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The combinatorial Riemann mapping theorem

by

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1. Introduction

The combinatorial Riemann mapping theorem is designed to supply a surface with local quasiconformal coordinates compatible with local combinatorial data. This theorem was discovered in an attempt to show that certain negatively curved groups have constant curvature. A potential application is that of finding local coordinates on which a given group acts uniformly quasiconformally. The classical Riemann mapping theorem may also be viewed as supplying local coordinates (take a ring and map it conformally, by the classical theorem, onto a right circular cylinder; pull the resulting flat coordinates back to the ring as canonical local coordinates). This coordinatization role is disguised in the classical case by the fact that a Riemann surface comes preequipped with local coordinates in the desired conformal class. In the combinatorial case we begin with a topological surface having no presupplied quasiconformal structure and our task is that of discovering the local coordinates (again by pulling coordinates back from an appropriate right circular cylinder).

The combinatorial data are supplied by coverings of the surface called *shinglings.* A *shingle* is a compact connected set. A *shingling* is a locally finite cover of the surface by shingles. (A shingling is like a tiling except that shingles are allowed to overlap while tiles usually do not overlap.) A shingling may be viewed as a combinatorial approximation to the surface. A given shingling, being locally finite, gives only a first approximation to a local quasiconformal structure on the surface. The total structure can only be determined by a sequence of finer and finer shinglings. The problem becomes that of

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determining when the approximate structures supplied by a sequence of shinglings are compatible so that there is a limiting quasiconformal structure. The method is that of *extremal length.* Extremal length has been studied both in the continuous [LV] and in the discrete [D] settings. We will mesh the two settings by taking the limit of discrete conformal optimizations.

Our approach to the problem is set out in Section 2 where we describe the classical theorem as the solution to a certain variational problem. For each shingling there is a corresponding finite variational problem which obviously has a solution. The remainder of the paper is then devoted to showing that, under appropriate conditions, solutions to the finite problems converge in the nicest possible way to a "combinatorial" Riemann mapping. The spirit of the undertaking is like that of Rodin and Sullivan, Beardon and Stephenson (see [RS], [Ro1], [Ro2], [BS]), with the difference that they assume that an underlying conformal structure is given from the start.

Acknowledgments. Matt Grayson helped in an early formulation of the axioms. Curt McMullen pointed out the importance of the Lipschitz condition in the proof of Theorem 5.9 and Corollary 5.10 and showed how the mapping theorem could be used to show that certain sequences of shinglings are not conformal. Peter Doyle showed me a number of references on combinatorial extremal length ([D], [M], [W]). Mladen Bestvina showed me the averaging trick of the proof of Theorem 7.1 so important in the final step of the mapping theorem. David Wright, Stephen Humphries, and William Floyd listened to many versions of tentative lemmas and propositions. Walter Parry [Pa] developed a nice list of properties of optimal weight functions. William Thurston helped me understand the difficulties, still unresolved, in making the combinatorial Riemann mapping theorem into a working tool. I thank them, and others forgotten for the moment, for their assistance.

The ideas that go into the proof of the theorem are very classical. I learned them primarily from [A], [DS], [Go], and [LV] but also in the classical treatments, [Ri] and [Hi]. Learning the classical arguments was a pleasure which extended over years. In making the arguments combinatorial, the clear connection with the classical was lost. In combinatorial form, the arguments are essentially self-contained and elementary, though lengthy.

Because I have opted to include almost all details, it is clearly possible to shorten the treatment in the following ways. Almost all of the propositions involving the approximate distance functions d_i would be obvious for true distance functions; once we convince ourselves that the d_i behave in approximately the same way, we can skip most of those details. Many of the geometric arguments break up into cases having the same result; the reader can work through one case with the confidence that the other cases work similarly.

Many steps of the numerical calculations are included; since they are essentially routine, they could be omitted. Skip what you will.

2. The Riemann mapping as a variational problem

The Riemann mapping theorem first appeared in [Ri]. The Riemann mapping theorem, in one of its versions, may be stated as follows. Suppose $\mathcal{R}$ is a closed topological annulus or *ring* in the complex plane. Then $\mathcal{R}$ inherits a natural Riemannian metric $|dz|$ and a natural area form $dx \cdot dy$. A conformal change of metric on $\mathcal{R}$ multiplies the metric by a positive function $\varrho = \varrho(z)$ and the area form by the positive function ϱ^2.

RIEMANN MAPPING THEOREM. *There is a positive continuous function ϱ such that the resulting Riemannian structure $\varrho|dz|$ on $\mathcal{R}$ is metrically a right circular cylinder, say of height H, circumference C, and area $A = HC$.*

It has always amazed me that Riemann could even have conjectured this theorem. Apparently what happened was this. (See [Poi, Chapter 1].) Think of $\mathcal{R}$ as a uniform conducting metal plate. Apply a voltage, maintaining one of the boundary curves at voltage H, the other at voltage 0. The current must flow and stabilize. Then the lines of equipotential form a family of simple closed curves filling up $\mathcal{R}$ and separating the ends of $\mathcal{R}$. The current flow lines also fill up $\mathcal{R}$ and are arcs joining the ends of $\mathcal{R}$. These two families of lines meet orthogonally, give flat coordinates to $\mathcal{R}$, and turn $\mathcal{R}$ into a right circular cylinder. The ratio (H/C) may be thought of as the resistance of the ring as a conducting plate to current flow between the ends. It is a conformal invariant. It is called the *analytic conformal modulus* of the ring $\mathcal{R}$. Note that

$$(H/C) = \frac{H^2}{A} = \frac{A}{C^2}.$$

There is a wonderful trick for creating conformal invariants. (See [A].) For a fixed Riemannian surface, one simply assigns a number to each metric conformally equivalent to the given one and then takes either the supremum or the infimum of those numbers over all of the metrics.

The resistance or modulus (H/C) is precisely such an invariant. (See [LV, Chapter 1].) It may be realized as follows. With each metric multiplier ϱ associate a ϱ-area $A(\varrho)$, a ϱ-height $H(\varrho)$, and a ϱ-circumference $C(\varrho)$ which gives respectively the area, the minimal distance between the ends, and the minimal distance around the ring with respect to the new Riemannian metric $\varrho \cdot |dz|$ and the new area form $\varrho^2 dx \cdot dy$. Then we have

$$(H/C) = \sup_{\varrho} \frac{H(\varrho)^2}{A(\varrho)} = \inf_{\varrho} \frac{A(\varrho)}{C(\varrho)^2}.$$

Furthermore, both the supremum and infimum are realized by that positive multiplier function ϱ which turns $\mathcal{R}$ into a right circular cylinder. That is, the optimal function ϱ is the absolute value of the derivative of the Riemann mapping. This circumstance will play a central role in our combinatorial theorem where the lack of local coordinates makes the definition of derivatives, let alone their use, difficult.

We shall be dealing with surfaces for which no Riemannian structure is given. Our principal potential application is to 3-manifold groups. According to [F], [BM], and observation, such groups often have a visual topological 2-sphere at infinity which has no obvious preferred Riemannian structure. The entire goal will be to find an appropriate Riemannian or quasiconformal structure on the surface on which the group of covering translations acts conformally. The group also often has a nice recursive combinatorial structure at infinity (see [C1], [C2], and [Gr]). This combinatorial structure at infinity creates shinglings at infinity which are of the type which we shall be studying in this paper. We lose no generality in assuming the existence of a topological metric on the surface. This metric will allow us to talk about the rough size of objects on the surface. Let $\mathcal{S}$ be an arbitrary shingling of our topological surface. Then $\mathcal{S}$ may be used to define an approximate metric and approximate area for subsets B of the surface. Simply define both the length and area of B to be the number of elements of the shingling that intersect B. That is, assume that each element of $\mathcal{S}$ has length and area equal to 1. It is then analogous to the classical case if we make a "conformal" change of approximate metric by changing the length of the element to ϱ and the area of an element to ϱ^2. The number ϱ may be an arbitrary nonnegative function on $\mathcal{S}$. The ϱ-length and ϱ-area of B are then simply the sums of the element lengths and areas for elements intersecting B. If $\mathcal{R}$ is a ring in our surface, we obtain heights, circumferences, and areas $H(\varrho)$, $C(\varrho)$, and $A(\varrho)$. Varying ϱ over all possibilities for which $A(\varrho)\neq 0$, we obtain two approximate conformal moduli,

$$M_{\sup}(\mathcal{R},\mathcal{S})=\sup_{\varrho}\frac{H(\varrho)^2}{A(\varrho)}$$

and

$$m_{\inf}(\mathcal{R},\mathcal{S})=\inf_{\varrho}\frac{A(\varrho)}{C(\varrho)^2}.$$

It is a fact which we shall prove elsewhere (Theorem 7.1) that, if the surface is Riemannian, if the elements of $\mathcal{S}$ are fairly round, fairly small relative to the size of $\mathcal{R}$, and do not overlap too much, rather like a slightly expanded circle packing, then the approximate conformal moduli will fairly closely approximate the analytic conformal modulus of $\mathcal{R}$.

What we have argued is that every shingling gives an approximate notion of conformal modulus to every ring in the surface. Now we pass to a sequence of such approximations.

Fix $K=K(1)>0$. (We use the modifier, (1), because we shall actually study a whole sequence of constants associated with a sequence of shinglings, of which K is the first constant on which the others will depend.) A *K-interval* is a real interval of the form $[r, K\cdot r]$, $r>0$. Let $\mathcal{S}_1, \mathcal{S}_2, \ldots$ denote a sequence of shinglings of a topological surface with mesh (largest element size) locally approaching 0. This property that mesh locally approaches 0 is independent of the particular topological metric with which the surface is endowed. We say that this sequence is a *conformal sequence* (K) if:

(i) for each ring $\mathcal{R}$, the approximate moduli $M_{\sup}(\mathcal{R}, \mathcal{S}_i)$ and $m_{\inf}(\mathcal{R}, \mathcal{S}_i)$, for all i sufficiently large, lie in a single K-interval $[r, K\cdot r]$; and

(ii) given a point x in the surface, a neighborhood N of x, and an integer I, there is a ring $\mathcal{R}$ in $N\setminus\{x\}$ separating x from the complement of N, such that for all large i the approximate moduli of $\mathcal{R}$ are all greater than I.

The intuitive content of these conditions is as follows. The first condition says that the approximate moduli defined by the shinglings are well-defined, at least asymptotically and up to multiplication by a uniform constant. This is a very strong condition that is difficult to recognize and generally depends on having the elements of the shinglings of uniform shape and placement. The multiplicative constant is unavoidable as an artifact of the combinatorial approximation. The second condition is the combinatorial analogue of the geometric property of a planar surface which says that around each point there is an arbitrarily small circular ring such that the ratio of the outer radius to the inner radius is arbitrarily large. This condition keeps points from exploding in the limit.

If approximate moduli are to approximate classical moduli in any sense, then conditions (i) and (ii) are clearly necessary. The combinatorial Riemann mapping theorem says that the conditions are also sufficient.

COMBINATORIAL RIEMANN MAPPING THEOREM. *Let $\mathcal{S}_1, \mathcal{S}_2, \ldots$ be a conformal sequence $K(1)$ of shinglings of a (metric) topological surface. Then there is a constant $K'=K'(K(1))$ satisfying the following condition. If $\mathcal{R}$ is any ring in the surface, then there is a metric D on $\mathcal{R}$ which makes $\mathcal{R}$, isometrically, a right circular cylinder and in which classical moduli and asymptotic approximate moduli are K'-comparable. That is, if $\mathcal{R}'$ is any ring in $\mathcal{R}$, then there are an integer I and a K'-interval $[r, K'\cdot r]$ such that the classical D-analytic modulus of $\mathcal{R}'$ in $\mathcal{R}$ and the approximate moduli $M_{\sup}(\mathcal{R}', \mathcal{S}_i)$ and $m_{\inf}(\mathcal{R}', \mathcal{S}_i)$ all lie in $[r, K'\cdot r]$ for each $i\geqslant I$.*

The content of the theorem is the existence of local analytic coordinates for which classical modulus is approximated by asymptotic combinatorial modulus.

COROLLARY. *There exists a quasiconformal structure on the surface whose analytic moduli are uniformly approximated by asymptotic combinatorial moduli. (Note that the* *quasiconformal structure is unique.)*

Proof of the corollary. Cover the surface by rings $\mathcal{R}_1, \mathcal{R}_2, \ldots$. Let $D_1, D_2, \ldots$ be metrics as promised by the theorem. We need to show that the transition functions

$$(\mathcal{R}_i \cap \mathcal{R}_j, D_i) \to (\mathcal{R}_i \cap \mathcal{R}_j, D_j)$$

are uniformly quasiconformal. Let $\mathcal{R}'$ be a ring in $\mathcal{R}_i \cap \mathcal{R}_j$. Let $[r_i, K' r_i]$ and $[r_j, K' r_j]$ be K' intervals and N a positive integer such that for $n > N$, the D_i-modulus $M_{\sup}(\mathcal{R}', D_i)$ and the approximate modulus $M_{\sup}(\mathcal{R}', \mathcal{S}_n)$ lie in $[r_n, K' r_n]$, while $M_{\sup}(\mathcal{R}', D_j)$ and $M_{\sup}(\mathcal{R}', \mathcal{S}_n)$ lie in $[r_j, K' r_j]$. Then, if $r = \min(r_i, r_j)$, the set

$$\{M_{\sup}(\mathcal{R}', D_i), M_{\sup}(\mathcal{R}', D_j)\}$$

lies in $[r, K'^2 r]$. By [LV, Theorem 7.2, p. 39], we conclude that the transition function is K'^2-quasiconformal. $\square$

The remainder of the paper is devoted to the proof of the combinatorial Riemann mapping theorem. We fix therefore a conformal sequence $(K(1))$

$$S_1, S_2, \ldots$$

with mesh going locally to 0 *and a ring* $\mathcal{R}$*. Our goal is to endow* $\mathcal{R}$ *with a special metric* D *which makes* $\mathcal{R}$ *a right circular cylinder and whose analytic moduli are in appropriate range.*

3. Optimal weight functions

For each shingling $\mathcal{S}$ of a ring $\mathcal{R}$ we shall prove (Proposition 3.1) the existence of a weight function $\varrho: \mathcal{S} \to [0, \infty)$ which is *optimal* in the sense that it realizes the supremum in the definition of $M_{\sup}(R, S)$. We can therefore associate with the shinglings $\mathcal{S}_1, \mathcal{S}_2, \ldots$ (of our conformal sequence) a sequence of optimal weight functions $\varrho_1, \varrho_2, \ldots$. Of course, with any weight function ϱ there are associated length and area functions $L(\varrho)$ and $A(\varrho)$. Therefore we obtain approximate length functions $L_1, L_2, \ldots$, and approximate area functions $A_1, A_2, \ldots$. While useful, the length functions $L_1, L_2, \ldots$ have potential defects that we have not been able to rule out. We therefore use a modified version of the functions L_i which give us approximate distance functions $d_1, d_2, \ldots$. The idea is to show that there is a subsequence such that the approximate distance functions and approximate area functions converge, respectively, to a true metric d on $\mathcal{R}$ (see Section 4) and a well-behaved area function A on $\mathcal{R}$ (see Section 5). The metric d and the area function A are then used to define flat coordinates on $\mathcal{R}$ as follows. The vertical or y-coordinate which measures the distance from the ends of $\mathcal{R}$ is simply the d-distance from one end of $\mathcal{R}$. The horizontal or x-coordinate is then the derivative with respect to y of the area function. (See Sections 5 and 6.)

Consider the classical Riemann mapping theorem as discussed in Section 2. There is no a priori reason for assuming the existence of a nonzero weight function $\sigma: \mathcal{R} \to [0, \infty)$ realizing the supremum

$$\frac{H(\sigma)^2}{A(\sigma)} = \sup_{\varrho} \frac{H(\varrho)^2}{A(\varrho)}.$$

The famous gap in Riemann's original proof (see [Ri] and [Hi]) lay essentially in his assumption that such a function existed (the Dirichlet principle). However, the corresponding problem for shinglings is easy.

3.1. Proposition (The existence of optimal weight functions). *There is a function* $\sigma = \varrho_i : \mathcal{S}_i \to [0, 1]$ *such that*

$$\frac{H(\sigma)^2}{A(\sigma)} = \sup_{\varrho} \frac{H(\varrho)^2}{A(\varrho)}.$$

If $\sigma = \varrho_i$ is normalized so that the associated area of the ring is 1, then we call ϱ_i an *optimal weight function* for $(\mathcal{R}, \mathcal{S}_i)$.

Proof. Pick a sequence $\varrho(1), \varrho(2), \ldots$ of weight functions on $\mathcal{S}_i$ such that $A(\varrho(j)) \neq 0$ and such that

$$\frac{H(\varrho(j))^2}{A(\varrho(j))}$$

converges to $M_{\sup}(\mathcal{R}, \mathcal{S}_i)$. Normalize each $\varrho(j)$ by scaling (H^2/A is scale invariant) so that $A(\varrho(j)) = 1$. Then, if $s \in \mathcal{S}_i$ intersects $\mathcal{R}$, $\varrho(j)(s) \in [0, 1]$. Hence, passing to a subsequence if necessary, we may assume that $\varrho(j)(s)$ converges, say to $\sigma(s) \in [0, 1]$. If $s \in \mathcal{S}_i$ and $s \cap \mathcal{R} = \varnothing$, define $\sigma(s) = 0$. Then $A(\sigma) = 1$ and

$$H(\sigma)^2 = \frac{H(\sigma)^2}{A(\sigma)} = M_{\sup}(\mathcal{R}, \mathcal{S}_i)$$

as desired. $\square$

There are of course optimal weight functions for the modulus $m_{\inf}(\mathcal{R}, \mathcal{S}_i)$ as well. Though we use such optimal weight functions only indirectly in this paper, it would be necessary to study them in determining that the axioms are satisfied by specific sequences of shinglings.

Optimal weight functions are fascinating. We will summarize here some of their abstract properties as developed by Walter Parry ([Pa]). Let $\mathcal{S}$ denote an abstract finite set. It corresponds to the shingling above. Let V denote a real vector space with $\mathcal{S}$ as basis. Then every element of V may be considered to be an $\mathcal{S}$-tuple of real numbers. We may think of the square of the length of an element of V as the area of that vector. An abstract path in V is a nonzero vector each coordinate of which is 0 or 1. Another way of viewing an abstract path is as a subset of $\mathcal{S}$; an element of $\mathcal{S}$ is in the path if its coordinate is 1. Let P denote a distinguished collection of paths. A weight vector w on $\mathcal{S}$ is a unit vector in V each coordinate of which is nonnegative. In terms of area, we are considering only those weight vectors of area 1. Define the height $H(w,P)$ of $\mathcal{S}$ with respect to w and P to be the minimal inner product of w with the elements of P. A weight vector w is said to be optimal (with respect to P) if, among all weight vectors, $H(w,P)^2$ is maximal.

Note that to maximize $H(w,P)^2$ is to minimize

$$\frac{1}{H(w,P)^2}.$$

Hence, if we had called $H(w)$ the circumference instead of the height, our single optimization would have included both of the optimizations considered in defining conformal moduli and approximate moduli above.

We prove here only the first of Parry's theorems, namely that there is a unique optimal weight vector. We will then summarize his other results.

3.2. PROPOSITION ([Pa]). *There is a unique optimal weight vector.*

Proof. Let w and w' be distinct nonnegative unit vectors such that $H(w)\geqslant H(w')$. If $t\in(0,1)$, then set $v=tw+(1-t)w'$. It is enough to show that

$$H\left(\frac{v}{\|v\|}\right) > H(w').$$

Let $p\in P$. Then

$$\begin{aligned}\frac{1}{\|v\|}v\cdot p &= \frac{1}{\|v\|}[t(w\cdot p)+(1-t)(w'\cdot p)]\\ &\geqslant \frac{1}{\|v\|}[tH(w)+(1-t)H(w')]\\ &\geqslant \frac{1}{\|v\|}H(w') > H(w'). \qquad \square\end{aligned}$$

We define a weight vector to be a *flow vector* if it is in the nonnegative cone of the paths. Intuitively it is then a bundle of paths. Every weight vector w determines a subset of paths, namely those paths $p\in P$ such that $w\cdot p=H(w,P)$. We call those paths *minimal* for w. We call w a *current flow vector* if it is in the nonnegative cone of its minimal paths. Parry characterizes the unique optimal weight vector as the unique current flow vector. He proves that some positive scalar multiple of this vector has integer entries.

In our situation, there are two obvious choices for the collection P of paths. For the first, call a subset of $\mathcal{S}$ an abstract path which joins the ends of $\mathcal{R}$ if it is the collection of elements of $\mathcal{S}$ which intersect a topological path in $\mathcal{R}$ joining the ends of $\mathcal{R}$. It is with respect to this collection of abstract paths that we have defined height. For the second collection, say that a topological simple closed curve *circles* $\mathcal{R}$ if it separates the ends of $\mathcal{R}$. Call a subset of $\mathcal{S}$ an abstract path circling $\mathcal{R}$ if it is the collection of elements of $\mathcal{S}$ which intersect a topological simple closed curve in $\mathcal{R}$ circling $\mathcal{R}$. It is with respect to this collection of abstract paths that we have defined circumference. Our two optimization problems then become that of finding a current flow vector joining the ends of $\mathcal{R}$ and a current flow vector circling $\mathcal{R}$.

Parry proves one more result, namely that since, in our situation, each path which circles $\mathcal{R}$ intersects each path which joins the ends of $\mathcal{R}$, $m_{\inf}(\mathcal{R},\mathcal{S})\leqslant M_{\sup}(\mathcal{R},\mathcal{S})$. The axioms for a conformal sequence then deal with the further relationship of $m_{\inf}(\mathcal{R},\mathcal{S}_i)$ to $M_{\sup}(\mathcal{R},\mathcal{S}_j)$ as $\mathcal{R}$, i, and j vary.

We will use Parry's results in discussing examples but in no other way.

We now leave the discussion of optimal weight functions in the abstract and take $\varrho_1,\varrho_2,\dots$ to be the unique optimal weight functions associated with our conformal sequence $\mathcal{S}_1,\mathcal{S}_2,\dots$. In addition to the optimal weight functions $\varrho_1,\varrho_2,\dots$, we have the associated length and area functions $L_1,L_2,\dots$ and $A_1,A_2,\dots$. We also obtain height functions $H_1,H_2,\dots$ and circumference functions $C_1,C_2,\dots$ for $\mathcal{R}$ and other rings contained in $\mathcal{R}$. Much of the remainder of the paper will be devoted to the limiting properties of the functions L_i and A_i.

Actually, the length functions $L_1,L_2,\dots$ are difficult to work with since there is no a priori reason for assuming that points of $\mathcal{R}$ have short L_i-length. We resolve this difficulty by following the admonition of G. Polya's "traditional mathematics professor" [Pol, p. 208]: "My method to overcome a difficulty is to go around it." We show that, though a point may have large length, it has nice neighborhoods with short frontiers in $\mathcal{R}$. This allows us to avoid and ignore points of long length. Our nice neighborhoods will be called *proper disks*. (See Figure 1.) A disk $D\subset\mathcal{R}$ is *proper* if it is closed and either lies in $\operatorname{Int}\mathcal{R}$ or intersects $\operatorname{Bd}\mathcal{R}$ along a boundary arc. The frontier of D in $\mathcal{R}$, denoted $\operatorname{Fr}D$, is either an arc α properly embedded in $\mathcal{R}$ or a simple closed curve β in $\operatorname{Int}\mathcal{R}$. The relative interior $D\backslash\operatorname{Fr}D$ is denoted by $\mathcal{R}\ \operatorname{Int}D$. The content of the next proposition is that each point has proper disk neighborhoods with short frontiers.

3.3. PROPOSITION (Existence of proper disk neighborhoods with short frontiers). *Given $x\in\mathcal{R}$, a neighborhood N of x, and $\varepsilon>0$, there is an annulus $\mathcal{R}(x,N,\varepsilon)$ in N*

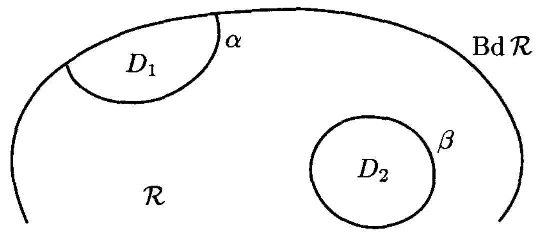

Fig. 1

separating x from the complement of N, and an integer $I(x,N,\varepsilon)$ such that, for each $i\geqslant I(x,N,\varepsilon)$, there is a proper disk neighborhood $D=D(x,N,\varepsilon,i)$ of x in N such that $\mathrm{Fr}\,D$ *lies in $\mathcal{R}(x,N,\varepsilon)$ and such that $L_i(\mathrm{Fr}\,D)<\varepsilon$. (See Figure 2.)*

Proof. We may assume that $N\cap\mathcal{R}$ is a proper disk $D(N)$. By property (ii) in the definition of conformal sequence, there is an annulus $\mathcal{R}(x,N,\varepsilon)$ in N separating x from the complement of N and an integer $I(x,N,\varepsilon)$ such that, for each $i\geqslant I(x,N,\varepsilon)$, the approximate modulus $m_{\inf}(\mathcal{R}(x,N,\varepsilon),\mathcal{S}_i)$ is greater than $1/\varepsilon^2$. Let J be a simple closed curve circling $\mathcal{R}(x,N,\varepsilon)$ of minimal L_i-length in general position with respect to $\mathrm{Bd}\,\mathcal{R}$. Let J' be a component of $J\cap\mathcal{R}$ separating x from $\mathrm{Fr}\,D(N)$. Let D be the proper disk neighborhood of x with $\mathrm{Fr}\,D=J'$. It remains to see that J, and hence $\mathrm{Fr}\,D=J'$, has L_i-length less than ε. If $L_i(J)=0$, we are done. Otherwise,

$$\frac{1}{\varepsilon^2}<m_{\inf}(\mathcal{R}(x,N,\varepsilon),\mathcal{S}_i)\leqslant\frac{A_i(\mathcal{R}(x,N,\varepsilon))}{C_i(\mathcal{R}(x,N,\varepsilon))^2}\leqslant\frac{1}{L_i(J)^2}.$$

That is, $L_i(J)<\varepsilon$. □

Having proved the existence of proper disk neighborhoods with short frontiers, we first pass to a subsequence of the ϱ_i's in order to ensure that these neighborhoods cover $\mathcal{R}$ uniformly. We then use these neighborhoods to define modified approximate distance functions d_i on $\mathcal{R}$ which behave better than the length functions L_i.

We shall use in what follows the notion of the *star* of a set in a collection. Let X be a set and Y a subset. Let C be a collection of subsets of X. The *star* $\mathrm{star}(Y,C)$ of Y in C is the union of the elements of C that intersect Y. The star operator may of course be iterated so that we have sets $\mathrm{star}^n(Y,C)\subset X$ for all $n>0$.

By Proposition 3.3, we may assume after passing to a subsequence that the following condition is satisfied. *For each $x\in\mathcal{R}$ and for each $i\in Z_+$, there exists an annulus $\mathcal{R}(x,i)$ of metric diameter $<1/i$ surrounding x having the following property. If $j\geqslant i$, then $\mathcal{R}(x,i)$ misses* $\mathrm{star}^2(x,\mathcal{S}_j)$ *and there is a proper-disk neighborhood $D=D(x,i,j)$ of* $\mathcal{R}\cap\mathrm{star}^2(x,\mathcal{S}_j)$ *whose frontier* $\mathrm{Fr}\,D$ *lies in $\mathcal{R}(x,i)$ and has length $L_j(\mathrm{Fr}\,D)<1/i$.*

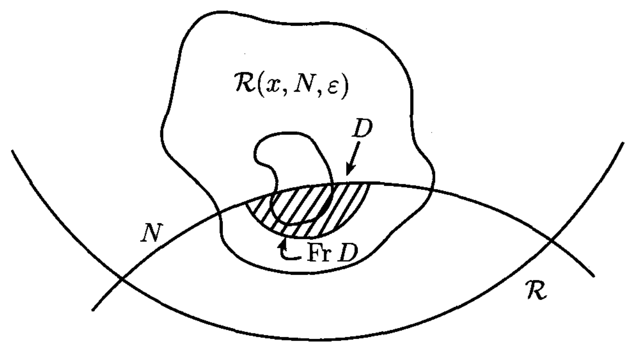

Fig. 2

We associate with L_i a modified approximate distance function $d_i: \mathcal{R} \times \mathcal{R} \to [0, \infty)$ as follows. Let $x \in \mathcal{R}$. An *i-approximation* to x in $\mathcal{R}$ is a proper disk $D(x)$ of metric diameter $<1/i$ such that $\mathcal{R} \cap \operatorname{star}^2(x, \mathcal{S}_i) \subset \mathcal{R} \operatorname{Int} D(x)$ and such that $L_i(\operatorname{Fr} D(x)) < 1/i$. By our choices in the previous paragraph, every $x \in \mathcal{R}$ has an i-approximation. An *i-approximate path* from x to y is a path in $\mathcal{R}$ joining i-approximations of x and y. The *approximate distance* $d_i(x, y)$ from x to y is the minimum L_i-length of all i-approximate paths from x to y in $\mathcal{R}$.

