PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

THE COMMUTANTS OF CERTAIN ANALYTIC TOEPLITZ OPERATORS[1]

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ABSTRACT. In this paper we characterize the commutants of two classes of analytic Toeplitz operators. We show that if F in H^∞ is univalent and nonvanishing, the $\{T_{F^2}\}' = \{T_z\}'$. When φ is the product of two Blaschke factors, we characterize $\{T_\varphi\}'$ in terms of algebraic combinations of Toeplitz and composition operators.

Introduction. Let H^2 denote the Hilbert space of functions f analytic in the open unit disk D which satisfy

$$\sup_{0 \leqslant r < 1} \int \left| f(re^{i\theta}) \right|^2 d\theta < \infty .$$

Let H^∞ denote the algebra of bounded analytic functions on D. For φ in H^∞, T_φ is the *analytic Toeplitz operator* defined by $T_\varphi f = \varphi f$. Let $\{T_\varphi\}'$ denote the commutant of T_φ, i.e. the algebra of operators which commute with T_φ. The study of analytic Toeplitz operators has been extensive and many of their properties are well known [**2**], [**4**].

In [**6**], Nordgren gave a sufficient condition for an analytic Toeplitz operator to have no nontrivial reducing subspaces. Since the projection onto a subspace commutes with an operator if and only if the subspace reduces the operator, the problem of finding reducing subspaces can be generalized to that of determining the commutant of an analytic Toeplitz operator. In a recent paper [**3**], Deddens and Wong study this latter problem. One of their results is that φ univalent implies $\{T_\varphi\}' = \{T_z\}'$, the algebra of analytic Toeplitz operators. We extend that result to the case where φ is the square of a nonvanishing univalent function. The extension generalizes and simplifies Nordgren's Example 2 in [**6**].

In certain special cases [**1**], [**3**], [**8**], analytic Toeplitz operators induced by inner functions play a significant role in commutant problems. Since these are unilateral shifts, their commutants can be characterized matricially [**3**]. On the other hand, the problem of finding more revealing function theoretic characterizations of their commutants is difficult. Our main result is a function theoretic characterization of $\{T_\varphi\}'$ when φ is the product of two Blaschke factors.

Received by the editors September 5, 1974 and, in revised form, November 5, 1974.

AMS (MOS) subject classifications (1970). Primary 47B35, 47B20, 30A76.

Key words and phrases. Analytic function, inner function, H^∞, H^2, analytic Toeplitz operator, commutant.

[1] This paper represents part of the author's doctoral dissertation written at the University of North Carolina under the direction of Joseph A. Cima.

Notation and preliminaries. $[X, Y, \dots]$ will denote the closed linear span of the vectors $X, Y, \dots$, and $(\cdot,\cdot)$ will denote the inner product on H^2. Twofunctions which we will often use are $K_z(\lambda) = (1 - \bar{z}\lambda)^{-1}$ and $B_z(\lambda) = (z - \lambda)(1 - \bar{z}\lambda)^{-1}$.

Note that if BG is the inner-outer factorization of F, then

$$\ker T_F^* = \ker T_G^* T_B^* = \ker T_B^* .$$

The last equality follows from the fact that T_G has dense range, which implies that T_G^* is one-to-one. For details on the inner-outer factorization, see **[5]**.

THEOREM 1. *Let F in H^∞ be univalent and nonvanishing. Let $\varphi = F^2$. Then* $\{T_\varphi\}' = \{T_z\}'$.

PROOF. First, we will show that the conclusion holds if there is a set V with accumulation point in D such that the inner part of $\varphi - \varphi(z)$ is B_z for all z in V. Suppose $T \in \{T_\varphi\}'$. Fix z in V. Since T^* commutes with $T^*_{\varphi-\varphi(z)}$, $\ker\ T^*_{\varphi-\varphi(z)}$ is invariant for T^*. But

$$\ker T^*_{\varphi-\varphi(z)} = \ker T^*_{B_z} = [K_z],$$

so K_z is an eigenvector for T^*. Hence, there exists a number $\psi(z)$ such that $T^*K_z = \overline{\psi(z)}K_z$. Since z was arbitrary, we can do the above for each z in V. For f in H^2 and z in V,

$$(Tf)(z) = (Tf, K_z) = (f, T^*K_z) = \left(f, \overline{\psi(z)}\,K_z\right) = \psi(z)f(z).$$

Therefore, $(T1)(z) = \psi(z)$ for all z in V. Equivalently, we have that $(Tf)(z) = (T1)(z)f(z)$ for all z in V. Hence, $Tf = (T1)f$. Since $(T1)f$ is in H^2 for any f in H^2, $T1$ is in H^∞ and $T = T_{T1}$. Thus, we have shown that $\{T_\varphi\}' \subseteq \{T_z\}'$ and the reverse inclusion holds for any analytic Toeplitz operator.

