THE COMPARABILITY GRAPH OF A TREE

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1. **Introduction.** By an *unoriented graph* we mean a pair (G, R), where G is a set (finite or infinite) and R is an irreflexive and symmetric relation defined on G. If R is a relation on G (i.e., a subset of $G\times G$), we write $x\,R\,y$ if and only if $(x, y)\in R$. If $(x, y)\in G\times G$ and $(x, y)\notin R$, then we write $x\,\tilde{R}\,y$. An *oriented graph* is a pair (G, S), where G is a set and S is a relation on G which is irreflexive and antisymmetric (i.e., $x\,S\,y$ and $y\,S\,x$ do not both hold for any $x\in G$, $y\in G$). An oriented graph (G, S) is *transitively oriented* if and only if, for all x, y, z in G, $x\,S\,y$ and $y\,S\,z$ imply $x\,S\,z$. A transitively oriented graph is a *partially ordered set*, and in this case S is called a *partial order* of G. (H, S) is a *subgraph* of (G, R) if and only if (i) $H\subseteq G$ and (ii) $x\,S\,y$ if and only if $x\,R\,y$, for all $x, y\in H$. For brevity we may sometimes suppress mention of the relations R or S, and refer only to a graph G or a subgraph H. Whenever possible we may form an intuitive picture of a graph (G, R) by imagining the elements of G to be points in the plane, and interpreting $x\,R\,y$ to mean that x and y are connected by a line segment (directed or undirected according as (G, R) is oriented or not).

The general problem which we propose to investigate may be described as follows. Let (G, R) be an unoriented graph. We say that a relation T on G is *a transitive orientation of* (G, R) if and only if (i) (G, T) is a transitively oriented graph, and (ii) $x\,R\,y$ if and only if $x\,T\,y$ or $y\,T\,x$, for all $x, y\in G$. Now we may ask the question: what are necessary and sufficient conditions on an unoriented graph (G, R) for (G, R) to possess a transitive orientation?

The problem may also be stated in a slightly different form. Let $(G, <)$ be a partially ordered set. We may construct from $(G, <)$ an unoriented graph (G, R) by defining $x\,R\,y$ if and only if $x<y$ or $y<x$; i.e., $x\,R\,y$ if and only if x and y are comparable elements of G with respect to the partial order $<$. We shall call (G, R) the *comparability graph* of $(G, <)$. Then our question is the following. Given an unoriented graph (G, R), under what conditions is (G, R) the comparability graph of a partially ordered set?

It is easy to discover a number of necessary conditions. For example, if (G, R) contains any closed polygon with an odd number of

Received by the editors August 3, 1961.

[1] This research was supported by a National Science Foundation grant (No. G16314).

sides (greater than 3), then (G, R) must also contain a diagonal of this polygon. Also, there are certain six-element graphs which possess no transitive orientation. For example, let $G=\{a, b, c, d, e, f\}$ and let

$$R_1 = \{(a,b), (b,c), (a,c), (a,d), (b,e), (c,f)\},$$
$$R_2 = \{(a,b), (b,c), (a,c), (a,d), (b,d), (b,e), (c,e), (a,f), (c,f)\}.$$

The reader may easily verify that neither (G, R_1) nor (G, R_2) admit a transitive orientation. Thus, the comparability graph of a partially ordered set may contain neither (G, R_1) nor (G, R_2) as a subgraph.

The problem we have formulated appears to be very difficult, in its most general form, and hence it seems natural to make a first attack by restricting the class of partially ordered sets under discussion. Let us say that a partially ordered set $(G, <)$ is a *tree* if and only if whenever x and y are incomparable elements of G there exists no $z \in G$ with $x<z$ and $y<z$. Our purpose in this paper is to characterize the comparability graph of a tree; i.e., we obtain a simple necessary and sufficient condition that an unoriented graph (G, R) possess a transitive orientation T such that (G, T) is a tree. We intend, in subsequent work, to attempt to extend our methods to larger classes of partially ordered sets.

2. Main result. Throughout this paper set inclusion will be denoted by $\subseteq$, and $\subset$ will be reserved for proper inclusion. $A-B$ denotes the relative complement of the set B with respect to the set A. If R is a relation on a set M and $A \subseteq M$, we define $R/A = R \cap (A \times A)$.

