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The collocation and meshless methods for differential equations in R(2)

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THE COLLOCATION AND MESHLESS METHODS FOR DIFFERENTIAL EQUATIONS IN R^2

by

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ABSTRACT

The Collocation and Meshless Methods for Differential Equations in R^2

by

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In recent years, meshless methods have become popular ones to solve differential equations. In this thesis, we aim at solving differential equations by using Radial Basis Functions, collocation methods and fundamental solutions (MFS). These methods are meshless, easy to understand, and even easier to implement.

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LIST OF ABBREVIATIONS

2D	two-dimensions
3D	three-dimensions
BEM	boundary element method
CS-RBF	compactly supported radial basis function
FDM	finite difference method
FEM	finite element method
MFS	method of fundamental solutions
MQ	Multiquadric
PDE	partial differential equation
RBF	radial basis function
TPS	thin plate spline

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CHAPTER 1

INTRODUCTION

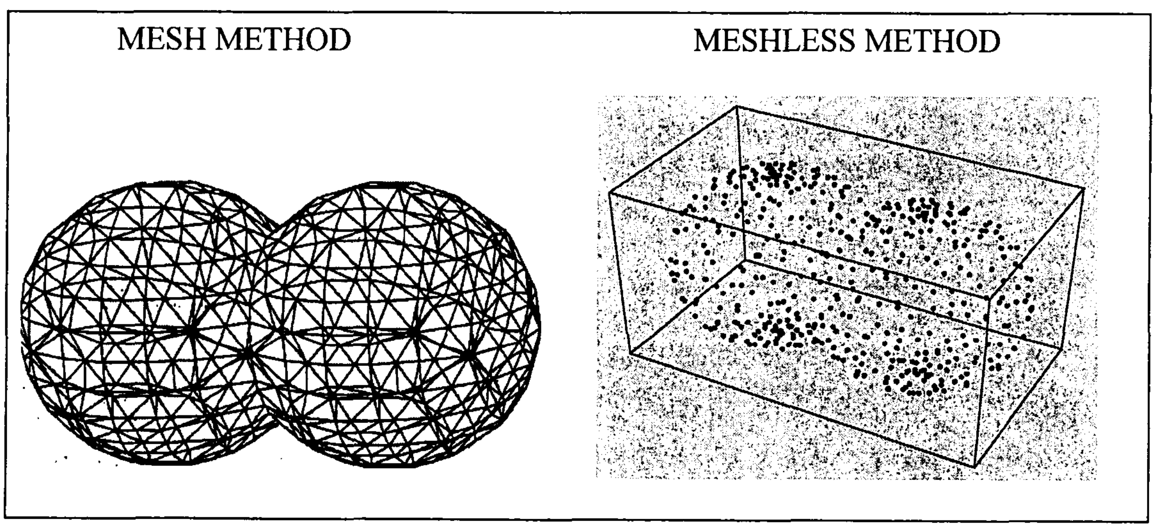
1.1 Meshless Methods

In the past several decades, the traditional methods for solving partial differential equations (PDEs) in various fields of science and engineering have been subject to the finite element methods (FEM), the finite difference methods (FDM), and boundary element method (BEM). In general, the FEM and FDM require a regular domain mesh generation to solve problems. Consequently, numerical results sometimes impair computational accuracy and the convergence rates of the methods. In spite of their great success, these conventional numerical methods still have some drawbacks that impair their computational efficiency and even limit their applicability to more practical problems. Regarding the BEM, even though it only requires mesh generation on the boundary of the domain, it involves quite sophisticated mathematics and some difficult mesh generation impedes the computational process (making it more labor intensive and time consuming) and poses an obstacle to solve more difficult, irregularly shape, and high dimensional problems. Hence, meshless methods provide an attractive alternative for solving certain problems.

In recent years there is an increasing interest in developing the Meshless Method. Most Meshless methods [Atluri and Shen 2002, Belytschko 1996, Duarte and Oden 1996] are still based on Finite Element Method (FEM), hence it is still quite complicated.

Meshless methods require the approximations of given differential equations from a set of unstructured nodes; i.e., without any pre-defined connectivity or relationship among the nodes. Instead of generating mesh, meshless methods use scattered nodes, which can be randomly distributed, throughout the computational domain. In 1990, Kansa [Kansa 1990] introduced a collocation method using radial basis function (RBFs) for solving PDEs. The advantage of this meshless method over its predecessors is the ability to use amorphous nodes that neither need to be a certain shape nor a certain pattern. Since there is no meshing required, a few hundred nodes in the meshless method would be the same as thousands of nodes required for meshing. In comparison, the meshless methods are computationally effective due to its simple implementation but with reasonable accuracy [Zerroukat, Power and Chen 1998; Li, Cheng and Chen 2003; Cheng 2003]. In this thesis, we will describe certain meshless methods and apply them to different problems.

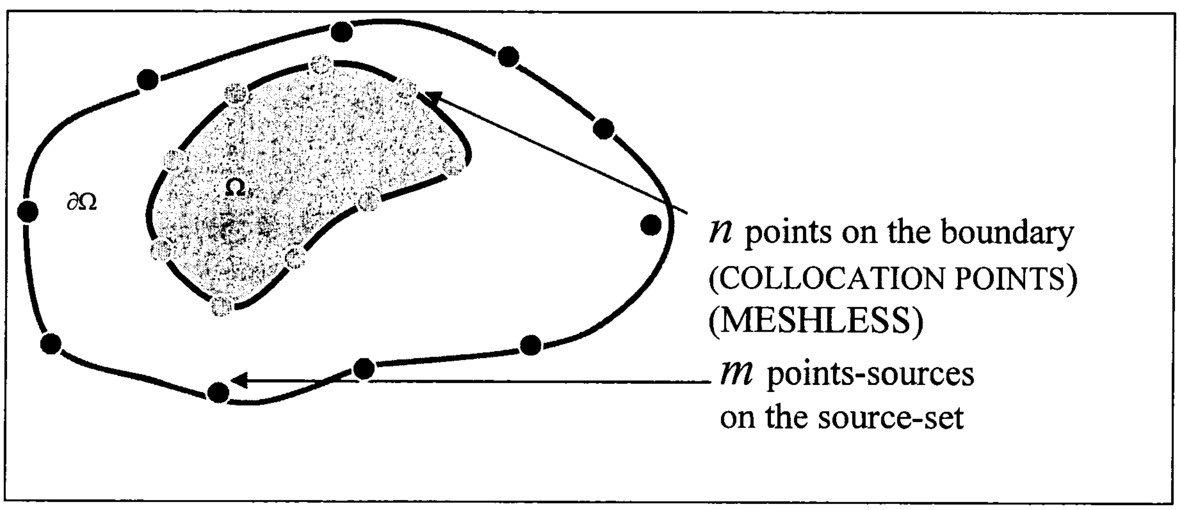
Figure 1.1: The comparison between Mesh Method and Meshless Method



1.2 Collocation Method

In mathematics, a collocation method is a method for the numerical solution of ordinary differential equations and partial differential equations and integral equations. The idea to choose a finite-dimensional space of candidate solutions (usually, polynomials up to a certain degree) and a number of points in the domain (called *collocation points*), and to select that solution which satisfies the given equation at the collocation points (see figure 1.2).

