PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

THE COMPLETE GENERATING FUNCTION FOR GESSEL WALKS IS ALGEBRAIC

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(Communicated by Jim Haglund)

ABSTRACT. Gessel walks are lattice walks in the quarter plane $\mathbb{N}^2$ which start at the origin $(0,0) \in \mathbb{N}^2$ and consist only of steps chosen from the set $\{\leftarrow, \swarrow, \nearrow, \rightarrow\}$. We prove that if $g(n;i,j)$ denotes the number of Gessel walks of length n which end at the point $(i,j) \in \mathbb{N}^2$, then the trivariate generating series $G(t;x,y) = \sum_{n,i,j\geq 0} g(n;i,j)x^i y^j t^n$ is an algebraic function.

1. INTRODUCTION

The starting question in lattice path theory is the following: How many ways are there to walk from the origin through the lattice $\mathbb{Z}^2$ to a specified point $(i,j) \in \mathbb{Z}^2$, using a fixed number n of steps chosen from a given set S of admissible steps. The question is not hard to answer. If we write $f(n;i,j)$ for this number and define the complete generating function

$$F(t;x,y) := \sum_{n=0}^{\infty}\Bigl(\sum_{i,j\in\mathbb{Z}} f(n;i,j)x^i y^j\Bigr)t^n \ \in \mathbb{Q}[x,y,x^{-1},y^{-1}][[t]]$$

then a simple calculation suffices to see that $F(t;x,y)$ is rational, i.e., it agrees with the series expansion at $t=0$ of a certain rational function $P/Q \in \mathbb{Q}(t,x,y)$. This is elementary and well-known.

Matters are getting more interesting if restrictions are imposed. For example, the generating function $F(t;x,y)$ will typically no longer be rational if lattice paths are considered which, as before, start at the origin, consist of n steps, end at a given point (i,j), but which, as an additional requirement, *never step out of the right half-plane.* In was shown in [8, Prop. 2] that no matter which set S of admissible steps is chosen, the complete generating function F for such walks is algebraic, i.e., it satisfies $P(F,t,x,y)=0$ for some polynomial $P \in \mathbb{Q}[T,t,x,y]$.

If the walks are not restricted to a half-plane but to a quarter plane, say to the first quadrant, then the generating function F might not even be algebraic. For some step sets it is, for others it is not [6]. Among the series which are not algebraic, there are some which are still D-finite with respect to t (i.e., they satisfy a linear

2000 *Mathematics Subject Classification.* Primary 05A15, 14N10, 33F10, 68W30; Secondary 33C05, 97N80.

A.B. was partially supported by the Microsoft Research-INRIA Joint Centre.

M.K. was partially supported by the Austrian Science Fund (FWF) grant P19462-N18.

differential equation in t with polynomial coefficients in $\mathbb{Q}[t, x, y]$), and others which are not even that [8, 20].

Bousquet-Mélou and Mishna [7] have systematically investigated all the walks in the quarter plane with step sets $S \subseteq \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\}$. By discarding trivial cases and applying symmetries, they reduce the 256 different step sets to 79 inherently different cases. They provide a unified way to prove that 22 of those are D-finite, and give striking evidence that 56 are not D-finite. Only a single step set sustained their attacks, and this is the step set that we consider here.

This critical step set is $\{\leftarrow, \swarrow, \nearrow, \rightarrow\}$. The central object of the present article are thus lattice walks in $\mathbb{Z}^2$ which

- start at the origin $(0, 0)$,
- consist of n steps chosen from the step set $\{\leftarrow, \swarrow, \nearrow, \rightarrow\}$, and
- never step out of the first quadrant $\mathbb{N}^2$ of $\mathbb{Z}^2$.