Some further definitions relating to unoriented graphs will be necessary. A subgraph H of an unoriented graph (G, R) will be called *complete* if and only if $x \in H$ and $y \in H$ imply $x\, R\, y$. A subgraph H of (G, R) is *empty* if and only if $x \in H$ and $y \in H$ imply $x\, \tilde{R}\, y$. H is a *maximal complete subgraph* of G if and only if H is complete and there exists no complete subgraph K of G with $H \subset K$. It is an immediate consequence of Zorn's Lemma that every complete subgraph of a graph G is contained in a maximal complete subgraph of G.

The following definition plays a central role in our discussion. A related but weaker condition has recently been considered by Hajnal and Surányi [1].

DEFINITION. An unoriented graph (G, R) *has the diagonal property* if and only if whenever x_1, x_2, x_3, and x_4 are elements of G with $x_1\, R\, x_2\, R\, x_3\, R\, x_4$, then we also have $x_1\, R\, x_3$ or $x_2\, R\, x_4$.

The reader may easily verify that the diagonal property is equivalent to the following condition: whenever $\{x_0, x_1, \cdots, x_k\}$ is a finite sequence of elements of G with $x_0\, R\, x_1\, R \cdots R\, x_k$, there exists an i such that $1 \leqq i \leqq k-1$ and $x_0 \; R \; x_i \; R \; x_k$.

We now state our main theorem. No restriction is placed on the cardinal number of G.

THEOREM. *An unoriented graph* (G, R) *is the comparability graph of a tree if and only if* (G, R) *has the diagonal property.*

We first prove the necessity of the diagonal property. Suppose that (G, R) is the comparability graph of a tree $(G, <)$. Then $x \; R \; y$ means that $x<y$ or $y<x$. Suppose that (G, R) does not have the diagonal property. Then there exist elements a, b, c, and d in G such that $a \; R \; b \; R \; c \; R \; d$, but $a \; \tilde{R} \; c$ and $b \; \tilde{R} \; d$. Let us consider two cases.

Case 1. Suppose that $a<b$. Then, since c is comparable with b but not with a, we must have $c<b$. But $a<b$ and $c<b$ contradicts the condition that $(G, <)$ is a tree.

Case 2. Suppose that $b<a$. Then we must have $b<c$, since c is not comparable with a. Likewise we must have $d<c$. But $b<c$ and $d<c$ is again a contradiction.

The proof of the sufficiency of the diagonal property is less simple. Given an unoriented graph (G, R) with the diagonal property, we are of course required to produce a transitive orientation T of (G, R) such that the oriented graph (G, T) is a tree. The construction of the relation T will be broken down into a sequence of lemmas. In each of the following lemmas (G, R) is assumed to be an unoriented graph with the diagonal property.

LEMMA 1. *Let N be a maximal complete subgraph of* (G, R). *Let $\mathfrak{A}$ be any family of maximal complete subgraphs of* (G, R) *with $N \notin \mathfrak{A}$. Then the family $\{M \cap N : M \in \mathfrak{A}\}$ is a nest of sets.*

PROOF. If the lemma is false, then there exist $M_1 \in \mathfrak{A}$, $M_2 \in \mathfrak{A}$ such that $M_1 \cap N$ and $M_2 \cap N$ are incomparable sets. Suppose that $x_1 \in (M_1 - M_2) \cap N$, $x_2 \in (M_2 - M_1) \cap N$. We have $x_1 \; R \; x_2$, since x_1, $x_2 \in N$. Also, by maximality of M_2, $x_1 \notin M_2$ implies that there exists $q \in M_2$ with $x_1 \; \tilde{R} \; q$. Likewise, $x_2 \notin M_1$ implies that there exists $p \in M_1$ with $x_2 \; \tilde{R} \; p$. Then $p \; R \; x_1 \; R \; x_2 \; R \; q$ but $p \; \tilde{R} \; x_2$ and $x_1 \; \tilde{R} \; q$, contradicting the diagonal property.

For any $m \in G$, let us define $G_m = \{x \in G : x \; R \; m \text{ or } x = m\}$.