Since we shall have many propositions that deal with the functions d_i, we take a moment to mention two fundamental techniques used in dealing with them.

The first technique is this. Suppose that i-approximations $D(x)$ and $D(y)$ intersect but that their frontiers $\operatorname{Fr} D(x)$ and $\operatorname{Fr} D(y)$ do not. Then one of the two sets contains the other, say $D(x) \subset D(y)$. Then $D(y)$ is an i-approximation to x. Any arc joining $D(x)$ to something also joins $D(y)$ to that same set. Thus we may in almost all cases replace $D(x)$ by $D(y)$ as an i-approximation to x. But then $\operatorname{Fr} D(x)$ and $\operatorname{Fr} D(y)$ do intersect, in fact coincide. In summary of the first technique, *if two i-approximations intersect, we may assume that their frontiers intersect.*

The second technique considers the nonadditivity of length; an arc α which is the concatenation of arcs α_0 and α_1 will in general not have L_i-length that is the sum of the L_i-lengths of the subarcs. In fact, the length of α may equal the length of one of the subarcs. The problem is that there may be a shingle which hits both subarcs. The weight of this shingle will be counted twice in the sum of the lengths, only once in the length of the sum. One can avoid this problem in developing lower bounds for the L_i-length of α by taking subarcs α_0 and α_1 of α which intersect no common shingle. We demonstrate the second technique by example. Let $\mathcal{R}_0$ and $\mathcal{R}_1$ denote the ends of $\mathcal{R}$. Let J denote a simple closed curve in the interior of $\mathcal{R}$ separating the ends of $\mathcal{R}$. Let α join the ends of $\mathcal{R}$. Let α_0 denote an open subarc of α whose closure is irreducible from $\operatorname{star}(\mathcal{R}_0, \mathcal{S}_i)$

to $\operatorname{star}(J, \mathcal{S}_i)$. We say that α_0 is *open-irreducible* from $\operatorname{star}(\mathcal{R}_0, \mathcal{S}_i)$ to $\operatorname{star}(J, \mathcal{S}_i)$. Let α_1 be open-irreducible from $\operatorname{star}(\mathcal{R}_1, \mathcal{S}_i)$ to $\operatorname{star}(J, \mathcal{S}_i)$. Then, clearly, no shingle hits both α_0 and α_1 so that

$$L_i(\alpha) \geqslant L_i(\alpha_0) + L_i(\alpha_1).$$

How long are α_0 and α_1? Let p_0 denote the endpoint of α_0 in $\operatorname{star}(\mathcal{R}_0, \mathcal{S}_i)$, p_1 the endpoint in $\operatorname{star}(J, \mathcal{S}_i)$. Let s_0 denote a shingle which contains p_0 and hits $\mathcal{R}_0$. Let s_1 denote a shingle which contains p_1 and hits J. Let q_0 denote a point of $\mathcal{R}_0$ in s_0. Let q_1 denote a point of J in s_1. Let $D(q_j)$ denote an i-approximation to q_j, $j=0,1$. Then p_j lies in the relative interior of $D(q_j)$ so that α_0 is an i-approximate arc from $\mathcal{R}_0$ to J. *In particular the d_i-distance from $\mathcal{R}_0$ to J is a lower bound on the L_i-length of α_0.* In order to examine the L_i-distance between the same two sets, we need to consider two cases. If the two i-approximations intersect, then by technique one we may assume that their frontiers intersect. In that case, the union of their frontiers joins $\mathcal{R}_0$ to J so that the L_i-distance from $\mathcal{R}_0$ to J is at most $2/i$ and the d_i-distance is at most $1/i$. If on the other hand the two i-approximations are disjoint, then α_0 hits both frontiers, the L_i-distance from $\mathcal{R}_0$ to J is therefore at most $L_i(\alpha_0)+2/i$. *In either case, $L_i(\alpha_0)$ is at least as large as the L_i-distance from $\mathcal{R}_0$ to J minus $2/i$. Note that the argument would have remained unchanged had we used* star^2 *instead of* star. Technique two deals with such arguments involving open-irreducible arcs and their lengths.

The next proposition observes that the function d_i repairs the apparent defect in the length function L_i and, in the limit, satisfies the triangle inequality.

3.4. PROPOSITION (The functions d_i and the triangle inequality). *For each $x \in \mathcal{R}$, $d_i(x,x) < 1/i$. For each $x, y, z \in \mathcal{R}$,*

$$d_i(x,z) \leqslant d_i(x,y) + d_i(y,z) + 2/i.$$

Proof. The inequality

$$d_i(x,x) < 1/i$$

is immediate since we may take an arbitrary i-approximation $D(x)$ to x and take an i-approximate path from $D(x)$ to itself in $\operatorname{Fr} D(x)$. As for the other, pick i-approximations $D(x)$, $D(y)$, $E(y)$, and $E(z)$ and paths α and β, α joining $D(x)$ and $D(y)$ with $L_i(\alpha) = d_i(x,y)$, and β joining $E(x)$ and $E(y)$ with $L_i(\beta) = d_i(y,z)$. By technique one above, we may assume that if any two of these i-approximations intersect, so also do their frontiers. It follows that the set

$$\alpha \cup \operatorname{Fr} D(y) \cup \operatorname{Fr} E(y) \cup \beta$$

contains an i-approximate path from x to z, and this i-approximate path necessarily has L_i-length $\leqslant L_i(\alpha) + 2/i + L_i(\beta)$. □

3.5. PROPOSITION. *If α is an L_i-minimal arc joining the ends $\mathcal{R}_0$ and $\mathcal{R}_1$ of $\mathcal{R}$, then*

$$L_i(\alpha \cap \operatorname{star}(\mathcal{R}_0, \mathcal{S}_i)) < 1/i.$$

Proof. Let $X = \alpha \cap \operatorname{star}(\mathcal{R}_0, \mathcal{S}_i)$. Let Y be a subarc of α open-irreducible from $\mathcal{R}_1$ to $\operatorname{star}^2(\mathcal{R}_0, \mathcal{S}_i)$. Note that no shingle hits both X and Y. Hence $L_i(X) \leqslant L_i(\alpha) - L_i(Y)$. By the argument of technique two above, $L_i(Y) > L_i(\alpha) - 1/i$ since α is L_i-minimal joining the ends of $\mathcal{R}$. Hence $L_i(X) < 1/i$ as claimed. □

4. Approximate distances and their limits

The aim of this section is to show that some subsequence of the approximate distance functions d_i converges uniformly to a limit d and that the limit d is a true metric on $\mathcal{R}$ compatible with the topological metric with which we began.

4.0.1. THEOREM (Existence of limit metric). *Some subsequence of the approximate metrics $d_1, d_2, \ldots$ converges to a true topological metric d on $\mathcal{R}$ that is compatible with the given topology on $\mathcal{R}$.*

The flat metric D whose existence is asserted by the combinatorial Riemann mapping theorem which makes $\mathcal{R}$ into a right circular cylinder will be a modification of d which takes into account not only the limit of the d_i-distances (the ϱ_i-lengths L_i) but also the limit of the ϱ_i-areas A_i. (After all, the optimal weight function takes both lengths and areas into account.) While d alone exhibits strange local and global irregularities, the mesh of length and area will be completely well-behaved, a miracle which deserves to be better understood.

Theorem 4.0.1 is a consequence of two propositions, Proposition 4.0.2 and Proposition 4.0.3. We need a classical theorem before we can appreciate Proposition 4.0.2. Functions $f_1, f_2, \ldots$ from a space X into a metric space (Y, d) are *asymptotically equicontinuous* if, for each $x \in X$ and for each $\varepsilon > 0$, there exist a neighborhood N of x and an integer I such that, for each $i \geqslant I$ and for each $y, y' \in f_i(N)$, $d(y, y') < \varepsilon$. The functions are *uniformly bounded* if all of the images lie in a single compact subspace of Y.

ARZELA–ASCOLI THEOREM. *Suppose X is separable and (Y, d) metric. If*

$$f_1, f_2, \ldots : X \to Y$$

is uniformly bounded and asymptotically equicontinuous, then some subsequence converges to a function $f: X \to Y$. The limit function f is continuous. If the space X is compact, then the convergence is uniform.

Proof. Let C, compact in Y, contain the union of the images. Let $x_1, x_2, \dots$ be a countable dense set in X. Passing first to a subsequence of f's, we may assume that, for each i, the sequence $f_1(x_i), f_2(x_i), \dots$ converges to a point $f(x_i) \in C$. We claim that $f_1(x), f_2(x), \dots$ now converges for every $x \in X$. Indeed, since

$$f_1(x), f_2(x), \dots \in C,$$

some subsequence

$$f_{i(1)}(x), f_{i(2)}(x), \dots$$

converges, say to $y \in C$. Let $\varepsilon > 0$. By equicontinuity, there exist a neighborhood N of x and an integer I such that, for each $x' \in N$ and for each $i \geqslant I$, $d(f_i(x), f_i(x')) < \varepsilon$. Choose $x_j \in N$. Choose $K \geqslant I$ so large that $k \geqslant K$ implies $d(f_k(x_j), f(x_j)) < \varepsilon$. Choose $L \geqslant K$ so large that $l \geqslant L$ implies $d(f_{i(l)}(x), y) < \varepsilon$. Then, for $l \geqslant L$ we find

$$\begin{aligned} d(f_l(x), y) \leqslant{} & d(f_l(x), f_l(x_j)) + d(f_l(x_j), f(x_j)) \\ & + d(f(x_j), f_{i(l)}(x_j)) + d(f_{i(l)}(x_j), f_{i(l)}(x)) + d(f_{i(l)}(x), y) < 5\varepsilon. \end{aligned}$$

Hence $f_1(x), f_2(x), \dots$ converges to y and we define $f(x) = y$. Thus $f_1, f_2, \dots$ converges to f. Finally we note that f is continuous at x. Indeed, if $x' \in N$ and $i \geqslant I$,

$$d(f(x), f(x')) \leqslant d(f(x), f_i(x)) + d(f_i(x), f_i(x')) + d(f_i(x'), f(x')).$$

The middle term is less than ε, the other terms approach 0 for $i \to \infty$. Uniformity of convergence is checked similarly for X compact. □

4.0.2. PROPOSITION. *The sequence $d_1, d_2, \dots : \mathcal{R} \times \mathcal{R} \to [0, \infty)$ is uniformly bounded and asymptotically equicontinuous.*

COROLLARY. *Some subsequence of $d_1, d_2, \dots$ converges to a limit function $d: \mathcal{R} \times \mathcal{R} \to [0, \infty)$. The function d is a continuous pseudometric.*

Proof of the corollary. Convergence to a continuous function $d: \mathcal{R} \times \mathcal{R} \to [0, \infty)$ is a consequence of the Arzela–Ascoli theorem. The triangle inequality for d is a consequence of Proposition 3.4. Symmetry $d(x, y) = d(y, x)$ follows from the symmetry of each d_i. □

4.0.3. PROPOSITION. *The limit pseudometric d separates closed sets X and points x in $\mathcal{R}$ in the sense that, if $x \notin X$, then $d(x, X) > 0$.*

Proof of Theorem 4.0.1. Proposition 4.0.2 and its corollary supply a subsequence of $d_1, d_2, \dots$ converging to a continuous pseudometric d. But a continuous pseudometric d satisfies the conclusion of Proposition 4.0.3 if and only if d is a true topological metric on $\mathcal{R}$ compatible with the given topology on $\mathcal{R}$. □

It remains to prove Propositions 4.0.2 and 4.0.3. The proofs appear in three subsections. Subsection 4.1 proves Proposition 4.0.2. Subsection 4.2 establishes a very important length-area inequality. Subsection 4.3 employs this inequality in the proof of Proposition 4.0.3.

4.1. Uniform boundedness and asymptotic equicontinuity of $d_1, d_2, \ldots$ (a proof of Proposition 4.0.2)

4.1.1. PROPOSITION. *The numbers $H_i(\mathcal{R})$ and $C_i(\mathcal{R})$ are uniformly bounded.*

Proof. Choose an interval $[r, K(1)r]$ and an integer I such that, for all $i \geqslant I$, the moduli $m_{\inf}(\mathcal{R}, \mathcal{S}_i)$ and $M_{\sup}(\mathcal{R}, \mathcal{S}_i)$ lie in $[r, K(1)r]$. Then for all weight functions ϱ on $\mathcal{S}_i$ yielding $A(\mathcal{R}, \varrho) \neq 0$, we have

$$r \leqslant m_{\inf}(\mathcal{R}, \mathcal{S}_i) \leqslant \frac{A(\mathcal{R}, \varrho)}{C(\mathcal{R}, \varrho)^2}$$

and

$$\frac{H(\mathcal{R}, \varrho)^2}{A(\mathcal{R}, \varrho)} \leqslant M_{\sup}(\mathcal{R}, \mathcal{S}_i) \leqslant K(1)r.$$

Hence $C(\mathcal{R}, \varrho)^2 \leqslant A(\mathcal{R}, \varrho)/r$ and $H(\mathcal{R}, \varrho)^2 \leqslant A(\mathcal{R}, \varrho) \cdot K(1)r$. Since $A(\mathcal{R}, \varrho_i) = A_i(\mathcal{R}) = 1$, the result follows. □

4.1.2. PROPOSITION. *If ϱ is an optimal weight function for $(\mathcal{R}, \mathcal{S})$ and if $s \in \mathcal{S}$ has positive ϱ-weight, then there is an $L(\varrho)$-minimal path joining the ends of $\mathcal{R}$ which intersects the shingle s.*

Proof. Suppose the contrary. Let L be the minimum ϱ-length of a path through s joining the ends of $\mathcal{R}$. Redefine the weight of s to be $\varrho(s) \cdot (1-\lambda)$ where

$$0 < \varrho(s) \cdot \lambda < L - H(\varrho).$$

Let H' and A' be the new height and area of $\mathcal{R}$. By the optimality of ϱ,

$$\frac{(H')^2}{A'} \leqslant \frac{H^2}{A}. \tag{1}$$

But direct calculation shows

$$H' = H \tag{2}$$

and

$$A' = A - 2\lambda \varrho(s)^2 + \lambda^2 \varrho(s)^2. \tag{3}$$

Substitution of (2) and (3) in (1) shows

$$-2\lambda\varrho(s)^2+\lambda^2\varrho(s)^2\geqslant 0 \tag{4}$$

which is absurd for λ sufficiently near 0. □

The proof of this proposition is a model which will be improved and used a number of times before we are through.

4.1.3. PROPOSITION (Every point is d_i-near to a point of positive weight). *If $x\in\mathcal{R}$, then there exist a point $x'\in\mathcal{R}$ of length $L_i(x')>0$, i-approximations $D(x)$ to x, and $D(x')$ to x', and a path α joining $\operatorname{Fr}D(x)$ to $\operatorname{Fr}D(x')$ such that $L_i(\alpha\cup\operatorname{Fr}D(x'))<1/i$.*

Proof. If at any point we find x', $D(x)$, $D(x')$ such that $\operatorname{Fr}D(x)\cap\operatorname{Fr}D(x')\neq\varnothing$, we are done since we may take $\alpha\subset\operatorname{Fr}D(x)\cap\operatorname{Fr}D(x')$ and $L_i(\alpha\cup\operatorname{Fr}D(x'))=L_i(\operatorname{Fr}D(x'))<1/i$.

If possible, choose x' of positive weight in some i-approximation to x. Then choose i-approximations D and E to x and x' respectively such that D and E intersect. By fundamental technique one, we may assume that $\operatorname{Fr}D\cap\operatorname{Fr}E\neq\varnothing$; and we are done.

If no i-approximation to x contains a point of positive weight, then choose $D(x)$ arbitrarily and let β be an arc irreducible from $D(x)$ to the points of positive weight. Let x' be the terminal endpoint of β and $D(x')$ an i-approximation to x'. If $D(x)\cap D(x')=\varnothing$, then a subarc α of β irreducible from $\operatorname{Fr}D(x)$ to $\operatorname{Fr}D(x')$ has ϱ_i-lenth 0; and we are done. Otherwise, since we can have neither $D(x)\subset D(x')$ nor $D(x')\subset D(x)$,

$$\operatorname{Fr}D(x)\cap\operatorname{Fr}D(x')\neq\varnothing.$$ □

4.1.4. PROPOSITION. *The approximate distance functions d_i are uniformly bounded.*

Proof. Let $x\in\mathcal{R}$. Choose x', $D(x)$, $D(x')$, and $\alpha(x)$ as in Proposition 4.1.3. Use Proposition 4.1.2 to choose a path $\beta(x)$ which joins the ends of $\mathcal{R}$, has ϱ_i-length $H(\varrho_i)$, and intersects $\operatorname{Fr}D(x')$. Given $y\in\mathcal{R}$, choose y', $D(y)$, $D(y')$, $\alpha(y)$, and $\beta(y)$ similarly. Let J be a simple closed curve of length C_i circling $\mathcal{R}$. Then the union of the sets $\alpha(x)$, $\operatorname{Fr}D(x')$, $\beta(x)$, J, $\beta(y)$, $\operatorname{Fr}D(y')$, and $\alpha(y)$ is connected, joins $D(x)$ to $D(y)$, and has L_i-length equal to or less than $2/i+2H_i+C_i$. From Proposition 4.1.1 it follows that the numbers $d_i(x,y)$ are uniformly bounded. □

4.1.5. PROPOSITION. *Let D denote a proper disk with $L_i(\operatorname{Fr}D)\leqslant\varepsilon$. Let α denote an L_i-minimal path joining the ends of $\mathcal{R}$. Then $L_i(\alpha\cap D)<\varepsilon+2/i$.*

Proof. Note that $L_i(\alpha)=H_i$. We may assume that α hits D. Let $\mathcal{R}_0$ and $\mathcal{R}_1$ denote the ends of $\mathcal{R}$. If $\mathcal{R}_0$ misses $\operatorname{star}D$, then let α_0 be a subarc of α open irreducible from

$\mathcal{R}_0$ to star D. Otherwise, let α_0 be empty. Define α_1 similarly. Since no shingle hits both $\alpha\cap D$ and the union $\alpha_0\cup\alpha_1$,

$$L_i(\alpha\cap D)\leqslant H_i-L_i(\alpha_0\cup\alpha_1).$$

By the second fundamental technique, H_i is no longer than

$$L_i(\alpha_0\cup\alpha_1)+L_i(\operatorname{Fr} D)+2/i.$$

Consequently,

$$L_i(\alpha\cap D)<L_i(\operatorname{Fr} D)+2/i\leqslant\varepsilon+2/i. \qquad \square$$

4.1.6. PROPOSITION. *Let D denote a proper disk in $\mathcal{R}$ with $L_i(\operatorname{Fr} D)<\varepsilon$. Assume every i-approximation to x in $\mathcal{R}$ lies in $\mathcal{R}$ $\operatorname{Int} D$. Then there is an i-approximation $D(x)$ to x in $\mathcal{R}$ such that $\operatorname{Fr} D$ and $\operatorname{Fr} D(x)$ can be joined by an arc of L_i-length $<\varepsilon+3/i$.*

Proof. By Proposition 4.1.3 there exist a point x' of positive L_i-length, i-approximations $D(x)$ to x and $D(x')$ to x', and an arc α joining $\operatorname{Fr} D(x)$ to $\operatorname{Fr} D(x')$,

$$L_i(\alpha\cup\operatorname{Fr} D(x'))<1/i.$$

By Proposition 4.1.2 there is an L_i-minimal path β joining the ends of $\mathcal{R}$ which intersects $\operatorname{Fr} D(x')$. Since $D(x)\subset\mathcal{R}$ $\operatorname{Int} D$, the connected set

$$\alpha\cup\operatorname{Fr} D(x')\cup(\beta\cap D)$$

joins $\operatorname{Fr} D(x)$ to $\operatorname{Fr} D$. But by Proposition 4.1.5

$$L_i[(\alpha\cup\operatorname{Fr} D(x'))\cup(\beta\cap D)]<1/i+(\varepsilon+2/i). \qquad \square$$

4.1.7. PROPOSITION. *The approximate metrics $d_1, d_2, \ldots$ are asymptotically equicontinuous.*

Proof. Given $(x,y)\in\mathcal{R}\times\mathcal{R}$ and $\varepsilon>0$, we must find a neighborhood $M\times N$ of (x,y) and an integer J such that, for each $(x',y')\in M\times N$ and for each $i\geqslant J$,

$$|d_i(x,y)-d_i(x',y')|<\varepsilon.$$

Pick I so large that $10/I<\varepsilon$. If $x\neq y$, then require that the $1/I$-neighborhoods of x and y be disjoint. As noted in the paragraphs following the proof of Proposition 3.3, there exist annuli, $\mathcal{R}(x,I)$ and $\mathcal{R}(y,I)$, in the $1/I$-neighborhoods of x and y having this property: if

$i \geqslant I$, there are proper-disk neighborhoods $D(x)=D(x, I, i)$ of $\mathcal{R} \cap \operatorname{star}^2(x, \mathcal{S}_i)$ and $D(y)=D(y, I, i)$ of $\mathcal{R} \cap \operatorname{star}^2(y, \mathcal{S}_i)$ such that $\operatorname{Fr} D(x) \subset \mathcal{R}(x, I)$, $\operatorname{Fr} D(y) \subset \mathcal{R}(y, I)$, $L_i(\operatorname{Fr} D(x)) < 1/I$, and $L_i(\operatorname{Fr} D(y)) < 1/I$. Pick connected neighborhoods M of x and N of y in $\mathcal{R}$ and pick $J > I$ so large that for each $i \geqslant J$, all i-approximations to all points of M miss $\mathcal{R}(x, I)$ and all i-approximations to all points of N miss $R(y, I)$. If $x=y$, then choose $N=M$. Take $x' \in M \cap \mathcal{R}$, $y' \in N \cap \mathcal{R}$, $i \geqslant J$ $(>I)$. Then we have proper-disk neighborhoods $D(x)$ of x and $D(y)$ of y as above with $L_i(\operatorname{Fr} D(x)) < 1/I$ and $L_i(\operatorname{Fr} D(y)) < 1/I$. Take i-approximations $E(x')$ to x' and $E(y')$ to y'. Then $E(x') \subset \mathcal{R} \operatorname{Int} D(x)$ and $E(y') \subset \mathcal{R} \operatorname{Int} D(y)$. If $D(x)$ and $D(y)$ are disjoint, then we may argue as follows. Any path joining $E(x')$ to $E(y')$ has a subpath α joining $\operatorname{Fr} E(x')$ to $\operatorname{Fr} D(x)$, a subpath β joining $\operatorname{Fr} D(x)$ to $\operatorname{Fr} D(y)$, and a subpath γ joining $\operatorname{Fr} D(y)$ to $\operatorname{Fr} E(y')$. Let δ denote an L_i-minimal path joining $D(x)$ to $D(y)$. It follows from Proposition 4.1.6 that

$$L_i(\delta) \leqslant d_i(x', y') < L_i(\delta) + 2 \cdot [1/I + 3/i] + 2/I$$

since α and γ can be chosen of lengths $< 1/I + 3/i$. On the other hand, if $D(x)$ and $D(y)$ do intersect, then one finds similarly that

$$0 \leqslant d_i(x', y') < 2 \cdot [1/I + 3/i] + 2/I.$$

In either case, the values for $d_i(x', y')$ are restricted to a real interval of width less than $4/I + 6/i < 10/I < \varepsilon$. We conclude that $d_1, d_2, \ldots$ is asymptotically equicontinuous. □

Propositions 4.1.4 and 4.1.7 prove Proposition 4.0.2. □

4.2. An area-length inequality

The following inequality will be useful in all that follows. It should be viewed as saying that the approximate metrics $d_1, d_2, \ldots$ have curvature $\geqslant 0$. For each $x \in \mathcal{R}$, $r > 0$, and $i \in Z_+$, we define

$$D(x, r, i) = \{y \in \mathcal{R} \mid d_i(x, y) \leqslant r\}.$$

When x and i are fixed, we use the shorthand notation,

$$D(r) = D(x, r, i).$$

4.2.1. THEOREM (Quadratic estimate on area). *There is a positive constant $K(2)$ such that, for each $x \in \mathcal{R}$, each $r > 0$, and each $i \in Z_+$ sufficiently large,*

$$A_i[D(x, r, i)] \leqslant K(2) \cdot r^2.$$

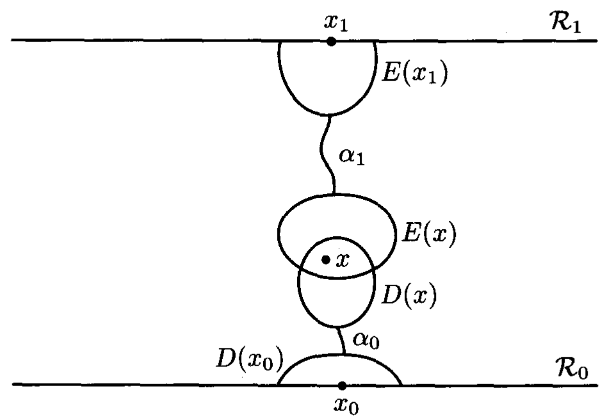

Fig. 3

Remark 1. In any Riemannian (= locally Euclidean) geometry defined on $\mathcal{R}$, this inequality would be satisfied automatically—in the small with $K(2)\approx\pi$ since the geometry is locally Euclidean, in the large because the area is bounded. But in hyperbolic space the inequality fails for large r, and a scaling of the hyperbolic metric to create large negative curvature forces the inequality to fail for small r.

Remark 2. The argument for this theorem is our first really serious variational argument using the optimality of the weight functions ϱ_i. The arguments are of standard calculus of variations type. One modifies the given weight function, observes the consequence of optimality, and takes a limit as the variation goes to zero. We had our first taste of such arguments in the simple proposition, Proposition 4.1.2. All such arguments have two parts, a geometric and an analytic. The geometric part studies paths in $\mathcal{R}$ and their lengths with respect to the new weight function. The intent of this is to be able to estimate the new height function on the ring. The geometric argument can be technical and lengthy. The analytic part simply calculates the new area explicitly from the new weight function. In general the analytic part is a straightforward calculation. We will prove a number of propositions first in support of the geometric part of the argument.

4.2.2. PROPOSITION. *Let $x\in\mathcal{R}$, and let $\mathcal{R}_0$ and $\mathcal{R}_1$ denote the ends of $\mathcal{R}$. Then*

$$|d_i(x,\mathcal{R}_0)+d_i(x,\mathcal{R}_1)-H_i|<6/i.$$

(*The reader can perhaps improve on the number* 6.)

Proof. We first prove that

$$H_i\leqslant d_i(x,\mathcal{R}_0)+d_i(x,\mathcal{R}_1)+4/i.$$

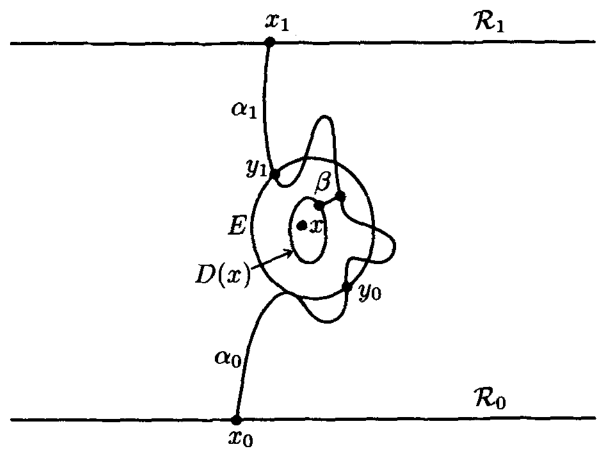

Fig. 4

Choose $x_0 \in \mathcal{R}_0$, $x_1 \in \mathcal{R}_1$, i-approximations $D(x_0)$, $D(x)$, $E(x)$ and $E(x_1)$ and arcs α_0 and α_1 joining $D(x_0)$ to $D(x)$ and $E(x)$ to $E(x_1)$ such that $L_i(\alpha_j)=d_i(x,\mathcal{R}_j)$, $j=0,1$. (See Figure 3.) If any two i-approximations intersect, we assume their frontiers intersect. Then the union of $\operatorname{Fr} D(x_0)$, α_0, $\operatorname{Fr} D(x)$, $\operatorname{Fr} E(x)$, α_1, and $\operatorname{Fr} E(x_1)$ contains a path from $\mathcal{R}_0$ to $\mathcal{R}_1$ of L_i-length less than

$$d_i(x,\mathcal{R}_0)+d_i(x,\mathcal{R}_1)+4/i.$$

We now prove that

$$d_i(x,\mathcal{R}_0)+d_i(x,\mathcal{R}_1) \leqslant H_i+6/i.$$

Let

$$E=\{y\in\mathcal{R} \mid d_i(x,y)\leqslant 2/i\}.$$

(See Figure 4.) Note that E is *open* (in opposition to what one would expect from *continuous* distance functions). By Propositions 4.1.2 and 4.1.3 there is an L_i-minimal path α joining the ends of $\mathcal{R}$ which is joined to an i-approximation $D(x)$ by a path $\beta=ab$, $b\in\alpha$, $L_i(\beta)<1/i$. In particular α intersects E so that it contains maximal subarcs $\alpha_j=x_jy_j$, $x_j\in\mathcal{R}_j$, α_j missing E.