Figure 1.2: The collocation points and the source points



Consider partial differential equations

$$Lu(x) = f(x), \quad x \in \Omega \qquad (1.1)$$

$$Bu(x) = g(x), \quad x \in \partial\Omega \qquad (1.2)$$

Where L,B are differential operators, and Ω is a bounded domain in R^2.

Let $M(r) = M(\|x\|)$ be a radial basis function. Choose points

$x_i \in \Omega$,$1 \le i \le N$, $x_i \in \partial\Omega$,$N+1 \le i \le n$, and form;

$$u_n(x) = \sum_{i=1}^{n} \alpha_i M(\|x - x_i\|) \tag{1.3}$$

We want to determine $\{\alpha_i\}$ such that u_n is an approximate solution of (1.1). We use the collocation method to calculate this solution.

Namely, we set

$$Lu_n(x_i) = f(x_i) \ , \ 1 \le i \le N \tag{1.4}$$

$$Bu_n(x_i) = g(x_i) \ , \ N+1 \le i \le n \tag{1.5}$$

Which results in a nxn linear system for $\{\alpha_i\}$. Hence $\{\alpha_i\}$ can be solved. Precise examples will be given later.

1.3 Thesis Overview

This thesis focuses on examining the numerical solutions of certain boundary value problems by meshless methods in 2D. All numerical results and figures are obtained through the mathematical computer software MATLAB. All examples in this thesis are limited to second order linear differential equations in 2D. While this thesis focuses on solving equations in 2D, the process for solving problems in 3D is straightforward and can be generalized.

Chapter 2 serves as an introduction to radial basis functions (RBFs), a meshless technique used to approximate multivariate functions or surfaces. And numerical examples for solving differential equations by using RBFs are presented.

In Chapter 3, the method of fundamental solutions (MFS) is described with same examples being given. In chapter 4 we outline some lines for future directions.

CHAPTER 2

RADIAL BASIS FUNCTION

Radial basis functions (RBFs) are primarily used to reconstruct unknown, multivariate functions from the given data, by using the Collocation Methods. Such functions can be used as solutions of partial differential equations on general domains. While other classical methods require the tedious task of generating mesh on the domain and the boundary, RBFs do not. Instead, RBFs reduce multivariate functions to scalar functions, which allow the computation to be efficient, accurate, stable, easy to implement, and truly meshless.

In recent decades, employing RBFs in the fields of numerical mathematics and scientific computing has been on the rise. In R.L. Hardy [Hardy 1971], RBFs were used for geophysical surface-fitting. R. Frank [Frank 1982] assessed 29 approximation methods concerning RBFs. Today the implementation of RBF approximation is widespread: computer graphics, surface reconstruction, neural networks, picture processing and scratch removal, medical applications, science and engineering problems, etc.

2.1 Definition

A radial basis function in R^n is expressed by using a univariate real-valued function φ.

$$\Phi(x,y)=\varphi(r) \quad \text{for all x,y} \in \mathbb{R}^d \tag{2.1}$$

as a symmetric multivariate function $\Phi:\mathbb{R}^d X\,\mathbb{R}^d \to \mathbb{R}$, where Φ is called the *associated kernel*, and r represents the radius*(or Euclidean norm)* among data points:

$$r=\|x-y\| \tag{2.2}$$

For convenience, let

$$\varphi_i(r)=\varphi(\|x-x_i\|) \tag{2.3}$$

Table 2.1 is a list of commonly used radial basis functions

Table 2.1: Commonly used radial basis functions

Name	$\varphi(r)$	CPD ORDER m	Max. Dimensions d
Thin Plate Spline(TPS)	$r^2 \log r$	2	∞
Cubic Spline	r^3	2	∞
Multiquadric MQ)	$\sqrt{r^2+c^2}$	1	∞
Inverse Multiquadric	$1/\sqrt{r^2+c^2}$	0	∞
Gaussian	e^{-cr^2}	0	∞

In the table above, CPD is "conditionally positive definite" and c *(the shape parameter)* represents some constant that can be chosen to increase accuracy.

2.2 Interpolation and Approximation

Given a 2D finite set of scattered n data points (x_i, y_i), often known as *centers (or interpolation points)*, it is assumed that some function values $f\,(x_i, y_i)$ are given on these points. Based on this information, the task is to approximate a function that will fit the function values. Using RBFs, one can find a linear combination that closely approximates the function f :

$$f(x,y) \approx \tilde{f}(x,y), \tag{2.4}$$

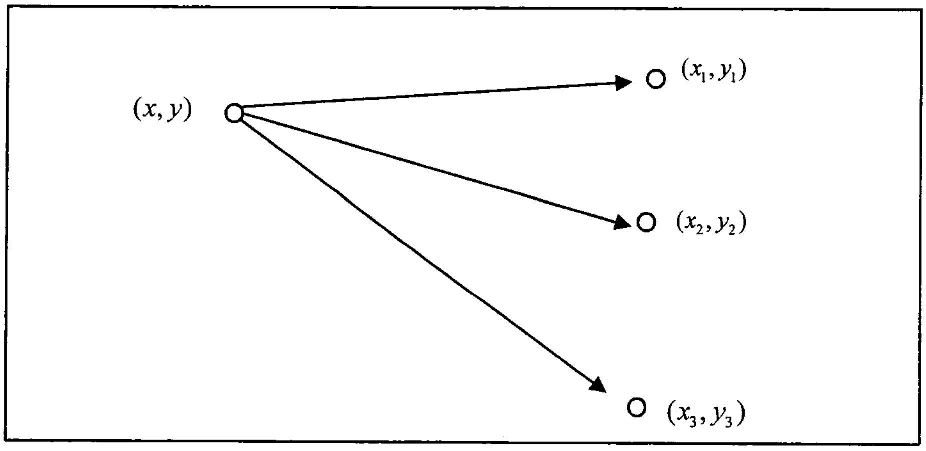
where

$$\tilde{f}(x,y) = \sum_{i=1}^{d} \alpha_i \varphi_i(x,y), \tag{2.5}$$

$$\varphi_i(x,y) = \varphi\left(\sqrt{(x-x_i)^2 + (y-y_i)^2}\right), \tag{2.6}$$

where $\{\alpha_i\}$ are unknown coefficients that are to be determined. There are measurable distances among the n centers (see Figure 2.1).