LEMMA 2. *Let C be any complete subgraph of* (G, R). *Then the family $\{G_m : m \in C\}$ is a nest of sets.*

PROOF. Suppose that G_m and G_n are incomparable sets for some m, $n \in C$. Let $x \in G_m - G_n$, $y \in G_n - G_m$. Then $x \; R \; m \; R \; n \; R \; y$, but $x \; \tilde{R} \; n$ and $m \; \tilde{R} \; y$, a contradiction.

The construction of the transitive orientation T for the graph (G, R) will be accomplished by an inductive procedure. By a linearly ordered set we mean a partially ordered set in which each two distinct elements are comparable. Let $\mathfrak{M}$ denote the set of all maximal complete subgraphs of (G, R). Roughly speaking, we shall define inductively a linear order on each $M \in \mathfrak{M}$ in such a way that, if R_M and R_N are the linear orders on M and N respectively, then R_M and R_N agree on $M \cap N$. The transitive orientation T will then be defined as the union of all the R_M's. The members of the family $\mathfrak{M}$ will of course be the maximal chains of the partially ordered set (G, T).

First we give two more definitions.

DEFINITION. Let $M \in \mathfrak{M}$. If R_M is a linear order on M, we say that R_M is *admissible for* M if and only if $m \in M$, $n \in M$, and $G_n \subset G_m$ imply $m\ R_M\ n$.

DEFINITION. Let $(A, <)$ be a linearly ordered set and let $B \subseteq A$. We say that B is an *initial section* of A if and only if $b \in B$, $x \in A$, and $x < b$ imply $x \in B$.

LEMMA 3. *Each $M \in \mathfrak{M}$ possesses an admissible linear ordering R_M.*

PROOF. Let $M \in \mathfrak{M}$. For $m, n \in M$, let us define $m\ \sigma_M\ n$ if and only if $G_m = G_n$. The relation σ_M is an equivalence relation on M. Let $\mathcal{E}$ denote the set of all equivalence classes with respect to σ_M. For any $E \in \mathcal{E}$, let R_E be an arbitrary linear ordering of E. Then we define

$$R_M = \bigcup \{R_E: E \in \mathcal{E}\} \cup \{(m, n) \in M \times M: G_n \subset G_m\}.$$

Since the family $\{G_m: m \in M\}$ is a nest, by Lemma 2, it follows immediately that R_M is an admissible linear ordering of M.

Now let $\mathfrak{M}$ be well-ordered, so that $\mathfrak{M} = \{M_\alpha: \alpha \in W\}$, where W is some ordinal $\{0, 1, \cdots, \alpha, \cdots\}$. For each $\alpha \in W$, we shall define by transfinite induction a relation R_α satisfying

(1) R_α is an admissible linear ordering of M_α,

(2) for each $\beta < \alpha$, we have $R_\alpha/(M_\alpha \cap M_\beta) = R_\beta/(M_\alpha \cap M_\beta)$,

(3) for each $\beta < \alpha$, $M_\alpha \cap M_\beta$ is an initial section of both M_α and M_β in the linear orders R_α and R_β respectively.

Choose R_0 as any admissible linear order on M_0. Let $\alpha \in W$, and assume that R_β has been defined satisfying (1), (2), and (3) for all $\beta < \alpha$. We shall show how to define R_α.

The family $\{M_\alpha \cap M_\beta: \beta < \alpha\}$ is a nest of sets, by Lemma 1. Let $Z = \bigcup \{M_\alpha \cap M_\beta: \beta < \alpha\}$, and let $S = \bigcup \{R_\beta/(M_\alpha \cap M_\beta): \beta < \alpha\}$.

LEMMA 4. *S is a linear order on Z.*

PROOF. We first check that S is transitive. Suppose that $x\ S\ y$

and $y\ S\ z$. Then there exist $\gamma < \alpha$ and $\delta < \alpha$ such that $(x, y) \in R_\gamma/(M_\alpha \cap M_\gamma)$ and $(y, z) \in R_\delta/(M_\alpha \cap M_\delta)$. Since the family $\{M_\alpha \cap M_\beta : \beta < \alpha\}$ is a nest, the sets $M_\alpha \cap M_\gamma$ and $M_\alpha \cap M_\delta$ are comparable. Assume that $M_\alpha \cap M_\gamma \subseteq M_\alpha \cap M_\delta$. Then x, y, and z are all elements of $M_\alpha \cap M_\delta$, and R_δ is a linear order on M_δ by our inductive hypothesis. Hence $(x, z) \in R_\delta/(M_\alpha \cap M_\delta)$, and thus $x\ S\ z$. The proof that $x\ S\ y$ or $y\ S\ x$ for all $x, y \in Z$, with $x \neq y$, is very similar and may be left to the reader.