These walks are also known as *Gessel walks*. By $g(n; i, j)$, we denote the number of Gessel walks of length n which end at the point $(i, j) \in \mathbb{Z}^2$. The complete generating function of this sequence is denoted by

$$G(t; x, y) = \sum_{n=0}^{\infty} \Big(\sum_{i,j \in \mathbb{Z}} g(n; i, j) x^i y^j \Big) t^n.$$

Since $g(n; i, j) = 0$ if $\min(i, j) > n$ or $\max(i, j) < 0$, the inner sum is a polynomial in x and y for every fixed choice of n, and thus $G(t; x, y)$ lives in $\mathbb{Q}[x, y][[t]]$.

Gessel [unpublished] considered the special end point $i = j = 0$, i.e., Gessel walks returning to the origin, so-called *excursions*. Their counting sequence $g(n; 0, 0)$ starts as

$$1,\ 0,\ 2,\ 0,\ 11,\ 0,\ 85,\ 0,\ 782,\ 0,\ 8004,\ 0,\ 88044,\ 0,\ 1020162,\ 0,\ \dots$$

He observed empirically that these numbers admit a simple hypergeometric closed form. His observation became known as the *Gessel conjecture* and remained open for several years. Only recently, it was shown to be true:

Theorem 1. [14] $G(t; 0, 0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 & \end{matrix} \,\middle|\, 16t^2\right) = \displaystyle\sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$

This result obviously implies that $G(t; 0, 0)$ is D-finite. Less obvious at this point, and actually overlooked until now, is the fact that the power series $G(t; 0, 0)$ is even *algebraic*. Because of the alternative representation

$${}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 & \end{matrix} \,\middle|\, 16t^2\right) = \frac{1}{t^2}\left(\frac{1}{2}\, {}_2F_1\left(\begin{matrix} -1/6 & -1/2 \\ 2/3 & \end{matrix} \,\middle|\, 16t^2\right) - \frac{1}{2}\right)$$

it was clear that algebraicity could be decided by reference to Schwarz's classification [24] of algebraic ${}_2F_1$'s, but as nobody recognized that the parameters $(-1/6, -1/2; 2/3)$ actually fit to Case III of Schwarz's table, the rumor started to circulate that $G(t; 0, 0)$ is not algebraic. In fact:

Corollary 2. *$G(t; 0, 0)$ is algebraic.*

With Theorem 1 and standard software packages like gfun [23, 18] at hand, discovering and proving Cor. 2 is an easy computer algebra exercise. Compared to a proof by table-lookup, the constructive proof given below has the advantage that it applies similarly also in situations where no classification results are available.

Proof. The idea is to come up with a polynomial $P(T,t)$ in $\mathbb{Q}[T,t]$ and prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n$ as a root. Using Thm. 1, this implies that $P(T,t^2)$ is an annihilating polynomial for $G(t;0,0)$, so that the latter series is indeed algebraic.

A suitable polynomial P can be *guessed* automatically (cf. Sections 2.1 and 3.1). This is how we discovered the polynomial P stated below.

By the implicit function theorem, this polynomial P admits a root $r(t) \in \mathbb{Q}[[t]]$ with $r(0) = 1$. Since $P(T,0) = T - 1$ has a single root in $\mathbb{C}$, the series $r(t)$ is the unique root of P in $\mathbb{C}[[t]]$. Now, $r(t)$ being algebraic, it is D-finite, and thus its coefficients satisfy a recurrence with polynomial coefficients. To complete the proof, it is then sufficient to type the following commands into Maple.

```
> with(gfun):
> P:=(t,T) -> -1+48*t-576*t^2-256*t^3+(1-60*t+912*t^2-512*t^3)*T
   +(10*t-312*t^2+624*t^3-512*t^4)*T^2+(45*t^2-504*t^3-576*t^4)*T^3
   +(117*t^3-252*t^4-288*t^5)*T^4+189*t^4*T^5+189*t^5*T^6
   +108*t^6*T^7+27*t^7*T^8:
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P(t,r), r(t)), r(t), g(n));
```