The argument is completed as follows. We show first that α_0 and α_1 cannot intersect a common shingle, for otherwise α could not be minimal, the portion of α near b in E being too long and unnecessary; it follows that $L_i(\alpha_0)+L_i(\alpha_1)\leqslant L_i(\alpha)=H_i(\mathcal{R})$. We next show that $d_i(x,\mathcal{R}_j)<L_i(\alpha_j)+3/i$. Our desired inequality follows.

Suppose that α_0 and α_1 did intersect a common shingle. Pick an i-approximation D to a point of that shingle. Then $\alpha_0\cup\alpha_1\cup\operatorname{Fr} D$ connects $\mathcal{R}_0$ to $\mathcal{R}_1$ so that

$$H_i(\mathcal{R})\leqslant L_i(\alpha_0\cup\alpha_1\cup\operatorname{Fr} D)<L_i(\alpha_0\cup\alpha_1)+1/i. \tag{1}$$

On the other hand, α contains a subarc $\gamma=(b,y)$ open-irreducible from b to $\operatorname{star}(\alpha_0\cup\alpha_1,\mathcal{S}_i)$. Let $D(y)$ be an i-approximation to y. Then $\beta\cup\gamma$ forms an i-approximate path from x to y so that

$$2/i < L_i(\beta\cup\gamma) < L_i(\gamma)+1/i. \tag{2}$$

But no shingle intersects both γ and $\alpha_0\cup\alpha_1$. Hence

$$L_i(\gamma)+L_i(\alpha_0\cup\alpha_1) \leqslant L_i(\alpha) = H_i(\mathcal{R}). \tag{3}$$

From (1), (2), and (3) we deduce that $H_i<H_i$, a contradiction.

It remains to show that $d_i(x,\mathcal{R}_j)<L_i(\alpha_j)+3/i$. Since α_j is maximal, there is a point z_j of $\alpha\cap E$ in a shingle which contains y_j. Let $D(z_j)$ be an i-approximation to z_j, $D_j(x)$ an i-approximation to x, and γ_j a path joining $\operatorname{Fr} D_j(x)$ to $\operatorname{Fr} D(z_j)$ of L_i-length $\leqslant 2/i$. Then the union of γ_j, $\operatorname{Fr} D(z_j)$, and α_j joins $\operatorname{Fr} D_j(x)$ to $\mathcal{R}_j$ and has L_i-length $<L_i(\alpha_j)+3/i$. □

4.2.3. Proposition. *Suppose $r<d_i(x,\mathcal{R}_j)$. Then*

$$d_i(D(r),\mathcal{R}_j)+r \leqslant d_i(x,\mathcal{R}_j)+1/i.$$

Proof. Recall that $D(r)=D(x,r,i)$. We treat two exceptional cases separately:

(1) $D(x)\cap D(y)\neq\varnothing$ for some i-approximations $D(x)$ of x and $D(y)$ of y, $y\in\mathcal{R}\setminus D(r)$. Then $r<d_i(x,y)<1/i$. Since $d_i(D(r),\mathcal{R}_j)\leqslant d_i(x,\mathcal{R}_j)$, we have

$$d_i(D(r),\mathcal{R}_j)+r < d_i(x,\mathcal{R}_j)+1/i$$

as desired.

(2) $D(r)\cap D(z)\neq\varnothing$ for some i-approximation $D(z)$ of z, $z\in\mathcal{R}_j$. Then $d_i(D(r),\mathcal{R}_j)\leqslant d_i(D(r),z)<1/i$. Since $r<d_i(x,\mathcal{R}_j)$ we have

$$d_i(D(r),\mathcal{R}_j)+r < 1/i+d_i(x,\mathcal{R}_j)$$

as desired.

Suppose therefore that neither special case is satisfied. Take an i-approximate path α of minimal length from x to $\mathcal{R}_j$, α joining i-approximations $D(x)$ of x and $D(z)$ of z, $z\in\mathcal{R}_j$. Let $\beta=(a,b)\subset\alpha$ be open irreducible from $D(z)$ to $\operatorname{Fr} D(r)$, $a\in D(z)$, $b\in\operatorname{Fr} D(r)$. Since we are not in case (2), β, though possibly degenerate, is not empty. Let $D(r)^c$ denote the complement of $D(r)$ in $\mathcal{R}$. Let $\gamma=(c,d)\subset\alpha$ be open-irreducible from $D(x)$ to $\operatorname{star} D(r)^c$, $c\in D(x)$, $d\in\operatorname{star} D(r)^c$. Let $D(y)$ i-approximate $y\in D(r)^c$ in such a way that $d\in\mathcal{R}$ $\operatorname{Int} D(y)$. Note that $D(x)$ and $D(y)$ are disjoint since we are not in case (1). Then γ joins $D(x)$ and $D(y)$ so that $r<d_i(x,y)\leqslant L_i(\gamma)$. Since no shingle intersects both β and γ, $L_i(\beta)+L_i(\gamma)\leqslant L_i(\alpha)$. Since $b\in\operatorname{Fr}(D(r))$, there is a point $w\in D(r)$ and i-approximation $D(w)$ such that $b\in\mathcal{R}$ $\operatorname{Int} D(w)$. Hence β joins $D(w)$ to $D(z)$ and $d_i(D(r),\mathcal{R}_j)\leqslant L_i(\beta)$. Hence $d_i(D(r),\mathcal{R}_j)+r\leqslant L_i(\alpha)=d_i(x,\mathcal{R}_j)$. □

4.2.4. PROPOSITION. *Let $\mathcal{R}_0$ and $\mathcal{R}_1$ denote the boundary components of $\mathcal{R}$. Let $x \in \mathcal{R}$. Let H_i denote the ϱ_i-height of $\mathcal{R}$. Then $d_i(x, \mathcal{R}_j) \leqslant H_i + 1/i$.*

Proof. By Propositions 4.1.2 and 4.1.3, there is an L_i-minimal path α joining the ends of $\mathcal{R}$, an i-approximation $D(x)$ to x, and a path β of L_i-length $<1/i$ joining $D(x)$ to α. Then $\alpha \cup \beta$ joins x to both $\mathcal{R}_0$ and $\mathcal{R}_1$ and has $L_i(\alpha \cup \beta) \leqslant L_i(\alpha) + L_i(\beta) < H_i + 1/i$. □

4.2.5. PROPOSITION. *Let $\mathcal{R}_0$ and $\mathcal{R}_1$ denote the boundary components of $\mathcal{R}$. Let $x \in \mathcal{R}$. Let α be an i-approximate path from x to $\mathcal{R}_1$ of minimal L_i-length. Then*

$$L_i(\alpha \cap \operatorname{star}(\operatorname{Bd} \mathcal{R})) < 3/i.$$

Proof. If $\operatorname{star}^2 \mathcal{R}_0 \cap \operatorname{star}^2 \mathcal{R}_1 \neq \varnothing$, then $H_i < 2/i$, and hence $L_i(\alpha) < 3/i$ by Propositions 4.1.2 and 4.1.3. Therefore we assume the intersection empty. Let $y \in \mathcal{R}_1$ be such that α joins i-approximations $D(x)$ to x and $D(y)$ to y. If $D(x) \cap (D(y) \cup \operatorname{star}^2 \mathcal{R}_1) \neq \varnothing$, then $D(x)$ intersects some i-approximation to a point of $\mathcal{R}_1$ so that $L_i(\alpha) < 1/i$. Hence we may assume the intersection empty. We consider two cases.

Case 1: $\alpha \cap \operatorname{star}^2 \mathcal{R}_0 \neq \varnothing$. Pick an arc β in α open-irreducible from $\operatorname{star}^2 \mathcal{R}_0$ to $D(y) \cup \operatorname{star}^2 \mathcal{R}_1$. Then

$$L_i(\alpha \cap \operatorname{star}(\operatorname{Bd} \mathcal{R})) + L_i(\beta) \leqslant L_i(\alpha) \leqslant H_i + 1/i < L_i(\beta) + 3/i.$$

Thus $L_i(\alpha \cap \operatorname{star}(\operatorname{Bd} \mathcal{R})) < 3/i$ as claimed.

Case 2: $\alpha \cap \operatorname{star}^2 \mathcal{R}_0 = \varnothing$. Pick an arc β in α open-irreducible from $D(x)$ to $D(y) \cup \operatorname{star}^2 \mathcal{R}_1$. Then

$$L_i(\alpha \cap \operatorname{star}(\operatorname{Bd} \mathcal{R})) + L_i(\beta) \leqslant L_i(\alpha) \leqslant L_i(\beta) + 1/i.$$

In this case $L_i(\alpha \cap \operatorname{star}(\operatorname{Bd} \mathcal{R})) < 1/i$. □

For the remainder of this subsection, we will generally be considering only one i and one $x \in \mathcal{R}$. Hence we retain the simplified notation,

$$D(r) = D(x, r, i) = \{y \in \mathcal{R} \mid d_i(x, y) \leqslant r\}.$$

Despite the closed inequality, $\leqslant$, $D(r)$ is an open set and not entirely well-behaved. We expand $D(r)$ slightly so as to make it a closed and path connected set $E(r)$ as follows. There exist finitely many i-approximations

$$D_1, D_2, \ldots, D_k, E_1, E_2, \ldots, E_k$$

and arcs

$$\alpha_1, \alpha_2, \ldots, \alpha_k$$

such that:

(1) For each $y \in D(r)$, there exists a j such that D_j is an i-approximation to y. For each j, D_j is an i-approximation to some $y \in D(r)$.

(2) For each j, E_j is an i-approximation to x.

(3) If two of

$$D_1, D_2, \ldots, D_k, E_1, E_2, \ldots, E_k$$

intersect, their frontiers intersect in general position. If $D_j \cap E_j \neq \varnothing$, then α_j is a constant path in $\operatorname{Fr} D_j \cap \operatorname{Fr} E_j$. Otherwise α_j irreducibly joins $\operatorname{Fr} D_j$ to $\operatorname{Fr} E_j$.

(4) For each j, if $\alpha_j \not\subset \operatorname{Fr} E_j$, then $L_i(\alpha_j) \leqslant r$.

Then $E(r)$ is the union of the disks D_j, the arcs α_j, and the disks E_j.

4.2.6. PROPOSITION. *The space $E(r)$ is compact, arcwise connected and satisfies*

$$D(r) \subset E(r) \subset D(r+1/i).$$

Proof. Only the set inclusion $E(r) \subset D(r+1/i)$ needs to be verified. Let $y \in E(r)$. If $y \in \alpha_j$ for some j, then $y \in D(r)$. If $y \in D_j \cup E_j$ for some j, then pick an i-approximation $D(y)$ for y. We assume that intersecting i-approximations have intersecting frontiers. If $D(y)$ intersects E_j, we find an i-approximate path from y to x in $\operatorname{Fr} D(y) \cap \operatorname{Fr} E_j$. If $D(y)$ hits D_j and D_j hits E_j, then we find an i-approximate path from y to x in $\operatorname{Fr} D_j$. Finally, if neither of these conditions holds, then $L_i(\alpha_j) \leqslant r$ and there is an i-approximate path from y to x in $\operatorname{Fr} D_j \cup \alpha_j$. In any case the path described has L_i-length $\leqslant r+1/i$. □

We lose no generality in assuming that, for all i and for all $x \in \mathcal{R}$, all i-approximations which intersect an i-approximation of x all lie in a single proper disk in $\mathcal{R}$.

4.2.7. PROPOSITION. *If $E(r)$ does not lift to the universal cover $\widetilde{\mathcal{R}}$ of $\mathcal{R}$, then there is a loop C in $E(r)$ which is noncontractible in $\mathcal{R}$ and has length $L_i(C) \leqslant 2r+4/i$.*

Proof. Recall that $E(r)$ is the union of finitely many sets of the form

$$D_j \cup \alpha_j \cup E_j.$$

Each of these individually lifts to $\widetilde{\mathcal{R}}$. This is clear when $D_j \cap E_j = \varnothing$, for in that case

$$D_j \cup \alpha_j \cup E_j$$

is contractible. Otherwise

$$D_j \cup \alpha_j \cup E_j = D_j \cup E_j$$

and, by our supposition in the paragraph preceding the statement of the proposition, $D_j \cup E_j$ lies in a proper disk in $\mathcal{R}$, hence lifts to $\widetilde{\mathcal{R}}$. We determine a specific lift of each set $D_j \cup \alpha_j \cup E_j$ as follows. Pick a lift for E_1. Every set $E_1 \cup E_j$ has a unique lift extending the lift of E_1. Then $D_j \cup \alpha_j \cup E_j$ has a unique lift extending the lift of E_j.

Now suppose that $E(r)$ does not lift. Take a noncontractible simple closed curve J in $E(r)$. Homotop J in turn out of $\operatorname{Int} D_1, ..., \operatorname{Int} D_k, \operatorname{Int} E_1, ..., \operatorname{Int} E_k$. Then the homotoped J lies in

$$X = \bigcup_j (\operatorname{Fr} D_j \cup \alpha_j \cup \operatorname{Fr} E_j).$$

Hence X is not liftable. Therefore there is a point z in X and indices j, which we may take to be 1 and 2, such that

$$z \in (\operatorname{Fr} D_1 \cup \alpha_1 \cup \operatorname{Fr} E_1) \cap (\operatorname{Fr} D_2 \cup \alpha_2 \cup \operatorname{Fr} E_2)$$

and such that the prescribed lifts z_1 and z_2 of z defined by lifting

$$D_1 \cup \alpha_1 \cup E_1$$

and

$$D_2 \cup \alpha_2 \cup E_2$$

are different. Pick $x' \in \operatorname{Fr} E_1 \cap \operatorname{Fr} E_2$. Let C_j, $j=1,2$, be a path from x' to z in

$$\operatorname{Fr} D_j \cup \alpha_j \cup \operatorname{Fr} E_j.$$

Then $C = C_1 * \overline{C}_2$ is noncontractible, lies in

$$\operatorname{Fr} D_1 \cup \alpha_1 \cup \operatorname{Fr} E_1 \cup \operatorname{Fr} E_2 \cup \alpha_2 \cup \operatorname{Fr} D_2 \subset E(r)$$

and therefore has length $L_i(C) \leqslant 2r + 4/i$. □

We wish to find a similar controlled simple closed curve circling $\mathcal{R}$ in the case where the set $E(r)$ does lift to the universal cover. We spend some time and prove two propositions before we are able to state and prove the desired result, Proposition 4.2.10. We will make successively greater restrictions on the integer i which we consider.

We will want our simple closed curve to lie in a certain horizontally controlled area B which we now define. We are still assuming that x, i, and r are fixed. Let H_i denote the L_i-height of $\mathcal{R}$. Let $a_0 = d_i(x, \mathcal{R}_0)$ and $a_1 = d_i(x, \mathcal{R}_1)$. Let $b_j = \max\{1/i, a_j - 11/i\}$. Define

$$B_j = \{y \in \mathcal{R} \mid d_i(y, \mathcal{R}_j) \leqslant b_j\}.$$

The sets B_0 and B_1 are open. Let B denote the complement of the union of their closures. Note that the open sets B_0 and B_1 contain $\mathcal{R}_0$ and $\mathcal{R}_1$, respectively.

4.2.8. PROPOSITION. *If* $6/i < H_i$, *then*

$$\operatorname{star}^2 B_0 \cap \operatorname{star}^2 B_1 = \varnothing.$$

For such i, *the open set* B *contains a simple closed curve which separates the ends of* $\mathcal{R}$.

Proof. If $\operatorname{star}^2 B_0 \cap \operatorname{star}^2 B_1 \neq \varnothing$, then there exist points $y_0 \in B_0$ and $y_1 \in B_1$ such that any two of their i-approximations $D(y_0)$ and $D(y_1)$ intersect. We may pick $x_0 \in \mathcal{R}_0$, $x_1 \in \mathcal{R}_1$, i-approximations $D(x_0)$, $D(y_0)$, $D(y_1)$, $D(x_1)$, and paths α_j with $L_i(\alpha_j) \leqslant b_j$ joining $D(x_j)$ to $D(y_j)$ such that if any two of these i-approximations intersect, their frontiers intersect. Then

$$\operatorname{Fr} D(x_0) \cup \alpha_0 \cup \operatorname{Fr} D(y_0) \cup \operatorname{Fr} D(y_1) \cup \alpha_1 \cup \operatorname{Fr} D(x_1)$$

joins the ends of $\mathcal{R}$. Hence

$$\begin{aligned} H_i &\leqslant L_i(\operatorname{Fr} D(x_0)) + L_i(\alpha_0) + L_i(\operatorname{Fr} D(y_0)) + L_i(\operatorname{Fr} D(y_1)) + L_i(\alpha_1) + L_i(\operatorname{Fr} D(x_1)) \\ &\leqslant 4/i + \max\{1/i, \alpha_0 - 11/i\} + \max\{1/i, \alpha_1 - 11/i\}. \end{aligned}$$

The possibilities, up to interchange of 0 and 1, are

$$H_i \leqslant 6/i, \tag{1}$$

$$H_i \leqslant a_0 - 6/i, \tag{2}$$

$$H_i \leqslant a_0 + a_1 - 18/i. \tag{3}$$

The first contradicts our assumption $6/i < H_i$. The second and third conflict with Proposition 4.2.2 which states that $a_0 + a_1 < H_i + 6/i$.

The closures $\bar{B}_0$ and $\bar{B}_1$ are therefore disjoint, closed sets containing $\mathcal{R}_0$ and $\mathcal{R}_1$ and separated by B. By the unicoherence of the 2-sphere $\mathcal{R}/\{\mathcal{R}_0, \mathcal{R}_1\}$, some component of B separates. And, in a 2-sphere, if an open set separates two points, it contains a simple closed curve separating those two points. □

4.2.9. PROPOSITION. *If* $r > 14/i$, *then there is an arc* α_j *in* $B_j \cup D(r)$ *joining* $\mathcal{R}_j$ *to an* i*-approximation* $D_j(x)$ *of* x.

COROLLARY. *If* $r > 14/i$, *then there is an arc* α *in* $B_0 \cup D(r) \cup B_1$ *joining the ends of* $\mathcal{R}$.

Proof of the corollary. Note that $\operatorname{Fr} D_j(x) \subset D(r)$. If $\operatorname{Fr} D_j(x)$ intersects $\mathcal{R}_j$, then replace α_j by the empty set. We may assume

$$\operatorname{Fr} D_0(x) \cap \operatorname{Fr} D_1(x) \neq \varnothing.$$

Thus

$$\alpha_0 \cup \operatorname{Fr} D_0(x) \cup \operatorname{Fr} D_1(x) \cup \alpha_1$$

joins the ends of $\mathcal{R}$ and lies in $B_0 \cup D(r) \cup B_1$. $\square$

Proof of the proposition. Suppose first that $D(r)$ intersects $\mathcal{R}_j$. Pick i-approximations $D(x)$ of x and $D(y)$ of y, $y \in \mathcal{R}_j \cap D(r)$, and an arc α joining $D(x)$ and $D(y)$, $L_i(\alpha) \leqslant r$. As always, we assume that i-approximations that intersect have intersecting frontiers. If $D(x) \cap D(y) \neq \varnothing$, we set $\alpha = \varnothing$. Then $\alpha \cup \operatorname{Fr} D(y)$ contains an arc α_j from $D(x)$ to $\mathcal{R}_j$. The arc α lies in $D(r)$. The arc $\operatorname{Fr} D(y)$ lies in the $1/i$-neighborhood of $\mathcal{R}_j$, hence lies in B_j.

Suppose next that $D(r) \cap \mathcal{R}_j = \varnothing$ so that $d_i(x, \mathcal{R}_j) > r$. By Proposition 4.2.3,

$$d_i(D(r), \mathcal{R}_j) \leqslant d_i(x, \mathcal{R}_j) - r + 1/i < d_i(x, \mathcal{R}_j) - 13/i.$$

In particular

$$1/i \leqslant d_i(D(r), \mathcal{R}_j) + 2/i < d_i(x, \mathcal{R}_j) - 11/i.$$

It follows from the definition of B_j that

$$B_j = \{y \in \mathcal{R} \mid d_i(y, \mathcal{R}_j) \leqslant d_i(x, \mathcal{R}_j) - 11/i\}.$$

With these estimates in hand, we are prepared to construct α_j and estimate its length. Let $x_j \in D(r)$ and $y_j \in \mathcal{R}_j$ satisfy $d_i(x_j, y_j) = d_i(D(r), \mathcal{R}_j)$. Let $D(x_j)$ and $D(y_j)$ be i-approximations to x_j and y_j, and let β_j be an arc from $D(x_j)$ to $D(y_j)$ with $L_i(\beta_j) = d_i(D(r), \mathcal{R}_j)$. Let $E(x_j)$ and $D_j(x)$ be i-approximations to x_j and x and γ_j an arc from $E(x_j)$ to $D_j(x)$ with $L_i(\gamma_j) = d_i(x, x_j)$. We may assume that any two of these i-approximations which intersect have intersecting frontiers. If $D(x_j)$ and $D(y_j)$ intersect, we replace β_j by the empty set. If $E(x_j)$ and $D_j(x)$ intersect, we replace γ_j by the empty set. Then

$$\operatorname{Fr} D(y_j) \cup \beta_j \cup \operatorname{Fr} D(x_j) \cup \operatorname{Fr} E(x_j) \cup \gamma_j$$

is connected and joins $\mathcal{R}_j$ to $\operatorname{Fr} D_j(x)$. The set γ_j lies in $D(r)$. The set

$$\operatorname{Fr} D(y_j) \cup \beta_j \cup \operatorname{Fr} D(x_j) \cup \operatorname{Fr} E(x_j)$$

lies in the $d_i(D(r), \mathcal{R}_j) + 2/i$-neighborhood of $\mathcal{R}_j$. As estimated above, this neighborhood lies in B_j. $\square$

4.2.10. PROPOSITION. *If* $6/i<H_i$, $14/i<r$, *and* $E(r)$ *lifts to the universal cover* $\widetilde{\mathcal{R}}$ *of* $\mathcal{R}$, *then there is a simple closed curve* J *in* B *separating the ends of* $\mathcal{R}$ *which can be divided into subarcs* A_0 *and* A_1 *with disjoint interiors such that*

$$\operatorname{Int} A_0 \subset B\setminus E(r), \tag{1}$$

$$A_1 \subset E(r), \tag{2}$$

$$L_i(A_1) \leqslant 2r+4/i. \tag{3}$$

Proof. By Proposition 4.2.8, there is a simple closed curve J_0 in B which separates the ends of $\mathcal{R}$. By the corollary to Proposition 4.2.9 there is an arc α in $B_0 \cup D(r) \cup B_1$ which connects the ends of $\mathcal{R}$. The lift J_1 of J_0 into $\widetilde{\mathcal{R}}$ is a line which must hit every lift of α. It can hit a lift of α only in a lift of $E(r)$. Since each lift of $E(r)$ is compact, J_1 must in fact hit more than one lift of $E(r)$. Let J_2 be an arc in J_1 irreducibly joining different lifts of $E(r)$. Let A_0 be the projection of J_2 to $\mathcal{R}$. Then (1) is clearly satisfied. Any two points in $\operatorname{Fr} E(r)$ are clearly joined by an arc in $E(r)$ of L_i-length $\leqslant 2r+4/i$. Let A_1 be such an arc joining the ends of A_0. This establishes (2) and (3). It remains only to show that $J=A_0\cup A_1$ separates the ends of $\mathcal{R}$. But that is clear because J lifts to an arc in $\widetilde{\mathcal{R}}$. □

4.2.11. PROPOSITION. *Suppose that* α *is a path, not necessarily* L_i*-minimal, joining the ends of* $\mathcal{R}$. *Then*

$$H_i \leqslant L_i(\alpha\setminus \operatorname{star}^2 D(r))+2r+4/i.$$

Proof. If $\operatorname{star} D(r)$ misses $\mathcal{R}_j$, then let α_j be a subarc of α irreducible from $\mathcal{R}_j$ to $\operatorname{star} D(r)$. Otherwise, let $\alpha_j=\varnothing$. Let $x_j \in D(r)$ be such that $\operatorname{star}^2 x_j$ intersects $\mathcal{R}_j \cup \alpha_j$. Let $D(x_j)$ and $D_j(x)$ be i-approximations to x_j and x, respectively, such that there is an arc β_j joining $D(x_j)$ and $D_j(x)$ of L_i-length $\leqslant r$. If any of $D(x_0)$, $D_0(x)$, $D_1(x)$, and $D(x_1)$ intersect, we may assume that their frontiers intersect. If $D(x_j)\cap D_j(x)\neq\varnothing$, we replace β_j by the empty set. Then

$$\alpha_1 \cup \operatorname{Fr} D(x_0) \cup \beta_0 \cup \operatorname{Fr} D_0(x) \cup \operatorname{Fr} D_1(x) \cup \beta_1 \cup \operatorname{Fr} D(x_1) \cup \alpha_1$$

is connected and joins $\mathcal{R}_0$ to $\mathcal{R}_1$. Hence

$$H_i \leqslant L_i(\alpha_0 \cup \alpha_1)+2r+4/i \leqslant L_i(\alpha\setminus \operatorname{star}^2 D(r))+2r+4/i. \qquad \square$$

We wish to define what we shall call a *barrier* for the set $D(r)$. The intuition of a barrier is that an arc joining the ends of $\mathcal{R}$ and hitting $D(r)$ has either to hit the barrier

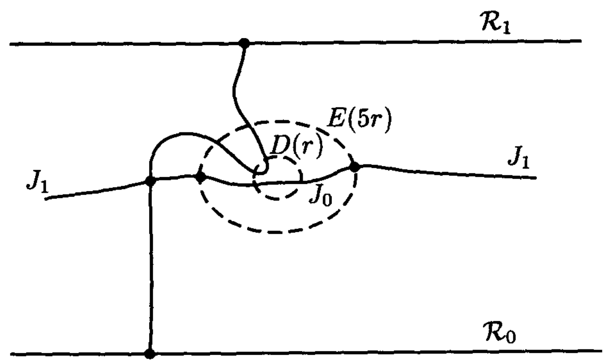

Fig. 5

or be very long. Our variational argument will show that the area of $D(r)$ is related to the length of a barrier for $D(r)$. First let us recall the setting:

$$x\in\mathcal{R},\ i\in Z_+,\ r>0,\ D(r),\ E(r);$$
$$a_j=d_i(x,\mathcal{R}_j);$$
$$b_j=\max\{1/i, a_j-11/i\};$$
$$B_j=\{y\in\mathcal{R}\mid d_i(y,\mathcal{R}_j)\leqslant b_j\};$$
$$B=\mathcal{R}\backslash(\bar{B}_0\cup\bar{B}_1).$$

Next let us recall our restrictions on i:

$$6/i<H_i,\quad 14/i<5r.$$

We add one further restriction:

$$19/i<r.$$

We already can conclude that there is a simple closed curve $J=J_0\cup J_1$ circling $\mathcal{R}$ such that, either $J_1=\varnothing$ or J_0 and J_1 are arcs with disjoint interiors; in any case $J_1\subset B\backslash E(5r)$ and $L_i(J_0)\leqslant 10r+4/i$.

Definition. We call the set star J_0 our *barrier* for $D(r)$.

4.2.12. PROPOSITION. *Suppose that α is an arc joining the ends of $\mathcal{R}$ such that $\alpha\cap\operatorname{star} J_0=\varnothing$. Then $L_i(\alpha\backslash\operatorname{star} D(r))\geqslant H_i$.*

Proof. (See Figure 5.) The simple closed curve $J=J_0\cup J_1$ separates $\mathcal{R}$ into two components with closures C_0 and C_1, $\mathcal{R}_0\subset C_0$ and $\mathcal{R}_1\subset C_1$. We examine each of the sets $\alpha_0=\alpha\cap C_0$ and $\alpha_1=\alpha\cap C_1$ separately. If $\alpha\cap\operatorname{star} D(r)=\varnothing$, then

$$L_i(\alpha\backslash\operatorname{star} D(r))=L_i(\alpha)\geqslant H_i.$$

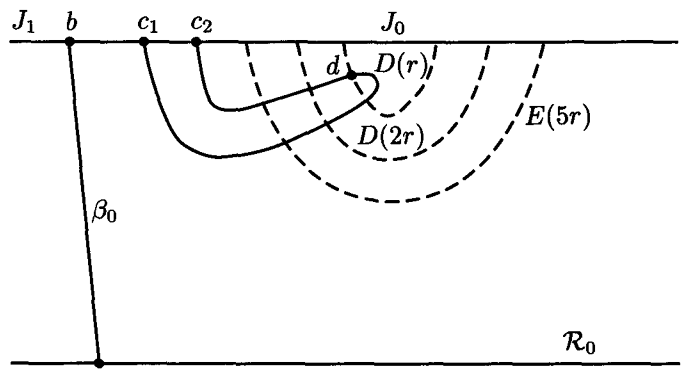

Fig. 6

Hence we may assume that at least one of α_0 and α_1, say α_0, intersects star $D(r)$. Our goal is to show the existence of sets $X_0 \subset \alpha_0 \backslash \operatorname{star} D(r)$ and $X_1 \subset \alpha_1 \backslash \operatorname{star} D(r)$ such that no shingle hits both, yet

$$L_i(X_0) \geqslant d_i(x, \mathcal{R}_0) + r - 4/i$$

and

$$L_i(X_1) \geqslant d_i(x, \mathcal{R}_1) - 11/i.$$

Hence, noting the proof of Proposition 4.2.2,

$$\begin{aligned} L_i(\alpha \backslash \operatorname{star} D(r)) &\geqslant L_i(X_0) + L_i(X_1) \\ &\geqslant d_i(x, \mathcal{R}_0) + d_i(x, \mathcal{R}_1) + r - 15/i \\ &\geqslant H_i + r - 19/i. \end{aligned}$$

Since $r > 19/i$, the proposition will follow.