Figure 2.1 : Representation of distances among n center (n=3) in 2D



These distances are then applied to a selected radial basis function (as indicated in Table 2.1) and written as n linear combination equations. The resulting system is

$$A\bar{\alpha} = \bar{f} \tag{2.7}$$

where A is the nxn coefficient matrix of the linear equations, $\bar{\alpha}$ is the vector of corresponding unknown coefficients, and $\bar{f}$ is a vector of the associated function values. Provided that matrix A is nonsingular, the unknown coefficients $\{\alpha_i\}$ are uniquely solvable:

$$\bar{\alpha} = A^{-1}\bar{f} \tag{2.8}$$

2.3 Solving PDEs by using RBFs

We next present an example of solving a boundary value problem by using RBFs.

Example 1) Poisson equation in $\mathbb{R}^2$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - e^y, (x, y) \in \Omega \\ \\ u(x, y) = x^2 - e^y, (x, y) \in \partial\Omega \end{cases}$$

where the exact solution is given by u(x, y)= $x^2 - e^y$

and let $\tilde{u}$ (x, y) be the approximate solution, given by

$$\tilde{u}(x, y) = \sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij}\sqrt{(x - x(i))^2 + (y - y(j))^2 + c},$$

where d is the number of mesh points evenly spaced on each axis from -1 to 1, $(x(i), y(j))$ is the chosen test point, then

$$\frac{\partial \tilde{u}}{\partial x} = \sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij} \frac{x - x(i)}{\sqrt{(x-x(i))^2 + (y-y(j))^2 + c}}$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} = \sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij} \frac{\sqrt{(x-x(i))^2 + (y-y(j))^2 + c} - \frac{(x-x(i))^2}{\sqrt{(x-x(i))^2 + (y-y(j))^2 + c}}}{(x-x(i))^2 + (y-y(j))^2 + c}$$

A similar equation can be derived for $\frac{\partial^2 \tilde{u}}{\partial y^2}$. And then we get

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = \sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij} \frac{(x-x(i))^2 + (y-y(j))^2 + 2c}{((x-x(i))^2 + (y-y(j))^2 + c)^{(3/2)}}$$

where $\{\alpha_{ij}\}$ are unknown coefficients that are to be determined. To determine the approximate value for the function inside the boundary, we set

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij} \frac{(x(m)-x(i))^2 + (y(k)-y(j))^2 + 2c}{((x(m)-x(i))^2 + (y(k)-y(j))^2 + c)^{3/2}} = 2 - e^{y(k)},$$

if $(x(m), y(k))$ is inside the square. And for $(x(m), y(k))$ on the boundary, we set

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \alpha_{ij} \sqrt{(x(m)-x(i))^2 + (y(k)-y(i))^2 C} = (x(m))^2 - e^{y(k)}$$

The system of above two sets of equations are used to determinate $\{\alpha_{ij}\}$.

Let

$$e(x, y) = |u(x, y) - \tilde{u}(x, y)| \qquad (2.9)$$

be the approximate error, evaluated at 225 points evenly distributed on the square. The following table presents the approximation error e for different choices of c and d

Table 2.2: Test Example 1, absolute maximum errors e

C	d=5	d=7	d=10	d=12	d=15
0.25	0.0564	0.0227	0.0070	0.0029	6.4491e-004
0.50	1.66e-004	0.0136	0.0032	0.0011	1.668e-004
0.75	0.0309	0.0019	0.0018	5.1901e-004	5.4779e-005
1.00	0.0077	0.0065	0.0011	2.8126e-004	0.0019
1.25	0.0202	0.0048	7.3823e-004	1.6890e-004	0.0048
1.50	0.0169	0.0037	5.1037e-005	9.3630e-005	0.025

After choosing different values of *c* and *d*, we notice that a better result (the maximum error to be very small) is achieved when c= 1.25, 1.50 and when d= 10,12.

We notice when we increase c, while increasing d up to d=15 and c=1 it does not gives us a good error anymore.