This outputs the first-order recurrence

$$(n+2)(3n+5)g_{n+1} - 4(6n+5)(2n+1)g_n = 0, \qquad g_0 = 1,$$

satisfied by the coefficients of $r(t) = \sum_{n=0}^{\infty} g_n t^n$. Its solution is $g_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n$, and therefore $g(t)$ and $r(t)$ coincide, and thus $g(t)$ is a solution of P, as claimed. $\square$

The aim in the present article is to lift Corollary 2 to the complete generating function, where x and y are kept as parameters. We are going to show:

Theorem 3. *$G(t;x,y)$ is algebraic.*

This twofold improvement of Thm. 1 is a surprising result. Until now, it was not even known whether $G(t;x,y)$ is D-finite with respect to t or not, and both cases seemed equally plausible in view of known results about other step sets. Thm. 3 implies that $G(t;x,y)$ is D-finite with respect to each of its variables, and in particular that the sequence $g(n;i,j)$ is P-finite (i.e., it satisfies a linear recurrence with polynomial coefficients in n) for any choice of $(i,j) \in \mathbb{N}^2$. This settles several conjectures made by Petkovšek and Wilf in [21, §2]. As noted in [21], even for simple values of (i,j) the sequence $g(n;i,j)$ is not hypergeometric, unlike the excursions sequence $g(2n;0,0)$. For instance, the sequence $g(2n+1;1,0)$ satisfies a third order linear recurrence, but it is not hypergeometric. Moreover, no closed formula seems to exist for $g(n;i,j)$, for arbitrary (i,j). All this indicates that counting general walks is much more difficult that just counting excursions.

Theorem 3 will be established by obstinately using the approach based on *automatic guessing and proof* promoted in [5], and by making heavy use of computer algebra. In contrast to Corollary 2, we manage in our proof of Theorem 3 to avoid exhibiting a polynomial that has $G(t;x,y)$ as a root. This is fortunate, since a posteriori estimates show that the minimal polynomial of $G(t;x,y)$ is huge, having a total size of about 30Gb.

Only annihilating polynomials of the section series $G(t;x,0)$ and $G(t;0,y)$ are produced and manipulated during our proof of Theorem 3. But even the computations with these have led to expressions far too large to be included in a printed publication; too large even to be processed efficiently by standard computer algebra systems like Maple or Mathematica. To complete our computations, we needed careful implementations of sophisticated special purpose algorithms, on computers with fast processors and large memory capacities. These computations were performed using the computer algebra system Magma [2]. Our result is therefore interesting not only because of its combinatorial significance, but it is also noteworthy because of the immense computational effort that was deployed to establish it.

2. A Dry Run: Kreweras walks

The computations which were needed for proving Thm. 3 were performed by means of efficient special purpose software running on fast hardware. It would not be easy to redo these calculations in, say, Maple or Mathematica on a standard computer. As a more easily reproducible calculation, we will show in this section how to reprove the classical result that the generating function of Kreweras walks is algebraic [17, 12, 6]. A slight variation of the very same reasoning, albeit with intermediate expressions far too large to be spelled out here, is then used in the next section to establish Thm. 3.

Kreweras walks differ from Gessel walks only in their choice of admissible steps. They are defined as lattice walks in $\mathbb{Z}^2$ which

- start at the origin $(0,0)$,
- consist only of steps chosen from the step set $\{\leftarrow, \downarrow, \nearrow\}$, and
- never step out of the first quadrant $\mathbb{N}^2$ of $\mathbb{Z}^2$.

