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The Complementary-Slackness Class of Hybrid Systems*

A. J. van der Schaft† and J. M. Schumacher‡

Abstract. In this paper we understand a "hybrid system" to be one that combines features of continuous dynamical systems with characteristics of finite automata. We study a special class of such systems which we call the complementary-slackness class. We study existence and uniqueness of solutions in the special cases of *linear* and *Hamiltonian* complementary-slackness systems. For the latter class we also prove an energy inequality.

Key words. Hybrid systems, Complementary slackness, Inequality constraints, Well-posedness.

1. Introduction

Recently the term "hybrid systems" has been used to refer to dynamical systems whose state has a continuous as well as a discrete component. Systems of this type arise in many different ways, and examples have been studied in the mathematical, engineering, and science literature for at least several decades, albeit mostly not under the heading of "hybrid systems." An incomplete list includes bang-bang control [B5], [L2], continuous systems with relays or bistable elements [PBGM, Section 22], [W3], piecewise analytic vector fields [S6], piecewise linear systems [G4], [S3] (see also [S4]), and mechanical systems subject to inequality constraints [P], [KR]. The 1966 paper by Witsenhausen [W3] may have been the first to use the term "hybrid" explicitly in connection with dynamical systems of a mixed continuous/discrete nature.

The recent surge of interest in hybrid systems is due to developments in several research areas, including theoretical computer science, control theory, and the theory of modeling and simulation. In computer science it has been acknowledged for some time that many computer systems operate in environments that are described by continuous rather than discrete variables, for instance, in the control

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of chemical processes or of robot mechanisms. In such cases the term *reactive systems* [MP] has been used. If timing constraints are taken into account *timed systems* [AD] are referred to, and if continuous dynamics come into play the term *hybrid systems* is used [MMP]. Such systems have recently drawn much interest in theoretical computer science [GNRR], [AKN], [PS], [AHS]. At the same time, there is a trend in control theory (see, for instance, [B3]) moving away from the standard paradigm that is formulated entirely in terms of differential and/or difference equations, toward formulations that also allow the presence of discrete elements, often described in this context as "switching logic." Actually this trend follows rather than precedes control engineering practice, since switching elements have already been used successfully although on an *ad hoc* basis in many control applications. The incorporation of discrete elements in continuous dynamical systems has also attracted the interest of researchers in the area of "classical" dynamical systems as a means for obtaining properties such as robust stability [G5]. In the area of modeling and simulation, the subjects of continuous system simulation and discrete-event simulation have been studied extensively (see, for instance, [C] and [DA]). Many modeling situations however call for a mixture of these two, and steps have been taken toward the construction of simulation languages for hybrid systems using object-oriented principles and bond graph methods [A1], [S5].

Proposals for defining the general class of hybrid systems or languages for it have been made in all the fields mentioned above (see, for instance, [GNRR], [A1], [S5], [BBM], and [BGM]). Such a definition or language usually calls for the specification of a number of items: (i) the *laws of motion* governing the continuous evolution in the intervals between events, (ii) the rules that determine the *event times* (times at which events will take place), (iii) the *transition rules* that determine the new discrete state after an event has taken place, and (iv) the *reinitialization rules* that determine a new value of the continuous state after an event. The most general framework would allow such rules to be specified rather arbitrarily. Problems that may be studied in this context include essentially all questions that may be asked for continuous systems and/or for discrete systems, so that in principle a huge research area lies ahead. However, perhaps it should be expected that most of the development in the area of hybrid systems will be concerned with systems that have some kind of special structure, as is the case for continuous dynamical systems where planar systems, Hamiltonian systems, singularly perturbed systems, and so on are studied.

The search for useful special structures in hybrid systems is of interest also from a *specification* point of view. Indeed, if items (i)–(iv) mentioned above have to be specified on a case-by-case basis as would be necessary in the general (unstructured) situation, then the sheer multitude of rules to be specified may quickly become prohibitive. It should be hoped that it will be possible to alleviate this problem by exploiting special structures, allowing the modeler to work in a high-level language.

In this paper we are concerned with a class of hybrid systems which does have a special structure, and which we have called the complementary-slackness class after a term used in optimization theory. For this class we study well-posedness, that is, existence and uniqueness of solutions. As noted by Sussmann [S6], these questions are more delicate for hybrid (discontinuous) systems than they are for continuous dynamical systems. Whereas in the latter case smoothness conditions (Lipschitz continuity) are well known to be sufficient for well-posedness, in the case of hybrid systems it has to be made sure, for instance, that no infinitely fast chattering will occur, in which the system keeps switching between several discrete states without finding a situation in which any continuous dynamics can be active. A sufficient condition to prevent such chattering is "strict separation between action sets and destination sets" (see, for instance, [BBM]). In this paper we consider sufficient conditions that are of a completely different nature and that do not assume such a strict separation.

Complementary slackness systems are defined precisely below, but they may be loosely described as systems that arise from variational principles in combination with inequality constraints. Real-world examples of complementary-slackness systems are in abundance. Electrical networks containing diodes, hydraulic systems containing one-way valves, and mechanical systems with stops can all be described as complementary-slackness systems. An advantage of the fact that these systems occur in nature is that a strong intuition is available about their operation. A second advantage that is employed below is the presence of the concept of *energy*, which allows us to make general statements about the trajectories of complementary-slackness systems. We also like to point out that "natural" structures often serve as guidelines for artificial designs; in this context reference may be made, for instance, to simulated annealing, or to the use of passivity in adaptive and nonlinear control.

In the mechanical context the study of systems subject to inequality constraints goes back to Fourier, who generalized the principle of virtual work in order to obtain equilibrium conditions for such systems (see p. 86 of [L1]). Further work in this direction was done among others by Farkas, whose results have later found widespread use in mathematical programming (see the historical survey on pp. 209–225 of [S1] for a detailed coverage). The idea of complementarity is already clear in Farkas' work, which took place at the end of the nineteenth century. Formulations of dynamic problems with inequality constraints were given in the books by Pérès [P] and Kilmister and Reeve [KR], and the well-posedness of the resulting equations was investigated by Lötstedt [L3]. In the case of electrical circuits, static problems (networks containing diodes together with resistors and voltage and current sources) have been studied extensively (see, for instance, [B1] and [VDV]) but the more involved dynamic problems appear to have drawn less attention. For the use of complementarity conditions in robotics see, for instance, [HM]. Within the bond graph modeling methodology that is suitable for a wide class of physical processes, the inclusion of inequality constraints that may alternate between active and inactive status can be realized by means of a switch element, as proposed by Strömberg [S5].

The structure of the present paper is as follows. We begin with some preliminaries on hybrid systems in general and on constrained differential equations. In Section 3 we introduce complementary-slackness systems and show how these describe a hybrid system in a very compact way. The next two sections are devoted to two special cases of special interest, namely, *linear* and *Hamiltonian* complementary-slackness systems, and we discuss the well-posedness problem for these cases. In particular we obtain sufficient conditions for well-posedness of so-called *bimodal* complementary-slackness systems. For Hamiltonian complementary-slackness systems, we moreover prove an energy inequality and give an interpretation for the law of conservation of momentum. Conclusions follow in Section 6.

2. Hybrid Systems and Differential-Algebraic Systems

It has already been noted in the Introduction that there are several ways to think about hybrid systems, which are largely similar but may put more or less emphasis on particular aspects. One possible approach is to view a hybrid system as *a family of continuous-time dynamical systems parametrized by the nodes of a transition graph.* The dynamical systems in the family may be described in a classical way by equations of the form

$$\dot{x}_i(t) = f_i(x_i(t), u(t)), \tag{2.1}$$

where $x_i(\cdot)$ is the *continuous state* attached to node i (which represents the *discrete state*) and $u(\cdot)$ represents a *continuous input.* A *continuous output* $y(\cdot)$ can be defined by adding the equation

$$y(t) = h_i(x_i(t)), \tag{2.2}$$

and a *discrete output* can be associated with each transition. The *timing* of transitions emanating from node i is determined by conditions that can be expressed in the continuous state of the system at node i and a *discrete input.* The *effect* of a transition is given by conditions that involve again the states and the inputs; in particular a new discrete state j should be specified as well as a new continuous state, which serves as an initial condition for the continuous dynamical system at node j. The nodes of the transition graph may also be referred to as the *modes,* in particular if the hybrid system in thought of as a continuous system which can be in various modes of operation (rather than as, for instance, a discrete system that is placed in a continuous environment).

In the types of hybrid systems that we are interested in, the modes of the system are connected to various algebraic constraints on the continuous states; therefore the most natural description of the continuous dynamics on the intervals between events is not the form (2.1) but rather the differential-algebraic form which uses a mixture of differential and algebraic equations. We, however, always work under conditions which ensure that these differential-algebraic equations (DAEs) may be *reduced* to equations of the form (2.1); actually we even only consider "autonomous" systems in this paper (in the sense defined below) so that there will be no input term in the reduced form. Not only is the differential-algebraic formulation more natural for the type of systems we have in mind, but, as is discussed in Section 3 of this paper, they also contain more information than the corresponding reduced forms and this extra information will be crucial in particular for the purposes of reinitialization.

We now briefly review some properties of DAEs, without striving for the greatest possible generality. In particular we only consider equations with constant coefficients and without forcing functions. A *vector DAE in fully implicit form* is a set of equations

$$f(z(t), \dot{z}(t)) = 0, \tag{2.3}$$

where f is a function from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}^N$. Often also encountered is the so-called *semiexplicit form*

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \\ 0 &= h(x(t), u(t)), \end{aligned} \tag{2.4}$$

where now f is a function from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ and h maps $\mathbb{R}^{n+k}$ to $\mathbb{R}^k$. It is clear that (2.4) may be viewed as a special form of (2.3) by identifying $z(t)$ with the vector having components $x(t)$ and $u(t)$. From a system theory perspective, it can be useful to introduce an output $y(t) = h(x(t), u(t))$ and to look at (2.4) as the "zero dynamics" of a system with state x, input u, and output y.

A *solution* of (2.3) is a continuously differentiable function $z\colon \mathbb{R} \to \mathbb{R}^N$ such that (2.3) holds for all t. We call $z_0 \in \mathbb{R}^N$ a *consistent point* of (2.3) if there is a solution $z(t)$ such that $z(0) = z_0$. It is not our purpose here to delve into the many singularities that may arise in the context of implicit systems, and in particular we always assume that the set of all consistent points forms a smooth manifold, which is denoted by $\mathscr{V}(f)$. The system (2.3) is called *autonomous* if for every point $z_0 \in \mathscr{V}(f)$ there is exactly one solution passing through z_0. (The terminology here follows that of system theory, in which an "autonomous system" is thought of as "a system with no inputs" [W2], rather than that of the theory of ordinary differential equations (ODEs). In computer-science terminology, autonomous systems might be referred to as *deterministic*.) For a simple example of a nonautonomous system, consider

$$\begin{aligned} z_1(t)\dot{z}_1(t) + z_2(t)\dot{z}_2(t) &= 0, \\ z_1^2(t) + z_2^2(t) &= 1. \end{aligned} \tag{2.5}$$

Note that the first equation is implied by the second one. Therefore, the consistent manifold for (2.5) consists of the points $z \in \mathbb{R}^2$ for which $z_1^2 + z_2^2 = 1$. The solution set of equations (2.5) consists of all trajectories $z(\cdot)$ of the form $z_1(t) = \sin(u(t) + \varphi)$, $z_2(t) = \cos(u(t) + \varphi)$ where $u(\cdot)$ is an arbitrary smooth function and φ is a constant; this representation explicitly shows that "the system has an input."

Sufficient conditions for the system (2.3) to be autonomous are provided by the various methods of reducing systems of the form (2.3) to a set of ODEs ("index reduction methods," see, for instance, [G3] and [BCP]). In the linear time-invariant case, necessary and sufficient conditions for (2.3) to be autonomous have been known for a long time and can be found, for instance, in the well-known text by Gantmacher [G2, Section XII.7]. We formulate these conditions in a way that is convenient below. Linear systems of the form (2.3) can be written as

$$E\dot{z}(t) = Az(t), \tag{2.6}$$

where E and A are matrices of size $N \times N$. In the linear case the set of consistent points forms a subspace which is denoted by $\mathscr{V}(E, A)$. A second subspace that is of interest is the space of *jump directions*. To avoid going into the technicalities of impulsive-smooth behaviors (for this see, for instance, [GS]), we here define the space of jump directions as the space spanned by the coefficients of the vector polynomials $z(s)$ that are such that $(sE - A)z(s)$ is constant. These polynomials correspond to impulsive solutions of the distributional version of the differential equation $E\dot{z} = Az$. The space of jump directions is denoted by $\mathscr{T}(E, A)$.