We have to distinguish four cases depending on the way α is constituted. (See Figure 6.)

Case 1 considers the situation where α_0 contains one subarc $[a, b]$ which begins with $a \in \mathcal{R}_0$, misses star $D(r)$, and ends with $b \in J_1$; and contains a second subarc $[c_1, c_2]$ which begins with $c_1 \in J_1$, intersects star $D(r)$ at d, and ends with $c_2 \in J_1$. If $a \notin \operatorname{star}(D(5r) \cup J_1)$, let

$$\beta_0 = [a, b') \subset [a, b]$$

be irreducible from $\mathcal{R}_0$ to $\operatorname{star}(D(5r) \cup J_1)$. Otherwise, let $\beta_0 = \varnothing$. Note that $L_i(\beta_0) \geqslant d_i(x, \mathcal{R}_0) - 5r - 2/i$. Let $\beta_j = (\gamma_j, \delta_j) \subset [c_j, d]$, $j = 1, 2$, be irreducible from $\operatorname{star}(\mathcal{R} \backslash D(5r))$ to star $D(2r)$. Note that $L_i(\beta_j) \geqslant 5r - 2r - 1/i$. Let

$$\beta_3 = (\varepsilon_1, \varepsilon_2) \subset (\delta_1, \delta_2)$$

be irreducible from $\operatorname{star}(\mathcal{R}\setminus D(2r))$ to $\operatorname{star} D(r)$. Note that $L_i(\beta_3)\geqslant 2r-r-1/i$. We lose no generality in assumming that no shingle hits both β_1 and β_2. For otherwise they could be joined by an arc α' in $\mathcal{R}\setminus\operatorname{star} D(r)$ of length $<1/i$. This would allow one to alter α in such a way as to throw away β_3, and more, yet add only α'. The result would be to decrease $L_i(\alpha\setminus\operatorname{star} D(r))$ by at least $r-1/i+1/i=r$. Hence for $X=\alpha_0\setminus\operatorname{star} J\setminus\operatorname{star} D(r)$,

$$\begin{aligned} L_i(\alpha_0\setminus\operatorname{star} J\setminus\operatorname{star} D(r)) &\geqslant L_i(\beta_0)+L_i(\beta_1)+L_i(\beta_2)\\ &\geqslant [d_i(x,\mathcal{R}_0)-5r-2/i]+2[5r-2r-1/i]\\ &= d_i(x,\mathcal{R}_0)+r-4/i, \end{aligned}$$

as desired.

The last three cases are slight modifications of one another. In all three cases there is an arc $[a,b]\subset\alpha_0$, $a\in\mathcal{R}_0$, $b\in J_1$, which intersects $\operatorname{star} D(r)$, say at d. The cases are distinguished by the placement of $a\in[a,b]$ relative to x.

Case 2 considers $a\in\mathcal{R}\setminus\operatorname{star} D(5r)$. This case is almost identical with Case 1: pick $\beta_0=[a,b')$ irreducible from $\mathcal{R}_0$ to $\operatorname{star}(D(5r)\cup J_1)$ if $a\notin\operatorname{star}(D(5r)\cup J_1)$, empty otherwise. Then define $\gamma_1=b'$ if $\beta_0\neq\varnothing$ and $\gamma_1=a$ otherwise. Define $\gamma_2=b$. Proceed to define β_1, β_2, and β_3 as before.

Case 3 considers

$$a\in D(5r)\setminus\operatorname{star} D(2r).$$

Then define $\beta_0=\varnothing$, define β_1 beginning in $D(5r)$ and ending in $\operatorname{star} D(2r)$ so that

$$L_i(\beta_1)\geqslant d_i(a,x)-2r-1/i\geqslant d_i(x,\mathcal{R}_0)-2r-1/i.$$

Define β_2 and β_3 as before. Then

$$\begin{aligned} L_i(X) &\geqslant L_i(\beta_1)+L_i(\beta_2)\\ &\geqslant d_i(x,\mathcal{R}_0)-2r-1/i+5r-2r-1/i\\ &= d_i(x,\mathcal{R}_0)+r-2/i. \end{aligned}$$

Case 4 considers $a\in\operatorname{star} D(2r)$. We define $\beta_0=\beta_1=\varnothing$, β_2, and β_3 as before. The result is the same.

Finally, we suppose that α_1 misses $\operatorname{star} D(r)$. There is an arc $[a,b]$ in α_1 which begins with $a\in\mathcal{R}_1$ and ends with $b\in J_1$. If $a\notin\operatorname{star} J_1$, then there is an arc $\beta_1=[a,b')$ irreducible from $a\in\mathcal{R}_1$ to $b'\in\operatorname{star} J_1$. Then $d_i(x,\mathcal{R}_1)-11/i\leqslant L_i(\beta_1)$. Let $X_1=\beta_1$. $\square$

We have now completed all of the preliminaries. We are ready to define a new weight function ϱ'. Let $\varrho=\varrho_i$, $H=H_i$, $A=A_i$, $\mathcal{S}=\mathcal{S}_i$, $L=L_i$. We derive our desired quadratic estimate on area from the fact that ϱ is $(\mathcal{R},\mathcal{S})$-optimal via a variational argument. We have constructed a barrier star J_0 for $D(r)$ of length $\leqslant 10r+4/i$. Let u be the ϱ-weight vector for those shingles of $\mathcal{S}$ hitting J_0. Let v be the ϱ-weight vector for those shingles hitting $D(r)$ but missing J_0. Let w be the ϱ-weight vector for the remaining shingles. Let I denote a vector having the same number of coordinates as u, each entry of which is 1. Define a new weight vector $\varrho'=(u',v',w')$ for $\mathcal{S}$ as follows for each $\lambda\in(0,\frac{1}{2})$:

$$u'=u+\lambda(4r+8/i)I, \tag{1}$$

$$v'=(1-\lambda)v, \tag{2}$$

$$w'=w. \tag{3}$$

Let $H'=H(\varrho')$, $A'=A(\varrho')$, $L'=L(\varrho')$.

4.2.13. PROPOSITION. $H'\geqslant H$.

Proof. Let α be a ϱ'-minimal path joining the ends of $\mathcal{R}$, so that $L'(\alpha)=H'$. If α misses star J_0, then $H'\geqslant H$ by Proposition 4.2.12. We suppose therefore that α hits star J_0. Let $\alpha_1=\alpha\setminus\text{star}^2 D(r)$. Let $\alpha_2=\alpha\cap\text{star}\, D(r)$. Then

$$\begin{aligned} H' &\geqslant L'(\alpha_1)+L'(\alpha_2)\geqslant L(\alpha_1)+(1-\lambda)L(\alpha_2)\\ &\geqslant [H-2r-4/i]+\tfrac{1}{2}L(\alpha_2). \end{aligned}$$

If $H'\geqslant H$, we are done. Otherwise $L(\alpha_2)\leqslant 4r+8/i$. The change from v to v' can therefore decrease the L-length of α by at most $\lambda\cdot(4r+8/i)$, while the change from u to u' must increase the L-length of α by at least $\lambda\cdot(4r+8/i)$. Hence $H'\geqslant H$. □

This completes the geometric part of the proof of Proposition 4.2.1. □

Analytic part of the proof of Proposition 4.2.1. Since ϱ is $(\mathcal{R},\mathcal{S})$-optimal,

$$(H')^2/A'\leqslant H^2/A=H^2.$$

By Proposition 4.2.13, $H\leqslant H'$. Hence

$$A\leqslant A'.$$

That is,

$$u\cdot u+v\cdot v+w\cdot w\leqslant u'\cdot u'+v'\cdot v'+w'\cdot w'.$$

Substituting (1), (2), (3), we obtain

$$(2\lambda-\lambda^2)v\cdot v\leqslant 2\lambda(4r+8/i)L(J_0)+\lambda^2(4r+8/i)^2 I\cdot I.$$

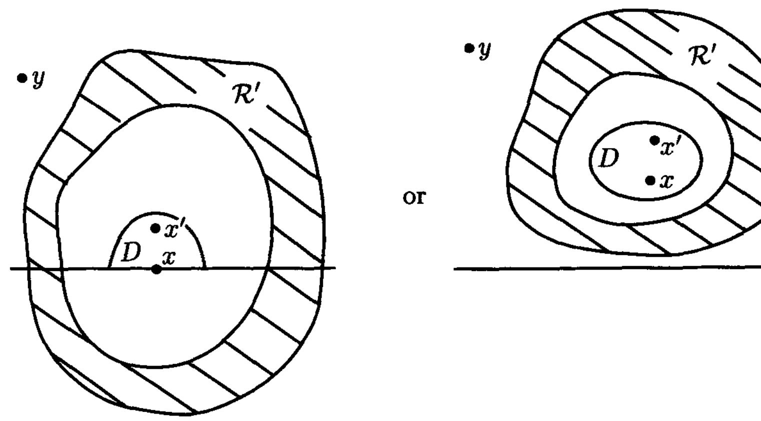

Fig. 7

Dividing by λ and letting $\lambda \to 0$, we obtain

$$v \cdot v \leqslant (4r+8/i)L(J_0).$$

But $L(J_0) \leqslant 10r+4/i$. Thus

$$v \cdot v \leqslant (4r+8/i)(10r+4/i) \leqslant 5r \cdot 11r.$$

Finally

$$A(D(r)) \leqslant u \cdot u + v \cdot v \leqslant 121r^2 + 55r^2 = 176r^2.$$

That is, we may take $K(2)=176$. Undoubtedly, this estimate can be improved substantially. This completes the analytic part of the proof of Proposition 4.2.1. □

4.3. Nondegeneracy of the limit pseudometric d

Proof of Proposition 4.0.3. The proof is a combinatorial analogue of [LV, Theorem 5.3, p. 74], which shows that the limit of K-quasiconformal mappings is K-quasiconformal. The difficulty in translating the classical proof to the combinatorial setting is that the classical proof implicitly uses the flatness of the coordinates in the complex plane. This ingredient is precisely the contribution of Theorem 4.2.1, our quadratic estimate on area.

We must show that the limit pseudo-metric d separates points from closed sets. Since d is continuous and $\mathcal{R}$ is compact, it suffices to show that, for each x and y in $\mathcal{R}$ with $x \neq y$, $d(x,y) > 0$. Suppose $d(x,y)=0$. We obtain a contradiction as follows. Since d is continuous and not identically 0, there exist a point x', $d(x,x') \neq 0$, and a proper disk D in $\mathcal{R}$ such that $x, x' \in \mathcal{R}$ Int D and $y \notin D$. Parallel to and very near to Bd D in the complement of D is a narrow ring $\mathcal{R}'$ which separates D from y, as in Figure 7.

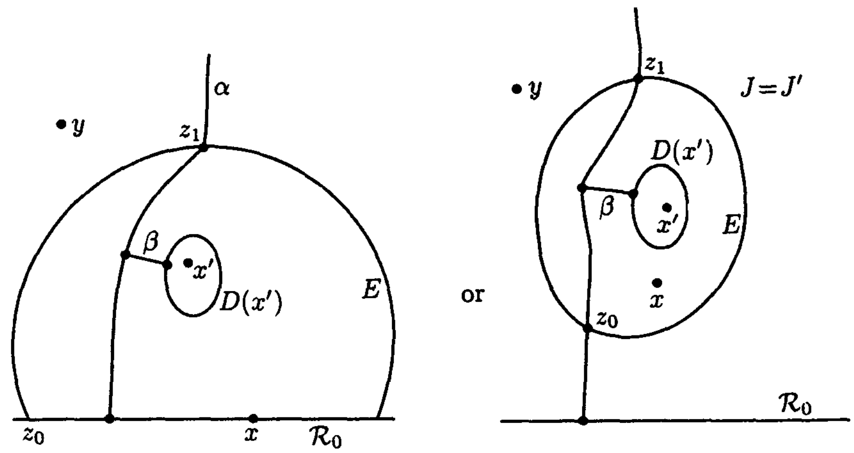

Fig. 8

We shall obtain a contradiction by showing that the approximate modulus of $\mathcal{R}'$ is 0. Note that any simple closed curve J circling $\mathcal{R}'$ will contain an arc or simple closed curve J' which is the frontier of a proper disk containing D and missing y. We need the following lemma.

LEMMA. *If* $0<8\varepsilon<d(x,x')$, *and if* J *is any simple closed curve circling* $\mathcal{R}'$, *then* $J\cap\mathcal{R}$ *contains an arc (or simple closed curve)* J' *joining a point at distance* $\geqslant 2\varepsilon$ *from* x *to a point at distance* 0 *from* x *in* $\mathcal{R}$.

Proof of the lemma. Pick i so large that any i-approximation to x' lies in D. By Propositions 4.1.2 and 4.1.3, there exist an i-approximation $D(x')$ to x', an L_i-minimal arc α joining the ends of $\mathcal{R}$, and an arc β of L_i-length $<1/i$ joining $\operatorname{Fr} D(x')$ to α . Let J' be an arc or simple closed curve in J which is the frontier of a proper disk E containing D and missing y. (See Figure 8.) Since J' separates D from y, J' has a point at distance 0 from x. We complete the proof by assuming every point of J' is at distance $<2\varepsilon$ from x and obtaining a contradiction. We may assume that E misses $\mathcal{R}_1$. If J' hits $\mathcal{R}_0$, let z_0 be a point of $J'\cap\mathcal{R}_0$ and define $\alpha_0=\{z_0\}$. Otherwise, let α_0 be an arc in $\alpha\cup\beta$ irreducible from $\mathcal{R}_0$ to E and let $z_0=\alpha_0\cap E$. Let α_1 be an arc in $\alpha\cup\beta$ irreducible from $\mathcal{R}_1$ to E and let $z_1=\alpha_1\cap E$. If $\operatorname{Fr} D(x')\cup\beta$ hits $\operatorname{star} J'$, let $\alpha_2=\varnothing$. Otherwise, let α_2 be irreducible in α from $\operatorname{Fr} D(x')\cup\beta$ to $\operatorname{star} J'$, α_2 half open with its missing endpoint in $\operatorname{star} J'$. Then

$$H_i\leqslant L_i(\alpha_0\cup\alpha_1)+d_i(z_0,x)+d_i(z_1,x)+4/i.$$

Since $L_i(\alpha\cap\operatorname{star}\mathcal{R}_0)<1/i$,

$$L_i(\alpha_0\cup\alpha_1)+L_i(\alpha_2)-1/i\leqslant H_i.$$

Hence

$$L_i(\alpha_2) \leqslant d_i(z_0, x) + d_i(x, z_1) + 5/i < 4\varepsilon + 5/i.$$

But $d_i(x', J') \leqslant L_i(\alpha_2) + 2/i$. Hence, for large i,

$$\begin{aligned} 8\varepsilon < d_i(x, x') &\leqslant \inf_{z \in J'} [d_i(x, z) + d_i(z, x')] + 2/i \\ &< 2\varepsilon + (4\varepsilon + 5/i) + 2/i = 6\varepsilon + 7/i < 8\varepsilon, \end{aligned}$$

a contradiction. □

Completion of the proof of Proposition 4.0.3. We are now ready to show that the approximate modulus of $\mathcal{R}'$ is 0. This will complete the proof of Proposition 4.0.3.

Pick an integer N very large. Then let i be even larger. Consider the sets $D_k = D(\varepsilon/e^k)$, $k = 0, 1, ..., N$, where

$$D(r) = \{y \in \mathcal{R} \mid d_i(x, y) \leqslant r\}.$$

Define

$$C_j = \{s \in \mathcal{S}_i \mid s \cap D_j \neq \varnothing \text{ but } s \cap D_{j+1} = \varnothing\}$$

for $j = 0, ..., N-1$. Put the remaining elements of $\mathcal{S}_i$ into collection C_{-1}. Assign each element $s \in C_j$ a new weight $\varrho'(s)$ as follows.

(1) If $s \in C_{-1}$, then $\varrho'(s) = 0$.

(2) If $s \in C_j$, $j \geqslant 0$, then

$$\varrho'(s) = \varrho_i(s) \cdot \frac{e^{j+1}}{\varepsilon(e-1)}.$$

Let J denote a simple closed curve circling the annulus $\mathcal{R}'$. By the lemma, there is an arc J' in $J \cap \mathcal{R}$ joining a point at d-distance 0 from x to d-distance $\geqslant \varepsilon$ from x. For large i, J' will join $D_0 = D(\varepsilon)$ and $D_N = D(\varepsilon/e^N)$. Hence, for $j = 0, 1, ..., N-1$, J' contains an open arc α_j irreducible from $\operatorname{star}(\mathcal{R} \setminus D_j)$ to $\operatorname{star} D_{j+1}$. No shingle of $\mathcal{S}_i$ hits two of these arcs $\alpha_0, \alpha_1, ..., \alpha_{N-1}$. Hence we may get a lower bound on the ϱ'-length $L'(J)$ as follows.

$$L'(J) \geqslant L'(\alpha_0 \cup ... \cup \alpha_{N-1}) = L'(\alpha_0) + ... + L'(\alpha_{N-1}).$$

For i very large, each of the $L'(\alpha_j)$ will be very close to 1, or larger. Indeed

$$L_i(\alpha_j) \geqslant \frac{\varepsilon}{e^j} - \frac{\varepsilon}{e^{j+1}} - 2/i = \frac{\varepsilon(e-1)}{e^{j+1}} - 2/i;$$

and, since only the shingles of C_k hit α_j with $k \geqslant j$,

$$L'(\alpha_j) \geqslant L_i(\alpha_j) \cdot \frac{e^{j+1}}{\varepsilon(e-1)}.$$

That is, N is an asymptotic lower bound for $L'(J)$ as $i \to \infty$.

We use Theorem 4.2.1 to obtain an upper bound on the ϱ'-area $A'(\mathcal{R}')$ of $\mathcal{R}'$. This theorem supplies an absolute constant $K(2)$ $(=176)$ such that, for large i,

$$A_i(D(r)) \leqslant K(2)r^2.$$

Hence, since every shingle of positive weight hits $D(\varepsilon)$,

$$\begin{aligned} A(\mathcal{R}', \varrho') = A'(D(\varepsilon)) &\leqslant \sum_{j=0}^{N-1} A(D(\varepsilon/e^j)) \cdot \left(\frac{e^{j+1}}{\varepsilon(e-1)} \right)^2 \\ &\leqslant \sum_{j=0}^{N-1} K(2) \left(\frac{\varepsilon e^{j+1}}{e^j \varepsilon (e-1)} \right)^2 \\ &= N \cdot K(2)(e/(e-1))^2. \end{aligned}$$

Thus we have

$$m_{\inf}(R', \mathcal{S}_i) \leqslant \frac{A(\mathcal{R}', \varrho')}{C(\mathcal{R}', \varrho')^2} \leqslant \frac{N \cdot K(2)(e/(e-1))^2}{N^2} \to 0$$

as $N \to \infty$. □

4.4. Topological properties of the limit metric d

It is reasonably easy at this point to prove that $\mathcal{R}$, with the limit metric d, is topologically nice. This subsection is devoted to that task.

4.4.1. PROPOSITION. *Let $x \in \alpha \subset \mathcal{R}$, α a connected set. Let*

$$v_0 < v_1 < v_2 < \ldots < v_n$$

be the values assumed by $d_i(x, y)$, $y \in \alpha$. Then

$$v_j - v_{j-1} < 1/i, \quad j = 1, \ldots, n.$$

Proof. Let

$$D = \{y \in \alpha \mid d_i(x, y) \leqslant v_{j-1}\}.$$

Then D is open in α and has nonempty complement

$$D^c = \{y \in \alpha \mid d_i(x, y) \geqslant v_j\}.$$

Hence D has a limit point $z \in D^c$. Some shingle hits z and an element $w \in D$. Let $D(x)$, $D(w)$, and $D(z)$ be i-approximations to x, w, and z such that $D(x)$ and $D(w)$ are joined by a path β of L_i-length $\leqslant v_j$. If two of these intersect, we assume their frontiers intersect. In particular, $\operatorname{Fr} D(w) \cap \operatorname{Fr} D(z) \neq \varnothing$. If $\operatorname{Fr} D(x) \cap \operatorname{Fr} D(w) \neq \varnothing$, then $\operatorname{Fr} D(w)$ contains an i-approximate path of L_i-length $< 1/i$ from x to z. Otherwise $\beta \cup \operatorname{Fr} D(w)$ is connected and contains an i-approximate path from x to z of L_i-length $< v_j + 1/i$. But $d_i(x, z) \geqslant v_{j+1}$.

□

4.4.2. PROPOSITION. *The limit metric d is a path metric.*

Proof. Let $x, y \in \mathcal{R}$ and let $d_0 = d(x, y)$. We need to show the existence of a path α from x to y in $\mathcal{R}$ of d-length d_0. Since $d_i \to d$, we may pick i-approximate paths α_i from x to y in $\mathcal{R}$ of L_i-length $L_i(\alpha_i) = d_i(x, y)$, $L_i(\alpha_i) \to d_0$. Since i-approximate paths can double back on themselves with no increase in L_i-length, we have no hope that the arcs α_i will converge homeomorphically to an arc α of d-length d_0. Instead we parametrize the arcs α_i by approximate distance from x and show that, so parametrized, a subsequence converges to the desired arc α.

The reparametrization. For each i and for each $t \in [0, d_0]$, let $u_i(t) \in [0, d_i(x, y)]$ be a value as near t as possible such that, for some $z \in \alpha_i$, $d_i(x, z) = u_i(t)$. Define $\beta_i: [0, d_0] \to \mathcal{R}$ by

$$\beta_i(t) = \{z \in \alpha_i \mid d_i(x, z) = u_i(t)\}.$$

Notice that $\beta_i(t) \neq \varnothing$ for each t; that is, β_i is everywhere defined. The sequence $\beta_1, \beta_2, \ldots$ is therefore a sequence of multi-valued functions from $[0, d_0]$ to $\mathcal{R}$, each everywhere defined.

Now note that the definitions of uniformly bounded and asymptotically equicontinuous apply without change to such multi-valued functions. The Arzela–Ascoli theorem also applies in this generality, again without change in the proof.

LEMMA. *The sequence $\beta_i: [0, d_0] \to \mathcal{R}$ is uniformly bounded and asymptotically equicontinuous.*

Completion of the proof of the proposition. Before proving the lemma, we will use it to prove the proposition. Some subsequence of $\beta_1, \beta_2, \ldots$, which we may take to be the whole sequence, converges to a continuous function $\alpha: [0, d_0] \to \mathcal{R}$. By Proposition 4.4.1, $|t - u_i(t)| < 1/i$. Hence

$$|d_i(x, \beta_i(0)) - 0| = |u_i(0) - 0| < 1/i.$$

Consequently, $\alpha(0) = x$. Similarly,

$$|d_i(x, \beta_i(t)) - t| = |u_i(t) - t| < 1/i$$

so that $d(x, \alpha(t)) = t$. By the triangle inequality $d(\alpha(t), \alpha(u)) = |t - u|$. Hence the length of α is d_0. It remains only to prove that α contains y. Pick $0 < t_0 < d_0$. Let γ_i be an arc in α_i irreducible from star $D(u_i(t_0))$ to the terminal endpoint of α_i on an i-approximation to y. Then $\limsup L_i(\gamma_i) \leqslant d_0 - t_0$. But γ_i intersects both $\beta_i([0, d_0])$ and an i-approximation to y. Hence α must contain a point of the $d_0 - t_0$ neighborhood of y. Letting $t_0 \to d_0$, we conclude that $y \in \alpha$. □

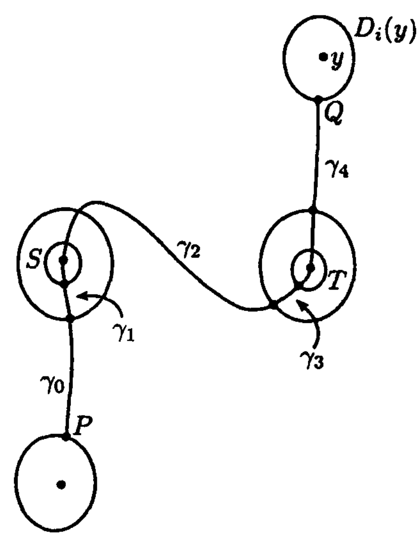

Fig. 9

Proof of the lemma. Let $0 \leqslant s \leqslant t \leqslant d_0$. Then

$$0 \leqslant |u_i(t) - u_i(s)| \leqslant t - s + 2/i.$$

Let $S, T \in \beta_i(s) \cup \beta_i(t)$. We will assume that α_i passes from i-approximations $D_i(x)$ through S, then through T, then to an i-approximation $D_i(y)$. We write α_i as $PSTQ$ with $P \in D_i(x)$, $Q \in D_i(y)$. We pick $r > 0$ and consider arcs as follows: $\gamma_0 \subset PS$ is irreducible from $D_i(x)$ to $\operatorname{star} D(S \cup T, 2r)$, $\gamma_1 \subset PS$ is irreducible from the star of the complement $D(S \cup T, 2r)^c$ to $D(S, r)$, $\gamma_2 \subset ST$ is irreducible from $\operatorname{star} D(S, r)$ to $\operatorname{star} D(T, r)$, $\gamma_3 \subset ST$ is irreducible from $\operatorname{star}(D(S \cup T, 2r)^c)$ to $D(T, r)$, $\gamma_4 \subset TQ$ is irreducible from $D(S \cup T, 2r)$ to $D_i(y)$. (See Figure 9.) If any shingle hits two of $\gamma_0, \gamma_2, \gamma_4$, then at least one of γ_1 and γ_3 can be discarded with a reduction in length of at least $r - 1/i$ and compensating increase of less than $1/i$, a net reduction of at least $r - 2/i$, and a contradiction to the minimality of α_i. Hence the three hit no common shingle. That is,

$$[u_i(s) - r] + [d_0 - u_i(t) - r] + L_i(\gamma_2) - 2/i \leqslant L_i(\gamma_0) + L_i(\gamma_4) + L_i(\gamma_2) \leqslant L_i(\alpha_i) = d_0.$$

Hence,

$$L_i(\gamma_2) \leqslant u_i(t) - u_i(s) + 2r + 2/i \leqslant t - s + 2r + 4/i,$$

and

$$d_i(S, T) \leqslant (2r + 2/i) + L_i(\gamma_2) \leqslant 4r + 6/i + t - s.$$

Since r was arbitrary, needed large with respect to i, and since $d_i \to d$ uniformly, we see that $\operatorname{diam}(\beta_i(t) \cup \beta_i(s))$ goes to 0 uniformly with $|t-s|$. The lemma follows. □

4.4.3. PROPOSITION. *Through every point x of $\mathcal{R}$ there is a path α joining the ends of $\mathcal{R}$ and having length $H = d(\mathcal{R}_0, \mathcal{R}_1)$.*

Proof. By Propositions 4.1.2 and 4.1.3 there is an i-approximate path β_i from $\mathcal{R}_0$ to $\mathcal{R}_1$ which comes within d_i-distance $1/i$ of x. By the proof of Proposition 4.4.2, some subsequence of

$$\beta_1, \beta_2, \ldots$$

converges to a path α from $\mathcal{R}_0$ to $\mathcal{R}_1$. This path has length

$$L(\alpha) = \lim_{i \to \infty} L_i(\beta_i) = \lim_{i \to \infty} H_i = H = d(\mathcal{R}_0, \mathcal{R}_1),$$

and it passes through x. □

Definition. We define

$$J(t) = \{x \in \mathcal{R} \mid d(x, \mathcal{R}_0) = t\}.$$

We call the sets $J(t)$ *level curves* with respect to the metric d.