Now let R_{M_α} be any admissible linear order on M_α. We define

$$R_\alpha = S \cup R_{M_\alpha}/(M_\alpha - Z) \cup \{(x, y) : x \in Z, y \in M_\alpha - Z\}.$$

LEMMA 5. *R_α satisfies* (1), (2), *and* (3).

PROOF. (1) It is clear that R_α is a linear order on M_α. Let us show that R_α is admissible. Suppose that $x, y \in M_\alpha$ and $G_y \subset G_x$. We must consider several cases.

Case I. $x \in Z$, $y \in Z$. In this case we have $x, y \in M_\alpha \cap M_\delta$ for some $\delta < \alpha$. By our inductive hypothesis R_δ is an admissible linear order on M_δ. Hence $(x, y) \in R_\delta/(M_\alpha \cap M_\delta) \subseteq S$. Hence $(x, y) \in R_\alpha$.

Case II. $x \in M_\alpha - Z$, $y \in M_\alpha - Z$. The result in this case follows immediately by the admissibility of R_{M_α}.

Case III. $x \in M_\alpha - Z$, $y \in Z$. We show that this case cannot occur. For suppose that $y \in M_\alpha \cap M_\beta$ for some $\beta < \alpha$. Then $x \notin M_\beta$. Hence, by maximality of M_β, there exists $p \in M_\beta$ with $x\ \tilde{R}\ p$. Hence $p \notin G_x$ but $p \in G_y$, contradicting $G_y \subset G_x$.

In the remaining case when $x \in Z$, $y \in M_\alpha - Z$, we have at once that $(x, y) \in R_\alpha$. Hence the proof of (1) is complete.

(2) Let $\beta < \alpha$. Then $R_\alpha/(M_\alpha \cap M_\beta) = S/(M_\alpha \cap M_\beta)$ by definition of R_α. But by definition of S we have $S/(M_\alpha \cap M_\beta) = R_\beta/(M_\alpha \cap M_\beta)$. This proves (2).

(3) Let $\beta < \alpha$. Let $x \in M_\alpha \cap M_\beta$, $y \in M_\beta - M_\alpha$, and $z \in M_\alpha - M_\beta$. We must prove that $x\ R_\beta\ y$ and $x\ R_\alpha\ z$. Since $y \notin M_\alpha$, by maximality of M_α there exists $p \in M_\alpha$ with $y\ \tilde{R}\ p$. Hence $p \notin G_y$, but $p \in G_x$. Since the family $\{G_m : m \in M_\beta\}$ is a nest, we have $G_y \subset G_x$. Hence $x\ R_\beta\ y$, since R_β is an admissible linear order on M_β. In a similar way we show that $z \notin M_\beta$ implies $G_z \subset G_x$, and hence $x\ R_\alpha\ z$.

We now define $T = \bigcup \{R_\alpha : \alpha \in W\}$.

LEMMA 6. *For all $x, y \in G$, we have $x\ R\ y$ if and only if $x\ T\ y$ or $y\ T\ x$.*

PROOF. $x\ T\ y$ implies $x\ R_\alpha\ y$ for some α. Thus x and y are elements of the complete subgraph M_α and hence $x\ R\ y$. Conversely, suppose that $x\ R\ y$. Then there exists a maximal complete subgraph M_α such that $x, y \in M_\alpha$. Then $x\ R_\alpha\ y$ or $y\ R_\alpha\ x$, since R_α is a linear order on M_α. Hence $x\ T\ y$ or $y\ T\ x$.