For the purposes of the proposition below, we define two sequences of subspaces by the following iterations (see [A2] and [K]):

$$\mathscr{V}^0 = \mathbb{R}^N, \qquad \mathscr{V}^{j+1} = A^{-1}E\mathscr{V}^j \quad (j = 0, 1, \ldots), \tag{2.7}$$

$$\mathscr{T}^0 = \{0\}, \qquad \mathscr{T}^{j+1} = E^{-1}A\mathscr{T}^j \quad (j = 0, 1, \ldots), \tag{2.8}$$

where $A^{-1}E\mathscr{V}^j$ stands for $\{z \mid Az \in E\mathscr{V}^j\}$, and similarly for $E^{-1}A\mathscr{T}^j$. Note that the first sequence is nonincreasing and the second is nondecreasing, so both must have limits.

Proposition 2.1. *Consider a system of linear algebraic and differential equations of the form* (2.6). *The subspace of consistent points of* (2.6) *can be computed as the limit of the recursion* (2.7), *so*

$$\mathscr{V}(E, A) = \lim \mathscr{V}^j(E, A). \tag{2.9}$$

The subspace of jump directions of (2.6) *can be computed as the limit of the recursion* (2.8), *so*

$$\mathscr{T}(E, A) = \lim \mathscr{T}^j(E, A). \tag{2.10}$$

Moreover, the system (2.6) *is autonomous if and only if*

$$\mathscr{V}(E, A) \oplus \mathscr{T}(E, A) = \mathbb{R}^N. \tag{2.11}$$

Proof. The statements can be inferred from the classical theory of systems of first-order linear differential equations in terms of the Kronecker canonical form [G2, Section XII.7], using the well-known connection between this canonical form and the recursions (2.7)–(2.8) (see, for instance, [D], [A3], and [S2]). ∎

The key importance of the direct-sum decomposition (2.11) will be clear later, when a change of mode will call for a projection of vectors in the space $\mathbb{R}^N$ to the consistent manifold of a new dynamics. Without the presence of a guiding complementary subspace, this projection is not well defined. Anticipating the generalization to the nonlinear case, we refer to the collection of planes parallel to $\mathscr{T}(E, A)$ as the *complementary foliation* that goes with the consistent manifold $\mathscr{V}(E, A)$ of (2.6).

3. Complementary-Slackness Systems

In this section we introduce complementary-slackness systems and show how they specify a hybrid system. To motivate the development, we begin with a simple example (see [B2]). Consider the physical system in Fig. 1. Two carts are connected

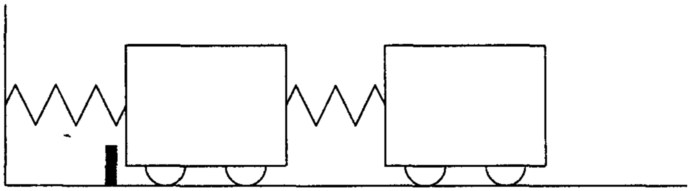

Fig. 1. Example of a complementary-slackness system.

to each other and to a fixed wall by springs. The motion of the left cart is restricted by a (purely nonelastic) stop. There are two modes, corresponding to the constraint being active or not. For simplicity, we assume that the masses of the carts are normalized to 1, that the springs are linear with spring constants equal to 1, and that the stop is placed at the equilibrium position of the left cart. The equations of motion may then be written as follows, where x_1 and x_2 denote the deviations of the left and the right cart, respectively, from their equilibrium positions, and $\lambda(t)$ represents the reaction force exerted by the stop when the constraint is active:

$$\begin{aligned} \dot{x}_1(t) &= x_3(t), \\ \dot{x}_2(t) &= x_4(t), \\ \dot{x}_3(t) &= -2x_1(t) + x_2(t) + \lambda(t), \\ \dot{x}_4(t) &= x_1(t) - x_2(t), \\ y(t) &= x_1(t), \\ u(t) &= \lambda(t), \\ y(t) \geq 0, \qquad & u(t) \geq 0, \qquad y(t)u(t) = 0. \end{aligned} \tag{3.1}$$

The two modes of the system correspond to situations in which either $y(t) = 0$ (active constraint) or $u(t) = 0$ (inactive constraint). By the physics of the system, the constraint force must be nonnegative and can only be positive if $y(t) = 0$, whereas the deviation of the left cart from its equilibrium position is always nonnegative, and the reaction force must be zero when this deviation is positive. These alternatives are expressed by the definitions of $y(t)$ and $u(t)$ and by the last line of (3.1).

Now, in general, a complementary-slackness system with n states and k side constraints is given by equations of the form

$$\begin{aligned} f(z(t), \dot{z}(t)) &= 0 \qquad (f\colon \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \to \mathbb{R}^n), \\ y(t) &= h_1(z(t)) \qquad (y(t) \in \mathbb{R}^k), \\ u(t) &= h_2(z(t)) \qquad (u(t) \in \mathbb{R}^k), \\ y(t) \geq 0, \qquad & u(t) \geq 0, \qquad y(t)^T u(t) = 0. \end{aligned} \tag{3.2}$$

The inequalities are understood componentwise. The conditions on $y(t)$ and $u(t)$ imply that for each index i and at each time t we must have either $y_i(t) = 0$ and $u_i(t) \geq 0$ or $u_i(t) = 0$ and $y_i(t) \geq 0$. Paired conditions of this type occur in optimization problems with inequality constraints and are known in this context as "complementary-slackness conditions"; the terminology is derived from saying that an inequality has "slack" if it is strict. A special form of (3.2) related to the semiexplicit form of DAEs is the following:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \qquad (f\colon \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n), \\ y(t) &= h(x(t), u(t)) \qquad (y(t) \in \mathbb{R}^k), \\ y(t) &\geq 0, \qquad u(t) \geq 0, \qquad y(t)^T u(t) = 0. \end{aligned} \tag{3.3}$$

Whereas the general formulation (3.2) treats the dual variables $y(t)$ and $u(t)$ on an equal footing, this is no longer true in the formulation above. Note in particular the double role played by $u(t)$. In (3.3) it is natural to think of $u(t)$ as an input and $y(t)$ as an output.

We now want to establish some terminology that is used throughout the rest of the paper. Write K for the index set $\{1, \ldots, k\}$. To each subset of K corresponds a particular set of DAEs, to wit

$$\begin{aligned} f(z(t), \dot{z}(t)) &= 0, \\ h_{1i}(z(t)) &= 0 \qquad (i \in I), \\ h_{2i}(z(t)) &= 0 \qquad (i \in K \backslash I). \end{aligned} \tag{3.4}$$

This is a differential-algebraic system of $n + k$ equations in $n + k$ unknowns. The *consistent manifold* of this system, in the sense of Section 2, is denoted by $\mathscr{V}_I$. The dynamics on $\mathscr{V}_I$ as defined by equations (3.4) is referred to as *mode I* of the system (3.2). The system (3.2) thus has 2^k modes. It is a standing assumption throughout this paper that each mode represents a well-defined autonomous dynamics so that for each initial point $z \in \mathscr{V}_I$ there exists a unique solution according to mode I which can be extended indefinitely.

Note that $\mathscr{V}_I$ is defined by only *equality* constraints. We also define the set of *feasible points of mode I*, denoted by $\mathscr{W}_I$, as the subset of $\mathscr{V}_I$ on which the *inequality* constraints corresponding to mode I are satisfied:

$$\mathscr{W}_I = \{z \in \mathscr{V}_I \mid h_{1i}(z) \geq 0 \ (i \in K \backslash I),\ h_{2i}(z) \geq 0 \ (i \in I)\}. \tag{3.5}$$

The possibility that $\mathscr{W}_I$ is empty for some index sets I is not *a priori* excluded. A point is said to be *feasible* for the entire system (3.2) if it is feasible for at least one of the modes of (3.2).

Consider now a point $z \in \mathscr{V}_I$. By the standing assumption mentioned above, there is a unique solution according to mode I starting at time 0 in z. Denote the point reached from z after time t in mode I by $\theta(t, z; I)$. If there is an $\varepsilon > 0$ such that $\theta(t, z; I) \in \mathscr{W}_I$ for all $t \in [0, \varepsilon]$, then we say that *smooth continuation is possible from z in mode I*. The set of such points is denoted by $\mathscr{S}_I$, so

$$\mathscr{S}_I = \{z \in \mathscr{V}_I \mid \exists \varepsilon > 0 \text{ s.t. } \forall t \in [0, \varepsilon]\ \theta(t, z; I) \in \mathscr{W}_I\}. \tag{3.6}$$

If smooth continuation in mode I is not possible, then at least one of the following two index sets is nonempty:

$$\begin{aligned}\Gamma_1(z; I) &:= \{i \in K\backslash I \,|\, \exists\varepsilon > \text{s.t. } \forall t \in (0, \varepsilon)\, (h_1(\theta(t, z; I)))_i < 0\},\\ \Gamma_2(z; I) &:= \{i \in I \,|\, \exists\varepsilon > \text{s.t. } \forall t \in (0, \varepsilon)\, (h_2(\theta(t, z; I)))_i < 0\}.\end{aligned} \tag{3.7}$$

These index sets play a role in the transition rules that we define next.

Equations (3.2) as such do not yet define a hybrid system. According to the conceptual framework of Section 2, we have to define a transition graph and associate a continuous dynamics to each node of this graph. An obvious choice is to let the *nodes* of the transition graph correspond to the *modes* of the system (3.2). In order to define *transitions* between nodes (more specifically the reinitialization rules) we need some additional information. This information is provided by a *complementary foliation* $\mathscr{T}_I$ associated with each mode I, which allows *projection* of points z onto the consistent manifold $\mathscr{V}_I$. The direct-sum decomposition (2.11) suggests that such a foliation is in some sense canonically given in the case of autonomous linear systems. We shall see below in Section 5 that in the case of Hamiltonian systems a canonical choice can also be made.

So assume now that we have equations (3.2) and that for each mode $I \subset K$ we have a foliation $\mathscr{T}_I$ that allows projection onto the consistent manifold $\mathscr{V}_I$ of mode I. We define the possible trajectories of the hybrid system associated to (3.2) and the foliations $\mathscr{T}_I$ by specifying all possible evolutions from an arbitrary initial point z, as follows:

(i) Smooth continuation from z is allowed according to any mode I such that $z \in \mathscr{S}_I$.
(ii) If no smooth continuation is possible, a jump is allowed for each I such that $z \in \mathscr{V}_I$ from z along $\mathscr{T}_J$ onto $\mathscr{V}_J$, where $J = J(z, I)$ is the index set determined by

$$J = (I\backslash\Gamma_2(z; I)) \cup \Gamma_1(z; I). \tag{3.8}$$

If a jump occurs, then from the new point $z' \in \mathscr{V}_J$ the same alternatives as above are taken into consideration. In particular, we do not exclude the possibility that a jump will again occur. Also we allow in principle multiple solutions starting from points z that belong to the consistent manifolds of more than one mode. We say that the complementary-slackness system is *well-posed* (as a closed dynamical system) if from each feasible point there exists a unique solution path starting with at most a finite number of jumps followed by a smooth continuation on an interval of positive length. We note that this is a "local" notion of well-posedness, in particular we do not discuss here the possibility of accumulation of event times ("Zeno trajectories"). Well-posedness may fail because of the following reasons:

(i) There exist points that belong to the feasible sets of two (or more) modes, smooth continuation is possible according to both modes, and the two solution paths are different.
(ii) There exist points that belong to the feasible sets of two (or more) modes, smooth continuation is possible according to neither of these modes, and the indicated jumps are different.

(iii) There exist feasible points that cause an infinite number of jumps to occur, for instance, by cycling between two points (from z one jumps to z', from z' one jumps back to z, and so on).

It is the aim of *well-posedness theorems*, several of which are presented below, to give conditions under which such obstructions do not occur.

Remark 3.1. Our formulation here is motivated by the level of generality of the "fully implicit" form (3.2). For systems written in the "semiexplicit" form (3.3), it may be tempting to let jumps take place in the space of x-variables $\mathbb{R}^n$ rather than in the space of z-variables $\mathbb{R}^{n+k}$. The relation between these two formulations for the linear case is discussed in more detail in the next section. It may be noted that the choice between the two alternatives is a classical one—it represents one of the differences between the setting used in the calculus of variations and the one used in optimal control theory.