Figure 2.2: Uniform distribution of 225 (x,y) pair of points

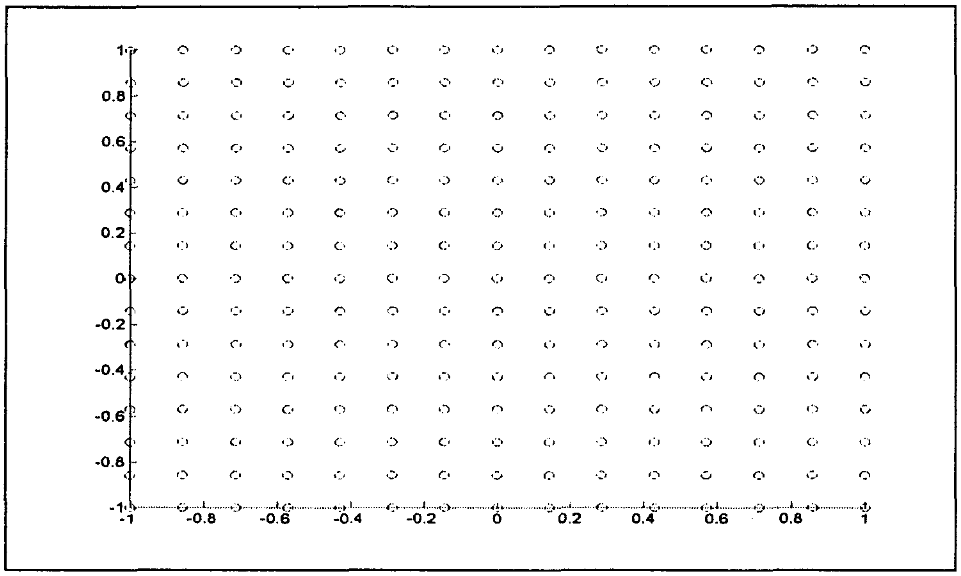


Figure 2.3: Exact solution $f(x, y) = x^2 - e^y$

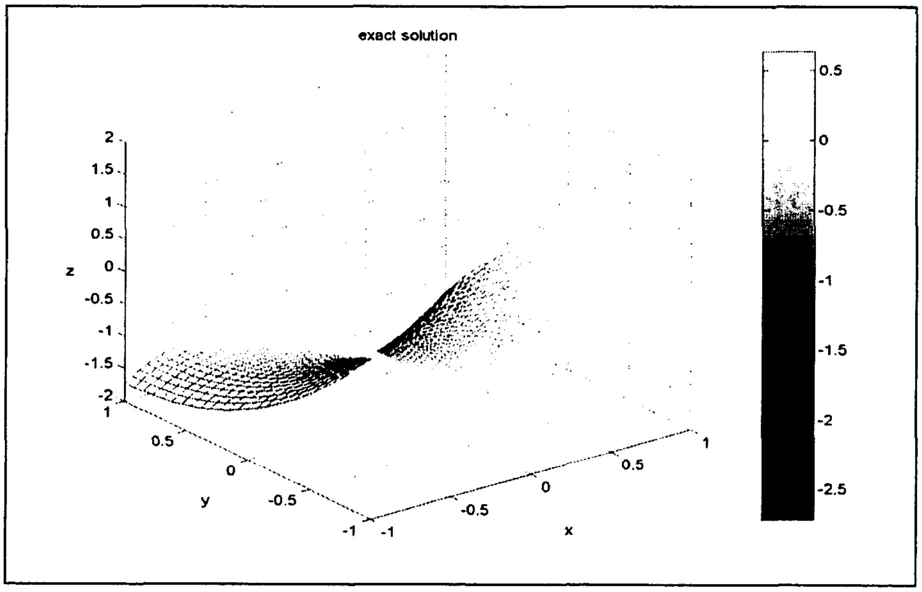


Figure 2.4: Approximate solution of $f(x,y) = x^2 - e^y$

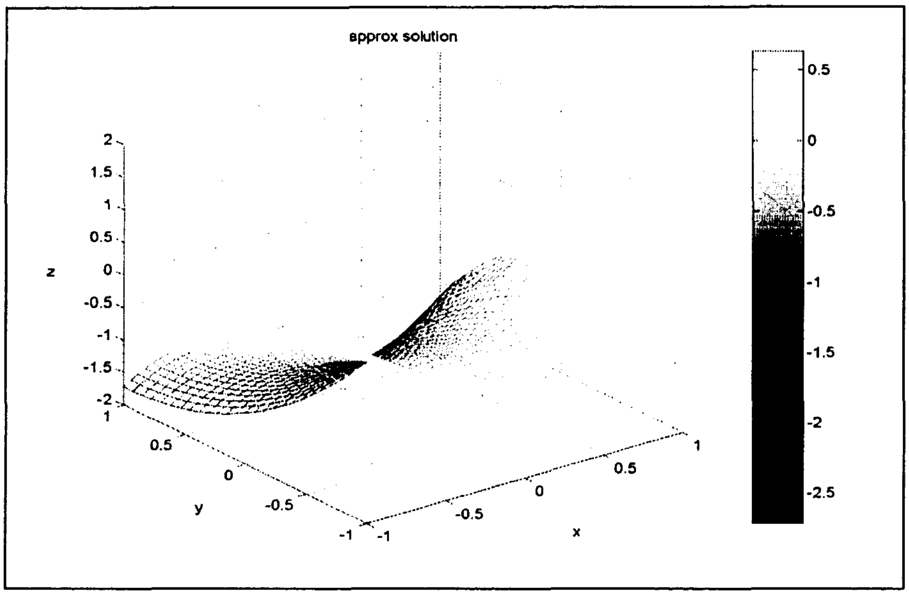
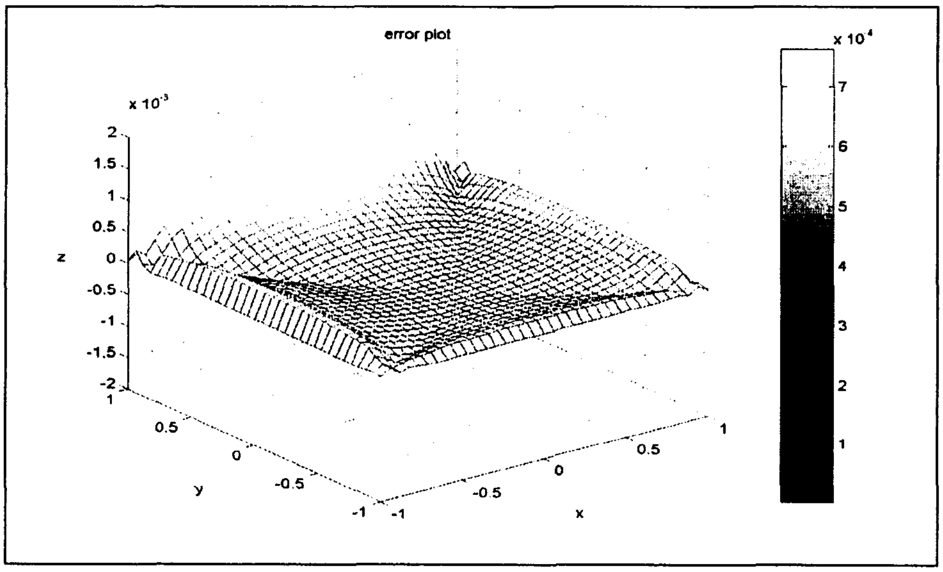


Figure2.5: Absolute error of approximate solution



Ex2) Poisson equation with Dirichlet and Neumann conditions on the boundaries

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=6y-4y\sin 2x \\ u(x,y)=y\sin 2x+y^3;(x,y)\in L1,L4 \\ \dfrac{\partial u}{\partial x}=2y\cos 2x;(x,y)\in L2 \\ \dfrac{\partial u}{\partial y}=\sin 2x+3y^2;(x,y)\in L3 \end{cases}$$

where the exact solution is given by $u(x,y)=y\sin 2x+y^3$

Similarly, we use

$$\tilde{u}(x,y)=\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\sqrt{(x-x(i))^2+(y-y(j))^2+c}$$

As an approximate solution if $(x(m),y(k))$ is inside the square, we have,

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\frac{(x(m)-x(i))^2+(y(k)-y(j))^2+2c}{((x(m)-x(i))^2+(y(k)-y(j))^2+c)^{3/2}}=6y(k)-4y(k)\sin 2x(m)$$

For $(x(m),y(k))$ on L2

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\frac{x(m)-x(i)}{\sqrt{x(m)-x(i))^2+(y(k)-y(j))^2+c}}=2y(k)\cos 2x(m)$$