If $f(n;i,j)$ denotes the number of Kreweras walks consisting of n steps and ending at the point $(i,j)\in\mathbb{Z}^2$, then it follows directly from its combinatorial definition that the sequence $f(n;i,j)$ satisfies the multivariate recurrence with constant coefficients

$$f(n+1;i,j) = f(n;i+1,j) + f(n;i,j+1) + f(n;i-1,j-1), \tag{1}$$

for all $n,i,j\geq 0$. Together with the boundary conditions $f(n;-1,0)=0$ and $f(n;0,-1)=0$ $(n\geq 0)$ and $f(0;i,j)=\delta_{i,j,0}$ $(i,j\geq 0)$, this recurrence equation implies the functional equation

$$F(t;x,y) = 1 + \bigl(\tfrac{1}{x}+\tfrac{1}{y}+xy\bigr)tF(t;x,y) - \tfrac{1}{y}tF(t;x,0) - \tfrac{1}{x}tF(t;0,y)$$

for the generating function

$$F(t;x,y) = \sum_{n=0}^{\infty}\Bigl(\sum_{i,j=0}^{\infty} f(n;i,j)x^i y^j\Bigr)t^n.$$

Noting that $F(t;0,y)$ and $F(t;y,0)$ are equal by the symmetry of the step set about the main diagonal of $\mathbb{N}^2$, the last equation becomes

$$F(t;x,y) = 1 + \bigl(\tfrac{1}{x}+\tfrac{1}{y}+xy\bigr)tF(t;x,y) - \tfrac{1}{y}tF(t;x,0) - \tfrac{1}{x}tF(t;y,0).$$

At the heart of our next arguments is the *kernel method*, a method commonly attributed to Knuth [15, Solutions of Exercises 4 and 11 in §2.2.1] which has already been used to great advantage in lattice path counting. After bringing the functional equation for $F(t;x,y)$ to the form

$$((x+y+x^2y^2)t - xy)F(t;x,y) = xtF(t;x,0) + ytF(t;y,0) - xy, \tag{K}$$

the kernel method consists of coupling x and y in such a way that this equation reduces to a simpler one, from which useful information about the *section series* $F(t;x,0)$ can be extracted. In the present case, the substitution

$$y \to Y(t,x) = \frac{x - t - \sqrt{-4t^2x^3 + x^2 - 2tx + t^2}}{2tx^2}$$
$$= t + \tfrac{1}{x}t^2 + \tfrac{x^3+1}{x^2}t^3 + \tfrac{3x^3+1}{x^3}t^4 + \tfrac{2x^6+6x^3+1}{x^4}t^5 + \cdots \in \mathbb{Q}[x,x^{-1}][[t]],$$

which is legitimate since the power series $Y(t,x)$ has positive valuation, puts the left hand side of (K) to zero, and therefore shows that $U = F(t;x,0)$ is a solution of the *reduced kernel equation*

$$(\mathrm{K_{red}}) \qquad U(t,x) = \frac{Y(t,x)}{t} - \frac{Y(t,x)}{x} U(t, Y(t,x)).$$

The key feature of Equation ($\mathrm{K_{red}}$) is that its unique solution in $\mathbb{Q}[[x,t]]$ is $U = F(t;x,0)$. This is a consequence of the following easy lemma. Here, and in the rest of the article, $\operatorname{ord}_v S$ denotes the valuation of a power series S with respect to some variable v occurring in S.

Lemma 4. *Let $A, B, Y \in \mathbb{Q}[x,x^{-1}][[t]]$ be such that $\operatorname{ord}_t B > 0$ and $\operatorname{ord}_t Y > 0$. Then there exists at most one power series $U \in \mathbb{Q}[[x,t]]$ with*

$$U(t,x) = A(t,x) + B(t,x) \cdot U(t, Y(t,x)).$$

Proof. By linearity, it suffices to show that the only solution in $\mathbb{Q}[[x,t]]$ of the homogeneous equation $U(t,x) = B(t,x) \cdot U(t,Y(t,x))$ is the trivial solution $U = 0$, for if U were non-zero, then the valuation of $B(t,x) \cdot U(t,Y(t,x))$ would be at least equal to $\operatorname{ord}_t B + \operatorname{ord}_t U$, thus strictly greater than the valuation of U, a contradiction. □

We are now ready to reprove the following classical result.