4.4.4. PROPOSITION. *Each of the level curves $J(t)$, $t \in (0, H)$, is a simple closed curve separating $\mathcal{R}_0$ from $\mathcal{R}_1$.*

Proof. A beautiful theorem from plane topology says that a compact set C separating two points p and q is a simple closed curve if each point of C is arcwise accessible from each of the two complementary domains of C containing p and q. Take $t \in (0, H)$ and set $C = J(t)$. It is clear that C is a compact set which separates $\mathcal{R}_0$ from $\mathcal{R}_1 = J(H)$ in $\mathcal{R}$. Let $p \in \mathcal{R}_0$ and $q \in \mathcal{R}_1$. Let α be an arc of length H passing through x and joining $\mathcal{R}_0$ to $\mathcal{R}_1$. Then α demonstrates that x is arcwise accessible from the complementary domain of p and the complementary domain of q. Hence $J(t)$ is a simple closed curve. □

4.4.5. PROPOSITION. *The ring $\mathcal{R}$ is a topological product of the form $J \times [0, H]$, where J is a simple closed curve and $J \times \{t\}$ corresponds to $J(t)$.*

LEMMA. *Suppose D is a 2-cell in $\mathcal{R}$ whose boundary consists of four arcs which we call B, for bottom, L for left, T for top, and R for right. Suppose B and T are horizontal in the sense that $B \subset J(a)$, $T \subset J(b)$, $a < b$. Suppose that L and R are monotone in the sense that each intersects each $J(t)$, $a \leqslant t \leqslant b$, precisely once. Suppose finally that $P \in \operatorname{Int} T$ and $Q \in \operatorname{Int} B$. Then there is a monotone arc PQ from P to Q with $\operatorname{Int} PQ \subset \operatorname{Int} D$.*

Proof of the lemma. It is an easy matter to construct an initial approximation to PQ which consists of alternate horizontal and monotone arcs. The only trick is to show

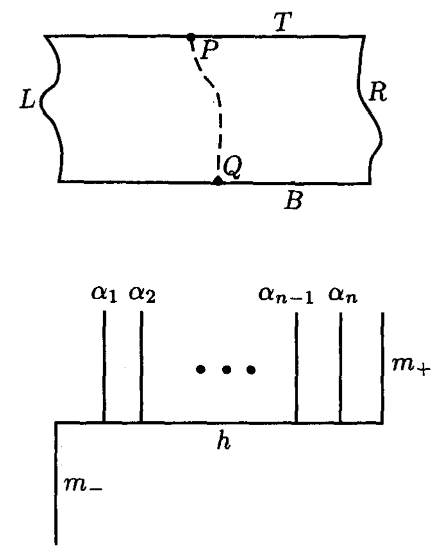

Fig. 10

that the horizontal arcs can be tipped slightly so as to be monotone. One does this a little at a time so that PQ is a uniform limit of approximations with horizontal bits of smaller and smaller diameter. Concentrate therefore on one horizontal bit h and the monotone bit m_+ rising from one of its ends. (See Figure 10.) From each point of $\operatorname{Int} h$ it is possible to construct a monotone rising arc α by Proposition 4.4.3. Pick finitely many of these, say $\alpha_1, \alpha_2, ..., \alpha_n$ such that the resulting subdivision of h is arbitrarily small. Cut the α_i back so that they are disjoint. Pick $J(t)$ above, but so near h, that the disks bounded by $h, \alpha_1, \alpha_2, ..., \alpha_n, m_+$, and $J(t)$ are roughly of the same diameter as the partition of h by the α_i's. Replace $h \cup m_+$ by a staggered curve as in Figure 11.

There is nothing that says that the new segments are short, but they will be of small diameter and close setwise to the original segments. That is, we may make the new approximation arbitrarily close to the previous approximation in the uniform topology and we can separate vertically any prescribed finite subset. As a consequence, the limit can be chosen to be a monotone arc arbitrarily near the first approximation. □

Proof of Proposition 4.4.5. We have two rings $\mathcal{R}$ and $J \times [0, H]$ in which we have a notion of horizontal and monotone arcs and in which both horizontal and monotone arcs are abundant. We call a disk in either *standard* if it is constituted as in the lemma with horizontal top and bottom, monotone sides. Either ring can be decomposed into finitely many standard disks (exercise—join nearby horizontal levels by small, disjoint monotone arcs). So decompose $\mathcal{R}$. The lemma, applied to $J \times [0, H]$, allows one to copy

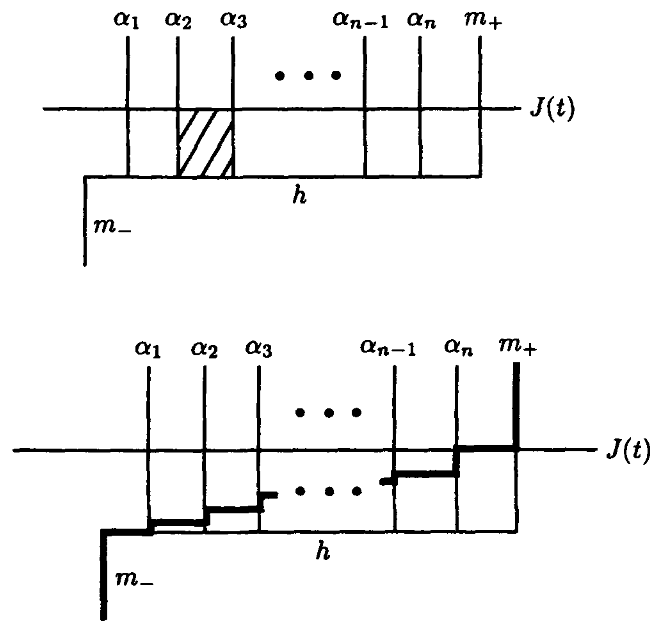

Fig. 11

the 1-skeleton of $\mathcal{R}$ in $J \times [0, H]$ in a level preserving fashion. Now subdivide each of the standard disks resulting from this copy into tiny standard cells in $J \times [0, H]$. The lemma, applied to $\mathcal{R}$, allows one to extend the original 1-skeleton of $\mathcal{R}$ to copy the new 1-skeleton of $J \times [0, H]$ in a level preserving fashion in $\mathcal{R}$. Iterate this back and forth subdivision. With only slight care, the procedure converges to a level-preserving homeomorphism. □

5. Approximate areas and their limits

The limiting metric d serves well as a measure of distance to the ends $\mathcal{R}_0 = J(0)$ and $\mathcal{R}_1 = J(H)$ of $\mathcal{R}$ but poorly as a measure of circumference. In order to measure circumference properly we need to carefully consider the limiting properties of the ϱ_i-areas defined on subsets of $\mathcal{R}$. The principal results of this section show that ϱ_i-area and the ϱ_i-height in $\mathcal{R}$ are, at least asymptotically, both locally and globally proportional. These results have at least three fascinating consequences. The first is disconcerting: the d-lengths of segments in the slices

$$J(t) = \{y \in \mathcal{R} \mid d(\mathcal{R}_0, y) = t\}$$

can vary discontinuously, both locally and globally; hence $(\mathcal{R}, d)$ is not generally a right circular cylinder as one would expect from the classical case. The second consequence is encouraging: the d-lengths of the curves $J(t)$ are uniformly bounded away from 0 and ∞; the discontinuity of d-lengths in horizontal slices is therefore fairly nicely controlled. The final and most important consequence is that the derivative of area with respect to height is completely smooth and uniform; this derivative may be used as horizontal measure in the curves $J(t)$ and assigns these curves identical length.

The proportionality of height and area under an optimal metric was discovered accidentally when we tried to prove the combinatorial analogue for $(\mathcal{R}, d)$ of the following classical result. The physical version of this theorem seems to follow from a general version of Kirchhoff's current law, where voltage is given by distance to the ends, power is given by area, and current is the derivative of power with respect to voltage.

LEMMA (see [LV, p. 35]). *Let $J_1, J_2, \ldots, J_{k-1}$ be horizontal simple closed curves in $(\mathcal{R}, d)$. Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ denote the rings into which $J_1, J_2, \ldots, J_{k-1}$ divide $\mathcal{R}$. If $\mathcal{R}$ is a right circular cylinder, then the classical moduli $M_1, M_2, \ldots, M_k, M$ of $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k, \mathcal{R}$ satisfy the equation*

$$M_1 + \ldots + M_k = M.$$

Expressed in terms of heights and areas, this lemma requires that

$$\frac{H_1^2}{A_1} + \ldots + \frac{H_k^2}{A_k} - \frac{H^2}{A} = 0.$$

However, we have the following easy proposition.

PROPOSITION. *Let $h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, a$ be positive numbers such that $a = a_1 + \ldots + a_k$. Then*

$$\sum_i \frac{h_i^2}{a_i} - \frac{(\sum_i h_i)^2}{a} = \frac{1}{a} \sum_{i<j} a_i a_j \left(\frac{h_i}{a_i} - \frac{h_j}{a_j} \right)^2.$$

Thus the proposition and lemma are compatible if and only if, for each i and j, $H_i / A_i = H_j / A_j$.

Proof of the proposition.

$$\begin{aligned} \sum_i \frac{h_i^2}{a_i} - \frac{(\sum_i h_i)^2}{a} &= \frac{1}{a} \left[\sum_i \frac{h_i^2}{a_i}(a - a_i) - \sum_{i \neq j} h_i h_j \right] \\ &= \frac{1}{a} \sum_{i \neq j} \left(\frac{h_i^2}{a_i} a_j - h_i h_j \right) \\ &= \frac{1}{a} \sum_{i<j} a_i a_j \left(\frac{h_i^2}{a_i^2} - \frac{2 h_i h_j}{a_i a_j} + \frac{h_j^2}{a_j^2} \right) \\ &= \frac{1}{a} \sum_{i<j} a_i a_j \left(\frac{h_i}{a_i} - \frac{h_j}{a_j} \right)^2. \end{aligned}$$

□

The annulus $(\mathcal{R}, d)$ has uniform height $H=\lim_{i\to\infty} H_i$; it is the union of horizontal curves

$$J(t)=\{y\in\mathcal{R} \mid d(y,\mathcal{R}_0)=t\}, \quad t\in[0,H].$$

If $X\subset\mathcal{R}$ and $A\subset[0,H]$, we define

$$X[A]=\{x\in X \mid d(x,\mathcal{R}_0)\in A\}.$$

Thus $J(t)=\mathcal{R}[t]$.

5.1. THEOREM. *If* $0\leqslant a<b\leqslant H$, *then*

$$\lim_{i\to\infty} A_i(\mathcal{R}[a,b])=\frac{(b-a)}{H}.$$

That is, area is asymptotically proportional to height.

Proof. Let i be so large that $|d-d_i|<\varepsilon$. Let T denote the set of shingles in $\mathcal{S}_i$ hitting $\mathcal{R}[a,b]$. Let U denote the set missing $\mathcal{R}[a,b]$. Let $\varrho_i=(t,u)$, where $t=\varrho_i|T$ and $u=\varrho_i|U$. Define $\varrho'=(t',u')$ where $t'=\lambda t$ and $u'=\mu u$. The weight functions ϱ_i and ϱ' give us areas

$$\begin{aligned}A&=\varrho_i\cdot\varrho_i=t\cdot t+u\cdot u=1.\\ A'&=\varrho'\cdot\varrho'=t'\cdot t'+u'\cdot u'=\lambda^2\cdot t\cdot t+\mu^2\cdot u\cdot u\end{aligned}$$

and heights h,h'. We use lower case letter h and h' to avoid conflict with the d-height $H=\lim_{i\to\infty} H_i$ of $\mathcal{R}$. The optimality of ϱ_i yields

$$\frac{(h')^2}{A'}\leqslant\frac{h^2}{A}. \tag{$*$}$$

We obtain a lower bound for h' in the following way. Let α be a path of minimal ϱ'-length joining the ends of $\mathcal{R}$ (so that $L'(\alpha)=h'$). Let $\alpha_0\subset\alpha$ be open-irreducible from star $\mathcal{R}_0$ to star $\mathcal{R}[a,H]$. Let $\alpha_1\subset\alpha$ be open-irreducible from star $\mathcal{R}[0,a]$ to star $\mathcal{R}[b,H]$. Let $\alpha_2\subset\alpha$ be open-irreducible from star $\mathcal{R}[H]$ to star $\mathcal{R}[0,b]$. In each case, if the sets being joined already intersect, let the corresponding α_i be empty. It is an easy matter to check that no shingle hits two of the α_i so that

$$h'=L'(\alpha)\geqslant L'(\alpha_0)+L'(\alpha_1)+L'(\alpha_2).$$

Even more, no shingle of T hits α_0 or α_2 and no shingle of U hits α_1, so that

$$\begin{aligned}L'(\alpha_0)&=\mu L(\alpha_0),\\ L'(\alpha_1)&=\lambda L(\alpha_1),\\ L'(\alpha_2)&=\mu L(\alpha_2).\end{aligned}$$

Either α_2 is empty,

$$\operatorname{star}\mathcal{R}[H] \cap \operatorname{star}\mathcal{R}[0,b] \neq \varnothing,$$

and $d_i(\mathcal{R}[H], \mathcal{R}[0,b]) < 1/i$; or α_2 is a nonempty i-approximate path from $\mathcal{R}[H]$ to $\mathcal{R}[0,b]$, hence has ϱ_i-length $\geqslant H-b-\varepsilon-1/i$. Similarly,

$$L(\alpha_1) \geqslant b-a-\varepsilon-1/i$$

and

$$L(\alpha_0) \geqslant a-\varepsilon-1/i.$$

Thus

$$h' \geqslant \mu(a-\varepsilon-1/i)+\lambda(b-a-\varepsilon-1/i)+\mu(H-b-\varepsilon-1/i).$$

We substitute all of our calculations into $(*)$ to obtain

$$\frac{[\mu(a-\varepsilon-1/i)+\lambda(b-a-\varepsilon-1/i)+\mu(H-b-\varepsilon-1/i)]^2}{\lambda^2 t\cdot t+\mu^2 u\cdot u} \leqslant (H+\varepsilon)^2.$$

As $i\to\infty$, we may assume that $\varepsilon\to 0$, $1/i\to 0$, and that $t\cdot t\to X$, $u\cdot u\to Y$, $X+Y=1$. Setting $b-a=x$ and $H-(b-a)=y$ so that $H=x+y$, we obtain in the limit, for all $\lambda,\mu>0$,

$$\frac{(\lambda x+\mu y)^2}{\lambda^2 X+\mu^2 Y} \leqslant (x+y)^2.$$

The trivial cases are $(X=0, x=0)$ and $(Y=0, y=0)$. Otherwise, the left hand side attains its maximum at $\lambda=x/X$, $\mu=y/Y$, where the value attained is

$$\frac{x^2}{X}+\frac{y^2}{Y}.$$

But, recalling that $X+Y=1$, we verify that

$$\frac{x^2}{X}+\frac{y^2}{Y}-(x+y)^2 = \left[x\left(\frac{Y}{X}\right)^{1/2} - y\left(\frac{X}{Y}\right)^{1/2}\right]^2 \geqslant 0$$

can be $\leqslant 0$ if and only if

$$x/X = y/Y.$$

This completes the proof of the theorem. □

Alternative proof. Here is a slight variant on the proof. It is this variant which we shall generalize in establishing the local version of the theorem.

As in the first proof, fix $\varepsilon>0$, choose i large, define $\varrho'=(\lambda t, \mu u)$, and estimate h' to find, with $x=b-a-\varepsilon-1/i$ and $y=b-a-3\varepsilon-2/i$,

$$\begin{aligned} h' &\geqslant \lambda(b-a-\varepsilon-1/i)+\mu(H-b+a-2\varepsilon-2/i) \\ &\geqslant \lambda(b-a-\varepsilon-1/i)+\mu(H_i-b+a-3\varepsilon-2/i) \\ &= \lambda x+\mu(H_i-y). \end{aligned}$$

Fix $\mu>1$ and define λ so that

$$H_i = \lambda x+\mu(H_i-y) \leqslant h'. \tag{1}$$

Note that if μ is fairly close to 1, ε and $1/i$ small, then $0<\lambda<1$. The optimality of ϱ_i then yields

$$\frac{H_i^2}{A'} \leqslant \frac{(h')^2}{A'} \leqslant \frac{H_i^2}{A_i}$$

so that

$$t\cdot t+u\cdot u = A_i \leqslant A' = \lambda_2\, t\cdot t+\mu^2\, u\cdot u. \tag{2}$$

We solve (1) for λ and substitute in (2) to obtain

$$0 \leqslant \frac{1}{x^2}[H_i^2(\mu-1)^2-2\mu(\mu-1)H_i y]\, t\cdot t+\left(\left(\frac{\mu y}{x}\right)^2-1\right) t\cdot t+(\mu^2-1)u\cdot u.$$

With μ fixed, we let $\varepsilon\to 0$ and $i\to\infty$. We see that $H_i\to H$, $x\to b-a$, $y\to b-a$. We may also assume $t\cdot t\to X$ and $u\cdot u\to Y$, $X+Y=1$. We obtain in the limit

$$0 \leqslant \left[\frac{H^2}{(b-a)^2}(\mu-1)^2-2\mu(\mu-1)\frac{H}{b-a}\right]X+[\mu^2-1][X+Y].$$

We divide by $\mu-1$ (>0) and take the limit as $\mu\to 1$ to obtain

$$0 \leqslant -\frac{2H}{b-a}X+2[X+Y]$$

or

$$\frac{H}{X+Y} \leqslant \frac{b-a}{X}. \tag{3}$$

Similarly

$$\frac{H}{X+Y} \leqslant \frac{H-b+a}{Y}. \tag{4}$$

Since $X+Y=1$, it follows from (3) and (4) that

$$H = HX+HY \leqslant (b-a)+(H-b+a) = H.$$

Hence both (3) and (4) must be equalities. □

With the global proportionality of area and height, we are prepared to show that, in general, the ring $\mathcal{R}$ with the limiting metric d is not a right circular cylinder. We give an explicit example. (See Figure 12.) The argument for the theorem was carried out in the limit rather than with $(\mathcal{R}, \mathcal{S}_i)$ and ϱ_i in order simply to avoid technical details involving ε and $1/i$. For our example, these technicalities do not arise at all so that we may apply a nonasymptotic version of the theorem.

We take $\mathcal{R}$ as the cylinder

$$\mathcal{R} = S^1 \times [0,1] = [\text{Reals}/\langle 1\rangle] \times [0,1]$$

and prescribe the shinglings $\mathcal{S}_i$ as follows. A shingle $s \in \mathcal{S}_i$ is either of the form

$$[t, t+(1/2i)] \times [u, u+(1/2i)]$$

with $t \in [0,1)$ and $u \in \left[0, \frac{1}{2}\right)$ multiples of $1/2i$ or of the form

$$[t, t+(3/2i)] \times [u, u+(1/2i)]$$

with $t \in [0,1)$ and $u \in \left[\frac{1}{2}, 1\right)$ multiples of $1/2i$. Thus the lower half of $\mathcal{R}$ is tiled by squares and the upper half of $\mathcal{R}$ is shingled three-deep by rectangles of width $3/2i$ and height $1/2i$. Note that this shingling is invariant under a rotation of $1/2i$. Let ϱ_i be an optimal weight function on $(\mathcal{R}, \mathcal{S}_i)$. Since the optimal weight function is unique, the optimal weight function is also invariant under that rotation. It follows that ϱ_i is constant on the shingles in a given horizontal strip. The argument of the previous section shows that the ratio of height to area is constant over all horizontal strips. It follows that weights are constant on the entire lower half. Similarly, weights are constant on the entire upper half. If one then compares a strip in the lower half with a strip in the upper half, one concludes that the weights are three times as large in the upper half as in the lower half. It follows that the heights, hence areas of the two "halves" differ by a factor of 9. Similarly, circumference differs by a factor of 3. There is, therefore, in the limit, a jump discontinuity in d-lengths of $J(t)$. In the lower half the area is the product of height and circumference. In the upper half the area is one third of the product of the height and circumference.

Before passing on to a local version of our area/height theorem, we deduce a consequence about the d-lengths of the curves

$$J(t) = \{y \in \mathcal{R} \mid d(y, \mathcal{R}_0) = t\}.$$

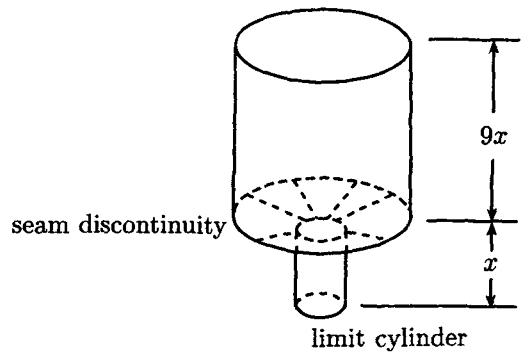

Fig. 12

We need a preliminary proposition deducing the approximate moduli of the annuli $\mathcal{R}[a,b]$, $0 \leqslant a < b \leqslant H$.

For each ring $\mathcal{R}$ we may define a more precise approximate modulus $\text{mod}(\mathcal{R},\mathcal{S})$ as follows, where $\mathcal{S}$ now refers to the entire conformal sequence $\mathcal{S}=\mathcal{S}_1,\mathcal{S}_2,\ldots$ of tilings:

$$\text{mod}(\mathcal{R},\mathcal{S}) = \limsup_{i\to\infty} M_{\text{sup}}(\mathcal{R},\mathcal{S}_i).$$

5.2. Proposition. *If* $0 \leqslant a < b \leqslant H$, *then*

$$\text{mod}(\mathcal{R}[a,b],\mathcal{S}) = (b-a)H.$$

Proof. Choose i so large that the ϱ_i-area α of $\mathcal{R}[a,b]$ is very close to $(b-a)/H$ (previous theorem) and that the ϱ_i-height η is very close to $b-a$. Then

$$M_{\text{sup}}(\mathcal{R}[a,b],\mathcal{S}_i) \geqslant \eta^2/\alpha \approx (b-a)H.$$

Thus $\text{mod}(\mathcal{R}[a,b],\mathcal{S}) \geqslant (b-a)H$. Similarly,

$$\text{mod}(\mathcal{R}[0,a]) \geqslant aH \quad \text{and} \quad \text{mod}(\mathcal{R}[b,H]) \geqslant (H-b)H.$$

Trim these three annuli to obtain $\mathcal{R}[0,a-\varepsilon]$, $\mathcal{R}[a+\varepsilon,b-\varepsilon]$, and $\mathcal{R}[b+\varepsilon,H]$. Choose i so large that no shingle hits any of these three together with one of the curves $J(a)$ and $J(b)$. Choose a weight function σ_i on $\mathcal{S}_i$ so that σ_i is 0 on any shingle missing

$$\mathcal{R}[0,a-\varepsilon]\cup\mathcal{R}[a+\varepsilon,b-\varepsilon]\cup\mathcal{R}[b+\varepsilon,H]$$

and such that σ_i is optimal when restricted to any of the three annuli. Let h_0,h_1,h_2 be the associated heights and A_0,A_1,A_2 the associated areas. By suitably scaling σ_i on the three rings, it is possible to ensure that

$$A_0/h_0 = A_1/h_1 = A_2/h_2.$$

Then if h is the σ_i (scaled) height and A the σ_i-area of $\mathcal{R}$,

$$h \geqslant h_0+h_1+h_2$$

and

$$A = A_0+A_1+A_2.$$

Hence

$$\begin{aligned}H^2 = M_{\sup}(\mathcal{R}, \mathcal{S}_i) &\geqslant \frac{h^2}{A} \geqslant \frac{(h_0+h_1+h_2)^2}{(A_0+A_1+A_2)} \\ &= \left(\frac{h_0^2}{A_0}\right)+\left(\frac{h_1^2}{A_1}\right)+\left(\frac{h_2^2}{A_2}\right) \\ &\approx \operatorname{mod}(\mathcal{R}[0, a-\varepsilon])+\operatorname{mod}(\mathcal{R}[a+\varepsilon, b-\varepsilon])+\operatorname{mod}(\mathcal{R}[b+\varepsilon, H]) \\ &\geqslant (a-\varepsilon)H+(b-a-2\varepsilon)H+(H-b-\varepsilon)H \approx H^2.\end{aligned}$$

Thus we must have approximate equality everywhere. That is,

$$\operatorname{mod}(\mathcal{R}[a, b], \mathcal{S}) = (b-a)H. \qquad \square$$

We characterize the d-length of curves in terms of L_i-lengths of curves.

5.3. PROPOSITION. *Let J be an arc (or simple closed curve) in $\mathcal{R}$. Then $L(J) \leqslant L$ if and only if, for each $\varepsilon>0$ and for each integer I, there exist a curve J' and an integer $i \geqslant I$ such that*

$$d(J(x), J'(x)) < \varepsilon \quad \text{for each } x,$$

and

$$L_i(J') < L+\varepsilon.$$

Proof. Assume first that $L(J)>L$. Pick $\delta>0$ so small that $L(J)>L+2\delta$. Then it is an easy matter to find disjoint arcs $\alpha_1, \alpha_2, \ldots, \alpha_n$ in J such that

$$\sum_1^n \operatorname{diam}(\alpha_j) > L+2\delta.$$

Choose $\varepsilon>0$ so small that if $1/i<\varepsilon$, then no shingle of $\mathcal{S}_i$ can hit two of the ε-neighborhoods $N(\alpha_j, \varepsilon)$, $j=1, \ldots, n$. Further, choose $\varepsilon>0$ so small that if J' is within ε of J, then there exist subarcs $\beta_1, \beta_2, \ldots, \beta_n$ of J' such that $\beta_j \subset N(\alpha_j, \varepsilon)$ and

$$\sum_1^n \operatorname{diam}(\beta_j) > L+2\delta.$$

Finally, choose I so large that $1/I<\varepsilon$ and, for each $i\geqslant I$, $|d-d_i|<\delta/n$. Now let J' be within ε of J and let $i\geqslant I$. Then

$$L_i(J')\geqslant\sum_1^n L_i(\beta_j)\geqslant\sum_1^n(\operatorname{diam}\beta_j-\delta/n)>(L+2\delta)-\delta=L+\delta>L+\varepsilon.$$

For the converse we assume J is a simple closed curve. Assume that $L(J)\leqslant L$. Let $\varepsilon>0$ and I be given. Pick any cycle

$$x_0,x_1,\ldots,x_n=x_0$$

in J such that

$$\operatorname{diam}(x_ix_j)<\varepsilon/k,$$

k large. Pick m large and $i\geqslant I$ so large that $|d-d_i|<1/mn$. Pick i-approximations

$$D(x_0),E(x_1),D(x_1),E(x_2),\ldots,D(x_{n-1}),E(x_n)$$

to $x_0,x_1,\ldots,x_n$ such that there is an arc α_j of ϱ_i-length $L_i(\alpha_j)=d_i(x_{j-1},x_j)$ joining $D(x_{j-1})$ to $E(x_j)$. As always, we assume frontiers intersect if approximations intersect. We replace α_j by the empty set if

$$\operatorname{Fr}D(x_{j-1})\cap\operatorname{Fr}E(x_j)\neq\varnothing.$$

Then

$$\bigcup_j(\operatorname{Fr}D(x_{j-1})\cup\alpha_j\cup\operatorname{Fr}E(x_j))$$

contains a closed curve J' into which J maps, sending $A_j=x_{j-1}x_j$ into

$$B_j=\operatorname{Fr}D(x_{j-1})\cup\alpha_j\cup\operatorname{Fr}E(x_j).$$

The sets A_j and B_j are at d_i-distance $<1/i$ from one another, hence d-distance $<1/i+1/mn$. The set A_j has d-diameter $<\varepsilon/k$. The set α_j has ϱ_i-length

$$d_i(x_{j-1},x_j)<\operatorname{diam}A_j+1/mn<\varepsilon/k+1/mn.$$

Hence B_j has

$$\operatorname{diam}B_j<L_i(B_j)+1/mn<(\varepsilon/k+1/mn+2/i)+1/mn.$$

Hence J and J' lie within

$$\operatorname{diam}A_j+d(A_j,B_j)+\operatorname{diam}B_j<\varepsilon/k+(1/i+1/mn)+(\varepsilon/k+2/mn+2/i)$$

of one another. Since k, m, and i may be chosen large, we find that J and J' may be made close. Finally, we calculate the ϱ_i-length of J':

$$L_i(J') \leqslant \sum_1^n L_i(\operatorname{Fr} D(x_{j-1}) \cup \alpha_j \cup \operatorname{Fr} E(x_j)) \leqslant 2n/i + \sum_1^n (\operatorname{diam} A_j + 1/mn)$$
$$= 2n/i + 1/m + \sum_1^n \operatorname{diam} A_j \leqslant 2n/i + 1/m + L(\alpha).$$

Since m is large, and since i may be chosen large *after* n is determined, we may require that

$$2n/i + 1/m < \varepsilon. \qquad \square$$

5.4. PROPOSITION. *If the sequence $\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2, \dots$ is K-conformal and if $H = \lim_{i\to\infty} H_i$, then*

$$1/H = L(J(t)) \leqslant \sqrt{K}/H.$$

Proof. We first establish the lower bound. If it were possible that $L(J(t)) < 1/H$, then by the previous proposition there would be curves J' near $J(t)$, hence circling R, and large i such that $L_i(J') < 1/H_i$. Let $\varrho_i = (u, v)$, where u is the weight vector corresponding to shingles of $\mathcal{S}_i$ hitting J' and v the vector corresponding to the remaining shingles of $\mathcal{S}_i$. Let I denote the vector having the length of u and having each entry equal to 1. Add $\eta > 0$ to the weight of each shingle hitting J' to form a new weight function $\varrho' = (u + \eta I, v)$. Let $h' \geqslant H_i + \eta$ be the new height and

$$A' = \varrho' \cdot \varrho' = u \cdot u + 2\eta|u| + \eta^2 I \cdot I + v \cdot v = A_i + 2\eta|u| + \eta^2 I \cdot I = 1 + 2\eta|u| + \eta^2 I \cdot I$$

the new area. Then by the optimality of ϱ_i,

$$\frac{(H_i + \eta)^2}{A'} \leqslant \frac{(h')^2}{A'} \leqslant \frac{H_i^2}{A_i},$$

or

$$H_i^2 + 2\eta H_i + \eta^2 \leqslant H_i^2 A' = H_i^2 (1 + 2\eta|u| + \eta^2 I \cdot I).$$

Subtract H_i^2, divide by $2\eta H_i^2$, and let $\eta \to 0$ to find

$$1/H_i \leqslant |u| = L_i(J'),$$

a contradiction.