Now if $(x(m),y(k))$ on L3, we have

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\frac{y(k)-y(j)}{\sqrt{(x(m)-x(i))^2+(y(k)-y(j))^2+c}}=\sin 2x(m)+3(y(k))^2 \qquad \text{for L3}$$

Finally, for $(x(m),y(k))$ on L1 and L4, we have

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\alpha_{ij}\sqrt{(x(m)-x(i))^2+(y(k)-y(j))^2}=y(k)\sin 2x(k)+(y(k))^3$$

The coefficient $\{\alpha_{ij}\}$ can be determined by $A\bar{\alpha}=\bar{f}\Rightarrow\bar{\alpha}=A^{-1}\bar{f}$

The error

$$e = |u(x,y) - \tilde{u}(x,y)| \tag{2.10}$$

where $u(x,y)$ is the exact solution and $\tilde{u}(x,y)$ is the approximate solution for u. e is calculated at 225 points evenly distributed on the square.

Table 2.3 : Test Example 2, absolute maximum error e

C	**d=5**	**d=8**	**d=10**	**d=13**	**d=15**	**d=17**
0.25	0.66	0.63	0.26	0.25	0.17	0.13
0.50	0.60	0.65	0.32	0.17	0.06	0.03
0.75	0.69	0.27	0.21	0.09	0.03	0.02
1.00	0.69	0.38	0.11	0.03	0.02	1.05
1.25	1.23	0.24	0.09	0.025	0.11	1.5
1.50	1.12	0.18	0.08	0.028	0.14	1.77

Notice that we got the best result when (c=1, d=15).

Figure 2.6 : Exact solution of $f(x, y) = y \sin 2x + y^3$

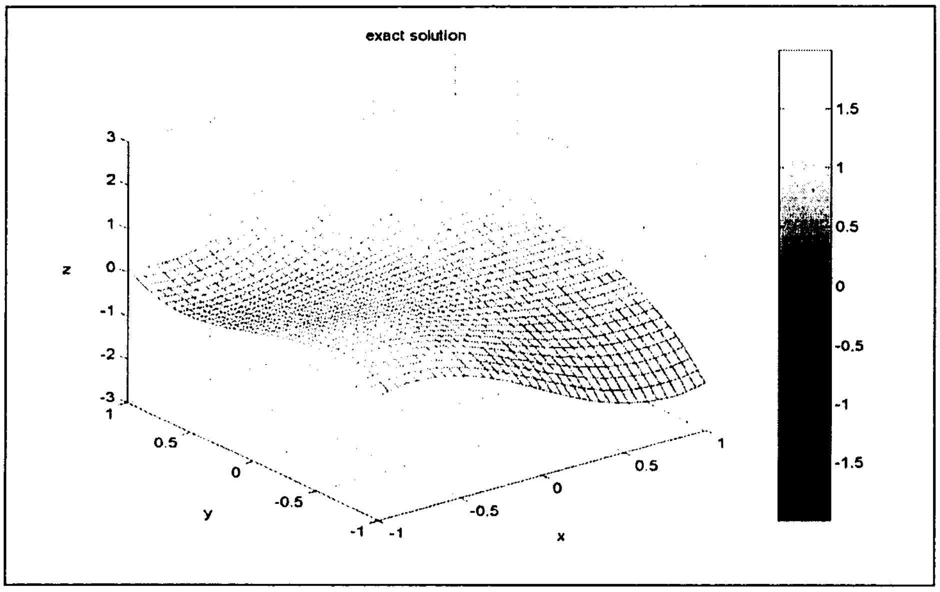


Figure 2.7: Approximate solution of $f(x, y) = y \sin 2x + y^3$

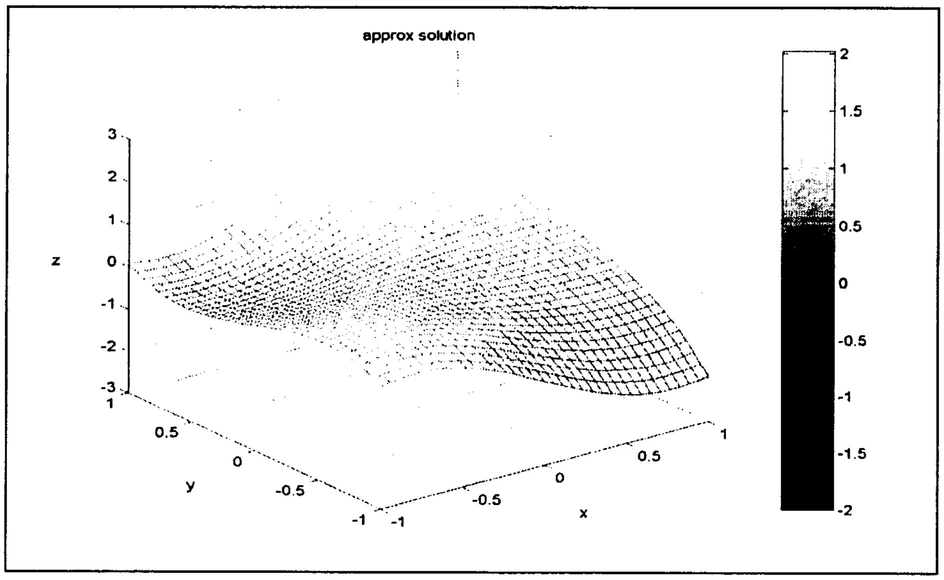
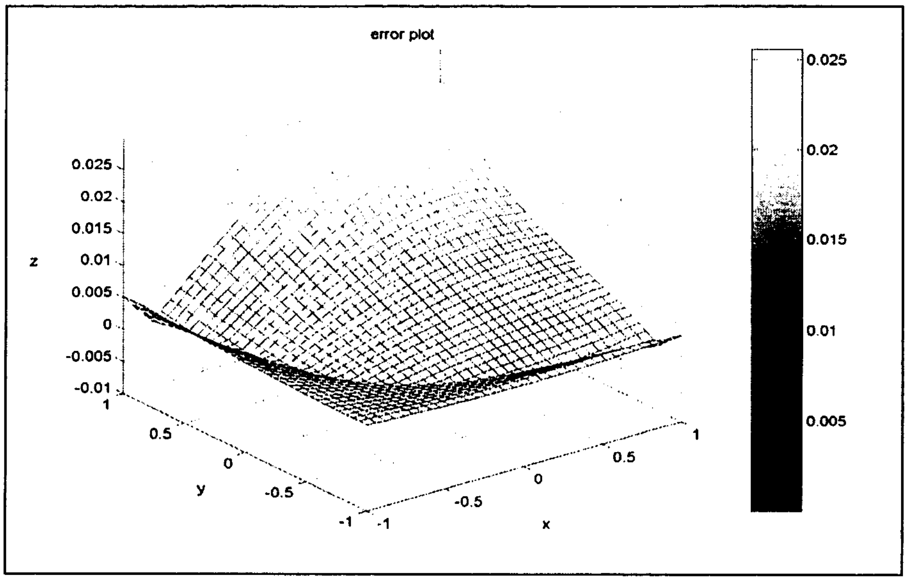


Figure2.8: Absolute error of approximate solution



2.4 Concluding Remarks

The only information being manipulated is the distance among centers. Distances are effortless to compute. For multidimensional problems, the degree of difficulty to compute scalars does not change. This component alone gives RBF approximation a huge computational advantage over the classical methods.