Remark 3.2. It may seem contrived to allow a sequence of jumps, but already from simple examples it can be seen that correct physical modeling calls for this. Consider, for instance, the example of Fig. 1. We let the jumps be determined by the "linear" foliation (2.11) (which in this case coincides with the "Hamiltonian" foliation to be discussed in Section 5). The consistent manifolds for the unconstrained mode and the constrained mode are denoted by $\mathscr{V}_0$ and $\mathscr{V}_1$, respectively, and the associated foliations (or complementary subspaces in the linear case) by $\mathscr{T}_0$ and $\mathscr{T}_1$. By computation using the algorithms (2.7) and (2.8) we find

$$\mathscr{V}_0 = \ker[0 \quad 0 \quad 0 \quad 0 \quad 1], \qquad \mathscr{V}_1 = \ker\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathscr{T}_0 = \operatorname{im}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathscr{T}_1 = \operatorname{im}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Take an initial point $z = (x_1, \ldots, x_4, \lambda)$ with $x_1 = 0$, $x_2 > 0$, $x_3 < 0$, $\lambda = 0$; this corresponds to a situation in which the left block hits the stop at a time when the right block is to the right of its equilibrium position. Note that z belongs to $\mathscr{V}_0$ but not to $\mathscr{V}_1$. Smooth continuation in the unconstrained mode is obviously not possible and so a jump will occur along $\mathscr{T}_1$ to $\mathscr{V}_1$. This produces a point z' with coordinates $z' = (0, x_2, 0, x_4, -x_2)$. In particular the λ-coordinate will be negative and so although z' belongs to $\mathscr{V}_1$ it is not even feasible for the constrained mode, and smooth continuation in this mode is therefore not possible. Note that z' does not belong to $\mathscr{V}_0$ because its final component is nonzero. We must now jump from z' along $\mathscr{T}_0$ to $\mathscr{V}_0$; a new point z'' is produced with coordinates $z'' = (0, x_2, 0, x_4, 0)$. From this point, smooth continuation is possible in the unconstrained mode. The solution that is obtained in this way corresponds to the physical insight which tells us that if the right block is to the right of its equilibrium position at the moment at which the left block hits the stop, it will immediately pull the left block away from the stop again. Another example of such a situation, in which certain constraints force a state jump but do not actually become active, is provided in Example 5.3.

Example 3.3. Consider an electrical network consisting of linear resistors, capacitors, inductors, transformers, and gyrators, and of k diodes. Replacing the diodes first by ports, equations for the network can be written in the "hybrid" (the term is overworked here) form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.9}$$

where u_i denotes either current or voltage at the ith port, and y_i denotes voltage or current at the ith port accordingly. Connecting the diodes will produce equations $u_i = -V_i$ and $y_i = I_i$ for voltage-controlled ports and equations $u_i = I_i$, $y_i = -V_i$ for current-controlled ports. By finally adding the ideal diode characteristics

$$V_i \leq 0, \qquad I_i \geq 0, \qquad V_i I_i = 0, \tag{3.10}$$

a set of equations in the semiexplicit form (3.3) is obtained. Complementary-slackness systems whose modes are linear, as in this example, are studied further in the next section.

Remark 3.4. In our formulation of the dynamics of a complementary-slackness system we have taken care not to give any independent status to the discrete state; in other words, we do not assume that the physical system "knows" which mode it is in. This formulation has been chosen on the basis of physical considerations. For instance, we believe that it would not be reasonable from a physical point of view to include in the initial conditions for an electrical circuit with diodes, any information as to which diodes are initially conducting or not; this information should be able to be derived from the continuous state components (assuming there are no hysteretic effects). From a mathematical point of view, however, there is in principle no objection against an alternative formulation in which the system does know which mode it is in. Such a formulation would lead to a simpler formulation of the dynamics. It will be a consequence of the well-posedness theorems to be proved below that the systems considered in these theorems behave *as if* they follow the alternative (mathematical) model; note however that these theorems are proved on the basis of the weaker (physical) assumptions. The well-posedness theorems may therefore be seen as a justification of a simulation methodology that chooses the initial discrete state consistently with the initial continuous state, but further on treats the discrete state as additional information.

Of course, in a general hybrid system formulation it will be desired to allow situations in which the discrete state does have an independent status. For instance, in the description of hysteretic systems it would be natural to use a mixture of discrete and continuous state variables in such a way that the discrete state is not completely determined by the current values of the continuous state variables.

On the other hand, one may expect to find situations in which the discrete state is completely determined by the continuous state not only in the type of physical systems discussed in this paper, but also for example in closed-loop systems that consist of a continuous process with a discrete controller.

4. Linear Complementary-Slackness Systems

In this section we study systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t) \in \mathbb{R}^n, \\ y(t) &= Cx(t) + Du(t), \qquad y(t) \in \mathbb{R}^k, \quad u(t) \in \mathbb{R}^k, \\ & y(t) \geq 0, \qquad u(t) \geq 0, \qquad y(t)^T u(t) = 0. \end{aligned} \tag{4.1}$$

This is a linear version of the semiexplicit form (3.3). Of course it would also be possible to consider a linear version of the fully implicit form (3.2), but we do not do that here. The vector $x(t)$ is referred to as a *state vector.*

The system (4.1) has 2^k modes. Letting $K = \{1, 2, \ldots, k\}$ as usual, each mode corresponds to a subset $I \subset K$ by the requirements $y_i = 0$ $(i \in I)$, $u_i = 0$ $(i \notin I)$. The dynamics in mode I can be characterized as follows. Define matrices $\hat{C}_I$ and $\hat{D}_I$ by

$$\begin{aligned} i\text{th row of } \hat{C}_I &= i\text{th row of } C \qquad \text{if} \quad i \in I \\ &= 0 \qquad\qquad\qquad\quad\ \text{if} \quad i \notin I \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} i\text{th row of } \hat{D}_I &= i\text{th row of } D \qquad \text{if} \quad i \in I \\ &= i\text{th row of } I_k \qquad \text{if} \quad i \notin I. \end{aligned} \tag{4.3}$$

Also define

$$E = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_I = \begin{bmatrix} A & B \\ \hat{C}_I & \hat{D}_I \end{bmatrix}, \qquad z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{4.4}$$

The dynamics in mode I is now given by

$$E\dot{z}(t) = F_I z(t). \tag{4.5}$$

We always assume in this paper that all modes are autonomous. We then *define* an (autonomous) linear complementary-slackness system by the dynamics (4.1) together with the transition rules as specified in Section 3 with projections determined by the following pairs of complementary subspaces (see Proposition 2.1):

$$\mathscr{V}_I := \mathscr{V}(E, F_I), \qquad \mathscr{T}_I := \mathscr{T}(E, F_I). \tag{4.6}$$

The special form of E in (4.4), which is of course due to the semiexplicit nature of (4.1), suggests a description of the consistent manifold and the complementary foliation based on subspace algorithms that take place in $\mathbb{R}^n$ rather than $\mathbb{R}^{n+k}$. For each $I \subset K$, define a sequence of subspaces of $\mathbb{R}^n$ by

$$\begin{aligned} V_I^0 &= \mathbb{R}^n, \\ V_I^{k+1} &= \{x \mid \exists u\colon Ax + Bu \in V_I^k,\ \hat{C}_I x + \hat{D}_I u = 0\}. \end{aligned} \tag{4.7}$$

This is a nonincreasing sequence of subspaces which therefore must reach a limit in a finite number of steps; the limit is denoted by $\mathscr{V}_I$. It is readily verified that

$$\mathscr{V}(E, F_I) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| x \in V_I,\ Ax + Bu \in V_I,\ \hat{C}_I x + \hat{D}_I u = 0 \right\}, \tag{4.8}$$

where $\mathscr{V}(E, F_I)$ is as defined in (2.7) and (2.9). Likewise, we have

$$\mathscr{T}(E, F_I) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| x \in T_I \right\}, \tag{4.9}$$

where T_I is the limit of the nondecreasing sequence of subspaces of $\mathbb{R}^n$ defined by

$$\begin{aligned} T_I^0 &= \{0\}, \\ T_I^{k+1} &= \{x \mid \exists \tilde{u}, \exists \tilde{x} \in T_I^k\colon x = A\tilde{x} + B\tilde{u},\ \hat{C}_I \tilde{x} + \hat{D}_I \tilde{u} = 0\}. \end{aligned} \tag{4.10}$$

The significance of the subspaces V_I and T_I can be described as follows. The dynamics in mode I is given by the differential-algebraic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ 0 &= \hat{C}_I x(t) + \hat{D}_I u(t). \end{aligned} \tag{4.11}$$

An element $x_0 \in \mathbb{R}^n$ is called a *consistent state for mode I* if there exists a smooth solution $(x(\cdot), u(\cdot))$ of the above system such that $x(0) = x_0$. It is not difficult to verify (see Proposition 2.1) that the set of consistent states is given by V_I; note in particular that, by construction of V_I, there exists for every $x \in V_I$ a vector such that $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{V}_I$. From (4.9) it is obvious that the subspace T_I contains precisely the x-components of the jump directions of the system (4.5). Again using Proposition 2.1, it is verified that the dynamics in mode I is autonomous if and only if V_I and T_I are complementary; for this, note that all vectors of the form $\begin{bmatrix} 0 \\ u \end{bmatrix}$ belong to $\mathscr{T}(E, F_I)$. In this case it follows that to each $x \in V_I$ there exists a *unique* u such that $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{V}_I$, so the condition $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{V}_I$ implicitly defines u as a (linear) function of x on V_I. Moreover, projecting a vector $\begin{bmatrix} x \\ u \end{bmatrix}$ along $\mathscr{T}_I$ onto $\mathscr{V}_I$ comes down to projecting x along T_I onto V_I, and letting u be determined by the requirement $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathscr{V}_I$.

Already in the case of linear complementary-slackness systems, solutions may be nonunique or may fail to exist. This is shown in the examples below.

Example 4.1. Consider the complementary-slackness system given by the following equations:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
\dot{x}_2(t) &= x_3(t),\\
\dot{x}_3(t) &= -u(t),\\
y(t) &= x_1(t) + x_2(t) + x_3(t),\\
y(t) \geq 0, \qquad u(t) &\geq 0, \qquad y(t)u(t) = 0.
\end{aligned}
\tag{4.12}
$$

This system has two modes, one in which $u(t) = 0$ (we call this mode 0) and one in which $y(t) = 0$ (mode 1). The set of consistent points for mode 0 is easily seen to consist of all points $\begin{bmatrix} x \\ u \end{bmatrix}$ whose u-component vanishes, and the dynamics in this mode is given by

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
\dot{x}_2(t) &= x_3(t),\\
\dot{x}_3(t) &= 0,\\
y(t) &= x_1(t) + x_2(t) + x_3(t)\\
u(t) &= 0.
\end{aligned}
\tag{4.13}
$$

In mode 1 we have the constraint $x_1 + x_2 + x_3 = 0$ which by differentiation leads to the constraint $x_2 + x_3 - u = 0$. These two equations define the set of consistent points for mode 1, as can be verified by means of Proposition 2.1. The consistent set can, for instance, be parametrized by the coordinates x_1 and x_2, and in terms of these coordinates the dynamics in mode 1 is given by

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
\dot{x}_2(t) &= -x_1(t) - x_2(t),\\
y(t) &= 0,\\
u(t) &= -x_1(t).
\end{aligned}
\tag{4.14}
$$

Now consider the initial data $(x_1(0), x_2(0), x_3(0), u(0)) = (0, 1, -1, 0)$. This point is consistent in both modes, so we have to check the inequality constraints to see in which of these modes smooth continuation is possible. In mode 0 we have

$$
\begin{aligned}
y(0) &= 0,\\
\dot{y}(0) &= x_2(0) + x_3(0) = 0,
\end{aligned}
$$

and

$$
\ddot{y}(0) = x_3(0) = -1,
$$

so that, if we follow the dynamics of mode 0, $y(t)$ would be negative for $t > 0$, which is not allowed. On the other hand, mode 1 produces

$$
u(0) = 0 \quad \text{and} \quad \dot{u}(0) = -x_2(t) = -1,
$$

which again leads to a violation of the inequality constraints. We must conclude that smooth continuation is possible in neither of the two modes. On the other hand, since the given initial condition is consistent for both modes, a jump will only take the initial point into itself and so we have a case of period-1 cycling. It might also be said that this system exhibits *deadlock.*

Example 4.2. Consider now the same example as above, but with the initial condition $(x_1(0), x_2(0), x_3(0), u(0)) = (0, -1, 1, 0)$, which is sign reversed with respect to the one considered above. Doing the same calculations as in the previous example, we now find that smooth continuation is possible *both* in mode 0 and in mode 1; moreover, these modes clearly produce different solutions. So whereas the previous example showed a case of *nonexistence* of solutions, here we have a case of *nonuniqueness* of solutions.