To conclude the proof of our theorem, we shall now show that (G, T) is a tree. We first prove that (G, T) is a partially ordered set. T is irreflexive, since all R_α have this property, so it is necessary only to check the transitivity of T. Suppose that $x\ T\ y$ and $y\ T\ z$ for some x, y, and z in G. We shall show that the set $\{x, y, z\}$ forms a complete subgraph of (G, R). By Lemma 6 we know that $x\ R\ y$ and $y\ R\ z$, so we need only to show that $x\ R\ z$. First note that if $G_x = G_y$, then it follows at once that $x\ R\ z$, since $z \in G_y$. So assume that $G_x \neq G_y$. In this case, since $x\ R_\alpha\ y$ for some α, and the linear order R_α is admissible, we must have $G_y \subset G_x$. Hence there exists $q \in G$ such that $q\ R\ x$ and $q\ \tilde{R}\ y$. Then $q\ R\ x\ R\ y\ R\ z$ but $q\ \tilde{R}\ y$. By the diagonal property we have $x\ R\ z$. Therefore $\{x, y, z\}$ is a complete subgraph of (G, R). Hence there exists $\gamma \in W$ such that $\{x, y, z\} \subseteq M_\gamma$. But $x\ T\ y$ implies $x\ R_\gamma\ y$, and $y\ T\ z$ implies $y\ R_\gamma\ z$. Since R_γ is transitive on M_γ, we have $x\ R_\gamma\ z$, and hence $x\ T\ z$.

Now to show that (G, T) is a tree, let x and y be elements of G with $x\ \tilde{T}\ y$ and $y\ \tilde{T}\ x$, and suppose that there exists $z \in G$ with $x\ T\ z$ and $y\ T\ z$. Then there exist $\alpha, \beta \in W$ with $x, z \in M_\alpha$ and $y, z \in M_\beta$. Since $x\ \tilde{R}\ y$, we have $x \notin M_\beta$, $y \notin M_\alpha$. Also, by the construction of T, we have $x\ R_\alpha\ z$ and $y\ R_\beta\ z$. But, since $z \in M_\alpha \cap M_\beta$, this means that $M_\alpha \cap M_\beta$ is not an initial section of M_α or of M_β, contradicting condition (3). Hence (G, T) is a tree, and the proof of the theorem is complete.

3. **Some remarks on Souslin's Problem.** If G is a partially ordered set and $H \subseteq G$, we say that H is *totally unordered* if and only if all pairs of distinct elements of H are incomparable. E. W. Miller [2] has shown that the well-known Problem of Souslin may be stated in the following equivalent form. *If* $(G, <)$ *is a tree in which all chains and all totally unordered subsets are countable, does it follow that G is countable*? Since we have shown that the diagonal property characterizes the comparability graph of a tree, we are now able to state Souslin's Problem in a "graph-theoretic" form as follows. *If G is an unoriented graph with the diagonal property in which all complete subgraphs and all empty subgraphs are countable, then must G be countable*? It should be noted that the affirmation of this proposition is a weaker statement than an open question relating to Souslin's Problem recently discussed by Hajnal and Surányi [1].

We may also give another formulation to Souslin's Problem in the following way. Let us call a partially ordered set $(G, <)$ a *pseudo-tree* if and only if the comparability graph of $(G, <)$ has the diagonal property. A simple example of a pseudo-tree which is not a tree is the partially ordered set consisting of a least element 0, a greatest element I, and a pair of incomparable elements x, y, with $0<x<I$, $0<y<I$. The reader may easily verify that a partially ordered set $(G, <)$ is a pseudo-tree if and only if it satisfies the following condition: whenever a, b, c, d are distinct elements of G such that a and b are incomparable, $c<a$, $c<b$, and $d<b$, then d is comparable with c. Now, according to our theorem of the preceding section, given any pseudo-tree P, there exists a mapping of P onto a tree which is 1:1 and which preserves the comparability relation. Thus we may state Souslin's Problem as follows. *Does there exist an uncountable pseudo-tree in which* (i) *every chain is countable and* (ii) *every totally unordered subset is countable?*

It is important to point out that there exist uncountable partially ordered sets satisfying conditions (i) and (ii) above. An example of such a partially ordered set is due, in essence, to Sierpinski [**3**]. Let G denote the set of all ordinals preceding the first uncountable ordinal, and let E_1 be the real numbers. Let $<$ denote the usual ordering of both G and E_1. Let f be a 1:1 mapping of G into E_1. For $x, y \in G$, we define $x\ T\ y$ if and only if $x<y$ and $f(x)<f(y)$. (G, T) is an uncountable partially ordered set, and Sierpinski's results [**3**] imply that (G, T) satisfies conditions (i) and (ii).

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