Solution stability and computational efficiency should always be considered when using RBFs in computation. RBFs can be adequately chosen to ensure a nonsingular interpolation matrix. The addition of polynomial terms for CPD RBFs, proper selection of the amount of interpolation points, or appropriate choice of compactly supported RBF can ensure this condition.

CHAPTER 3

THE METHOD OF FUNDAMENTAL SOLUTIONS

Lately meshless methods, such as the method of fundamental solutions (MFS) have attracted much attention in the engineering literature[Fairweather and Karageorghis 1998, Golberg and Chen 1994] for more than three decades to solve the boundary value problems (BVPs) of differential equations. Many boundary value problems in science or engineering involve domains that are irregular in shape. Such problems can be very difficult (and relatively expensive) to solve with domain meshing methods like the FEM or FDM. Instead, it is prefer to use the less costly BEM, since it only requires boundary meshing. However, this method also has its disadvantages: (1) it may involve the evaluation of singular integrals; (2) it becomes complicated when meshing surfaces in 3D; and (3) it has a slow rate of convergence due to the use of low order polynomial approximations.

Improving upon the discretization of the domain or boundary, meshless methods have evolved as the most advantageous means to solve boundary value problems of any shape. As an indirect extension of the BEM, the *method of fundamental solutions* (MFS) has emerged as a powerful meshless method for solving boundary value problems. In 1964, the MFS was initially proposed by Kupradze and Aleksidze [Kupradze and Aleksidze 1964]. The MFS has been also known as the *superposition method, desingularized method, and the charge simulation method.* Extensively studied by Cheng

[Cheng 1987], Katsurada and Okamoto [Katsurada and Okamoto 1988], Fairweather and A. Karageorghis [Fairweather and Karageorghis 1998], Goldberg and Chen [Golberg and Chen 1998] in the 1990's, and [X.Li, 2005] in the 2000's, the MFS has been established as a useful method in solving homogeneous, elliptic PDE's.

3.1 Methodology and Fundamental Solutions

The MFS approximates the solution of a PDE by using the linear combinations of the fundamental solution of the major differential operator. A generalized homogeneous PDE problem with mixed boundary conditions in 2D is

$$Lu(x,y) = 0, \qquad x, y \in \Omega \tag{3.1}$$

$$u(x,y) = g_1(x,y), \qquad x, y \in \Gamma_1 \tag{3.2}$$

$$\frac{\partial}{\partial n} u(x,y) = g_2(x,y), \qquad x, y \in \Gamma_2 \tag{3.3}$$

L is a second order linear differential operator with a known fundamental solution. Ω is simply connected, bounded and nonempty domain in $R^d, d = 2,3$. $\partial\Omega$ is the boundary, where $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. g_1 and g_2 are known functions and $\frac{\partial u}{\partial n}$ is the outward normal derivative of u to the boundary. A PDE problem having (3.2) as the only boundary condition on $\partial\Omega$ is called *Dirichlet* boundary condition. The presence of $\frac{\partial u}{\partial n}$ as the only boundary condition is called the *Neumann* boundary condition. The *Robin* boundary condition has both (3.2) and (3.3).

An approximate solution to the above PDE problem with Dirichlet boundary condition is represented by

$$\tilde{u}(x,y)=\sum_{i=1}^{m}\alpha_i G\left(\sqrt{(x-x_i)^2+(y-y_i)^2}\right), \tag{3.4}$$

where $\{\alpha_i\}$ are unknown coefficients, $G(r)$ is the associated *fundamental solution* of the governing differential operator, and r is the Euclidean norm as defined in (2.2). According to the MFS, *m collocation points* on the boundary $\partial\Omega$ and *m source points* on a fictitious boundary outside of the physical domain (see Figure 3.1) are used to approximate (3.4). the rational behind using a fictitious boundary is to ensure that there will be no singularities in the linear combinations of the selected fundamental solution. The Euclidean distance between a point and itself on the boundary $\partial\Omega$ would yield $r=0$; this may pose a problem for calculations involving the fundamental solution. To approximate the solution to the PDE problem with mixed boundary conditions $\{\alpha_i\}$ must satisfy the following linear equations:

$$\tilde{g}_1 \approx \sum_{i=1}^{\hat{m}}\alpha_i G\left(\sqrt{(x-x_i)^2+(y-y_i)^2}\right), \tag{3.5}$$

$$\tilde{g}_2 \approx \sum_{i=\hat{m}+1}^{m}\alpha_i \frac{\partial}{\partial n} G\left(\sqrt{(x-x_i)^2+(y-y_i)^2}\right), \tag{3.6}$$

where $\hat{m}$ is the number of collocation points on Γ_1 and m is the total number of source points. (cf. X. Li [12-14] for details)