Remark 4.3. It is of interest to consider the above examples also in reverse time. It is easily calculated that from the initial condition of Example 4.1 there are *two* solutions for $t < 0$, while there is *no* solution emanating in negative time from the initial condition in Example 4.2.

So the question arises: under what conditions is a linear complementary-slackness system well-posed in the sense that there is a unique solution starting from each feasible point? We provide an answer to this question only for the case of systems with a single constraint ($k = 1$). We return to the case of systems with multiple constraints at the end of the section. We now first present some preparatory material.

For brevity, complementary-slackness systems with a single constraint are referred to as *bimodal* systems. With an obvious terminology, the two modes of a system of the specific form (3.3) are referred to as the *input-constrained mode* ($u(t) = 0$) and the *output-constrained mode* ($y(t) = 0$). These modes are also denoted as mode 0 and mode 1, respectively. A bimodal system is said to be *degenerate* if the state/input/output trajectories of one of the modes form a subset of the state/input/output trajectories of the other mode. In other words, a degenerate bimodal system is one that does not really have two distinct modes. For an example of such a system, consider the equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= u(t), \\ y(t) &= x_1(t), \\ y(t) \geq 0, \qquad u(t) &\geq 0, \qquad y(t)u(t) = 0. \end{aligned} \tag{4.15}$$

The output-constrained mode allows only the zero trajectory for states, inputs, and outputs; but this trajectory is of course also allowed by the input-constrained mode.

In the following lemma we record some basic facts about linear bimodal systems.

Lemma 4.4. *A bimodal system of the form* (4.1) *is autonomous if and only if the transfer function* $g(s) := D + C(sI - A)^{-1}B$ *is nonzero, or, equivalently, if the Markov parameters defined by*

$$g_0 := D, \qquad g_j := CA^{j-1}B \quad (j \geq 1) \tag{4.16}$$

do not all vanish. Assuming this, let κ *be the "relative degree" defined by*

$$g_j = 0 \quad (0 \leq j < \kappa), \qquad g_\kappa \neq 0. \tag{4.17}$$

We then have

$$V_0 = \mathbb{R}^n, \qquad T_0 = \{0\}, \qquad \mathscr{V}_0 = V_0 \times \{0\}, \qquad \mathscr{T}_0 = T_0 \times \mathbb{R}, \tag{4.18}$$

and

$$\begin{aligned} V_1 &= \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\kappa-1} \end{bmatrix}, \qquad T_1 = \operatorname{im}[B \quad AB \quad \cdots \quad A^{\kappa-1}B], \\ \mathscr{V}_1 &= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| \, x \in V_1, u = -g_\kappa^{-1}CA^k x \right\}, \qquad \mathscr{T}_1 = T_1 \times \mathbb{R}, \end{aligned} \tag{4.19}$$

where the indices 0 *and* 1 *refer to the input-constrained mode and the output-constrained mode, respectively. The dynamics in mode* 0 *and mode* 1 *are given by*

$$\dot{x}(t) = Ax(t), \qquad y(t) = Cx(t), \qquad u(t) = 0, \tag{4.20}$$

and

$$\dot{x}(t) = (A - g_\kappa^{-1}BCA^\kappa)x(t), \qquad u(t) = -g_\kappa^{-1}CA^\kappa x(t), \qquad y(t) = 0, \tag{4.21}$$

respectively.

Proof. As is well known, in a neighborhood of infinity we have

$$g(s) = \sum_{j=0}^{\infty} g_j s^{-j}, \tag{4.22}$$

so that the transfer function is indeed nonzero if and only if not all Markov parameters vanish. First assume that this holds; then (4.18)–(4.21) are readily verified and in particular it is clear that the system is autonomous. On the other hand, if the transfer function is zero, then starting from the initial state $x(0) = 0$ any input may be applied while the output $y(t)$ will remain at zero for all time, and so the output-constrained mode is nonautonomous. ∎

The coefficient g_κ appearing in (4.17) is referred to as the *leading Markov parameter* of the system (4.1). By a standard convention, in case $\kappa = 0$ the formulas in (4.19) mean that $V_1 = \mathbb{R}^n$ and $T_1 = \{0\}$. For $\kappa \geq 1$, we may write $A - g_\kappa^{-1}BCA^\kappa = PA$ where

$$P = I - g_\kappa^{-1}BCA^{\kappa-1} \tag{4.23}$$

is the projection along im B onto ker $CA^{\kappa-1}$.

We next give a characterization of nondegeneracy for bimodal systems.

Lemma 4.5. *An autonomous bimodal system of the form* (4.1) *is degenerate if and only if the set of consistent states for the output-constrained mode coincides with the set of unobservable points for the pair* (C, A).

Proof. Degeneracy will occur either when $y(\cdot) = 0$ implies $u(\cdot) = 0$ or vice versa. The latter case is a rather trivial one; it can occur only when $C = 0$ and the transfer function is a nonzero constant. In this case V_1 is the whole state space $\mathbb{R}^n$ and it does coincide with the space of unobservable points. We now consider the other possibility, that $y(\cdot) = 0$ implies $u(\cdot) = 0$. From (4.19) and (4.21) we see that degeneracy occurs if and only if $Cx = 0, \ldots, CA^{\kappa-1}x = 0$ implies that $CA^{\kappa}x = 0$ also. An equivalent formulation is that $x \in V_1$ should imply that $Ax \in V_1$ also. If this holds, then V_1 is an A-invariant subspace of $\ker C$ and so all points in V_1 are unobservable. Since the reverse inclusion holds in general, the proof of the lemma is complete. ∎

Remark 4.6. In the situation in which the pair (C, A) is observable, the condition for nondegeneracy comes down to $V_1 \neq \{0\}$. The triviality of V_1 corresponds to the situation in which the transfer function $g(s)$ is of the form $g(s) = (p(s))^{-1}$ where $p(s)$ is a polynomial. For instance, in the example (4.15) we have $g(s) = s^{-2}$.

In the proof of the theorem below we use the "lexicographic inequality" for vectors which is defined as follows: if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then $x \prec y$ if there is an index $i \in \{1, \ldots, n\}$ such that $x_j = y_j$ for all $j < i$, and $x_i < y_i$. We use $x \preceq y$ for the situation in which either $x \prec y$ or $x = y$; the expressions $x \succ y$ and $x \succeq y$ have the obvious meanings. We also need the following lemma; the simple proof is omitted.

Lemma 4.7. *If L is a lower triangular matrix with positive diagonal elements, then $x \succ 0$ if and only if $Lx \succ 0$ and $x \prec 0$ if and only if $Lx \prec 0$.*

In other words the lemma states that lower triangular matrices with positive diagonal elements are lexicographically sign-preserving. We now come to the main result of this section.

Theorem 4.8. *An autonomous bimodal system of the form* (4.1) *is well-posed as a closed dynamical system if its leading Markov parameter is positive. For nondegenerate systems without feedthrough* ($D = 0$), *this condition is also necessary.*

Proof. Let κ be the index of the leading Markov parameter. For brevity of notation, define

$$C_1 = -g_\kappa^{-1}CA^{\kappa}, \qquad A_1 = A - g_\kappa^{-1}BCA^{\kappa}. \tag{4.24}$$

Introduce the observability matrices of mode 0 and mode 1:

$$\mathcal{O}_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad \mathcal{O}_1 = \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix}. \tag{4.25}$$

Note that, by the Cayley–Hamilton theorem, $\mathcal{O}_0 x = 0$ implies $CA^j x = 0$ for all $j \geq 0$ and likewise for mode 1. The sets of consistent points of mode 0 and of mode 1 are denoted by $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, whereas the sets of consistent states are written as V_0 and V_1.

Now assume that the leading Markov parameter is positive; we have to show that there exists a unique solution from each feasible point, consisting of smooth continuation after at most a finite number of jumps. We distinguish a number of cases, starting from a given point $\begin{bmatrix} x \\ u \end{bmatrix}$.

1. $x \notin V_1$. This condition excludes smooth continuation in mode 1. We distinguish between two possibilities:

1.1. $\mathcal{O}_0 x \succeq 0$. Smooth continuation in mode 0 is the only option and leads to a unique solution.

1.2. $\mathcal{O}_0 x \prec 0$. In this case a jump to mode 1 must occur. After the jump we have a new initial point which belongs to $\mathcal{V}_1$ and so we end up in case 2 which is treated next.

2. $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{V}_1$. Now we consider three possible situations.

2.1. $\mathcal{O}_0 x \prec 0$. Under this condition smooth continuation in mode 0 is excluded. We have to determine the lexicographic sign of $\mathcal{O}_1 x$ to see whether we can have continuation in mode 1. It is easily shown by induction that, for all $j \geq 0$,

$$C_1 A_1^j = -g_\kappa^{-1} CA^{\kappa+j} + \sum_{i=0}^{j-1} \gamma_{ji} CA^{\kappa+i}, \tag{4.26}$$

where the γ_{ji} are real coefficients whose exact values are irrelevant to the proof. It follows that

$$\mathcal{O}_1 x = \begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} x = -g_\kappa^{-1} L \begin{bmatrix} CA^\kappa \\ CA^{\kappa+1} \\ \vdots \\ CA^{\kappa+n-1} \end{bmatrix} x, \tag{4.27}$$

where L is a lower triangular matrix having 1's on its diagonal. Because $x \in V_1$ we have $CA^j x = 0$ for $j = 0, \ldots, \kappa - 1$, and so it follows from the above relation and Lemma 4.7, together with the assumption $g_\kappa > 0$, that $\mathcal{O}_1 x \succ 0$. Therefore smooth continuation in mode 1 is the only available option and it provides a unique solution.

2.2. $\mathcal{O}_0 x \succ 0$. By the same reasoning as above we find that $\mathcal{O}_1 x$ must be lexicographically negative so that smooth continuation in mode 1 is not possible. It also follows that $u \leq 0$. Distinguish two cases:

2.2.1. $u = 0$. We are in a point that is consistent for mode 0 and smooth continuation in this mode provides the unique solution.

2.2.2. $u < 0$. In this case we must jump along $\mathcal{T}_0$ to $\mathcal{V}_0$. Because $\mathcal{T}_0$ consists of all vectors of the form $\begin{bmatrix} 0 \\ u \end{bmatrix}$, the jump does not affect the value of x and we end up in case 2.2.1.

2.3. $\mathcal{O}_0 x = 0$. With $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{V}_1$, this implies $u = 0$. In this case it follows from the relation (4.27) that $\mathcal{O}_1 x = 0$ also and so smooth continuation is possible *both* in mode 0 and in mode 1. Note however that $\ker \mathcal{O}_0 \cap \ker \mathcal{O}_1 = \ker \mathcal{O}_0$ is both A- and A_1-invariant, and that the restrictions of A and A_1 to this subspace are identical. This implies that, for initial points in $\ker \mathcal{O}_0$, the trajectory according to mode 0 as produced by (4.20) coincides with the trajectory according to mode 1 following (4.21). We conclude that even in this case there is a unique solution.

This concludes the sufficiency part of the proof. For the necessity part, assume that the leading Markov parameter is negative. By the assumption of nondegeneracy and by Lemma 4.4 there exists a vector $x \in V_1$ such that $\mathcal{O}_0 x \neq 0$, and so (by changing sign if necessary) we can find a vector $x \in V_1$ such that $\mathcal{O}_0 x \prec 0$. Because $g_\kappa < 0$, the reasoning applied under case 2.1 of the sufficiency part implies here that $\mathcal{O}_1 x \prec 0$ also. Suppose now that we consider the initial point

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{V}_0;$$

this point is feasible for mode 0 because $x \in V_1 \subset \ker C$ by the assumption $D = 0$. Projecting the given point to $\mathcal{V}_1$ and back will only change u_0 to $-g_\kappa^{-1} C A^\kappa x$ and back to 0, whereas the state component will remain the same. Because both $\mathcal{O}_0 x$ and $\mathcal{O}_1 x$ are lexicographically negative, smooth continuation never becomes possible. ∎

Remark 4.9. The proof shows that, under the conditions stated, there is actually unique continuation from every *consistent* point (rather than just each feasible point). We shall see below (see Remark 5.8) that for nonlinear Hamiltonian complementary-slackness systems it is *not* possible in general to define a solution starting from an arbitrary consistent point.