We now establish the upper bound. Fix $a \leqslant t \leqslant b$, $a < b$. We shall show the existence of a simple closed curve J' in $\mathcal{R}[a, b]$ circling $\mathcal{R}[a, b]$ with $L_i(J') \leqslant \sqrt{K}/H$. By an argument

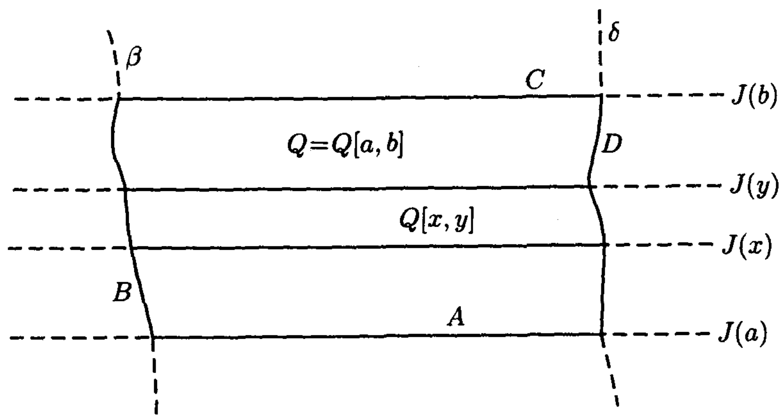

Fig. 13

that is only a slight modification of that given above (first half of the proof of Proposition 5.3), letting $a, b \to t$, we find that $L(J(t)) \leqslant \sqrt{K}/H$. Let c be the ϱ_i-circumference of $\mathcal{R}[a,b]$. Then

$$\begin{aligned}\frac{1}{K}(b-a)H &= \frac{1}{K}\cdot\frac{(b-a)^2}{(b-a)/H} \approx \frac{1}{K}\cdot\frac{H_i(\mathcal{R}[a,b])^2}{A_i(\mathcal{R}[a,b])}\\ &= \frac{1}{K}M_{\sup}(\mathcal{R}[a,b],\mathcal{S}_i) \leqslant m_{\inf}(\mathcal{R}[a,b],\mathcal{S}_i)\\ &\leqslant A_i(\mathcal{R}[a,b])/c^2 \approx (b-a)/Hc^2.\end{aligned}$$

Hence $c^2 \leqslant\approx K/H^2$ or $c \leqslant\approx \sqrt{K}/H$. □

Definition. A *quadrilateral* Q in $\mathcal{R}$ is a disk having the following form. (See Figure 13.)

The boundary arcs A and C, called the *bottom* and *top* of Q, respectively, are *horizontal* in the sense that they lie in level curves $J(a)$ and $J(b)$, $0 \leqslant a < b \leqslant H$, of $\mathcal{R}$. The boundary arcs B and D, called the *left* and *right sides* of Q, respectively, are *vertical* in the sense that they lie in arcs β and δ of minimal length H joining the ends of $\mathcal{R}$. We sometimes write $Q[a,b]$ to indicate the bottom level a and the top level b of Q. If $[x,y] \subset [a,b]$, then we write $Q[x,y] = \mathcal{R}[x,y] \cap Q$ for the quadrilateral formed by truncating Q below level x and above level y. The *height* $h(Q)$ of Q is $b-a$, the d-distance between A and C. The *width* $w(Q)$ of Q is the d-distance between B and D.

5.5. THEOREM. *There is a subsequence of i's such that, for each quadrilateral $Q \subset \mathcal{R}$, the areas $A_i(Q)$ converge.*

Proof. Define an arc α to be *weakly monotone* if $x, y \in \alpha \cap J(t)$ implies $xy \subset J(t)$.

Define α to be HV if it is a concatenation

$$h_1 * v_1 * \ldots * h_k * v_k$$

with each h_i horizontal and each v_i vertical (possibly degenerate). Among the weakly monotone arcs joining the ends of $\mathcal{R}$ there is a countable set

$$A = \alpha_1, \alpha_2, \ldots$$

of HV arcs that is dense. Now pick a countable dense set B in the interval $[0, H]$, 0 and H in B. Consider the family C of quadrilaterals Q such that the top and bottom of Q lie in levels $J(a)$ and $J(b)$, $a<b$, $a, b \in B$, and the sides of Q lie in arcs of A. Then C is countable. We may therefore pick our sequence of i's such that, for each $Q \in C$, $A_i(Q)$ converges.

We claim that, for each quadrilateral $Q \subset \mathcal{R}$, $A_i(Q)$ converges. Given $\varepsilon > 0$, there exist $Q', Q'' \in C$ such that

$$Q' \subset Q \subset Q''$$

and such that the ends of Q' and Q'' are homeomorphically within ε of the bottom $Q(a)$ and top $Q(b)$ of Q and the sides are homeomorphically within ε of the sides L and R of Q. We compare the areas of Q' and Q'' as follows. The set $Q'' \setminus Q'$ lies in the union of four sets; the ε-neighborhoods of $Q(a)$, $Q(b)$, L, and R. The first two have limiting areas bounded above by $2\varepsilon/H$ by Theorem 5.1. The latter two have limiting areas bounded above by $8(b-a)K(2)\varepsilon$ by the corollary to the quadratic area estimate, Theorem 4.2.1. If we now let $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \lim_{i \to \infty} A_i(Q'(\varepsilon)) \leqslant \liminf_{i \to \infty} A_i(Q) \leqslant \limsup_{i \to \infty} A_i(Q) \leqslant \lim_{\varepsilon \to 0} \lim_{i \to \infty} A_i(Q''(\varepsilon)).$$

But

$$\left(\lim_{i \to \infty} A_i(Q''(\varepsilon))\right) - \left(\lim_{i \to \infty} A_i(Q'(\varepsilon))\right) = 0,$$

as we have just seen. Hence $\lim_{i \to \infty} A_i(Q)$ exists as claimed. □

5.6. PROPOSITION. *There exist constants $K(3)$ and $K(4)$ such that, if Q is a quadrilateral and $2h(Q) < w(Q)$, then*

$$\frac{1}{K(3)} A(Q) \leqslant h(Q) w(Q) \leqslant K(4) A(Q).$$

Remarks. Recall that h denotes height, w denotes width, both measured by the limit metric d, and that A denotes area, measured by the limit assured by the previous proposition. Proposition 5.6 is a very important proposition conceptually. Parry [Pa] proves that, for an optimal weight function associated with a shingling of a quadrilateral (or ring), the first inequality is satisfied with $K(3)=1$. The first inequality depends principally on optimality. Our method of proof will not yield $K(3)=1$ but will depend instead on the quadratic estimate on area which involves the constant $K(2)$. The second inequality does not depend on optimality of the weight function at all. It is satisfied for all weight functions. The constant $K(4)$ depends only on the original constant $K(1)$ used in defining conformal sequence. The constant $K(4)$ depends on the geometry of the shingling. It is with the inequalities of Proposition 5.6 that most of the work must reside necessary in proving that the axioms of a conformal sequence are satisfied.

Proof. The proof we give connects the modulus of quadrilaterals and rings. Compare [LV, pp. 36–37]. Let α be a path joining the sides of Q of minimal length $(=w(Q))$. Realize α as a concatenation of paths $\alpha=\alpha_1 * \ldots * \alpha_n * \alpha_{n+1}$, where each α_i, $i<n+1$, has d-length $h(Q)$ and $0 \leqslant L(\alpha_{n+1})<h(Q)$. Note that $n \geqslant 2$ since, by hypothesis, $2h(Q)<w(Q)$. Then every point of Q lies within $h(Q)$ of some α_i. Hence Q is the union of $n+1$ sets of radius $\leqslant 2h(Q)$. Hence, by the quadratic area estimate,

$$\begin{aligned} A(Q) &\leqslant (n+1)\cdot K(2)\cdot 4h(Q)^2 = 4\frac{n+1}{n}\cdot K(2)\cdot n\cdot h(Q)\cdot h(Q) \\ &\leqslant 4\cdot\tfrac{3}{2}\cdot K(2)\cdot w(Q)\cdot h(Q). \end{aligned}$$

This proves the first inequality.

Next let β be the horizontal path in Q joining the midpoints of the sides of Q. Note that

$$L(\beta) \geqslant w(Q) > 2h(Q).$$

Let $p_1 \in \beta$ be the last point of β at distance $\frac{1}{2}h$ from the left side l of Q. Let p_2 be the first point of β at distance $\frac{1}{2}h$ from the right-hand side r of Q. Then p_1 precedes p_2 in the natural order on β which begins at $\beta\cap l$ and ends at $\beta\cap r$. For otherwise it is an easy matter to prove that the distance from l to r is $\leqslant 2h$, a contradiction. Let γ_1 denote a path of length $\frac{1}{2}h$ from l to p_1, γ_2 a path of length $\frac{1}{2}h$ from r to p_2. Let $\beta_0 = p_1p_2 \subset \beta$. Let Q' be a quadrilateral of height $\varepsilon>0$ such that β_0 is the arc joining the midpoints of the sides of Q'. We consider the ring $\mathcal{R}'=Q\setminus \operatorname{Int} Q'$. We use the weight functions ϱ_i restricted to the shingles of $\mathcal{S}_i$ hitting $\mathcal{R}'$. The height is, for large i, approximately $\frac{1}{2}h-\varepsilon$. The circumference is, for large i, at least $w(Q)-h$ since some subarc of a circumference has to join γ_1 and γ_2. Hence we have, for large i,

$$m_{\inf}(\mathcal{R}', \mathcal{S}_i) \leqslant \frac{A_i(\mathcal{R}')}{c_i^2(\mathcal{R}')} \leqslant \frac{A_i(\mathcal{R}')}{(w(Q)-h+\varepsilon)^2},$$

and

$$\frac{h_i^2}{A_i(\mathcal{R}')} \leqslant M_{\sup}(\mathcal{R}', \mathcal{S}_i) \leqslant K(1) m_{\inf}(\mathcal{R}', \mathcal{S}_i).$$

Consequently

$$(w(Q)-h+\varepsilon)^2 h_i^2(\mathcal{R}') \leqslant K(1) A_i(\mathcal{R}')^2 \leqslant K(1) A_i(Q)^2.$$

But $(w(Q)-h) \geqslant \frac{1}{2} w(Q)$ and

$$h_i(\mathcal{R}') \geqslant \tfrac{1}{2} h - \varepsilon.$$

That is,

$$\tfrac{1}{4} w(Q) \cdot h(Q) \leqslant \sqrt{K(1)}\, A(Q).$$

This proves the second inequality. □

5.7. EXAMPLE. *For each $L>0$, there exist examples of rings $\mathcal{R}$ and conformal sequences $(K(L))$ $\mathcal{S}_1, \mathcal{S}_2, \ldots$ covering $\mathcal{R}$ such that, for all i,*

$$L \cdot A_i(\mathcal{R}) \leqslant h_i(\mathcal{R}) \cdot c_i(\mathcal{R}).$$

Proof. We realize $\mathcal{R}$ as a square $[0,1] \times [0,1]$ with sides identified. We begin with an integer $j>L$ which will determine $K(L)$. We let $\mathcal{S}_i$ shingle $\mathcal{R}$ with shingles that are rectangles of the form $A \times B$, where the lower left corners are $(k/i, l/i)$, $k=0, \ldots, j-1$, $l=0, \ldots, i-1$; A has width j/i; B has height $1/i$. An optimal weight function assigns each shingle the same weight, say 1. Then $h_i(\mathcal{R})=ij$, $c_i(\mathcal{R})=i$, $A_i(\mathcal{R})=i^2$. Hence,

$$L \cdot A_i(\mathcal{R}) < j \cdot A_i(\mathcal{R}) = h_i(\mathcal{R}) \cdot c_i(\mathcal{R}).$$

We shall prove later (Theorem 7.1) that the sequence $\mathcal{S}_i$ is conformal $K(j(L))$ for suitable K. □

The reader might enjoy doing the corresponding calculation for a rectangle. The optimal weight functions are interesting.

5.8. PROPOSITION. *Suppose that X is a subset of $\mathcal{R}[0,a]$ and Y is a subset of $\mathcal{R}[a,H]$, and assume that $A_i(X)$ and $A_i(Y)$ converge, say to $A(X)$ and $A(Y)$. Then $A_i(X \cup Y)$ converges to $A(X)+A(Y)$.*

Proof. Clearly,

$$A_i(X \cup Y) \leqslant A_i(X) + A_i(Y)$$

so that

$$A(X \cup Y) \leqslant A(X) + A(Y).$$

Consider the ring $\mathcal{R}[a-\varepsilon, a+\varepsilon]$. Its asymptotic area is $\leqslant 2\varepsilon/H$ by Theorem 5.1. Hence the area overlap between X and Y is asymptotically $\leqslant 2\varepsilon/H$. Hence

$$\begin{aligned} A_i(X\cup Y) &\geqslant A_i(X[0, a-\varepsilon]) + A_i(Y[a+\varepsilon, H]) \\ &\geqslant A_i(X) + A_i(Y) - 4\varepsilon/H \\ &\to A(X) + A(Y). \end{aligned} \qquad \square$$

We are now ready to state and prove a most important theorem, namely the local version of the area/height theorem.

5.9. THEOREM. *There is a positive number $K(5)=16K(2)\cdot K(4)$ having the following property. Let $Q=Q[a,b]$ denote a quadrilateral wide enough that $2h(Q)<w(Q)$ and narrow enough that, for all large i, there is a ϱ_i-minimal arc joining the ends of $\mathcal{R}$ which misses a uniform neighborhood of Q. Define*

$$k = k(K(5), Q) = 1 + K(5)\cdot(h(Q)/w(Q)).$$

Let $Q[x,y]\subset Q$ be any quadrilateral whose sides lie in those of Q. Then

$$\frac{1}{k}\cdot\frac{A(Q)}{h(Q)} \leqslant \frac{A(Q[x,y])}{h(Q[x,y])} \leqslant k\frac{A(Q)}{h(Q)}.$$

Remark. Strictly speaking, the slight difficulty in the proof which arises if Q is *too* wide is inessential. Wide rectangles can be cut into narrower rectangles for which the argument and estimates apply, perhaps with a slight change in $K(5)$.

5.10. COROLLARY. *For each $t\in[a,b]$, define $A(t)=A(Q[a,t])$. Then the derivative $A'(t)$ exists, is Lipschitz as a function of t, and continuous as a function of $Q[t]$. (Recall that $Q[t]$ is the arc which is the slice in Q at level t.)*

Remark. Note that
$$\frac{A(Q[x,y])}{h(Q[x,y])} = \frac{(A(y)-A(x))}{(y-x)}$$
is the difference quotient used in evaluating $A'(t)$. Note that if $Q[x,y]$ were a true rectangle, then
$$\frac{A(Q[x,y])}{h(Q[x,y])}$$
would be the width of $Q[x,y]$. Thus we may think of $A'(t)$ as a "width" for Q at level t. The corollary says that this width varies continuously, even in a Lipschitz fashion. The necessity of this Lipschitz condition was pointed out to us by Curt McMullen (after we had proved the theorem but had failed to realize the significance of the multiplier).

Proof of the corollary. Define

$$k[x,y] = 1+K(5)\frac{h(Q[x,y])}{w(Q[x,y])}.$$

Let

$$a \leqslant s_0 \leqslant t_0 \leqslant t \leqslant t_1 \leqslant s_1 \leqslant b, \quad t_0 \neq t_1.$$

Then

$$\frac{1}{k[x,y]}\cdot\frac{A(Q[s_0,s_1])}{h(Q[s_0,s_1])} \leqslant \frac{A(Q[t_0,t_1])}{h(Q[t_0,t_1])} \leqslant k[s_0,s_1]\frac{A(Q[s_0,s_1])}{h(Q[s_0,s_1])}.$$

Hence, as t_0 and $t_1 \to t$, the values

$$\frac{A(Q[t_0,t_1])}{h(Q[t_0,t_1])}$$

are restricted to an interval of the size

$$\left[k[s_0,s_1] - \frac{1}{k[s_0,s_1]}\right]\frac{A(Q[s_0,s_1])}{h(Q[s_0,s_1])}.$$

The first factor can be made arbitrarily small by picking s_0 and s_1 close to t. The second factor is bounded by $k\cdot A(Q)/h(Q)$. Hence the oscillation can be made small and the limit $A'(t)$ therefore exists.

Now consider the derivative at two values $A'(t)$ and $A'(u)$, $t<u$. Pick

$$a \leqslant s_0 \leqslant t_0 \leqslant t \leqslant t_1 < u_0 \leqslant u \leqslant u_1 \leqslant s_1 \leqslant b, \quad t_0 \neq t_1,\ u_0 \neq u_1.$$

Then both

$$\frac{A(Q[t_0,t_1])}{h(Q[t_0,t_1])}$$

and

$$\frac{A(Q[u_0,u_1])}{h(Q[u_0,u_1])}$$

lie in the interval whose left hand endpoint is

$$\frac{1}{k[s_0,s_1]}\cdot\frac{A(Q[s_0,s_1])}{h(Q[s_0,s_1])}$$

and right hand endpoint is

$$k[s_0,s_1]\frac{A(Q[s_0,s_1])}{h(Q[s_0,s_1])}.$$

Therefore the derivatives lie in the same interval so that as $s_0 \to t$ and $s_1 \to u$ we find

$$|A'(t)-A'(u)| \leqslant \left(k[t,u] - \frac{1}{k[t,u]}\right)\frac{A(Q[t,u])}{h(Q[t,u])}.$$

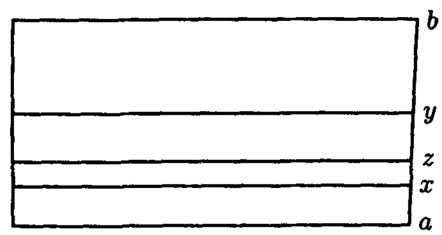

Fig. 14

For the moment we abbreviate all of these numbers: $k=k[t,u]$, $A=A(Q[t,u])$, $h=h(Q[t,u])$, $w=w(Q[t,u])$. Then

$$\left[k-\frac{1}{k}\right]\frac{A}{h}\leqslant\left[1+K(5)\frac{h}{w}-\frac{1}{1+K(5)h/w}\right]\frac{K(3)hw}{h}$$
$$=\frac{K(5)K(3)h(2w+K(5)h)}{w+K(5)h}\leqslant 2K(5)K(3)h.$$

Thus we may take $2K(5)K(3)$ as a Lipschitz constant for $A'(t)$.

We finally consider the continuity of $A'(t)$ as a function of the slice $Q[t]$ of Q at horizontal level t. To that end we let $P[a,b]$ be a second quadrilateral so that $Q[t]$ and $P[t]$ differ from one another by a distance $<\varepsilon$ and that at only one endpoint. Then if $s_0\leqslant t\leqslant s_1$, $s_0\neq s_1$, and s_1-s_0 is much smaller than ε, then the difference between $Q[s_0,s_1]$ and $P[s_0,s_1]$ is contained in a rectangle of width $<\varepsilon$ and height s_1-s_0 to which Proposition 5.6 applies and yields

$$\left|\frac{A(Q[s_0,s_1])}{s_1-s_0}-\frac{A(P[s_0,s_1])}{s_1-s_0}\right|\leqslant K(3)\cdot\varepsilon.$$

Letting $s_0,s_1\to t$, we find

$$|A'(t,Q)-A'(t,P)|\leqslant K(3)\cdot\varepsilon.$$

Continuity with respect to $Q[t]$ follows. □

Proof of Theorem 5.9.

Reduction 1. It suffices to prove the case where $x=a$ (or the symmetric case $y=b$). Indeed, suppose the theorem true for the case $x=a$ and the case $y=b$ and consider the more general case

$$a<x<y<b.$$

There is a number z, $x<z<y$ such that

$$\frac{b-z}{y-z}=\frac{z-a}{z-x}=\frac{b-a}{y-x}.$$

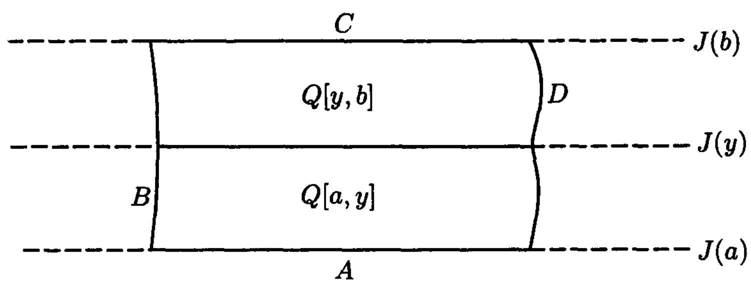

Fig. 15

(See Figure 14.) Consider the two pairs $Q[z,y]\subset Q[z,b]$ and $Q[x,z]\subset Q[a,z]$, both handled by the special case. Hence

$$\frac{A(Q[a,z])}{k}\leqslant\frac{z-a}{z-x}A(Q[x,z])\leqslant kA(Q[a,z]),$$

and

$$\frac{A(Q[z,b])}{k}\leqslant\frac{b-z}{y-z}A(Q[z,y])\leqslant kA(Q[z,b]).$$

Adding, we obtain the desired result

$$\frac{A(Q[a,b])}{k}\leqslant\frac{b-a}{y-x}A(Q[x,y])\leqslant kA(Q[a,b]).$$

Reduction 2. It suffices to prove that for $a<y<b$,

$$\frac{1}{k}\cdot\frac{A(Q[y,b])}{b-y}\leqslant\frac{A(Q[a,y])}{y-a}\leqslant k\frac{A(Q[y,b])}{b-y}.$$

(See Figure 15.) Indeed, we then calculate as follows. Let the heights of the two subrectangles be h_1, h_2, the areas A_1, A_2, so that we assume

$$\frac{1}{k}\cdot\frac{A_1}{h_1}\leqslant\frac{A_2}{h_2}\leqslant k\frac{A_1}{h_1}.$$

We want to prove

$$\frac{1}{k}\cdot\frac{A_1+A_2}{h_1+h_2}\leqslant\frac{A_1}{h_1}\leqslant k\frac{A_1+A_2}{h_1+h_2}.$$

We prove the second inequality first:

$$\frac{A_1+A_2}{h_1+h_2}\cdot\frac{h_1}{A_1}\geqslant\frac{A_1+(1/k)(h_2A_1/h_1)}{h_1+h_2}\cdot\frac{h_1}{A_1}=\frac{h_1+(h_2/k)}{h_1+h_2}\geqslant\frac{1}{k},\quad\text{since } k>1.$$

The first follows similarly:

$$\frac{A_1+A_2}{h_1+h_2}\cdot\frac{h_1}{A_1}\leqslant\frac{A_1+k(h_2A_1/h_1)}{h_1+h_2}\cdot\frac{h_1}{A_1}=\frac{h_1+kh_2}{h_1+h_2}\leqslant k,\quad\text{since } k>1.$$

Main body of the proof. Following the reductions, we consider a single rectangle $Q=Q[a,b]$ divided by a horizontal level $J[y]$ into two subrectangles $Q[a,y]$ and $Q[y,b]$. The proof is identical in spirit with the alternative proof to Theorem 5.1, where we increase weights near $Q[a,y]$ and decrease them near $Q[y,b]$, or vice versa. The difficulty is that the height of $\mathcal{R}$ changes radically under that operation unless one takes special care near the left and right sides of Q; one has to create a substantial penalty for paths that slide in and out of Q along the sides of Q. The use of penalty strips along the sides of Q makes both the geometric and the analytic part of the proof a bit delicate.

Penalty strip and $\varepsilon>0$. We must surround Q with a highly weighted penalty strip S_ε in order to localize the effects of weight variation. We fix a small positive number ε and declare x to be a point of S_ε if $x\in J[a,b]\setminus Q$ and there exists an arc in $J[a,b]$ from x to Q having d-length $\leqslant 2\varepsilon$. We also assume that for all i sufficiently large, there will be a ϱ_i-minimal arc joining the ends of $\mathcal{R}$ which misses star Q. Hence, any adjustment of weights on the shingles which hit Q will not increase the height of $\mathcal{R}$.

Weight multipliers λ and μ. We choose $\lambda<1$ very close to 1 and define $\mu>1$ by the formula

$$\mu(b-y)+\lambda(y-a)=b-a.$$

The positive integer i and the positive number $\delta=\delta(i)$. Since the approximate metrics d_i converge uniformly to the limit metric d, the difference $\delta=\delta(i)=\sup|d_i-d|$ converges to 0 as $i\to\infty$. Consequently by the proof of Proposition 4.4.2, for all i sufficiently large, any arc in $J[a,b]$ joining $J[a,b]\setminus(Q\cup S_\varepsilon)$ to star Q has ϱ_i-length $>\varepsilon$. We assume that i is that large and, furthermore, that $\delta=\delta(i)$ and $1/i$ are small with respect to ε.

The penalty function $f\colon[a,b]\to[1,\infty)$. For $x\in[a,y]$ we define

$$f(x)=1+(1-\lambda)(x-a)/\varepsilon.$$

For $x\in[y,b]$ we define

$$f(x)=1+(\mu-1)(b-x)/\varepsilon.$$

Note that the two definitions agree at $x=y$.

The new weight function $\varrho'=\varrho'(\varepsilon,\lambda,i)$, height $h'=h'(\varepsilon,\lambda,i)$, and area $A'=A'(\varepsilon,\lambda,i)$. We set $\varrho_i=(t,u,v,w)$, where the shingles (= coordinates) corresponding to t, u, v, and w are denoted $\mathcal{S}_i(t),\mathcal{S}_i(u),\mathcal{S}_i(v),\mathcal{S}_i(w)\subset\mathcal{S}_i$ and are defined as follows. Let $s\in\mathcal{S}_i$. If s hits $Q[a,y]$, then $s\in\mathcal{S}_i(t)$. If s misses $Q[a,y]$ but hits $Q[y,b]$, then $s\in\mathcal{S}_i(u)$. If s misses Q but hits S_ε, then $s\in\mathcal{S}_i(v)$. Otherwise $s\in\mathcal{S}_i(w)$. If $s\in\mathcal{S}_i$, then $\varrho'(s)=m(s)\cdot\varrho_i(s)$ where the multiplier $m(s)$ is defined as follows. If $s\in\mathcal{S}_i(t)$, $m(s)=\lambda$. If $s\in\mathcal{S}_i(u)$, then $m(s)=\mu$. If $s\in\mathcal{S}_i(w)$, then $m(s)=1$. If $s\in\mathcal{S}_i(v)$, then the definition is more complicated and is reserved for the next paragraph.

Suppose $s \in \mathcal{S}_i(v)$. Project $s \cap J[a,b]$ horizontally into either the left-hand side B of Q or the right-hand side D of Q, whichever is nearer, say into B. Let $p(s)$ denote that point of B (or D) nearest $J[y] \cap B$ which is also no further than ε from the projection image. Note that

$$p(s) \in J[\eta \circ p(s)],$$

$\eta \circ p(s) \in [a+\varepsilon, b-\varepsilon]$. Define $m(s) = f \circ \eta \circ p(s)$, where $f: [a,b] \to [1,\infty)$ is the penalty function described earlier.

In summary $\varrho' = m\varrho_i = (\lambda t, \mu u, (f \circ \eta \circ p) \cdot v, w)$. We let $h' = h'(\varepsilon, \lambda, i)$ and $A' = A'(\varepsilon, \lambda, i)$ denote the height and area associated with ϱ'.

Preliminary calculations. We shall need to compare ϱ', ϱ_i, and limit lengths and areas. Here are three little lemmas which will be helpful.

LEMMA P1. *Let α be an open path whose closure A is irreducible from* star $J[r]$ *to* star $J[s]$, $s>r$. *Then* $L_i(\alpha) \geqslant (s-r)-\delta$.

Proof. Since A joins star $J[r]$ to star $J[s]$, α contains an i-approximate path from $J[r]$ to $J[s]$. Hence $L_i(\alpha) \geqslant d_i(J[r], J[s])$. Since $d(J[r], J[s]) = s-r$ and $|d-d_i| < \delta = \delta(i)$, the result follows.

LEMMA P2. *If $s \in \mathcal{S}_i(v)$ intersects $J[y,b]$, then $m(s) \geqslant \mu$.*

Proof. It is clear that $\eta = \eta(s) \in [y, b-\varepsilon]$ so that $b-\varepsilon-\eta \geqslant 0$. Hence

$$\mu - m(s) = \mu - f(\eta) = \mu - 1 - \frac{(\mu-1)(b-\eta)}{\varepsilon} = \frac{\mu-1}{\varepsilon}(\varepsilon - b + \eta) \leqslant 0.$$

LEMMA P3. *If $s \in \mathcal{S}_i(v)$, then $d(s, p(s)) \leqslant 3\varepsilon + \delta + 1/i$.*

Proof. Let α denote a path in $J[a,b]$ of d-length $\leqslant 2\varepsilon$ joining s to Q. Project α horizontally into B or D, say into B, whichever is closer. Then the image α^* has d-length $\leqslant 2\varepsilon$, and it intersects the horizontal projection of $s \cap J[a,b]$ which has d_i-diameter $\leqslant 1/i$, hence d-diameter $\leqslant \delta + 1/i$. The point $p(s)$ lies within d-distance ε of the latter projection. The lemma follows.