3.2 Numerical Implementation

Let us consider the following PDE problem with Dirichlet boundary condition:

$$\Delta u = 0, \qquad x, y \in \Omega, \tag{3.7}$$

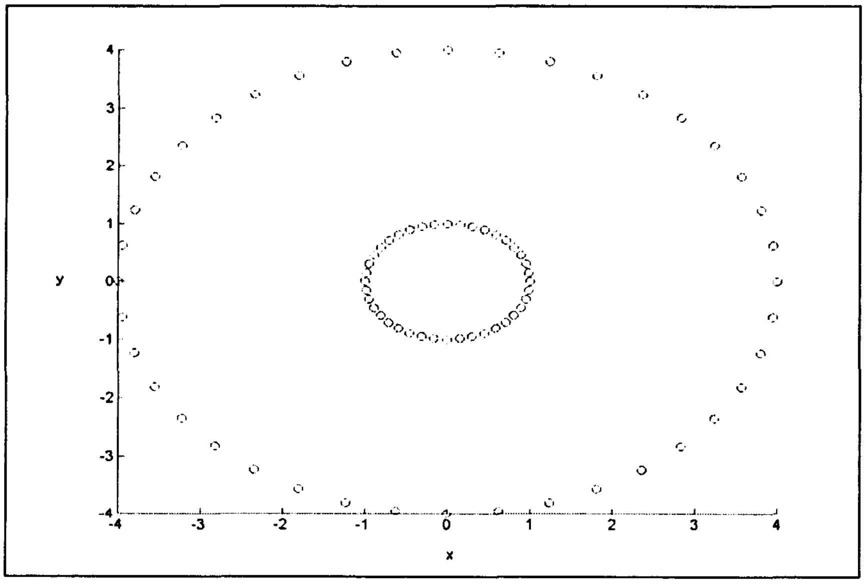
$$u = e^x \sin y, \qquad x, y \in \partial\Omega, \tag{3.8}$$

defined on domain (a circle):

$$\Omega \cup \partial\Omega = \left\{ x^2 + y^2 \le 1 \right\}. \tag{3.9}$$

For this problem, we use 30 uniformly distributed collocation points and 30 uniformly distributed source points on a fictitious circle with radius $r = 4$ (see Figure 3.1). The approximation is then tested on 30 uniformly distributed collocation points.

Figure 3.1: Distribution of collocation and source points



For the MFS, the shape of the fictitious boundary can copy the shape of the original boundary. However, for irregularly shaped boundaries, it may be difficult to construct a fictitious boundary with the same shape. For simplicity in computer coding, it is easier to use a circle as a fictitious boundary. Despite their shapes, fictitious boundaries serve the same purpose: to cancel out singularities that may occur on the original boundaries. Therefore, it is best to use a fictitious circle in 2D and a fictitious sphere in 3D

When the domain and the fictitious boundary are both circles, research by Cheng [Cheng 1987] has proven that the accuracy of the MFS improves as the fictitious radius enlarges. However, when the domain is not circular, this result is not necessarily applicable for many boundary value problems, the condition number of the collocation matrix increases exponentially and the MFS equations can become highly ill-conditioned as the fictitious radius increases [Golberg and Chen 1998]. When testing for MFS accuracy, it appears that there will be a critical fictitious radius that will minimize absolute maximum error.

As a result, the more collection/ source points used in the MFS equations, the more accurate the approximation. Accuracy will increase with an increase in collocation/source points, but there is a critical amount of points after which accuracy will no longer improve. Regarding the previous example, accuracy was relatively high for as little as 30 Collocation/source points.

The placement of collocation and source points is another significant factor in the accuracy of the MFS. So far, there is no theoretical result regarding the optimal selection of these points. In general, we choose uniformly distributed source and collocation points. Source points are normally equally spaced on the fictitious circle. Because the domain can be oddly-shaped, it may seem tricky to have equally-spaced collocation points along the same angles at which source points are placed.

Example 3) Solving PDE equation with fundamental solution method

$$\begin{cases} \Delta u = 0, (x, y) \in \Omega \\ f(x, y) = e^x \sin y, (x, y) \in \partial\Omega \end{cases} \tag{3.10}$$

where the exact solution $u = e^x \sin y$, and $\Omega = \{\|(x, y)\| \le r_1\}, \partial\Omega = \{x^2 + y^2 = r_1\}$

Let $G(x, y) = \frac{1}{2\pi}\ln(x^2 + y^2)$, where G is the fundamental solution of $\Delta u = 0$.

Choose $\tilde{\Omega} = \{x^2 + y^2 \le r_2\}, r_2 > r_1$

Choose m points $(\tilde{x}_i, \tilde{y}_i)$ on $\partial\Omega$, and $m = 20$ points $(\tilde{x}_i, \tilde{y}_i)$ on $\partial\tilde{\Omega}$, which are equally distributed on $\partial\Omega, \partial\tilde{\Omega}$, respectably.

From

$$u_m(x, y) = \sum_{i=1}^{m} \alpha_i \ln((x - \tilde{x}_i)^2 + (y - \tilde{y}_i)^2), \tag{3.11}$$

which satisfies $\Delta u = 0$ in Ω

And the boundary value function

$$f(x, y) = e^x \sin y. \tag{3.12}$$

Set

$$u_m(x_j, y_j) = f(x_j, y_j), 1 \le j \le m, \tag{3.13}$$

or

$$\sum_{i=1}^{m} \alpha_i \ln((x_j - \tilde{x}_i)^2 + (y_j - \tilde{y}_i)^2) = f(x_j, y_j), 1 \le j \le m, \tag{3.14}$$

where $\{\alpha_i\}$ is the coefficient and can be determined by $A\bar{\alpha}_i = \bar{f}$ where A is mxm matrix and f is the function given above.

Now we can estimate the error in Ω by:

$$e(x,y) = u(x,y) - u_m(x,y) \tag{3.15}$$

Where the maximum error will be determined:

$$\max|e(x,y)| \tag{3.16}$$

Table 3.1: Test Example 3, absolute maximum error e (rin= r_1, the radius of the inside circle and rout= r_2, the radius of outside circle (fictitious))

	rin=2	**rin=3**	**rin=4**	**rin=5**
rout=4	6.2598e-009	4.2229e-005	NaN	1.499
rout=5	2.7473e-010	2.5560e-007	4.1277e-004	NaN
rout=6	5.8520e-009	1.0965e-008	2.0781e-006	9.1783e-004
rout=7	9.5123e-009	3.7895e-010	1.0784e-007	1.8860e-005
rout=8	1.3209e-007	2.1434e-009	3.6866e-009	3.6866e-009
rout=9	3.8813e-007	1.6973e-008	2.1361e-009	3.1267e-008
rout=10	6.9148e-006	1.3821e-007	1.3553e-010	1.3553e-010
rout=11	7.3803e-005	4.2323e-007	8.1454e-009	6.5172e-010
rout=12	2.1410e-004	1.1867e-007	2.4174e-007	2.4174e-007

absolute maximum error, which refers to the greatest error among these test points and the exact solution u. The best error is when increasing radius of the inside circle and on the same time increasing the radius of the outside circle we will get the best result but the difference between these two radiuses should not be more than 10.