Remark 4.10. An example of a nondegenerate bimodal system that is well-posed in the sense that we defined although its leading Markov parameter is negative is provided by the following equations: $\dot{x}(t) = x(t) + u(t)$, $y(t) = x(t) - u(t)$.

Example 4.11. Consider again the electrical network equations of Example 3.3. If the representation is minimal and the network is passive, then there exists a symmetric matrix $Q > 0$ such that

$$\begin{bmatrix} A^T Q + QA & QB - C^T \\ B^T Q - C & -(D + D^T) \end{bmatrix} \leq 0 \tag{4.28}$$

(see [W1]). Consider now a passive network with one diode, so that in particular D is a scalar. It follows from the above equation that $D \geq 0$. Moreover, if $D = 0$ the equation implies that $C = B^TQ$ and so in this case we have $CB = B^TQB > 0$. In either case the leading Markov parameter is positive and hence the above theorem guarantees that the complementary-slackness equations are well-posed. Clearly, this is only a partly satisfactory result since it should be possible to prove that passive networks with more than one diode also give rise to well-posed equations.

The proof technique that we used above is not well suited to systems with more than one constraint. For such systems, the search for a mode in which smooth continuation is possible leads to a dynamic variant of the *linear complementarity problem* (LCP) that has been extensively studied in mathematical programming (see, for instance, [CPS]). Treating the dynamic LCP requires a substantial further development that will be taken up in forthcoming work. Here we just give a proposition for systems with multiple constraints that can be proved easily, but that already shows to some extent the role of the principal minors which also figure prominently in the theory of the LCP. Recall that the *principal minors* of a square matrix M are the determinants of the submatrices of M that are obtained by selecting rows and columns with the same indices [G1, p. 2].

Proposition 4.12. *Consider a complementary-slackness system given by* (4.1). *We have $V_I = \mathbb{R}^n$ for all $I \subset K$ (i.e., all states are consistent in all modes) if and only if all principal minors of the matrix D are nonzero.*

Proof. It follows immediately from (4.7) that $V_I = \mathbb{R}^n$ if and only if the matrix $\hat{D}_I$ defined in (4.3) is nonsingular. Requiring that $\hat{D}_I$ is nonsingular for all $I \subset K$ is, by a standard matrix argument, the same as requiring that all principal minors of D are nonzero. ■

5. Hamiltonian Complementary-Slackness Systems

In this section we consider complementary-slackness systems arising from Hamiltonian systems with geometric inequality constraints. Consider a conservative mechanical system, with n degrees of freedom $q_1, \ldots, q_n$ and total energy

$$\tfrac{1}{2}\dot{q}^T M(q)\dot{q} + V(q), \qquad M(q) > 0, \tag{5.1}$$

where $\frac{1}{2}\dot{q}^T M(q)\dot{q}$ is the kinetic energy corresponding the generalized mass matrix $M(q)$ and where $V(q)$ denotes the potential energy. In general, $q = (q_1, \ldots, q_n)$ gives local coordinates for an n-dimensional configuration manifold Q. The Hamiltonian equations of motion are obtained by defining the generalized momenta

$$p = M(q)\dot{q} \tag{5.2}$$

and are given as (with $\partial H/\partial p$ and $\partial H/\partial q$ denoting column vectors of partial derivatives)

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p), \end{aligned} \tag{5.3}$$

where the Hamiltonian $H(q, p)$ is the total energy expressed in the state variables $x = (q, p)$, i.e.,

$$H(q, p) = \tfrac{1}{2}p^T P(q)p + V(q), \qquad P(q) = M^{-1}(q) > 0. \tag{5.4}$$

The vector $x = (q, p)$ gives canonical local coordinates for the cotangent bundle T^*Q, with the classical Poisson bracket (F and G being functions on T^*Q)

$$\{F(q, p), G(q, p)\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right)(q, p). \tag{5.5}$$

(Coordinates are called "canonical" if $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0$, $i, j = 1, \dots, n$.) The Hamiltonian equations of motion (5.3) on $\mathscr{X} = T^*Q$ will also be succinctly written as

$$\dot{x} = X_H(x), \qquad x = (q, p) \in T^*Q, \tag{5.6}$$

where X_H is called a Hamiltonian vector field on T^*Q. For any function $F\colon T^*Q \to \mathbb{R}$ we have the following identity concerning the derivative of F along the Hamiltonian dynamics (5.6):

$$\frac{dF}{dt} = L_{X_H}F = \{H, F\}, \tag{5.7}$$

where L_{X_H} denotes the Lie derivative along X_H.

Now suppose k geometric constraints are imposed on the system, that is,

$$C(q) = 0, \qquad C\colon Q \to \mathbb{R}^k. \tag{5.8}$$

Then the constrained mechanical system is described by the constrained Hamiltonian equations of motion, see, e.g., [L1]:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + \frac{\partial C^T}{\partial q}(q)\lambda, \qquad \lambda \in \mathbb{R}^k, \\ C(q) &= 0, \end{aligned} \tag{5.9}$$

where $(\partial C^T/\partial q)(q)$ denotes the $n \times k$ matrix with the jth column given by $(\partial C_j/\partial q)(q)$, and where $\lambda(t) \in \mathbb{R}^k$ are the *constraint forces* needed to satisfy the geometric constraints $C(q(t)) = 0$ for all t (or the constraint forces *resulting* from imposing the geometric constraints).

By considering geometric *inequality* constraints $C_i(q) \geq 0$, $i = 1, \dots, k$ (as in the first example considered in Section 4), this leads to the equations of a *Hamiltonian complementary-slackness system*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + \frac{\partial C^T}{\partial q}(q)\lambda, \\ y &= C(q), \\ u &= \lambda, \\ y(t) \geq 0, &\qquad u(t) \geq 0, \qquad y^T(t)u(t) = 0, \end{aligned} \tag{5.10}$$

where the inequalities are understood componentwise as in (3.2). In fact, the inequalities $u_i = \lambda_i \geq 0$, $i \in K$, have the following physical interpretation. The constraint force λ_i is the generalized force corresponding to the generalized velocity $\dot{y}_i$. Since in the case of inequality constraints $y_i \geq 0$ the constraint forces will be always pushing in the direction of rendering y_i nonnegative this implies that $\lambda_i \geq 0$. For brevity, see (5.6), we also write (5.10) as

$$\begin{aligned} \dot{x} &= X_H(x) - X_C(q)\lambda, \qquad x = (q, p) \in T^*Q, \\ y &= C(q), \\ u &= \lambda, \\ & \qquad y \geq 0, \qquad u \geq 0, \qquad y^T u = 0. \end{aligned} \tag{5.11}$$

Also for brevity we denote for every $I \subset K$ the vector with components y_i, $i \in I$, by y_I, and the map with components C_i for $i \in I$, by C_I. Furthermore, we denote the vector with components λ_i, $i \in I$, by λ_I.

Throughout we assume that the geometric constraints $C_i(q)$, $i \in K$, are *independent*, i.e., the following assumption holds.

Assumption 5.1. For all subsets $I \subset K$,

$$\operatorname{rank} \frac{\partial C_I^T}{\partial q}(q) = \text{cardinality of the set } I, \tag{5.12}$$

for all q with $C_I(q) = 0$.

(Note that (5.12) is implied by, but not equivalent to, the condition rank $(\partial C^T/\partial q)(q) = k$ for all q.)

Each subset I of $K = \{1, \dots, k\}$ corresponds to a "mode" of the Hamiltonian complementary-slackness system, given by (5.10) with $y_i = 0$, $i \in I$, and $u_i = 0$, $i \in K \backslash I$. So for any $I \subset K$ we can consider the constrained system

$$\begin{aligned} \dot{x} &= X_H(x) - X_{C_I}(q)\lambda_I, \\ 0 &= y_I = C_I(q). \end{aligned} \tag{5.13}$$

The set of consistent states and the corresponding constraint forces are computed by differentiating y_I along (5.13):

$$\begin{aligned} y_I &= C_{I(q)} = 0, \\ \dot{y}_I &= L_{X_H - X_{C_I}\lambda_I}(C_I) = \{H, C_I\}(q, p) = \left[\frac{\partial C_I^T}{\partial q}(q)\right]^T P(q)p = 0, \\ \ddot{y}_I &= L_{X_H - X_{C_I}\lambda_I}\{H, C_I\} = \{H, \{H, C_I\}\}(q, p) - \{C_I^T, \{H, C_I\}\}\lambda_I = 0. \end{aligned} \tag{5.14}$$

(Here $\{H, C_I\}$ denotes the column-vector with components $\{H, C_i\}$, $i \in I$; similarly for $\{H, \{H, C_I\}\}$ and the $k \times k$ *matrix* $\{C_I^T, \{H, C_I\}\}$.) By Assumption 5.1 and the fact that $P(q) > 0$, we have

$$-\{C_I^T, \{H, C_I\}\}(q, p) = \left[\frac{\partial C_I^T}{\partial q}(q)\right]^T P(q) \frac{\partial C_I^T}{\partial q}(q) =: R_I(q) > 0 \tag{5.15}$$

for all q with $C_I(q) = 0$. So the constrained state space (set of consistent states) for mode I is given as

$$V_I = \left\{(q, p) \in T^*Q \,|\, C_I(q) = 0, \left[\frac{\partial C_I^T}{\partial q}(q)\right]^T P(q)p = 0\right\} \tag{5.16}$$

and the following constraint force, uniquely determined as

$$\lambda_I^c(q, p) := -R_I^{-1}(q)\{H, \{H, C_I\}\}(q, p), \tag{5.17}$$

renders V_I invariant (since $\ddot{y}_I$ is kept equal to zero). (Note that for $I = \varnothing$ we have $V_\varnothing = \mathscr{X} = T^*Q$ and $\lambda^c_\varnothing = 0$.) We thus arrive at the following conclusion.

Proposition 5.2. *Let Assumption* 5.1 *be satisfied. Then for all $I \subset K$ the differential-algebraic system* (5.13) *is autonomous and*

$$\mathscr{V}_I = \{(q, p, \lambda) \,|\, (q, p) \in V_I, \lambda_{K\setminus I} = 0, \lambda_I = \lambda_I^c(q, p)\}, \tag{5.18}$$

$$\mathscr{W}_I = \{(q, p, \lambda) \in V_I \,|\, C_{K\setminus I} \geq 0, \lambda_I^c(q, p) \geq 0\}, \tag{5.19}$$

where V_I is given by (5.16) *and $\lambda_I^c(q, p)$ by* (5.17).

Example 5.3. Consider a pendulum with a massless rope of length 1, with a unit mass attached to the end of the rope. Set $g = 1$. The Hamiltonian (total energy) is

$$H(q_1, q_2, p_1, p_2) = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + q_2, \tag{5.20}$$

where $(q_1, q_2) \in Q = \mathbb{R}^2$ are the cartesian coordinates of the unit mass. The (single) geometric inequality constraint is

$$C(q) = 1 - q_1^2 - q_2^2 \geq 0. \tag{5.21}$$

Thus I is either $\varnothing$ or the singleton $K = \{1\}$. Trivially $V_\varnothing = T^*Q$. Clearly, Assumption 5.1 is satisfied, implying that

$$V_K = \{(q_1, q_2, p_1, p_2) \,|\, q_1^2 + q_2^2 = 1, p_1q_1 + p_2q_2 = 0\}. \tag{5.22}$$

Furthermore, for all (q_1, q_2) such that $C(q) = 0$ we have $R_K(q) = 4(q_1^2 + q_2^2) = 4 > 0$. Finally, the constraint force which renders V_K invariant is given as

$$\lambda_K^c = \tfrac{1}{2}(p_1^2 + p_2^2 - q_2). \tag{5.23}$$

It is an important fact (provided Assumption 5.1 holds) that the dynamics of the autonomous differential algebraic systems (5.13) is *Hamiltonian* for every $I \subset K$. So not only the mode corresponding to $I = \varnothing$ is Hamiltonian, but every other mode is Hamiltonian as well. Indeed, we recall, for example, from Chapter 12 of [NS], that the dynamics (5.13) for any I is also given as the dynamics

$$\dot{x} = \{H - C_I^T\lambda_I^c, x\} \tag{5.24}$$

restricted to V_I; in fact, the Hamiltonian dynamics (5.24) leaves V_I invariant. Furthermore, the dynamics (5.13) for $I \neq \varnothing$ on V_I is also given as the Hamiltonian dynamics

$$\dot{x}_I = \{H_I, x_I\}_I, \qquad x_I \in V_I, \tag{5.25}$$

where x_I are local coordinates for V_I, $H_I\colon V_I \to \mathbb{R}$ is the Hamiltonian $H\colon T^*Q \to \mathbb{R}$ *restricted* to V_I, and $\{\ ,\ \}_I$ is the so-called *Dirac bracket* on V_I, given as

$$\{F, G\}_I = \{F, G\} + \{F, C_I^T\}R_I^{-1}(q)\{\{H, C_I\}, G\} - \{F, \{H, C_I\}\}^T R_I^{-1}(q)\{C_I, G\} \tag{5.26}$$

for any functions F, $G\colon V_I \to \mathbb{R}$, arbitrarily extended to functions on T^*Q. (Note that if we define for completeness $\{\ ,\ \}_\varnothing$ as the original Poisson bracket on $V_\varnothing = T^*Q$, then in fact (5.25) describes the Hamiltonian mode I for *every* subset $I \subset K$.)