COROLLARY TO LEMMA P3. *Let α denote an arc in B or D of length $\leqslant \varepsilon$. Let $S_\varepsilon(\alpha)$ denote the set of shingles of $\mathcal{S}_i(v)$ such that $p(s) \in \alpha$. Then any set covered by $S_\varepsilon(\alpha)$ has ϱ_i-radius $< 4\varepsilon$, hence ϱ_i-area $< K(6) = 16\varepsilon^2 \cdot K(2)$.*

Proof. Let s be a shingle of $\mathcal{S}_i(v)$ in $S_\varepsilon(\alpha)$. By Lemma P3, the d-distance from s to the midpoint of α is $\leqslant \frac{7}{2}\varepsilon + \delta + 1/i$. the d_i-distance is therefore $\leqslant \frac{7}{2}\varepsilon + 2\delta + 1/i$. The d_i-radius of any set covered by $S_\varepsilon(\alpha)$ is therefore no more than $\frac{7}{2}\varepsilon + 2\delta + 2/i < 4\varepsilon$. The quadratic area estimate therefore implies that the ϱ_i-area is no more than $K(6)=16\varepsilon^2\cdot K(2)$.

Estimating h' below. Let α denote a ϱ'-minimal arc joining the ends of $\mathcal{R}$, so that $h'=L'(\alpha)$. There is a ϱ_i-minimal arc missing $\text{star}(Q\cup S_\varepsilon)$ by hypothesis so that $h'\leqslant H_i$. Our goal is to show that h' is not much smaller than H_i. We express α as the concatenation $\alpha_0*\alpha_1$ of two arcs, where α_0 is the smallest subarc of α, α_0 possibly degenerate, containing

$$\alpha\cap(J[0]\cup\text{star}\,Q[a,y]).$$

We now spawn subcases according to the structure of α_1:

If α_1 intersects $\text{star}\,J[a]$, we have *Case* 1.

Otherwise $\alpha_0\cap\alpha_1$ lies in the interior of $Q\cup S_\varepsilon$ at a point of $\text{star}\,Q[a,y]$. In a subarc of α_1 irreducible from $\text{star}\,Q[a,y]$ to $\text{Bd}(Q\cup S_\varepsilon)\backslash\text{star}\,J[a]$ there is a subarc β of α_1, possibly degenerate, which is irreducible from $\text{star}\,Q$ to $\text{Bd}(Q\cup S_\varepsilon)\backslash\text{star}\,J[a]$ and meets the latter set at a point p. Note that no point of α_1 between α_0 and β lies outside $Q\cup S_\varepsilon$. If $p\in J[b]$, we have *Case* 2.

If $p\notin J[b]$, then β lies entirely in the penalty strip and joins a lateral side of S_ε with $\text{star}\,Q$. By our choice of i, $L_i(\beta)>\varepsilon$. We trim β if necessary in the following fashion so that β has vertical extent less than ε. If the horizontal projection of β into the open interval (a,b) lies in an interval $[r,s]\subset(a,b)$ of length $s-r<\varepsilon$, then we need do nothing. Otherwise we pick $[r,s]\subset(a,b)$ of length $s-r=\varepsilon$ which is contained in the projection image, and we replace β by a subarc irreducible between $\text{star}\,J[r]$ and $\text{star}\,J[s]$. By Lemma P1, $L_i(\beta)\geqslant\varepsilon-\delta$. If $r\in(a,y]$, we have *Case* 3. Otherwise, $r\in(y,b)$ and we have *Case* 4.

We treat only Case 4 since it contains all of the ideas needed in the other cases.

Case 4. $y<r<b$. The following horizontal levels are critical to this case:

$$0\leqslant a<y<r<s<H.$$

There are in $\alpha\backslash\beta$ open arcs

$$\beta_0,\ \beta_1,\ \beta_2,\ \text{and}\ \beta_3$$

whose closures are irreducible, respectively, from $\text{star}\,J[0]$ to $\text{star}\,J[a]$, $\text{star}\,J[a]$ to $\text{star}\,J[y]$, $\text{star}\,J[y]$ to $\text{star}\,J[r]$, and from $\text{star}\,J[s]$ to $\text{star}\,J[H]$. Conceivably one of these arcs may degenerate to a single point, in which case we replace the arc by the empty set. The choice of β_2 is the only critical choice. We take β_2 between α_0 and β. Consequently β_2 lies in $(Q\cup S_\varepsilon)\backslash\text{star}\,Q[a,y]$, so that by Lemma P2 all shingles intersecting β_2 have weight $\geqslant\mu$. Therefore, applying Lemmas P1 and P2 and the principle that

$$L'(\beta_j)\geqslant(\inf\{m(s)\mid s\cap\beta_j\neq\varnothing\})\cdot L_i(\beta_j),$$

we calculate:

$$\begin{aligned}h' = L'(\alpha) &\geqslant L'(\beta_0)+L'(\beta_1)+L'(\beta_2)+L'(\beta)+L'(\beta_3)\\ &\geqslant (a-\delta)+\lambda(y-a-\delta)+\mu(r-y-\delta)+f(r)\cdot(\varepsilon-\delta)+(H-s-\delta)\\ &\geqslant H-(2+\lambda+\mu+f(r))\delta \geqslant H_i-(3+\lambda+\mu+f(r))\delta.\end{aligned}$$

Remark. The coefficients of δ corresponding to $3+\lambda+\mu+f(r)$ are in cases 1, 2, and 3, respectively, 3, $3+\lambda+\mu$, and $3+\lambda+f(r)$. Hence the estimate of Case 4 is valid in all cases.

Area estimates in the penalty strip. In order to complete the proof of the theorem, we will need to compare ϱ_i and ϱ'-areas in $\mathcal{S}_i(v)$. To that end we choose integers m and n such that $(m-1)\varepsilon<y-a\leqslant m\varepsilon$ and $(n-1)\varepsilon<b-y\leqslant n\varepsilon$. Let $\eta\colon\mathcal{R}\to[0,H]$ be horizontal projection. Cover $B[a,y]$ and $D[a,y]$ by subarcs α of B and D which are precisely of length ε such that $\eta(\alpha)$ has form

$$\eta(\alpha)=[a+(j-1)\varepsilon, a+j\varepsilon], \quad j\in[1,m].$$

Associate with each such arc α the set $S_\varepsilon(\alpha)$ of shingles $s\in\mathcal{S}_i(v)$ whose penalty point $p(s)$ lies in α. Note that the ϱ'-multiplier $m(s)=f\circ\eta\circ p(s)$ for s is no larger than $1+(1-\lambda)j$ (definition of f), that the ϱ_i-radius of $\bigcup\{s\,|\,s\in S_\varepsilon(\alpha)\}$ is less than 4ε (Lemma P3), hence that

$$\sum_{s\in S_\varepsilon(\alpha)}\varrho_i(s)^2=\sum_{s\in S_\varepsilon(\alpha)}v(s)^2\leqslant 16\varepsilon^2\cdot K(2)=K(6)$$

(quadratic area estimate—see the corollary to Lemma P3). We similarly cover $B[y,b]$ and $D[y,b]$ with arcs β of length ε such that $\eta(\beta)=[b-k\varepsilon, b-(k-1)\varepsilon]$, $k\in[1,n]$. The ϱ'-multiplier of $s\in S_\varepsilon(\beta)$ is no larger than $1+(\mu-1)k$, and there are diameter and area estimates identical to those for α.

The critical area estimate is the following:

$$\begin{aligned}H_i^2\cdot A(\mathcal{S}_i(v),\varrho')-(h')^2A(\mathcal{S}_i(v),\varrho_i) &= H_i^2\sum_{s\in\mathcal{S}_i(v)}m(s)^2\cdot v(s)^2-(h')^2\sum_{s\in\mathcal{S}_i(v)}v(s)^2\\ &\leqslant\sum_\alpha[H_i^2(1+(1-\lambda)j(\alpha))^2-(h')^2]\sum_{s\in S_\varepsilon(\alpha)}v(s)^2\\ &\quad+\sum_\beta[H_i^2(1+(1-\lambda)k(\beta))^2-(h')^2]\sum_{s\in S_\varepsilon(\beta)}v(s)^2.\end{aligned}$$

The summations over the index s have been bounded before by $K(6)$. When those summations are replaced by their upper bound, the remaining summations can be evaluated explicitly. One needs to note that there are exactly 2 arcs α with the same $j(\alpha)$, 2 arcs β with the same $k(\beta)$, that $j(\alpha)$ runs from 1 to m, and $k(\beta)$ runs from 1 to n. We thus find our original difference bounded above by

$$\begin{aligned}\theta(\varepsilon,\lambda,i)=2K(6)\cdot\{&(m+n)(H_i^2-(h')^2)\\&+H_i^2(1-\lambda)[m(m+1)+(1-\lambda)\cdot\tfrac{1}{6}m(m+1)(2m+1)]\\&+H_i^2(\mu-1)[n(n+1)+(\mu-1)\cdot\tfrac{1}{6}n(n+1)(2n+1)]\}.\end{aligned}$$

Final estimates and proof of the theorem. By the optimality of ϱ_i on the shingling $\mathcal{S}_i$ we have

$$\begin{aligned}0&\leqslant H_i^2A'-(h')^2A_i\\&\leqslant[H_i^2\lambda^2-(h')^2](t\cdot t)+[H_i^2\mu^2-(h')^2](u\cdot u)+[H_i^2-(h')^2](w\cdot w)+\theta(\varepsilon,\lambda,i).\end{aligned}$$

We may assume that as $i\to\infty$, $t\cdot t\to T$, $u\cdot u\to U$, $w\cdot w\to W$,

$$H_i\geqslant h'\geqslant H_i-(3+\lambda+\mu+f(r))\delta,$$

and $H_i,h'\to H$. We divide by H^2 and take the limit as $i\to\infty$ to find

$$\begin{aligned}0\leqslant{}&(\lambda^2-1)T+(\mu^2-1)U+0\cdot W\\&+32\varepsilon^2K(2)\{(1-\lambda)[m(m+1)+(1-\lambda)\cdot\tfrac{1}{6}m(m+1)(2m+1)]\\&+(\mu-1)[n(n+1)+(\mu-1)\cdot\tfrac{1}{6}n(n+1)(2n+1)]\}.\end{aligned}$$

We note that $(\mu-1)/(1-\lambda)=(y-a)/(b-y)$. We divide our inequality by $2(\lambda-1)$ and then take the limit as $\lambda\to1$ to obtain

$$0\leqslant -T+\frac{y-a}{b-y}U+K(6)\left\{m(m+1)+\frac{y-a}{b-y}n(n+1)\right\}.$$

We now let $\varepsilon\to0$. Simultaneously εm and $\varepsilon(m+1)$ approach $y-a$, εn and $\varepsilon(n+1)$ approach $b-y$. We obtain after simplifying the last factor

$$T\leqslant\frac{y-a}{b-y}U+16K(2)(y-a)(b-a)=\frac{y-a}{b-y}(U+16K(2)(b-a)(b-y)).$$

We apply Proposition 5.6 to the product $(b-a)(b-y)$ to obtain

$$[(b-a)(b-y)]\leqslant[h(Q)\cdot h(Q[y,b])]\cdot\frac{w(Q[y,b])}{w(Q)}\leqslant\frac{h(Q)}{w(Q)}K(4)A(Q[y,b]).$$

Using the fact that $T=A(Q[a,y])$ and $U=A(Q[y,b])$, we may combine the last two inequalities to find that

$$A(Q[y,b])\leqslant\frac{y-a}{b-y}\left(U+16K(2)K(4)\frac{h(Q)}{w(Q)}U\right)=\frac{y-a}{b-y}\left(1+K(5)\frac{h(Q)}{w(Q)}\right)A(Q[y,b])$$

if $K(5)=16K(2)K(4)$. But this final inequality is precisely the one required by Reduction 2. The analytic part of Theorem 5.9 is proved. $\square$

Implications of area/height comparisons for width comparisons in a quadrilateral

5.11. PROPOSITION. *Horizontal lengths in a quadrilateral are locally comparable.*

Proof. Let $Q=Q[a,b]$ be a quadrilateral with $2h(Q)\leqslant w(Q)$. Let $t\in[a,b]$. Let $a_i\to t$ from below, and let $b_i\to t$ from above. Recall the function

$$k=k(K(5),Q)=1+K(5)\cdot\frac{h(Q)}{w(Q)}$$

from Theorem 5.9. By that theorem,

$$\frac{1}{k}\cdot\frac{A(Q[a,b])}{b-a}\leqslant\frac{A(Q[a_i,b_i])}{b_i-a_i}\leqslant k\frac{A(Q[a,b])}{b-a}.$$

But

$$\frac{1}{K(3)}\cdot\frac{A(Q[a_i,b_i])}{b_i-a_i}\leqslant w(Q[a_i,b_i])\leqslant K(4)\frac{A(Q[a_i,b_i])}{b_i-a_i}.$$

Hence

$$\frac{1}{kK(3)}\cdot\frac{A(Q[a,b])}{b-a}\leqslant w(Q[a_i,b_i])\leqslant kK(4)\frac{A(Q[a,b])}{b-a}.$$

Since $L(Q[t])=\lim w(Q[a_i,b_i])$, we find that, for $\tau_0,\tau_1\in[a,b]$,

$$\frac{1}{k^2K(3)K(4)}L(Q[\tau_0])\leqslant L(Q[\tau_1])\leqslant k^2K(3)K(4)L(Q[\tau_0]).$$

That is, horizontal distances are locally comparable. □

This result about the local comparability of horizontal distances is attractive. But we have seen that horizontal distance can vary discontinuously. A more attractive function, the one on which the comparability is based, is area/height. Corollary 5.10 dealt with the limiting version of area/height, namely the Lipschitz derivative $A'(t)$. Here is a slight refinement to add to Corollary 5.10.

5.12. COROLLARY. *In a quadrilateral Q, the length $L(Q[t])$ and the derivative $A'(t)$ are uniformly comparable. If one uses the entire ring $\mathcal{R}$ instead of the arbitrary quadrilateral Q of Corollary* 5.10 *in defining $A(t)$ and $A'(t)$, then $A'(t)$ is constant and equal to $1/H$.*

Proof. Let $a_i\to t$ from below and $b_i\to t$ from above, as before. Then $L(Q[t])$ is the limit of the widths $w(Q[a_i,b_i])$. By the argument in the paragraphs above, these widths are uniformly comparable with the difference quotients

$$\frac{A(Q[a_i,b_i])}{b_i-a_i}.$$

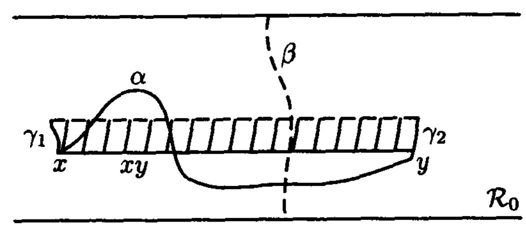

Fig. 16

But these quotients approach the derivative. We conclude that the widths and derivatives are uniformly comparable. If one uses the entire ring $\mathcal{R}$ instead of the arbitrary quadrilateral Q, then all of the area/height inequalities can be replaced by exact equality according to Theorem 5.1. The quotient

$$\frac{A(\mathcal{R}[a,b])}{b-a}$$

is by that theorem exactly $1/H$ for every a and b. It follows that $A'(t)=1/H$ as claimed. □

5.13. PROPOSITION. *The distance function $d(x,y)$ and horizontal distance $L(x,y)$ are uniformly comparable for x and y in the same level curve $J(t)$. More precisely, there is a uniform constant $K(7)$ such that, for all choices of x and y,*

$$L(x,y) \leqslant d(x,y) \leqslant K(7)\cdot L(x,y).$$

Proof. Let α be a path of length $d(x,y)$ joining x to y. (See Figure 16.) We use xy to indicate the minimal horizontal path from x to y. Let β denote a d-minimal path joining the ends of $\mathcal{R}$ which passes through a point p of $\operatorname{Int} xy$. Pick ε smaller than the distance from β to $\{x,y\}$. We assume that the level $J(t)$ containing xy is not $\mathcal{R}_1$. Let $Q[t,t+\varepsilon]$ be a quadrilateral of height ε such that $Q[t]=xy$. Let γ_1 and γ_2 be the sides of $Q[t,t+\varepsilon]$.

LEMMA. *Let β' denote an arbitrary path joining the ends of $\mathcal{R}$. Then either β' hits $\alpha\cup\gamma_1\cup\gamma_2$ or β' contains three subpaths β_0,β_1,β_2 in $\mathcal{R}\backslash Q[t,t+\varepsilon]$ such that:*

(1) *β_0 is open-irreducible from* $\operatorname{star}\mathcal{R}_0$ *to* $\operatorname{star} J(t)$,

(2) *β_1 is open-irreducible from* $\operatorname{star} J(t)$ *to* $\operatorname{star} J(t+\varepsilon)$,

(3) *β_2 is open irreducible from* $\operatorname{star} J(t+\varepsilon)$ *to* $\operatorname{star}\mathcal{R}_1$.

Proof of the lemma. Consider the homotopy classes of paths

$$\gamma\colon([0,1],0,1)\to(R\backslash(\gamma_1\cup\gamma_2),\mathcal{R}_0,\mathcal{R}_1).$$

Among those classes, only the class of β fails to have the desired subarcs. But since β' misses α, β' is not homotopic to β. This argument proves the lemma. $\square$

Completion of the proof of Proposition 5.13. Set $\varrho_i=(t,u,v)$, where $\mathcal{S}_i(t)$ is the set of shingles in $\mathcal{S}_i$ which hit the 2ε-neighborhood of $\gamma_1\cup\alpha\cup\gamma_2$, $\mathcal{S}_i(u)$ is the set of shingles not in $\mathcal{S}_i(t)$ but lying in $Q[t,t+\varepsilon]$; and $\mathcal{S}_i(v)$ is the set of remaining shingles. Define a new weight function $\varrho=(\lambda t,\mu u,v)$, $\lambda>1$, $\mu<1$, $\lambda+\mu=2$. Hence $\mu=2-\lambda$, $1-\mu=\lambda-1$, $\mu+1=3-\lambda$.

The geometric part of the argument requires that we estimate the new height function h'. Let β' be a ϱ'-minimal path joining the ends of $\mathcal{R}$. There exist in β' paths β_0,β_1,β_2, β_0 open-irreducible from star $\mathcal{R}_0$ to star $J(t)$, β_1 open-irreducible from star $J(t)$ to star $J(t+\varepsilon)$, β_2 open-irreducible from star $J(t+\varepsilon)$ to star $\mathcal{R}_1$. If β' misses $\alpha\cup\gamma_1\cup\gamma_2$, then we may assume by the lemma that each of β_0, β_1, and β_2 misses $Q[t,t+\varepsilon]$, hence $\mathcal{S}_i(u)$. Therefore

$$\begin{aligned} h'=L'(\beta') &\geqslant L'(\beta_0)+L'(\beta_1)+L'(\beta_2)\geqslant L(\beta_0)+L(\beta_1)+L(\beta_2)\\ &\geqslant (t-\gamma-2/i)+(\varepsilon-\gamma-2/i)+(H_i-t-\varepsilon-\gamma-2/i)=H_i-3\gamma-6/i.\end{aligned}$$

If β' hits $\gamma_1\cup\alpha\cup\gamma_2$, then β' has a subarc of length $>\varepsilon$ hitting $\mathcal{S}_i(t)$. Hence

$$\begin{aligned} h'=L'(\beta') &\geqslant L(\beta_0)+\mu L(\beta_1)+L(\beta_2)+(\lambda-1)\varepsilon\\ &\geqslant (t-\gamma-2/i)+\mu(\varepsilon-\gamma-2/i)+(H_i-t-\varepsilon-\gamma-2/i)+(\lambda-1)\varepsilon\\ &=H_i+(\mu+\lambda-2)\varepsilon-(2+\mu)\gamma-(2+\mu)2/i\\ &\geqslant H_i-3\gamma-6/i.\end{aligned}$$

This completes the geometric part of the proof. We turn to the analytic.

By the optimality of ϱ_i, we have

$$A_i(h')^2\leqslant A'(H_i)^2,$$

or

$$(t\cdot t+u\cdot u+v\cdot v)(H_i-3\gamma-6/i)^2\leqslant(\lambda^2t\cdot t+\mu^2u\cdot u+v\cdot v)H_i^2.$$

Letting $i\to\infty$ and assuming $t\cdot t\to T$, $u\cdot u\to U$, we obtain

$$(1-\mu)(1+\mu)U\leqslant(\lambda-1)(\lambda+1)T$$

or

$$(\lambda-1)(3-\lambda)U\leqslant(\lambda-1)(\lambda+1)T.$$

Dividing by $\lambda-1$ and letting $\lambda\to 1$ we obtain

$$U \leqslant T$$

and consequently

$$U+T \leqslant 2T.$$

But $U+T$ is at least as large as the area of $Q[t,t+\varepsilon]$, which in turn is at least as large as

$$\frac{1}{K(4)}\cdot\varepsilon\cdot w(Q[t,t+\varepsilon]).$$

On the other hand, utilizing the quadratic estimate on area as in the second lemma which appeared in the midst of the proof of Theorem 5.9,

$$T \leqslant K(6)\cdot 2\varepsilon\cdot d(x,y).$$

Thus

$$\frac{1}{K(4)}\varepsilon\cdot w(Q[t,t+\varepsilon]) \leqslant 2K(6)\cdot 2\varepsilon\cdot d(x,y).$$

Dividing by ε and letting $\varepsilon\to 0$, we obtain

$$\frac{1}{K(6)}L(x,y) \leqslant 4K(6)d(x,y).$$

This completes the proof of Proposition 5.13. □

The *taxicab metric* d' on $\mathcal{R}$ is defined as follows. Consider only paths that are piecewise vertical or horizontal. Sum the arclengths of the pieces. Take the infimum over all such paths. This gives a new metric $d' \geqslant d$.

5.14. PROPOSITION. *The metrics d' and d are comparable. That is, there exists a constant $K(8)$ such that*

$$d \leqslant d' \leqslant K(8)d.$$

Proof. It suffices to consider a single vertical-horizontal pair $v*h$, $v=xy$, $h=yz$. Let $w=xz$ denote a path of minimal d-length joining x to z. Then

$$\begin{aligned} L(v)+L(h) &= L(xy)+L(yz) \leqslant L(xy)+K(7)\cdot d(y,z) \\ &\leqslant L(xy)+K(7)[L(xy)+L(xz)] \leqslant [1+2K(7)]L(xz) \\ &= [1+2K(7)]L(w), \end{aligned}$$

where the inequality $L(yz) \leqslant K(7)d(y,z)$ is the comparability of horizontal length with distance, the inequality $d(y,z) \leqslant L(xy)+L(xz)$ is the triangle inequality for d, and the inequality $L(xy) \leqslant L(xz)$ follows since xy is vertical. □

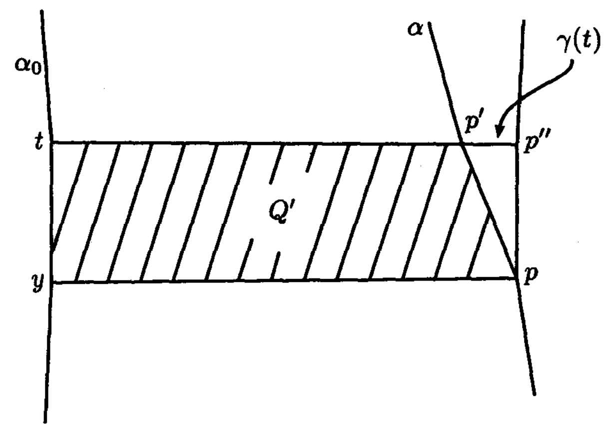

Fig. 17

6. Coordinates

We equip $\mathcal{R}$ with coordinates as follows. We fix one vertical arc α_0 joining the ends of $\mathcal{R}$. We use $\mathcal{R}_0$ as the x-axis and α_0 as the y-axis. For $p \in \mathcal{R}$, we define

$$x(p) = d(p, \mathcal{R}_0),$$

and we define $y(p) = dA(x(p), Q(p))/dx$, where $Q(p)$ is any quadrilateral which has its left side on α_0 and whose right side goes through p. By the fundamental propositions on A', Corollaries 5.10 and 5.12, $y(p)$ is well-defined, a continuous function of p, monotone increasing on each $J(t)$ cut at α_0, and assigns each $J(t)$ the length $1/H$.

We equip $\mathcal{R}$ with the Euclidean metric D which arises from the Riemannian metric

$$ds^2 = dx^2 + dy^2.$$

This metric is at long last the flat metric whose existence is asserted by the combinatorial Riemann mapping theorem.

6.1. PROPOSITION. *There is a constant $K(9)$ such that, if $p=(x,y) \in \mathcal{R}$, α rises vertically from p in the d-metric, and β rises vertially from p in the D-metric, and $\gamma(t)$ is the horizontal D-distance from $\alpha[t]$ to $\beta[t]$, then*

$$\frac{\gamma(t)}{t-y} \leqslant K(9).$$

Proof. (See Figure 17.) After all of the definitions have been sorted out, this is simply a restatement of the fact that $A'(t)$ is a Lipschitz function. Here are the details.

Let Q' be a quadrilateral with α_0 on the left side, α on the right. Then Q' may be used to calculate $A'(y,Q')$ and $A'(t,Q')$. Let $\{p'\}=\alpha\cap J(t)$ and $p''=(x,t)$. Let Q'' be a quadrilateral used to calculate the x-coordinate x of p'' so that $x=A'(t,Q'')$. Then

$$\gamma(t)=|A'(t,Q')-A'(t,Q'')|=|A'(t,Q')-A'(y,Q')|\leqslant 2K(5)K(3)(t-y)$$

since A' is Lipschitz with Lipschitz constant $2K(5)K(3)$ by the proof of Corollary 5.10. □

6.2. PROPOSITION. *The metrics d and D are comparable.*

Proof. Each is comparable with its taxicab metric. We proved this for d and it is well-known for the Euclidean metric D. We may therefore compare taxicab metrics.

Suppose $h*v$ is a Euclidean path, h horizontal (with respect to both d and D), and v D-vertical. Then the endpoints of this Euclidean path may be joined by a concatenation of three paths $h*\gamma*v'$, where γ is horizontal and v' is d-vertical. We recall that horizontal distances measured with respect to d and D are comparable by Corollary 5.12 and the discussion preceding it. Let $L(1)$ be a constant of comparability. Let $L(2)$ be a Lipschitz constant for $A'(t)$. Then

$$\begin{aligned}L_d(h)+L_d(\gamma)+L_d(v')&\leqslant L(1)L_D(h)+L(1)L_D(v')+L_D(v)\\&\leqslant L(1)L_D(h)+L(1)L(2)L_D(v)+L_D(v)\\&\leqslant (L(1)L(2)+1)[L_D(h)+L_D(v)].\end{aligned}$$

Suppose $h*v$ is a d-path, h horizontal, and v d-vertical. Then the endpoints may be joined by a concatenation of three paths $h*\gamma*v'$, where γ is horizontal and v' is D-vertical. Then

$$\begin{aligned}L_D(h)+L_D(\gamma)+L_D(v')&\leqslant L(1)L_d(h)+L(2)L_d(v)+L_d(v)\\&\leqslant (L(1)L(2)+1)[L_d(h)+L_d(v)].\end{aligned}$$ □

7. Quasiconformality of the transition functions

We have just assigned $\mathcal{R}$ Euclidean flat coordinates in such a way that the distance from the ends of $\mathcal{R}$ is given by the limit metric d and such that the Euclidean metric D is comparable with d. Our final task is to show that analytic moduli as defined by these Euclidean coordinates and combinatorial moduli as defined by the shinglings $\mathcal{S}_1,\mathcal{S}_2,\ldots$ are comparable. We accomplish this final task in two steps. The first shows that the analytic moduli assigned by Euclidean coordinates are uniformly comparable with the combinatorial moduli defined by almost square tilings $T_1, T_2, \ldots$ of $\mathcal{R}$ (square in the Euclidean flat coordinates). The second shows that the combinatorial moduli defined by $\mathcal{S}_1, \mathcal{S}_2, \ldots$ are comparable with the combinatorial moduli defined by almost square tilings $T_1, T_2, \ldots$ of $\mathcal{R}$.

In order to deal with the first step, we first need to develop some of the properties of the classical analytic modulus.

Let $\mathcal{R}_\infty$ be a ring in the complex plane C and let

$$\mathcal{R}_1 \supset \mathcal{R}_2 \supset \ldots \supset \mathcal{R}_\infty$$

be a sequence of rings in C converging to $\mathcal{R}_\infty$ homeomorphically. We assume that $\mathcal{R}_{i+1}$ circles $\mathcal{R}_i$ for $i=1,2,\ldots$. Fix $z_0 \in \operatorname{Int} \mathcal{R}_\infty$ and let

$$f_i\colon \mathcal{R}_i \to C$$

be the unique embedding which is conformal on $\operatorname{Int} \mathcal{R}_i$, takes z_0 to 0, has positive derivative at z_0, and has image with circular concentric boundaries, the inner boundary of radius 1 which is the image of the inner boundary of $\mathcal{R}_i$.