Now, we will establish the existence of a set V with accumulation point in D such that the inner part of $\varphi - \varphi(z)$ is B_z for all z in V. First, we claim that there exists z_0 in D such that $\varphi^{-1}(\varphi(z_0)) = \{z_0\}$. Suppose not. Then $-F(z)$ is in $F(D)$ for all z in D. Fix a in D. Since $F(D)$ is pathwise connected, there is a path P parameterized by $[0,1]$ such that

(i) $P(0) = F(a)$,

(ii) $P(1) = -F(a)$,

(iii) $P(t) \in F(D)$ for all t in $[0,1]$.

Let $Q(t) = -P(t)$. We claim that $P \cup Q$ is a closed path in $F(D)$ with nonzero winding number about zero. Since $F(D)$ is simply connected, there is a single-valued analytic branch of the logarithm defined in $F(D)$. Hence, there is an odd integer k such that

$$\int_P \frac{dz}{z} = \log F(a) - \log(-F(a)) = ik\pi.$$

By a change of variable, we then have $\int_Q (dz/z) = ik\pi$. Hence, the claim is established, but that contradicts the fact that zero is not in the simply connected open set $F(D)$.

Now, let z_0 be such that $\varphi^{-1}(\varphi(z_0)) = \{z_0\}$. Then $-F(z_0)$ is not in $F(D)$. Since $F(D)$ is simply connected, there exists an infinite sequence $\{u_n\}$ such that $-u_n \notin F(D)$ and $u_n \to F(z_0)$. But $F(D)$ is open, so there exists $N > 0$

such that $n \geqslant N$ implies u_n is in $F(D)$. We can assume $N = 1$. Hence, for every n, there exists z_n such that $F(z_n) = u_n$. Since F is a homeomorphism, $z_n \to z_0$.

$$\varphi - \varphi(z_n) = (F - F(z_n))(F + F(z_n)).$$

Since $F - F(z_n)$ is univalent and has a simple zero at z_n, its inner part is B_{z_n}. $F + F(z_n)$ is univalent and nonvanishing, so its inner part is trivial. Let $V = \{z_n\}$, and the proof is complete.

For B: $D \to D$ analytic, let C_B be the composition operator on H^2 defined by $C_B f = f \circ B$. J. Ryff [7] first showed that composition operators define bounded operators.

In the next theorem, $T_{1/(z-b)}$ will denote multiplication by $1/(z - b)$. This is a bounded linear operator from $B_b H^2$ onto H^2. We shall only apply it to functions in $B_b H^2$.

THEOREM 2. *Let* $B(z) = B_a(z) = (a - z)/(1 - \bar{a}z)$ *for some* $|a| < 1$. *If* $a \neq 0$, *let*

$$b = \frac{1}{\bar{a}}\left(1 - \sqrt{1 - |a|^2}\right).$$

If $a = 0$, *let* $b = 0$. *Suppose* $\varphi(z) = zB(z)$. *Then*

$$\begin{aligned}\{T_\varphi\}' &= \left\{T_{1/(z-b)}(T_{F_1} + T_{G_1}C_B)\colon F_1, G_1 \in H^\infty, F_1(b) = -G_1(b)\right\}\\ &= \left\{T_{F_2} + T_{G_2}C_B + \alpha T_{1/(z-b)}(1 - C_B)\colon F_2, G_2 \in H^\infty, \alpha \in \mathbf{C}\right\}.\end{aligned}$$

PROOF. First, note that $\varphi \circ B = \varphi$, so the zeros of

$$\varphi_c = (\varphi(c) - \varphi)/\left(1 - \overline{\varphi(c)}\varphi\right)$$

are c and $B(c)$. Second, $B(b) = b$, so b is the unique fixed point of B in D.

Suppose $T \in \{T_\varphi\}'$. Then $T \in \{T_{\varphi_c}\}'$ for all $c \in D$, and, equivalently, $T^* \in \{T^*_{\varphi_c}\}'$ for all $c \in D$. Hence T^* leaves invariant ker $T^*_{\varphi_c}$. Since the zeros of φ_c are c and $B(c)$, ker $T^*_{\varphi_c} = [K_c, K_{B(c)}]$ for $c \neq b$. Thus, for $c \neq b$, there exist $F(c)$ and $G(c)$ such that $T^*K_c = \overline{F(c)}K_c + \overline{G(c)}K_{B(c)}$. For $f \in H^2$ and $z \neq b$,

$$(*) \qquad (Tf)(z) = (Tf, K_z) = (f, T^*K_z) = F(z)f(z) + G(z)f(B(z)).$$

Let $i(z) = z$. Let $g = T1$ and $h = Ti$. For $z \neq b$, $g(z) = F(z) + G(z)$ and $h(z) = zF(z) + B(z)G(z)$. Solving for F and G, we find

$$F(z) = \frac{h(z) - B(z)g(z)}{z - B(z)} \quad \text{and} \quad G(z) = \frac{h(z) - zg(z)}{z - B(z)}.$$

Thus, each is analytic on $D - \{b\}$ and may have, at worst, a simple pole at b.