Later we recall that for every mode I the submanifold V_I can be identified with the cotangent bundle T^*Q_I, where $Q_I \subset Q$ is the constrained configuration space, and the Dirac bracket $\{\ ,\ \}_I$ with the natural Poisson bracket on T^*Q_I.

Example 5.3 (continued). The Dirac bracket on V_K is given as (F and G being arbitrary smooth functions on V_K)

$$\begin{aligned}\{F, G\}_K = \{F, G\} &+ \tfrac{1}{4}\{F, 1 - q_1^2 - q_2^2\}\{-2q_1p_1 - 2q_2p_2, G\} \\ &- \tfrac{1}{4}\{F, -2q_1p_1 - 2q_2p_2\}\{1 - q_1^2 - q_2^2, G\}.\end{aligned} \tag{5.27}$$

Later, see (5.78), we compute canonical coordinates with regard to the Dirac bracket as

$$q := \arctan\frac{q_2}{q_1}, \qquad p := q_1p_2 - q_2p_1. \tag{5.28}$$

The constrained dynamics in these coordinates is then given as

$$\begin{aligned}\dot{q} &= \{H_K, q\}_K = \frac{\partial H_K}{\partial p}, \\ \dot{p} &= \{H_K, p\}_K = -\frac{\partial H_K}{\partial q},\end{aligned} \tag{5.29}$$

where

$$H_K(q, p) = \tfrac{1}{2}p^2 + \sin q, \tag{5.30}$$

which are the Hamiltonian equations of motion for the mathematical pendulum with *rigid* link.

Now we come to the full description of the dynamics of the Hamiltonian complementary-slackness system as a hybrid system. The Hamiltonian modes (5.13) will correspond to the nodes of the transition graph of the hybrid system; and we have to specify the transition rules as in Section 4. So, consider a point $(q, p, \lambda) \in \mathscr{V}_I \cap Q^+$, with $Q^+ := \{(q, p, \lambda) | C_i(q) \geq 0, i \in K\}$. By Proposition 5.2, it is possible to integrate forward from this point in $\mathscr{V}_I$ according to the Hamiltonian dynamics of mode I. Denote the point reached from (q, p, λ) after time t by $(q(t), p(t), \lambda(t))$. Performing the same analysis as in Section 3 we obtain if (q, p, λ) is a jump point the critical index set

$$\Gamma(q, p, \lambda; I) = \Gamma_1(q, p, \lambda; I) \cup \Gamma_2(q, p, \lambda; I) \tag{5.31}$$

with

$$\begin{aligned}\Gamma_1(q, p, \lambda; I) &= \{i \in K \backslash I \mid \exists \varepsilon > 0 \text{ s.t. } C_i(q(t)) < 0 \text{ for all } t \in (0, \varepsilon)\},\\ \Gamma_2(q, p, \lambda; I) &= \{i \in I \mid \exists \varepsilon > 0 \text{ s.t. } \lambda^c_{I,i}(q(t), p(t)) < 0 \text{ for all } t \in (0, \varepsilon)\},\end{aligned} \tag{5.32}$$

where $\lambda^c_{I,i}$ denotes the component of λ^c_I corresponding to $i \in I$, with λ^c_I given by (5.17). We define the new index set

$$J := (I \backslash \Gamma_2(q, p, \lambda; I)) \cup \Gamma_1(q, p, \lambda; I) \tag{5.33}$$

as in (3.8). Note that $\Gamma_1(q, p, \lambda; I)$ can be interpreted as the set of those indices for which the inequality constraints are going to be violated, while on the other hand $\Gamma_2(q, p, \lambda; I)$ denotes the indices corresponding to active (equality) constraints for which the constraint forces are going to be negative.

Before specifying the projection rule to $\mathscr{V}_J$, we describe the projection rule to V_J, the set of consistent *states*. This projection is specified by the distribution $\bar{D}_J$ on T^*Q spanned by the constraint force vector fields, that is,

$$\bar{D}_J(q, p) := \operatorname{span}\{X_{C_j}(q, p) \mid j \in J\}, \qquad (q, p) \in T^*Q. \tag{5.34}$$

Since the Lie bracket $[X_{C_i}, X_{C_j}]$ equals $X_{\{C_i, C_j\}}$, it is zero for every $i, j \in K$. Hence the distribution $\bar{D}_J$ is *involutive*. Moreover, because of the standing Assumption 5.1, the distribution $\bar{D}_J$ has constant dimension equal to the cardinality of J on the submanifold

$$\ker C_J := \{(q, p) \in T^*Q \mid C_j(q) = 0, j \in J\}. \tag{5.35}$$

Since $L_{X_{C_i}} C_j = 0$ for every $i, j \in K$, it follows that we may *restrict* the distribution $\bar{D}_J$ to a distribution D_J on $\ker C_J$, which is involutive and of constant dimension. By Frobenius' theorem (see, e.g., [NS]) D_J therefore defines a *foliation* of $\ker C_J$, with leaves given by the integral manifolds of D_J, and the projection rule to V_J is *to project* (q, p) *along this foliation to a point* $(\bar{q}, \bar{p}) \in V_J$. Actually this projection turns out to be linear, as stated in the following proposition.

Proposition 5.4. *Let $J \subset K$. Then, for every $(q, p) \in \ker C_J$,*

$$D_J(q, p) = \operatorname{span}\left\{\sum_{i=1}^{n} \frac{\partial C_j}{\partial q_i}(q) \frac{\partial}{\partial p_i} \,\middle|\, j \in J\right\}, \tag{5.36}$$

and consequently $D_J(q, p)$ does not depend on p. Moreover, projection along the foliation corresponding to D_J is the linear projection along the linear subspace

$$\operatorname{span}\left\{\frac{\partial C_j}{\partial q}(q) \,\middle|\, j \in I\right\} = \operatorname{im} \frac{\partial C_I^T}{\partial q}(q) \qquad \textit{of } T_q^*Q.$$

Furthermore, for every $(q, p) \in V_J$,

$$T_{(q,p)}(\ker C_J) = T_{(q,p)} V_J \oplus D_J(q, p), \tag{5.37}$$

and so every $(q, p) \in \ker C_J$ projects along the foliation corresponding to D_J to a unique point $(q, \pi^q_J(p)) \in V_J$, where $\pi^q_J\colon T_q^*Q \to T_q^*Q$ denotes the linear projection *along* $\mathrm{im}(\partial C_J^T/\partial q)(q)$ *onto the subspace*

$$\left\{p \in T_q^*Q \,\middle|\, \left[\frac{\partial^T C_J}{\partial q}(q)\right]^T P(q)p = 0\right\} \subset T_q^*Q.$$

The above projection rule to V_J yields a well-defined projection rule to $\mathscr{V}_J \cap Q^+$ as follows. Let $(q, p, \lambda) \in \mathscr{V}_I \cap Q^+$ be a jump point leading to the new index set J. Necessarily $C_i(q) = 0$ for $i \in \Gamma_1(q, p, \lambda; I)$, and thus $(q, p) \in \ker C_J$. Hence by Proposition 5.4 (q, p) is projected in a unique manner to $(q, \pi_J^q(p)) \in V_J$. Then (q, p, λ) will be projected to the point $(q, \pi_J^q(p), \bar{\lambda}) \in \mathscr{V}_J$, where $\bar{\lambda}_{K\setminus J} := 0$ and $\bar{\lambda}_J := \lambda_J^c(q, \pi_J^q(p))$. (In particular, if $\Gamma_1(q, p, \lambda; I)$ is empty, then the projection will only consist of setting $\bar{\lambda}_{K\setminus J} := 0$ and $\bar{\lambda}_J := \lambda_J^c(q, \pi_J^q(p))$.) This projection rule to $\mathscr{V}_J$ corresponds to the foliation of $\mathscr{C}_J \subset \{(q, p, \lambda) | (q, p) \in T^*Q, \lambda \in \mathbb{R}^k\}$, with $\mathscr{C}_J(q, p, \lambda) := C_J(q)$, given by the leaves of the product distribution $D_J \times \mathrm{span}\{\partial/\partial\lambda_1, \ldots, \partial/\partial\lambda_k\}$ on (q, p, λ)-space. This foliation plays a role analogous to that of $\mathscr{T}_J$ in the linear case; although actually only $\ker \mathscr{C}_J$ is foliated rather than the entire (q, p, λ)-space, this is enough since jumps to $\mathscr{V}_J$ will only occur from points already in $\ker \mathscr{C}_J$. Furthermore, since q is left invariant by the projection rule, it follows that we have actually obtained a projection rule from $\mathscr{V}_I \cap Q^+$ to $\mathscr{V}_J \cap Q^+$.

The projection rule for Hamiltonian complementary-slackness systems has the following important property, which we call the *energy inequality*.

Proposition 5.5. *Let* $(q, p, \lambda) \in \mathscr{V}_I \cap Q^+$ *be a jump point leading to the new index set* J. *Then*

$$H(q, \pi_J^q(p)) \le H(q, p). \tag{5.38}$$

Proof. By (5.16) the tangent space to V_J at a point $(q, p) \in V_J$ may be identified with the linear space

$$\ker\left[\frac{\partial C_J^T}{\partial q}(q)\right]^T \times \ker\left[\frac{\partial C_J^T}{\partial q}(q)\right]^T P(q) \subset T_qQ \times T_q^*Q. \tag{5.39}$$

The second linear space on the left-hand side, that is $\ker[(\partial C_J^T/\partial q)(q)]^T P(q)$, is orthogonal to the linear subspace $(\partial C_J^T/\partial q)(q)$ of T_q^*Q with regard to the inner product determined by $P(q)$ on T_q^*Q. Therefore by Proposition

$$\tfrac{1}{2}[\pi_J^q(p)]^T P(q)\pi_J^q(p) \le \tfrac{1}{2}p^T P(q)p, \tag{5.40}$$

and (5.38) results. ∎

Remark 5.6. Note that since a jump point $(q, p, \lambda) \in \mathscr{V}_I \cap Q^+$ already satisfies $(q, p) \in V_I$, the projection of (q, p) along $D_J(q, p)$ onto V_J is actually performed along the smaller subspace (see (5.33))

$$\mathrm{span}\left\{\frac{\partial C_i}{\partial q}(q) \,\middle|\, i \in \Gamma_1(q, p, \lambda; I)\right\}. \tag{5.41}$$

Furthermore, by the definition of a jump point (q, p, λ)

$$\dot{y}_i(0) = \{H, C_i\}(q, p) \le 0, \qquad i \in J,$$

and so the momentum coordinates of a jump point (q, p, λ) necessarily satisfy

$$\left[\frac{\partial C_J^T}{\partial q}(q)\right]^T P(q)p \geq 0. \tag{5.42}$$

Example 5.3 (continued). Suppose the system is in mode $I = \varnothing$ (no active constraint and hence no constraint force, i.e., $\lambda = 0$). Let (q, p, λ) with $\lambda = 0$ be a jump point. Necessarily $C(q) = 0$. Project (q_1, q_2, p_1, p_2) along the forward integral curves of $-X_C(q) = \mathrm{col}(0, 0, -2q_1, -2q_2)$ to V_K given by (5.22). This is done by computing τ satisfying

$$p_1(\tau)q_1 + p_2(\tau)q_2 = 0, \tag{5.43}$$

where

$$\begin{aligned} p_1(\tau) &= p_1 - \tau \cdot 2q_1, \\ p_2(\tau) &= p_2 - \tau \cdot 2q_2. \end{aligned} \tag{5.44}$$

Since $C(q) = 1 - q_1^2 - q_2^2 = 0$, this yields $\tau = \frac{1}{2}(p_1q_1 + p_2q_2)$, and hence

$$\pi_K^q(p_1, p_2) = (p_1 - q_1(p_1q_1 + p_2q_2), p_2 - q_2(p_1q_1 + p_2q_2)). \tag{5.45}$$

The resulting point $(q, \pi_K^q(p), \lambda_K^c(q, \pi_K^q(p)))$, with λ_K^c given by (5.23), may or may not be feasible for mode K. For instance, we start from an initial point (q_0, p_0) with $q_{01} = 0$, $q_{20} = 1$, $p_{01} = 0$, $p_{02} > 0$. (The mass has been thrown upward to meet the constraint $q_1^2 + q_2^2 = 1$.) This jump point $(0, 1, 0, p_{02})$ will be projected to $(0, 1, 0, 0) \in V_K$. However, since $\lambda_K^c(0, 1, 0, 0) = -\frac{1}{2} < 0$, the projection is not feasible for mode K. Hence there is a subsequent jump in the constraint force to $\lambda = 0$, and from this there is smooth continuation in mode $\varnothing$; the mass will fall down again. Notice that if instead of $q_{20} = 1$ we have, e.g., $q_{20} = -1$, while $p_{02} < 0$, then the projection is feasible for mode K, and there is smooth continuation in this mode.

