}{m}$$

for $m \geq 4$ even. For $m = 1$ or 2, z may be taken to be any zero of f_m. Concerning the cohomology of the groups Θ_m, we can report the following computations:

$\mathbf{m = 1}$: The group Θ_1 is (up to conjugacy) the famous figure-eight-knot group. It is conjugate to a congruence subgroup of $\mathrm{SL}(2,\mathcal{O}_{-3})$. For $n \leq 120$ we have

$$\dim H^1(\Theta_1, E_n) = \begin{cases} n/3, & \text{if } n \equiv 0 \pmod 3, \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3, \\ (n+1)/3+1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$\mathbf{m = 2}$: The group Θ_2 is isomorphic to the fundamental group of the lens space with fundamental group of order 2 with a knot removed. It is conjugate to a group commensurable with $\mathrm{SL}(2,\mathcal{O}_{-7})$. For $n \leq 120$ we have

$$\dim H^1(\Theta_2, E_n) = \begin{cases} n/3, & \text{if } n \equiv 0 \pmod 3, \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3, \\ (n+1)/3, & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

except in the case $n = 12$, where we have

$$\dim H^1(\Theta_2, E_{12}) = 6.$$

$\mathbf{m \geq 3}$: Here we have

$$\dim H^1(\Theta_m, E_n) = 1$$

for all $3 \leq m \leq 150$ and $1 \leq n \leq 30$. This means that in this range we have $H^1_{\mathrm{cusp}}(\Theta_m, E_n) = 0$.

7.3 A Cocompact Tetrahedral Group

Let $\mathbf{CT}(26)$ be the tetrahedral hyperbolic reflection group constructed (for example) in [Elstrodt et al. 98, Section 10] and let $\Gamma_{26} \subseteq \mathrm{PSL}(2,\mathbb{C})$ be its unique subgroup of index 2. The quotient $\mathrm{PSL}(2,\mathbb{C})/\Gamma_{26}$ is compact. The group Γ_{26} is nonarithmetic and has the presentation

$$\Gamma_{26} = \langle a, b, c \mid a^3, b^2, c^5, (ac^{-1})^2, (bc^{-1})^3, (ab)^4 \rangle.$$

Using the data from [Elstrodt et al. 98] we infer that Γ_{26} can be generated (up to conjugacy) by the matrices

$$a = \begin{pmatrix} \frac{2t^3+t^2+t+2}{5} & 1 \\ \frac{-t^3+t^2-2}{5} & \frac{-t^3-t^2-t+3}{5} \end{pmatrix},$$

$$b = \begin{pmatrix} \frac{-3t^3+t^2-4t+2}{5} & b_2 \\ c_2 & \frac{3t^3-t^2+4t-2}{5} \end{pmatrix},$$

$$c = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

where $t \in \mathbb{C}$ is a primitive tenth root of unity and c_2 is one of the two complex roots of the polynomial

$$x^4 + \frac{-6t^3+6t^2+8}{5}x^3 + \frac{-t^3+t^2-3}{5}x^2 + \frac{-4t^3+4t^2+2}{25}x + \frac{3t^3-3t^2+2}{25}.$$

The entry b_2 is determined by

$$b_2 = (-20t^3+20t^2+35)c_2^3 + (-50t^3+50t^2+80)c_2^2 + (9t^3-9t^2-17)c_2 - 4t^3 + 4t^2 + 6.$$

We have found that $H^1(\Gamma_{26}, E_n) = 0$ for $0 \leq n \leq 90$. The group Γ_{26} has 222 conjugacy classes of subgroups of index less than or equal to 24.

We have also determined the dimensions of some cohomology spaces of these subgroups. Of the 222 subgroups, 191 satisfied $H^1(\Gamma, E_n) = 0$ in the range $0 \leq n \leq 10$, thirty subgroups had $\dim_{\leq 1000} H^1(\Gamma, E_n) = 1$ in the range $0 \leq n \leq 10$. One of the 222 had $\dim_{\leq 1000} H^1(\Gamma, E_n) = 2$, again in this range.

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