Next, we claim that $F(z)$ and $G(z)$ are bounded as $|z| \to 1$. Suppose $F(z)$ is not. Then there exists $\{z_n\}$ with $|z_n| \to 1$ such that $|F(z_n)| \to \infty$. By passing to a subsequence, if necessary, we can assume that $z_n \to w$ for some $|w| = 1$. Possibly taking another subsequence, we can assume $\{z_n\}$ is uniformly separated [5, p. 148]. Since B is an automorphism of D, $\{B(z_n)\}$ is uniformly separated and $B(z_n) \to B(w)$. Recalling that $B(b) = b$ and $|b| < 1$, we have $B(w) \neq w$ and we can assume that $\{z_n\}$ and $\{B(z_n)\}$ are disjoint. Hence, we can assume that $\{w_n\}$ is uniformly separated where $w_{2n-1} = z_n$ and w_{2n}

$= B(z_n)$. Let

$$I(z) = \prod_{n=1}^{\infty} \frac{\bar{w}_{2n}}{|w_{2n}|} \frac{w_{2n} - z}{1 - \bar{w}_{2n} z} .$$

Then $I(B(z_n)) = 0$ and there exists $\delta > 0$ such that $|I(z_n)| \geqslant \delta$. Let

$$f_n(z) = I(z)\left(1 - |z_n|^2\right)^{1/2}(1 - \bar{z}_n z)^{-1}.$$

Then $\|f_n\|_2 = 1$, $f_n(z_n) = I(z_n)(1 - |z_n|^2)^{-1/2}$, and $f_n(B(z_n)) = 0$. By $(*)$,

$$(Tf_n)(z_n) = F(z_n)f_n(z_n) + G(z_n)f_n(B(z_n)) = F(z_n)I(z_n)\left(1 - |z_n|^2\right)^{-1/2}.$$

Hence, $|(Tf_n)(z_n)| \geqslant \delta|F(z_n)|(1 - |z_n|^2)^{-1/2}$. Since point evaluation at z_n is bounded by $(1 - |z_n|^2)^{-1/2}$,

$$|(Tf_n)(z_n)| \leqslant \|Tf_n\|_2\left(1 - |z_n|^2\right)^{-1/2} \leqslant \|T\|\left(1 - |z_n|^2\right)^{-1/2}.$$

Combining this with the above inequality, we have

$$\delta|F(z_n)|\left(1 - |z_n|^2\right)^{-1/2} \leqslant \|T\|\left(1 - |z_n|^2\right)^{-1/2},$$

and, thus, $|F(z_n)| \leqslant \|T\|/\delta$. But this contradicts the assumption that $|F(z_n)| \to \infty$, so $F(z)$ is bounded as $|z| \to 1$. By a similar argument we can show that $G(z)$ is bounded as $|z| \to 1$.

Since $F + G = T1$, we know $F + G \in H^2$. Combining this with the fact that $F + G$ is bounded near ∂D, we have $F + G \in H^\infty$. Hence, $(z - b)(F(z) + G(z))$ has a zero at b; and therefore, the H^∞ functions $F_1 = (z - b)F$ and $G_1 = -(z - b)G$ are equal at b. Thus, $(*)$ becomes

$$\begin{aligned} T &= T_{1/(z-b)}(T_{(z-b)F} + T_{(z-b)G}C_B) = T_{1/(z-b)}(T_{F_1} + T_{G_1}C_B) \\ &= T_{1/(z-b)}(T_{F_1 - F_1(b)} + T_{G_1 - G_1(b)}C_B) + T_{1/(z-b)}(F_1(b) + G_1(b)C_B) \\ &= T_{F_2} + T_{G_2}C_B + F_1(b)T_{1/(z-b)}(1 - C_B) \\ &= T_{F_2} + T_{G_2}C_B + \alpha T_{1/(z-b)}(1 - C_B) \end{aligned}$$

where

$$F_2 = \frac{F_1 - F_1(b)}{z - b} \quad \text{and} \quad G_2 = \frac{G_1 - G_1(b)}{z - b}$$

and $\alpha \in \mathbf{C}$.

It is a straightforward computation to show that any operator of the form, $T_{1/(z-b)}(T_{F_1} + T_{G_1}C_B)$ with F_1 and G_1 in H^∞ and $F_1(b) = -G_1(b)$, commutes with T_φ.

Corollary. *If* $\varphi = B_a B_c$ *for* $a, c \in D$, *then* $\{T_\varphi\}' = \{T_{zB_d(z)}\}'$ *where*

$$d = \frac{a + c - |a|^2 c - |c|^2 a}{1 - |a|^2|c|^2} .$$

Proof.

$$\{T_\varphi\}' = \{T_{(ac - \varphi)/(1 - \bar{a}\bar{c}\varphi)}\}' = \{T_{zB_d(z)}\}'$$

since the zeros of $(ac - \varphi)/(1 - \overline{ac}\varphi)$ are zero and d.

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