Suppose on the other hand that the system is already in mode $I = K$ (that is, the constraint is active). Let $(q, p, \lambda_K^c(q, p))$ be a jump point, implying that

$$\lambda_K^c(q(t), p(t)) = \tfrac{1}{2}(p_1^2(t) + p_2^2(t) - q_2(t)) < 0 \qquad \text{for all} \quad t \in (0, \varepsilon), \tag{5.46}$$

while $\lambda_K^c(q(0), p(0)) = 0$. Then the system continues in mode $I = \varnothing$. Physically, the state will leave the constrained state space V_K, and $C(q)$ will become positive—the rope will not be fully stretched anymore.

Remark 5.7. The projection rule given for Hamiltonian complementary-slackness systems agrees with the rule given in Section 5 for *linear* complementary-slackness systems in cases where both are applicable, i.e., for linear Hamiltonian systems. Indeed for a such a system given by equations

$$\begin{aligned} \begin{bmatrix} \dot q \\ \dot p \end{bmatrix} &= \begin{bmatrix} 0 & P \\ -Q & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ C^T \end{bmatrix} u, \qquad P = P^T > 0, \qquad Q = Q^T, \\ y &= Cq, \qquad \operatorname{rank} C = \dim y, \end{aligned} \tag{5.47}$$

the subspaces T_I and V_I corresponding to the constraints $C_{Iq} = 0$ are easily seen to be equal to

$$T_I = \operatorname{im}\begin{bmatrix} 0 & PC_I^T \\ C_I^T & 0 \end{bmatrix}, \tag{5.48}$$

$$V_I = \ker\begin{bmatrix} C_I & 0 \\ 0 & C_I P \end{bmatrix}. \tag{5.49}$$

On the other hand, given a point (q, p) satisfying the constraints $C_{Iq} = 0$, projection along T_I onto V_I will really be only a projection along the subspace

$$\operatorname{span}\begin{bmatrix} 0 \\ C_I^T \end{bmatrix} \tag{5.50}$$

(since $C_I PC_I^T > 0$), which corresponds exactly to the distribution D_I (see (5.36)).

Remark 5.8. Note that for a *nonlinear* Hamiltonian system

$$\begin{aligned} \dot{x} &= X_H(x) - \sum_{j=1}^{m} X_{C_j}(q)u_j, \qquad x = (q, p), \\ y_j &= C_j(q), \qquad j = 1, \ldots, m, \end{aligned} \tag{5.51}$$

the analog of the *whole* subspace T_I would be the *distribution*

$$T_I(x) := \operatorname{span}\{X_{C_i}(x), [X_H, X_{C_i}](x) = X_{\{H,C_i\}}(x), i \in I\} \tag{5.52}$$

(which in fact is known as the *minimal conditioned invariant distribution* containing the input vector fields X_{C_i}, $i \in I$, see [IKGM]). However, in general (without imposing extra conditions on the matrix $R_I(q)$) this distribution T_I will *not* be involutive, in which case it is *not* possible to integrate it to a foliation of the whole of T^*Q. As already noted above, the nonlinear analog of the subspace $\mathscr{T}_I$ for a nonlinear Hamiltonian system (5.51) is the following product distribution on (q, p, u)-space:

$$D_I \times \operatorname{span}\left\{\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_k}\right\}, \tag{5.53}$$

which is complementary to $\mathscr{V}_I$ in $\ker \mathscr{C}_I$.

Remark 5.9. The projection rule given for Hamiltonian complementary-slackness systems corresponds mechanically to the idealized situation in which the mechanical stops corresponding to the geometric inequality constraints are all assumed to be purely nonelastic; that is, there is no bouncing back. In principle, elasticity of (some of) the boundaries could be modeled by projecting $p \in T^*Q$ along $\operatorname{im}(\partial C_J^T/\partial q)(q)$ not *onto* the plane $\ker[(\partial C_J^T/\partial q)(q)]^T P(q)$ but instead to some point on the other side of this plane, for example, to its mirror point if the boundaries are perfectly elastic (see, e.g., [KR] and [BNO]). Note that this implies that, although the mode J plays a role in determining the direction of the projection, the jump is not to mode J but instead to the originating mode I.

Remark 5.10. It should be noted that in this paper our proposals for the formulation of complementary-slackness systems are mainly just tested in the case of bimodal systems. This means that no strong conclusions can be drawn with respect to the effectiveness or the "physical" validity of these proposals in a context where the interaction of several constraints is of importance. This remark applies in particular to the rule (5.33) for the selection of the "target mode" J (see [KR] and [B4]).

Remark 5.11. From a mechanical point of view the projection $p \mapsto \pi_J^q(p)$ can be interpreted as the application of impulsive forces (*impulses*) to the system. Note that in our formulation of a Hamiltonian complementary-slackness system impulses may only occur if there is no smooth continuation in any of the modes I corresponding to the (constraint) forces $\lambda_I^c(q, p)$, see (5.17). This is in accordance with the *Principle of Constraints* formulated by Kilmister and Reeve [KR, p. 79]: "Constraints shall be maintained by forces, so long as this is possible; otherwise, and only otherwise, by impulses."

Recall that the set of feasible points for a complementary-slackness system with index set K is given as $\bigcup_{I \subset K} \mathscr{W}_I$. In Section 4 it has been shown that *linear* complementary-slackness systems with a *scalar* constraint are well-posed, in the sense that there are *unique* solutions starting from each feasible point, provided a certain positivity condition is satisfied. For Hamiltonian complementary-slackness systems with a scalar constraint we derive the following nonlinear analog.

Theorem 5.12. *Consider the Hamiltonian complementary-slackness system with scalar constraint*

$$\begin{aligned} \dot{x} &= X_H(x) - X_C(q)\lambda, \qquad x = (q, p) \in T^*Q, \\ y &= C(q), \\ u &= \lambda, \end{aligned} \tag{5.54}$$

$$y(t) \geq 0, \qquad u(t) \geq 0, \qquad y(t)u(t) = 0, \qquad u, y \in \mathbb{R},$$

with H as in (5.4). *Let Assumption* 5.1 *be satisfied, i.e.,* $\operatorname{rank}(\partial C/\partial q)(q) = 1$ *for every q with $C(q) = 0$, and assume that H and C are real-analytic functions on T^*Q. Then the system is well-posed, in the sense that there are unique solutions starting from each feasible point.*

Proof. Denote for simplicity $\mathscr{V}_{\{1\}} = \mathscr{V}_1$, $\mathscr{W}_{\{1\}} = \mathscr{W}_1$, $V_{\{1\}} = V_1$, $R_{\{1\}} = R$, $\pi_{\{1\}} = \pi$, and $\lambda_{\{1\}}^c = \lambda^c$. We refer to the mode corresponding to $I = \varnothing$, that is,

$$\dot{x} = X_H(x), \qquad x = (q, p) \in T^*Q, \tag{5.55}$$

as *mode* 0. The mode corresponding to $I = \{1\}$, that is,

$$\dot{x} = X_H(x) - X_C(x)\lambda^c(x), \qquad x \in V_1, \tag{5.56}$$

with

$$\lambda^c(q, p) = -R^{-1}(q)\{H, \{H, C\}\}(x), \tag{5.57}$$

and

$$R(q) = \left[\frac{\partial C}{\partial q}(q)\right]^T P(q)\frac{\partial C}{\partial q}(q), \qquad V_1 = \{(q, p) \in T^*Q \,|\, C(q) = 0, \{H, C\}(q, p) = 0\},$$

is called *mode* 1.

Now consider an arbitrary point $(q_0, p_0) \in T^*Q$. Denote solutions $(q(t), p(t))$ starting from (q_0, p_0) at $t = 0$ corresponding to mode 0 as $(q(t; 0), p(t; 0))$, and solutions corresponding to mode 1 (when existing) as $(q(t; 1), p(t; 1))$. Recall the last equation of (5.14), that is,

$$\ddot{y} = \{H, \{H, C\}\}(q, p) + R(q)\lambda. \tag{5.58}$$

Obviously, if $C(q_0) < 0$, then (q_0, p_0) is not feasible. On the other hand, if $C(q_0) > 0$, then there is unique smooth continuation in mode 0.

Thus, let $C(q_0) = 0$. Then there are two possibilities (compare the proof of Theorem 4.8):

1. $(q_0, p_0) \notin V_1$.
2. $(q_0, p_0) \in V_1$.

For possibility 1 there are two subcases.

1.1. *The solution* $(q(t; 0), p(t; 0))$ *starting from* (q_0, p_0) *satisfies* $C(q(t; 0)) \geq 0$ *for* t *small.* In this case there is unique smooth continuation in mode 0.

1.2. $\exists \varepsilon > 0$ *such that* $C(q(t; 0), 0) < 0$, *for all* $t \in (0, \varepsilon)$.

In the latter case, 1.2, a jump to mode 1 must occur and (q_0, p_0) is projected to $(q_0, \pi_1^{q_0}(p_0)) \in V_1$, and we end up in possibility 2.

So, let $(q_0, p_0) \in V_1$. Then there are three possibilities:

2.1. $\lambda^c(q_0, p_0) > 0$. Then there is smooth continuation in mode 1, while there is no smooth continuation in mode 0.

2.2. $\lambda^c(q_0, p_0) < 0$. Then there is no smooth continuation in mode 1. In this case we must jump to mode 0 by setting $\lambda = 0$. Because of (5.57) and positivity of $R(q)$ we have $\{H, \{H, C\}\}(q_0, p_0) > 0$, and so by (5.58) with $\lambda = 0$ we conclude $\ddot{y}(0) > 0$, and hence $C(q(t; 0)) > 0$ for t small, implying smooth continuation in mode 0.

2.3. $\lambda^c(q_0, p_0) = 0$. This is the hard case. By definition of $\lambda^c(q, p)$:

$$\{H, \{H, C\}\}(q(t; 1), p(t; 1)) + R(q(t; 1))\lambda^c(q(t; 1), p(t; 1)) = 0 \tag{5.59}$$

for all t. Differentiation of (5.59) to t yields, by (5.24),

$$\{H - \lambda^c C, \{H, \{H, C\}\}\} + \{H - \lambda^c C, R\}\lambda^c + R\{H - \lambda^c C, \lambda^c\} = 0, \tag{5.60}$$

where everything is evaluated in $(q(t; 1), p(t; 1))$. Substituting $t = 0$, that is, $q(0; 1) = q_0$, $p(0; 1) = p_0$, writing out the Poisson brackets, and using $C(q_0) = 0$, $\lambda^c(q_0, p_0) = 0$, we obtain

$$\{H, \{H, \{H, C\}\}\}(q_0, p_0) + R(q_0)\{H - \lambda^c C, \lambda^c\}(q_0, p_0) = 0. \tag{5.61}$$

Now suppose $\{H, \lambda^c\}(q_0, p_0) = \{H - \lambda^c C, \lambda^c\}(q_0, p_0) > 0$. Then $\lambda(q(t; 1), p(t; 1)) > 0$ for t small, and there is continuation in mode 1. Furthermore, since $\{H, \{H, \{H, C\}\}\} = y^{(3)}$ in mode 0, it follows by (5.61) that in this case $y^{(3)}(0) < 0$ and thus $y(t) < 0$, for t small in mode 0. So the only continuation is in mode 1.

On the other hand, suppose $\{H, \lambda^c\}(q_0, p_0) < 0$, then $\lambda(q(t; 1), p(t; 1)) < 0$ for t small, and so no continuation in mode 1 is possible. Moreover, in this case, by (5.61), $y^{(3)}(0) > 0$, and therefore $y(t) > 0$, for t small in mode 0. As a consequence the *only* continuation is in mode 0.