THEOREM. *Some subsequence of the f_i's converges to f_∞ on $\mathcal{R}_\infty$. For that subsequence, the domains $f_i(\mathcal{R}_i)$ converge to $f_\infty(\mathcal{R}_\infty)$.*

Proof. By the monotonicity of the classical modulus ([LV, Lemma 6.3, p. 35]),

$$0 < M(\mathcal{R}_\infty) \leqslant \ldots \leqslant M(\mathcal{R}_2) \leqslant M(\mathcal{R}_1) < \infty.$$

The classical modulus $M(\mathcal{R}_i)$ of $\mathcal{R}_i$ is the modulus of $f_i(\mathcal{R}_i)$, which is

$$\log M_i - \log \mu_i = \log M_i$$

where M_i is the radius of the large boundary of $f_i(\mathcal{R}_i)$ and $m_i=1$ is the radius of the small or inner boundary of $f_i(\mathcal{R}_i)$. That is,

$$M(\mathcal{R}_\infty) \leqslant \log M_i \leqslant M(\mathcal{R}_1),$$

so that there is a uniform bound, both upper and lower, on the outer radius of $f_i(\mathcal{R}_i)$. There is therefore clearly a subsequence of f_i's, which we take to be the entire sequence, such that the inner and outer boundaries converge, necessarily to concentric circles, the inner of radius 1, the outer of radius

$$M = \lim_{i \to \infty} M_i > 1.$$

We claim that 0 must lie strictly between the two limit circles, that is, that 0 is not on either circle. Suppose the contrary. Assume, for example, that 0 lies on the inner limit circle. Let $\mathcal{R}'$ be a ring which circles $\mathcal{R}_\infty$, misses z_0, and separates z_0 from the inner circle of $\mathcal{R}_\infty$. Then $M(\mathcal{R}')=M(f_i(\mathcal{R}'))$. We estimate $M(f_i(\mathcal{R}'))$ by [LV, Lemma 6.2, p. 34]. The boundary components of $f_i(\mathcal{R}')$ have diameter >1 and mutual distance from one another going to 0 as $i\to\infty$. [LV, Lemma 6.2] implies that

$$M(\mathcal{R}') = \lim_{i\to\infty} M(f_i(\mathcal{R}')) \leqslant 0,$$

a contradiction. It follows that 0 lies in the interior of the limit annulus.

Thus we may apply [Go, Chapter V, Section 5, Theorem 2, pp. 232–233] to conclude that the functions f_i converge uniformly on $\mathcal{R}_\infty$ to a homeomorphism having the defining properties of f_∞, hence equal to f_∞. The same theorem implies that the domains $f_i(\mathcal{R}_i)$ converge to $f_\infty(\mathcal{R}_\infty)$. □

COROLLARY. *The classical modulus of $\mathcal{R}_\infty$ may be realized as a limit,*

$$M(\mathcal{R}_\infty) = \lim_{i\to\infty} M(\mathcal{R}_i);$$

and, for large i, the ring $f_i(\mathcal{R}_\infty)$ almost fills the ring $f_i(\mathcal{R}_i)$.

The most glaring defect of this paper is that we do not develop truly good techniques for recognizing whether or not a sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots$ of shinglings is conformal. This section gives the most powerful sufficient condition that we know. This condition allows us to connect analytic and combinatorial moduli.

Definition. A shingling $\mathcal{S}$ covering a compact set U in the complex plane C is said to be *almost round* (K) if, for each shingle $s\in\mathcal{S}$, there is a pair of concentric circular disks $C(s)$ and $D(s)$ satisfying the following conditions:

$$C(s)\subset s\subset D(s), \tag{1}$$

$$\text{radius}\, D(s) \leqslant K\cdot \text{radius}\, C(s), \tag{2}$$

$$C(s_1)\cap C(s_2)=\varnothing \quad \text{for } s_1\neq s_2. \tag{3}$$

7.1. THEOREM. *Let $\mathcal{S}_1, \mathcal{S}_2, \ldots$ denote a sequence of shinglings covering a compact subset U of the complex plane C, each shingle intersecting U, the mesh (= largest element diameter) going locally to 0, each of which is almost round (K). Then the sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots$ is conformal. Furthermore, the approximate moduli assigned by $\mathcal{S}_1, \mathcal{S}_2, \ldots$ to rings in U are comparable with the classical analytic moduli in C.*

Remark. If $\mathcal{S}_1, \mathcal{S}_2, \ldots$ is such a sequence of shinglings of U and $f\colon U\to C$ is a conformal embedding (that is, f is a conformal embedding on a neighborhood of U), then the sequence $f(\mathcal{S}_1), f(\mathcal{S}_2), \ldots$ shingles $f(U)$, has mesh going locally to 0, and, for each $\varepsilon>0$, there is an integer $I=I(\varepsilon)$ such that $i\geqslant I$ implies $f(\mathcal{S}_i)$ is almost round $(K+\varepsilon)$. In other words, the hypothesis of the theorem is for all practical purposes conformally invariant on compact subsets of C. Hence the hypothesis could be stated in such a way as to apply to Riemann surfaces. Furthermore, it is permissible in using the hypothesis to map a given ring conformally before applying the hypothesis. The price one pays for doing this is that one must pass to larger values of i and values of K just slightly bigger than the given value.

Proof. Let $\mathcal{R}$ denote a ring in U. Let $\mathcal{R}'$ be concentric with, but slightly larger than, $\mathcal{R}$; and let $f:\mathcal{R}'\to C$ denote a homeomorphism from $\mathcal{R}'$ onto a right circular cylinder C of circumference 1 and height $M(\mathcal{R}')$, $f\,|\operatorname{Int}\mathcal{R}'$ conformal. By our theorem on the continuity of the classical modulus, we may assume that $f(\mathcal{R})$ almost fills $f(\mathcal{R}')=C$. That is, if we take $C=S\times[0,M(\mathcal{R}')]$ with S a circle of length 1, then we may assume that

$$S\times[\varepsilon, M(\mathcal{R}')-\varepsilon]\subset f(\mathcal{R})\subset S\times[0,M(\mathcal{R}')]=C.$$

In particular,

$$M(\mathcal{R})=M(f(\mathcal{R}))\in[M(\mathcal{R}')-2\varepsilon, M(\mathcal{R}')].$$

In order to simplify notation, we identify $\mathcal{R}$ with $f(R)$, $\mathcal{R}'$ with C, $\mathcal{S}_1,\mathcal{S}_2,\ldots$ with $f(\mathcal{S}_1), f(\mathcal{S}_2),\ldots$ (where we consider only those shingles which hit $\mathcal{R}$). As noted in the remark, the property of being almost round (K) is, at least asymptotically, essentially a conformal invariant so that we see that our identifications are permissible.

We now fix i and consider three weight functions ϱ, σ, and τ on the shingling $\mathcal{S}_i$ of $\mathcal{R}$. The functions ϱ and σ satisfy

$$\frac{H^2(\varrho)}{A(\varrho)}=M_{\sup}(R,\mathcal{S}_i)\quad\text{and}\quad\frac{A(\sigma)}{C^2(\sigma)}=m_{\inf}(R,S_i).$$

The function τ assigns to each shingle its Euclidean diameter. Our idea is now to compare $H^2(\varrho)/A(\varrho)$ with $H^2(\tau)/A(\tau)$, $H^2(\tau)/A(\tau)$ and $A(\tau)$ with $M(\mathcal{R}')$, $M(\mathcal{R}')$ with $M(\mathcal{R})$, then to compare $A(\sigma)/C^2(\sigma)$ with $A(\tau)/C^2(\tau)$, $A(\tau)/C^2(\tau)$ with $M(\mathcal{R}')$, and $M(\mathcal{R}')$ with $M(\mathcal{R})$. These comparisons will complete the proof.

We first deal with ϱ and τ. We are given the inequality

$$\frac{H^2(\tau)}{A(\tau)}\leqslant\frac{H^2(\varrho)}{A(\varrho)}$$

by definition of ϱ as optimal. We get an upper bound on

$$\frac{H(\varrho)^2}{A(\varrho)}$$

by means of the Cauchy–Schwarz inequality. We divide $C=\mathcal{R}'$ up into n vertical strips $v_1, v_2, \ldots, v_n$ of width $1/n$ joining the ends of $\mathcal{R}'$. Each contains a path joining the ends of $\mathcal{R}$. Hence $H(\varrho)$ is bounded above by the minimal ϱ-length of these strips. In turn, at least one strip has ϱ-length $\leqslant$ the average ϱ-length. We bound $H(\varrho)$ therefore by averaging the ϱ-length of these strips. Let $l_1, l_2, \ldots, l_n$ denote the lengths of $v_1, v_2, \ldots, v_n$. If $s\in\mathcal{S}_i$, then let $a(s)$ denote the number of strips s intersects. Then

$$a(s) \leqslant n\tau(s)+1.$$

Hence we may calculate:

$$\begin{aligned} H(\varrho) &\leqslant \frac{1}{n}\sum_j l_j = \frac{1}{n}\sum_j \sum_{s\cap v_j\neq\varnothing} \varrho(s) = \frac{1}{n}\sum_s a(s)\cdot\varrho(s) \\ &\leqslant \frac{1}{n}\sum_s [n\cdot\tau(s)+1]\varrho(s) = \sum_s \tau(s)\cdot\varrho(s) + \frac{1}{n}\sum_s \varrho(s). \end{aligned}$$

Letting $n\to\infty$, we obtain

$$H(\varrho) \leqslant \sum_s \tau(s)\varrho(s);$$

or, applying the Cauchy–Schwarz inequality,

$$\frac{H(\varrho)^2}{A(\varrho)} \leqslant \frac{\left(\sum_s \tau(s)\varrho(s)\right)^2}{\sum_s \varrho(s)^2} \leqslant \frac{\sum_s \tau(s)^2\cdot\sum_s \varrho(s)^2}{\sum_s \varrho(s)^2} = \sum_s \tau(s)^2 = A(\tau).$$

We now bound $A(\tau)$ above and $H(\tau)$ below.

$$\begin{aligned} A(\tau) &= \sum_s \tau(s)^2 \leqslant 4\sum_s (\text{radius } D(s))^2 \\ &\leqslant \frac{4K^2}{\pi}\sum_s \pi\cdot(\text{radius } C(s))^2 \leqslant \frac{4K^2}{\pi}\cdot\text{Area}(\mathcal{R}') \\ &= \frac{4K^2}{\pi}M(\mathcal{R}') \leqslant \frac{4K^2}{\pi}[M(R)+2\varepsilon]. \end{aligned}$$

Let α be a path joining the ends of $\mathcal{R}$. Then

$$H(\tau) \geqslant \sum_{s\cap\alpha\neq\varnothing} \tau(s) \geqslant M(\mathcal{R}')-2\varepsilon \geqslant M(R)-4\varepsilon.$$

Thus we obtain the inequalities

$$\frac{(M(\mathcal{R})-4\varepsilon)^2}{(4K^2/\pi)(M(\mathcal{R})+2\varepsilon)} \leqslant M_{\sup}(\mathcal{R},\mathcal{S}_i) \leqslant \frac{4K^2}{\pi}(M(\mathcal{R})+2\varepsilon).$$

If we choose ε so small that it is a small fraction of $M(\mathcal{R})$, say $\varepsilon=\delta\cdot M(\mathcal{R})$, we obtain

$$\frac{\pi(1-4\delta)^2}{4K^2(1+2\delta)}M(\mathcal{R})\leqslant M_{\sup}(\mathcal{R},\mathcal{S}_i)\leqslant\frac{4K^2(1+2\delta)}{\pi}M(\mathcal{R}).$$

Finally, we deal with the functions σ and τ. We are given the inequality

$$\frac{A(\sigma)}{C(\sigma)^2}\leqslant\frac{A(\tau)}{C(\tau)^2}$$

by the optimality of σ. We divide $S\times[\varepsilon,M(\mathcal{R}')-\varepsilon]$ up into n horizontal circular rings $h_1,h_2,\dots,h_n$ of height $[M(\mathcal{R}')-2\varepsilon]/n$ circling $\mathcal{R}$. Each contains a simple closed curve circling $\mathcal{R}$. Hence $C(\sigma)$ is bounded above by the average σ-length of these rings. Let $m_1,m_2,\dots,m_n$ denote the lengths of $h_1,h_2,\dots,h_n$. If $s\in\mathcal{S}_i$, let $b(s)$ denote the number of rings s intersects. Then

$$b(s)\leqslant\{n\cdot\tau(s)/[M(\mathcal{R}')-2\varepsilon]\}+1.$$

Hence we may calculate:

$$\begin{aligned}C(\sigma)&\leqslant\frac{1}{n}\sum_j m_j=\frac{1}{n}\sum_j\sum_{s\cap h_j\neq\varnothing}\sigma(s)=\frac{1}{n}\sum_s b(s)\cdot\sigma(s)\\&\leqslant\frac{1}{n}\sum_s\{n\cdot\tau(s)/[M(\mathcal{R}')-2\varepsilon]\}\sigma(s)+\frac{1}{n}\sum_s\sigma(s).\end{aligned}$$

Letting $n\to\infty$, we obtain

$$C(\sigma)\leqslant\frac{1}{M(\mathcal{R}')-2\varepsilon}\sum_s\tau(s)\cdot\sigma(s).$$

Applying the Cauchy–Schwarz inequality we obtain

$$\frac{A(\sigma)}{C(\sigma)^2}\geqslant\frac{[M(\mathcal{R}')-2\varepsilon]^2\cdot\sum_s\sigma(s)^2}{\sum_s\tau(s)^2\cdot\sum_s\sigma(s)^2}=\frac{(M(\mathcal{R}')-2\varepsilon)^2}{\sum_s\tau(s)^2}.$$

As before, we have

$$\sum_s\tau(s)^2=A(\tau)\leqslant\frac{4K^2}{\pi}[M(\mathcal{R})+2\varepsilon],$$

and we have $C(\tau)\geqslant1$. Thus we obtain the inequalities

$$\frac{(M(\mathcal{R})-4\varepsilon)^2}{(4K^2/\pi)(M(\mathcal{R})+2\varepsilon)}\leqslant m_{\inf}(\mathcal{R},\mathcal{S}_i)\leqslant\frac{(4K^2/\pi)[M(\mathcal{R})+2\varepsilon]}{1^2}$$

or

$$\frac{\pi}{4K^2}\cdot\frac{(1-4\delta)^2}{(1+2\delta)}M(\mathcal{R})\leqslant m_{\inf}(\mathcal{R},\mathcal{S}_i)\leqslant\frac{4K^2}{\pi}(1+2\delta)M(\mathcal{R}).$$

This calculation completes the proof of the theorem. □

We shall apply the theorem to a specific sequence of shinglings that are almost round. We view $\mathcal{R}$ as the Euclidean rectangle

$$[0,1/H]\times[0,H]$$

with sides identified,

$$(0,y)=(1/H,y), \quad y\in[0,H].$$

Let i be an integer such that $1/i\leqslant 1/H$ $(H\leqslant i)$. Let $m(i)$ be the largest integer in $1/(Hi)$, and let $n(i)$ be the largest integer in H/i. Partition $[0,1/H]$ by the points $x_0,x_1,\dots,x_{m(i)}$, with $x_j=j/i$ for $j<m(i)$, and $x_{m(i)}=1/H$. Partition $[0,H]$ by the points $y_0,y_1,\dots,y_{n(i)}$, with $y_k=k/i$ for $k<n(i)$, and $y_{n(i)}=H$. Define a tiling T_i of $\mathcal{R}$ by considering all of the rectangles of the form

$$[x_{j-1},x_j]\times[y_{k-1},y_k]$$

with $1\leqslant j\leqslant m(i)$ and $1\leqslant k\leqslant n(i)$. Most of these rectangles will be squares, but in any case the ratio of height to width will be in the interval $\left[\frac{1}{2},2\right]$. For $1\leqslant i<H$, we define T_i to consist of the single tile $\mathcal{R}$.

COROLLARY. *The sequence $T_1,T_2,\dots$ is conformal.*

As the final step in the proof of the combinatorial Riemann mapping theorem, we need to show that the combinatorial moduli assigned rings in $\mathcal{R}$ by the two conformal sequences $\mathcal{S}_1,\mathcal{S}_2,\dots$ and $T_1,T_2,\dots$ are uniformly comparable. First we need to compare the limiting areas in $\mathcal{R}$ and the Euclidean areas in $\mathcal{R}$.

7.2. PROPOSITION. *Limiting areas and Euclidean areas are comparable.*

Proof. Let Q denote a Euclidean rectangle in $\mathcal{R}$ with sides vertical, top and bottom horizontal. Approximate Q from within by a vertical stack of d-quadrilaterals each having width at least twice as large as its height, each having its sides within ε of the sides of Q. Each quadrilateral in this stack can be associated naturally with a subrectangle of Q whose top and bottom contain the top and bottom of the quadrilateral. Then each corresponding quadrilateral and subrectangle will have comparable heights and widths by the comparability of d and D. The corresponding areas are uniformly comparable to the product of height and width by Proposition 5.6 in the case of d and equal to the product of height and width in the Euclidean case. Hence the Euclidean area of Q and the limiting area of the stack of quadrilaterals are uniformly comparable. An approximation from outside by a stack of quadrilaterals shows that Q is assigned a limiting area comparable to its Euclidean area.

An arbitrary d-quadrilateral, on the other hand, can be approximated from within and from without by stacks of Euclidean rectangles in like manner. By a similar argument, the quadrilateral is assigned Euclidean area comparable to its limiting area.

The proposition follows. $\square$

7.3. THEOREM. *There is a uniform constant $K(10)$ depending only on the constant $K(1)$ such that, if $\mathcal{R}'$ is a ring contained in $\mathcal{R}$, then the approximate combinatorial moduli assigned $\mathcal{R}'$ by the two conformal sequences $\mathcal{S}_1, \mathcal{S}_2, \ldots$ and $T_1, T_2, \ldots$ are $K(10)$-comparable.*

Proof. Let $\mathcal{R}'$ be a ring contained in $\mathcal{R}$. Our idea is to show that we can approximate $m_{\text{inf}}(\mathcal{R}', T_I)$ by the modulus $A(\varrho)/C(\varrho)^2$ and $M_{\text{sup}}(\mathcal{R}', T_I)$ by the modulus $H(\sigma)^2/A(\sigma)$ for certain weight functions $\varrho = \varrho(i)$ and $\sigma = \sigma(i)$ associated with the pair $(\mathcal{R}', \mathcal{S}_i)$ for all i sufficiently large. This will complete the proof according to the following logic.

We will find a constant K depending only on $K(1)$ and weight functions ϱ and σ on $(\mathcal{R}', \mathcal{S}_i)$ such that

$$\frac{A(\varrho)}{C(\varrho)^2} \leqslant K m_{\text{inf}}(\mathcal{R}', T_I) \tag{1}$$

and

$$M_{\text{sup}}(\mathcal{R}', T_I) \leqslant K \frac{H(\sigma)^2}{A(\sigma)}. \tag{2}$$

The sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots$ is $K(1)$-conformal, and we may take the $T_1, T_2, \ldots$ to be $K(1)$-conformal as well. Hence, by taking I and i large, we may assume the inequalities,

$$\frac{1}{K(1)} M_{\text{sup}}(\mathcal{R}', \mathcal{S}_i) \leqslant m_{\text{inf}}(\mathcal{R}', \mathcal{S}_i) \leqslant K(1) M_{\text{sup}}(\mathcal{R}', \mathcal{S}_i) \tag{3}$$

and

$$\frac{1}{K(1)} M_{\text{sup}}(\mathcal{R}', T_I) \leqslant m_{\text{inf}}(\mathcal{R}', T_I) \leqslant K(1) M_{\text{sup}}(\mathcal{R}', T_I). \tag{4}$$

By definition we have the inequalities

$$m_{\text{inf}}(\mathcal{R}', \mathcal{S}_i) \leqslant \frac{A(\varrho)}{C(\varrho)^2} \tag{5}$$

and

$$\frac{H(\sigma)^2}{A(\sigma)} \leqslant M_{\text{sup}}(\mathcal{R}', \mathcal{S}_i). \tag{6}$$

From these six inequalities we deduce

$$\begin{aligned} m_{\text{inf}}(\mathcal{R}', \mathcal{S}_i) &\leqslant K(1) m_{\text{inf}}(\mathcal{R}', T_I) && \text{(by (5) and (1))} \\ &\leqslant K(1)^2 M_{\text{sup}}(\mathcal{R}', \mathcal{S}_i) && \text{(by (4), (2), and (6))} \\ &\leqslant K(1)^5 M_{\text{sup}}(\mathcal{R}', T_I) && \text{(by (3),(5),(1), and (4)).} \end{aligned}$$

Since $m_{\inf}(\mathcal{R}',\mathcal{S}_i)$ and $M_{\sup}(\mathcal{R}',\mathcal{S}_i)$ are $K(1)$ comparable ((3)), and $m_{\inf}(\mathcal{R}',T_I)$ and $M_{\sup}(\mathcal{R}',\mathcal{S}_i)$ are $K(1)$ comparable ((4)), the desired result follows.

We now set out to find ϱ and σ.

We fix I large and first expand each tile

$$t=[x_{j-1},x_j]\times[y_{k-1},y_k]$$

of T_I to the shingle

$$t'=[x_{j-1}-1/3I,x_j+1/3I]\times[y_{k-1}-1/3I,y_k+1/3I]$$

to form a new shingling T_I' of $\mathcal{R}$. Note that if two tiles were disjoint, so also are their expansions. No point lies in more than four tiles (shingles). Disjoint shingles lie at a D (Euclidean flat) distance at least $1/3I$ from one another. In particular, if i is very large, then no shingle of $\mathcal{S}_i$ will hit two shingles of T_I' that do not already intersect. Thus we may use the expanded shingles of T_I' as a sorting device for the shingles of $\mathcal{S}_i$.

Let ϱ' be an optimal weight function for the pair $(\mathcal{R}',T_I)$ realizing the equality

$$\frac{A(\varrho')}{C(\varrho')^2}=m_{\inf}(\mathcal{R}',T_I),$$

$\varrho'=0$ on each tile missing $\mathcal{R}'$. If i is large relative to I, then every shingle s of $\mathcal{S}_i$ which hits $\mathcal{R}'$ will lie in a shingle of T_I', and each will intersect at most four such. Assign s to a tile $t(s)\in T_I$ of maximal ϱ'-weight such that s intersects the expansion $t'(s)$ of $t(s)$. Define

$$\varrho(s)=\varrho'(t(s))\cdot I\cdot\varrho_i(s)$$

where ϱ_i is the optimal weight function for the pair $(\mathcal{R},\mathcal{S}_i)$. The function ϱ_i is used to make the shingles hitting a tile t act collectively like an approximate square in R, a square of side $1/I$. The multiplier I makes the square have side 1. The multiplier $\varrho'(t(s))$ then scales the square to the approximate size determined by ϱ'. If s misses $\mathcal{R}'$, we assign s the weight $\varrho(s)=0$. This defines the desired weight function ϱ on $(\mathcal{R}',\mathcal{S}_i)$.

We now bound $A(\varrho)/C(\varrho)^2$ from above. To this end we bound $A(\varrho)$ above and $C(\varrho)$ below. The area estimate is as follows.

$$A(\varrho)=\sum_{s\in\mathcal{S}_i}[\varrho'(t(s))\cdot I\cdot\varrho_i(s)]^2=\sum_{t\in T_I}\varrho'(t)^2\left[I^2\cdot\sum_{t(s)=t}\varrho_i^2(s)\right]\leqslant\sum_{t\in T_I}\varrho'(t)^2[I^2\cdot A_i(t')].$$

The Euclidean area of t' is at most

$$\left(\frac{8}{3I}\right)\cdot\left(\frac{8}{3I}\right).$$

By Proposition 7.2, the Euclidean area is comparable with the limit area $A(t')$. Hence there is a constant L such that, for i large,

$$I^2 \cdot A_i(t') \leqslant I^2 \cdot L \cdot \frac{64}{9I^2} = \frac{64}{9} L.$$

That is,

$$A(\varrho) \leqslant \sum_{t \in T_I} \varrho'(t)^2 \cdot \frac{64}{9} L = \frac{64}{9} L \cdot A(\varrho').$$

We estimate $C(\varrho)$ as follows. Let α be any path circling $\mathcal{R}'$. Let U be the set of tiles of T_I intersecting α. Let U' be the corresponding shingles of T_I'. For each $t' \in U'$, there is a path $\alpha(t')$ in α open-irreducible from star t to star$(\mathcal{R} \backslash \operatorname{Int} t')$, where stars are taken relative to the cover $\mathcal{S}_i$ of $\mathcal{R}'$. The ϱ_i-length of $\alpha(t)$ is at least as large as the d_i-distance from t to $\mathcal{R} \backslash \operatorname{Int} t'$ which approaches $1/3I$ in the limit. No shingle of $\mathcal{S}_i$ hits more than four of the arcs $\alpha(t)$, $t \in U$. Hence

$$\begin{aligned} L(\alpha, \varrho) &\geqslant \frac{1}{4} \sum_{t \in U} L(\alpha(t), \varrho) = \frac{1}{4} \sum_{t \in U} \sum_{s \cap \alpha(t) \neq \varnothing} \varrho'(t(s)) \cdot I \cdot \varrho_i(s) \\ &\geqslant \frac{I}{4} \sum_{t \in U} \varrho'(t) \sum_{s \cap \alpha(t) \neq \varnothing} \varrho_i(s) \geqslant \frac{I}{4} \left(\frac{1}{3I} - \varepsilon \right) \sum_{t \in U} \varrho'(t) \\ &= \frac{I}{4} \left(\frac{1}{3I} - \varepsilon \right) \cdot L(\alpha, \varrho') \geqslant \frac{I}{4} \left(\frac{1}{3I} - \varepsilon \right) \cdot C(\varrho'), \end{aligned}$$

where ε can be made as small as desired by making i large. In particular, we may make

$$\frac{1}{3I} - \varepsilon > \frac{1}{4I}.$$

Then we have

$$L(\alpha, \varrho) \geqslant \frac{C(\varrho')}{16}.$$

By choosing α so that $L(\alpha, \varrho) = C(\varrho)$, we obtain

$$C(\varrho) \geqslant \frac{C(\varrho')}{16}.$$

Hence

$$\frac{A(\varrho)}{C(\varrho)^2} \leqslant \frac{64}{9} L \cdot A(\varrho') \cdot \frac{16^2}{C(\varrho')^2} = \frac{2^{14}}{3^2} L m_{\inf}(\mathcal{R}', T_I).$$

This establishes (1). A similar argument establishes (2). $\square$

8. Questions and problems

The most pressing problem is this:

(1) How can one show that a particular sequence of shinglings is conformal?

The best result to date is Theorem 7.1. As we shall show in another paper, shinglings of the type covered by Theorem 7.1 arise from the combinatorics of every cocompact discrete hyperbolic group. (See [C2] for a preliminary discussion.) With more care, one can find similar theorems for finite-volume hyperbolic groups. The combinatorial Riemann mapping theorem allows one to show that such happens only with hyperbolic groups.

The general problem is delicate. If one iterates a random subdivision rule, it soon becomes apparent when the sequence of shinglings is not conformal; for there seem to develop stresses and strains in the subdivision which create eigendirections in the limit that are incompatible with conformality. It seems difficult to make this precise.

It is generally much easier to prove that a particular sequence is *not* conformal. Either direct calculation or the indirect assumption that the modulus can be calculated and that it converges allows one to show that the modulus of some particular ring is 0 or ∞. Sometimes one can show that rings surrounding a point must have bounded modulus, either directly or indirectly. Etc.

There is some hope of getting help in making actual calculations from [M] or [W]. We thank Peter Doyle for calling these papers to our attention.

Here are some closely related problems on which the combinatorial Riemann mapping theorem might shed some light.

(2) Must every closed 3-manifold with negatively curved fundamental group admit a hyperbolic structure?

Such spaces seem to have a visual 2-sphere at infinity, so that there is a topological surface at infinity to which one might apply the mapping theorem (see [BF] and [BM]). Such spaces have a natural recursive structure at infinity so that there is a natural sequence of shinglings at infinity (see [C1], [C2], and [Gr]).

(3) Can one develop deformation properties for conformal sequences so as to handle certain surgeries on hyperbolic knot and link spaces?

Thurston [T] has shown that most surgeries on a hyperbolic link are hyperbolic. What is the analogous theorem for conformal sequences?

Our constructions act like electrical networks with resistance at the nodes rather than on the connecting wires (see [DS]).

(4) What is the exact connection between our work and the classical work on electrical networks?

(5) What is the connection between our work and the classical work on the finite element method?

Our results give only quasiconformal conclusions. However, the coordinates defined are very nearly unique. The work of Rodin, Sullivan, Beardon, Stephenson, and He ([RS], [Ro1], [Ro2], [BS], [He]) give rise to conformal mappings by means of combinatorial data. Hence we have the following questions.

(6) What conditions need to be added to our hypotheses to ensure that the coordinates supplied by the combinatorial Riemann mapping theorem are in the conformal class of given conformal coordinates? Is it enough to use approximately round elements with small overlap whose position is asymptotically random?

Notes added (August, 1994). The Parry notes [Pa] on optimal weight functions have been subsumed in [CFP].

The theory connecting the combinatorial Riemann mapping theorem with the recognition of cocompact Kleinian groups is exposited in [CS].

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