Figure 3.2: Exact solution of $f(x,y) = e^x \sin y$

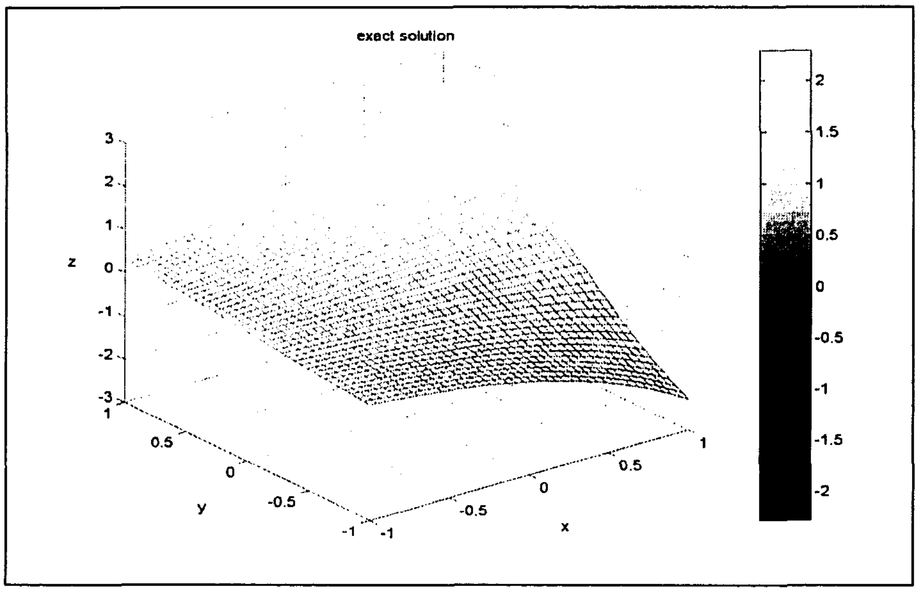


Figure 3.3: Approximate solution of $f(x,y) = e^x \sin y$

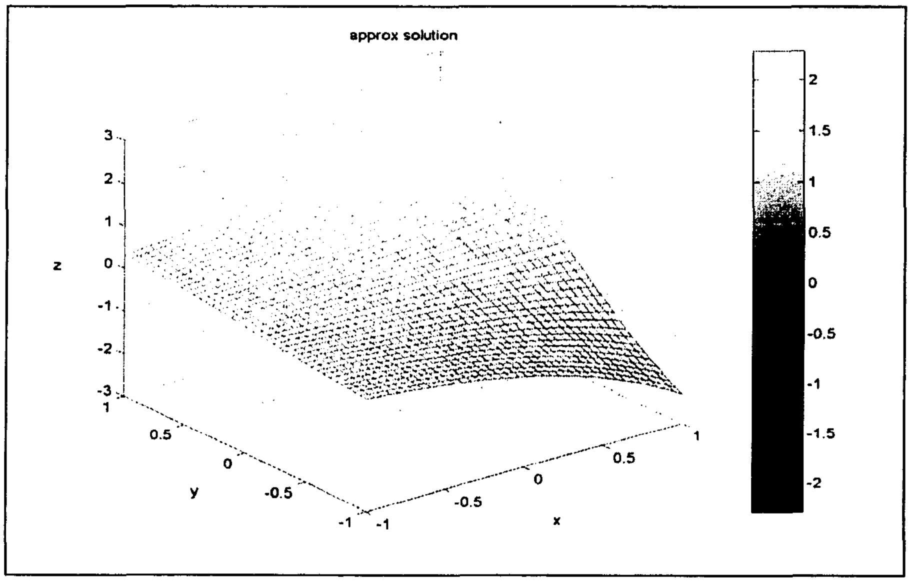
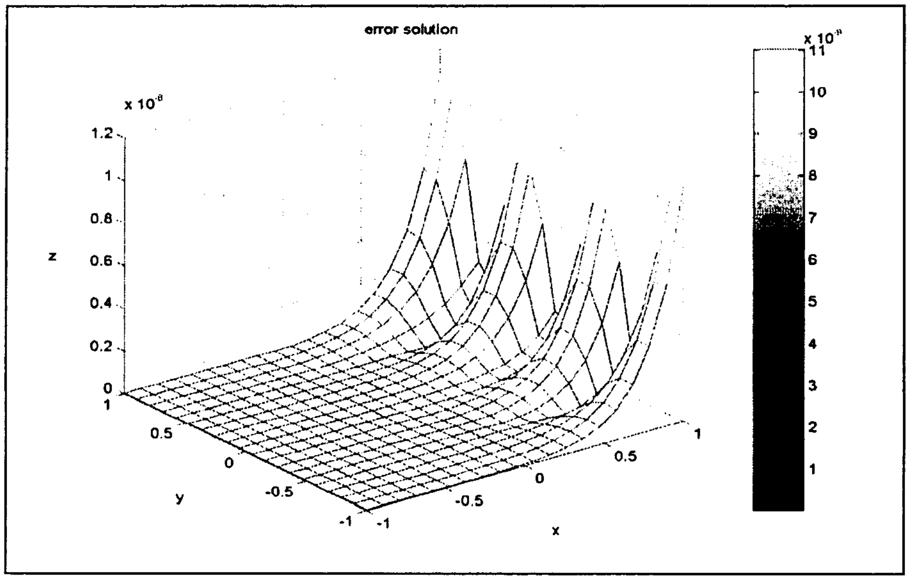


Figure 3.4: Error of $f(x,y) = e^x \sin y$



3.3 Concluding Remarks

From a numerical standpoint, the MFS has many benefits. The MFS is truly meshless; neither domain nor boundary discretization is needed. Calculations within this method are singularity-free due to the application of fictitious boundary. Requiring no numerical integration, the MFS produces low-cost computation and works well for irregularly shaped domains or high dimension problems. Furthermore, accuracy for this method can be optimized by taking into account the number and placement of collocation points and the size of the fictitious radius.

CHAPTER 4

4.1 CONCLUSION

As we have gone through a few problems, we can see that a lot of the various problems can be done using RBFs. We have gone through elliptic, square. In all these cases, we can see that the algorithms used can be easily understood and programmed. Furthermore, the boundaries of the domains in these problems need not be presented in any special way. Because the algorithms only care about how far the points are away form each other and not how they are placed, this give this method incredible flexibility. Also, when the number of points becomes so large that it will hinder the time to calculate the answer, a simple domain splitting technique can be used to reduce that number. So it seems that RBFs have massive potential in the field of PDEs.

4.2 Future Research

This thesis serves as a stepping stone toward a more improved solution to a 3D PDE. When dealing with Laplace transform for the 3D equation, the main challenge is to overcome the sensitivity and ill-posed condition. This subject is an excellent topic for future research.

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