Therefore, suppose

$$\{H - \lambda^c C, \lambda^c\}(q_0, p_0) = 0 = \{H, \{H, \{H, C\}\}\}(q_0, p_0). \tag{5.62}$$

The idea is now to compute the *second* time-derivative of (5.59), or, what is the same, the first time-derivative of (5.60):

$$\begin{aligned}&\{H - \lambda^c C, \{H - \lambda^c C, \{H, C\}\}\} + \{H - \lambda^c C, \{H - \lambda^c C, R\}\}\lambda^c \\ &\quad + 2\{H - \lambda^c C, R\}\{H - \lambda^c C, \lambda^c\} + R\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\} = 0\end{aligned} \tag{5.63}$$

and to evaluate this expression in $t = 0$, yielding, since $C(q_0) = 0$, $\lambda^c(q_0, p_0) = 0$ and, because of (5.62),

$$\{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0) + R(q_0)\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) = 0. \tag{5.64}$$

The reasoning now follows the same lines as above. Suppose $\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) > 0$. Then $\lambda(q(t; 1), p(t; 1)) > 0$ for t small. Furthermore, since

$$y^{(4)}(0) = \{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0) < 0,$$

we have $y(t) < 0$ for t small in mode 0. Therefore the only continuation is in mode 1.

On the other hand, suppose $\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) < 0$, then $\lambda(q(t; 1), p(t; 1)) < 0$ for t small, while $y(t) > 0$ for t small in mode 0. So the only continuation is in mode 0.

Therefore we are left with the case that apart from $C(q_0) = 0$, $\lambda^c(q_0, p_0) = 0$, not only (5.62) holds, but also

$$\{H - \lambda^c C, \{H - \lambda^c C, \lambda^c\}\}(q_0, p_0) = 0 = \{H, \{H, \{H, \{H, C\}\}\}\}(q_0, p_0). \tag{5.65}$$

In this case we differentiate (5.63) once more in $t = 0$, and apply the same reasoning. Define inductively, for $k \in \mathbb{N}$,

$$\begin{aligned}\mathrm{ad}_H^k\, C &= \{H, \mathrm{ad}_H^{k-1} C\}, & \mathrm{ad}_H^0\, C &= C, \\ \mathrm{ad}_{H-\lambda^c C}^k\, \lambda^c &= \{H - \lambda^c C, \mathrm{ad}_{H-\lambda^c C}^{k-1} \lambda^c\}, & \mathrm{ad}_{H-\lambda^c C}^0\, \lambda^c &= \lambda^c.\end{aligned} \tag{5.66}$$

It follows that either, for some $\ell < \infty$, $\mathrm{ad}_{H-\lambda^c C}^\ell\, \lambda^c(q_0, p_0) \neq 0$ or $\mathrm{ad}_{H-\lambda^c C}^k\, \lambda^c(q_0, p_0) = 0$ for all $k \in \mathbb{N}$.

In the first case, by taking the smallest ℓ such that $\mathrm{ad}_{H-\lambda^c C}^\ell\, \lambda^c(q_0, p_0) \neq 0$, we obtain by induction

$$\{\mathrm{ad}_H^{\ell+2}, C\}(q_0, p_0) + R(q_0)\, \mathrm{ad}_{H-\lambda^c C}^\ell\, \lambda^c(q_0, p_0) = 0, \tag{5.67}$$

while $\{\mathrm{ad}_H^{k+2}, C\}(q_0, p_0) = \mathrm{ad}_{H-\lambda^c C}^k \lambda^c(q_0, p_0) = 0$ for $k < \ell$. If $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) > 0$, then as above it follows that there is unique continuation in mode 1, while if $\mathrm{ad}_{H-\lambda^c C}^{\ell} \lambda^c(q_0, p_0) < 0$, then there is unique continuation in mode 0.

In the second case, if $\mathrm{ad}_{H-\lambda^c C}^k \lambda(q_0, p_0) = 0$, for all k, then $\mathrm{ad}_H^k C(q_0, p_0) = 0$ also, for all k, while already $C(q_0) = \{H, C\}(q_0, p_0) = 0$. From analyticity it follows that

$$\begin{aligned} \lambda^c(q(t;1), p(t;1)) &= 0 \qquad \text{for } t \text{ small}, \\ C(q(t;0)) &= 0 \qquad \text{for } t \text{ small}. \end{aligned} \tag{5.68}$$

This implies that there is continuation in mode 0 as well as in mode 1. However, because of (5.67) it follows that in mode 1 not only $C(q(t; 1))$ but also $\lambda^c(q(t; 1), p(t; 1))$ is zero for t small. Since, in all points (q, p) for which both $C(q) = 0$ and $\lambda^c(q, p) = 0$,

$$X_H(q, p) = X_{H-\lambda^c C}(q, p), \tag{5.69}$$

it thus follows that the continuation in mode 1 is the *same* as the continuation in mode 0. ■

As also mentioned after the proof of Theorem 4.8, the proof technique of Theorem 5.10, being basically exhaustive, is not well suited to Hamiltonian complementary-slackness systems with more than one geometric inequality constraint. As in the linear case, for such systems recourse should be taken to a dynamic version of the linear complementarity problem (LCP). This will be explored in a future paper. The idea of using the LCP in the context of mechanical systems with multiple inequality constraints has already been advocated in [L3].

It has already been remarked that for every $I \subset K$ the resulting mode is Hamiltonian with regard to the Dirac bracket $\{\ ,\ \}_I$ on the constrained state space V_I, and with regard to the restriction H_I of H to V_I. In fact, we can be more explicit about this Hamiltonian dynamics, due to the fact that the phase space is T^*Q, and the Hamiltonian has the special form (5.4). First, we define for every $I \subset K$ the *constrained configuration space*

$$Q_I := \{q \in Q \mid C_I(q) = 0\} \tag{5.70}$$

which is (because of Assumption 5.1) a submanifold of Q, of dimension $n_I := n - |I|$. It follows that we may choose local coordinates $q = (q_1, \ldots, q_n)$ for Q, such that locally $C_I = (q_{n_I+1}, \ldots, q_n)$, and $q_I = (q_1, \ldots, q_{n_I})$ are local coordinates for Q_I.

Associated with $q = (q_1, \ldots, q_n)$ natural momentum coordinates $p = (p_1, \ldots, p_n)$ are defined. Now it has been shown in Example 12.43 of [NS] that

$$(q_I, p_I) := (q_1, \ldots, q_{n_I}, p_1, \ldots, p_{n_I}) \tag{5.71}$$

are local coordinates for V_I, and moreover that they are *canonical* with regard to the Dirac bracket $\{\ ,\ \}_I$, that is,

$$\{p_i, q_j\}_I = \delta_{ij}, \qquad \{p_i, p_j\}_I = \{q_i, q_j\}_I = 0, \qquad i, j = 1, \ldots, n_I. \tag{5.72}$$

So V_I together with the Dirac bracket $\{\ ,\ \}_I$ can be identified with T^*Q_I with its natural Poisson bracket. Hence the Hamiltonian dynamics of every mode I given by (5.22) can be identified with the standard Hamiltonian dynamics

$$\begin{aligned}\dot{q}_I &= \frac{\partial H_I}{\partial p_I}(q_I, p_I),\\ \dot{p}_I &= -\frac{\partial H_I}{\partial q_I}(q_I, p_I),\end{aligned} \qquad (q_I, p_I) \in T^*Q_I, \tag{5.73}$$

where $H_I(q_I, p_I)$ is the Hamiltonian H *restricted* to V_I, and subsequently expressed in the local coordinates q_I, p_I for V_I. The following relation exists between the momentum variables p_I for different modes I, including the mode $I = \varnothing$. Note that

$$T^*_{q_I}Q_I \simeq \frac{T^*_{q_I}Q}{(T_{q_I}Q_I)^{\perp}}, \qquad q_I \in Q_I \tag{5.74}$$

where $\simeq$ denotes the canonical isomorphism $V^* \simeq W^*/(V^{\perp})$ for any subspace V of a linear space W, while furthermore $T_{q_I}Q_I \subset T_{q_I}Q$ is given as $\ker[(\partial C_I^T/\partial q)(q_I)]^T$. Hence

$$T^*_{q_I}Q_I \simeq \frac{T^*_{q_I}Q}{\mathrm{im}(\partial C_I^T/\partial q)}(q_I), \qquad q_I \in Q_I, \tag{5.75}$$

and so a momentum vector $p_I \in T_{q_I}Q_I$ can be identified with the equivalence class $[p]$, where

$$[p^1] = [p^2] \quad \Leftrightarrow \quad p^1 - p^2 \in \mathrm{im}\left(\frac{\partial C_I^T}{\partial q}\right)(q_I), \qquad p^1, p^2 \in T^*_{q_I}Q. \tag{5.76}$$

Comparing this to the *projection rule* for Hamiltonian complementary-slackness systems as given above (see, e.g., Proposition 5.4) we conclude that at the occasion of a *jump* in the state space (projection along $\mathrm{im}(\partial C_I^T/\partial q)(q_I)$) there is *conservation of momentum* in the sense that the equivalence class given in (5.76) remains the same.

Example 5.3 (continued). Define new coordinates adapted to $C(q) = 1 - q_1^2 - q_2^2$:

$$\tilde{q}_1 = \arctan\left(\frac{q_2}{q_1}\right), \qquad \tilde{q}_2 = 1 - q_1^2 - q_2^2, \tag{5.77}$$

for $Q = \mathbb{R}^2\backslash\{(q_1, q_2)|q_1 = 0)\}$. Carrying out the canonical transformation (see [L1]), the corresponding new momentum variables are found:

$$\begin{bmatrix}\tilde{p}_1\\ \tilde{p}_2\end{bmatrix} = \begin{bmatrix} -q_2 & q_1 \\ -\dfrac{q_1}{2(q_1^2+q_2^2)} & -\dfrac{q_2}{2(q_1^2+q_2^2)}\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}. \tag{5.78}$$

Therefore $\tilde{q}_1 = \arctan(q_2/q_1)$, $\tilde{p}_1 = q_1p_2 - q_2p_1$ are canonical coordinates for T^*Q_K, with $Q_K = \{(q_1, q_2)|q_1^2 + q_2^2 = 1\}$; this is in accordance with (5.28). Furthermore, as in (5.30), $H_K(\tilde{q}_1, \tilde{p}_1) = \frac{1}{2}\tilde{p}_1^2 + \sin\tilde{q}_1$.

6. Conclusions

We have studied a class of dynamical systems which we have called the complementary-slackness class. One interesting feature of these dynamical systems is that they combine continuous and discrete characteristics so that they can be considered as a subclass of the class of hybrid systems. Considered in this way the complementary-slackness systems form a rather small class but nevertheless there are already interesting conclusions to be drawn which may also be of relevance to other classes of hybrid systems. In particular we have seen that a simple transition rule does not suffice, and instead we have used a rule allowing for multiple jumps. We have also begun a study of existence and uniqueness of solutions, which has led us to some nontrivial problems. The description of the transitions between modes has been in terms of a complementary foliation associated to the consistent manifold of each mode. We have seen that such foliations can be given in a natural way for linear and for Hamiltonian systems, and it may be asked under what conditions complementary foliations can also be obtained for other types of constrained dynamics.

Although we have obtained some first results, clearly there are many questions still to be answered. Both in the linear and in the Hamiltonian case, the issue of well-posedness has only been resolved for bimodal systems. In particular, for Hamiltonian complementary-slackness systems and linear passive networks with diodes this is unsatisfactory since physical intuition strongly suggests (at least if our modeling is correct, see Remark 5.10) that such systems should be well-posed when there are also multiple constraints. A suitable extension of the linear complementarity problem of mathematical programming will be helpful in the systematic treatment of such situations.

Several extensions of the class of systems studied in this paper could be considered. Take, for instance, electrical or hydraulic networks, or mechanical systems with friction, which lead to complementary-slackness systems that are neither linear nor Hamiltonian. In many situations it will be of interest to add external inputs, and in particular control problems may be formulated for complementary-slackness systems. However, there are also other connections with control theory. An obvious relation is the one via optimal control problems with state inequality constraints, but using the Hamiltonian complementary-slackness structure, and in particular the energy inequality that we proved, may also be thought of as a means of finding nonsmooth stabilizing controllers for certain nonlinear systems. In many applications two-sided constraints (Coulomb friction, relays) are encountered, and it will be of interest to see whether these can be handled within the complementary-slackness framework. A further generalization would be to include hysteretic effects. We plan to address these issues in future work.

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