Theorem 5. [12] *$F(t;x,y)$ is algebraic.*

Proof. The computer-assisted part of the proof consists of three steps:

(1) *Guess* an algebraic equation for the series $F(t;x,0)$, by inspection of its initial terms.
(2) *Prove* that the equation guessed in Step (1) admits exactly one solution in $\mathbb{Q}[[x,t]]$, denoted $F_{\mathrm{cand}}(t;x,0)$.
(3) *Prove* that the power series $U = F_{\mathrm{cand}}(t;x,0)$ satisfies ($\mathrm{K_{red}}$).

Once this has been accomplished, the fact that $U = F(t;x,0)$ also satisfies Equation ($\mathrm{K_{red}}$), in conjunction with Lemma 4 (with the choice $A(t,x) = Y(t,x)/t$ and $B(t,x) = -Y(t,x)/x$), implies that $F_{\mathrm{cand}}(t;x,0)$ and $F(t;x,0)$ coincide.

In particular, $F(t;x,0)$ satisfies the guessed equation, and this certifies that $F(t;x,0)$ is an algebraic power series. Since $Y(t,x)$ is algebraic as well, and since the class of algebraic power series is closed under addition, multiplication and inversion, it follows from (K) that $F(t;x,y)$ is algebraic, too. This concludes the proof. □

In the rest of this section, we supply full details on the automated guessing step (1) and on the proving steps (2) and (3).

2.1. **Guessing.** Given the first few terms of a power series, it is possible to determine potential equations that the power series may satisfy, for example by making a suitable ansatz with undetermined coefficients and solving a linear system. In practice, either Gaussian elimination or fast special purpose algorithms based on Hermite-Padé approximation [1] are used. The computation of such candidate equations is known as *automated guessing* and is one of the most widely known features of packages such as Maple's gfun [23, 18].

If sufficiently many terms of the series are provided, automated guessing will eventually find an equation whenever there is one. In principle, it may return false equations, so that—in order to provide fully rigorous proofs—equations discovered by this method must be subsequently proven by an independent argument. In practice, if the method is applied properly, it virtually never delivers false equations, but it may happen that existing equations cannot be recovered because the computation would require too many resources.

In the Kreweras case, the computations are feasible in Maple. We now provide commented code which enables the discovery of an algebraic equation potentially satisfied by $F(t;x,0)$. First, a function f is defined which computes the numbers $f(n;i,j)$ via the multivariate recurrence (1).

```
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
```

Using this function, we compute "sufficiently many" coefficients of $F(t;x,0)$; by trial and error we found that 80 terms are sufficient in our case. The terms are polynomials in x with integer coefficients. We assign the truncated power series to the variable S.

```
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):
```

Next, starting from S, the gfun guessing function seriestoalgeq discovers a candidate for an algebraic equation satisfied by $F(t;x,0)$. For efficiency reasons, we do not use the built-in version of gfun, but a recent one which can be downloaded from http://algo.inria.fr/libraries/papers/gfun.html

```
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T);
```

The guessed polynomial reads:

$$\begin{aligned}
P(T,t,x) = {} & (16x^3t^4 + 108t^4 - 72xt^3 + 8x^2t^2 - 2t + x)\\
& + (96x^2t^5 - 48x^3t^4 - 144t^4 + 104xt^3 - 16x^2t^2 + 2t - x)T\\
& + (48x^4t^6 + 192xt^6 - 264x^2t^5 + 64x^3t^4 + 32t^4 - 32xt^3 + 9x^2t^2)T^2\\
& + (192x^3t^7 + 128t^7 - 96x^4t^6 - 192xt^6 + 128x^2t^5 - 32x^3t^4)T^3\\
& + (48x^5t^8 + 192x^2t^8 - 192x^3t^7 + 56x^4t^6)T^4\\
& + (96x^4t^9 - 48x^5t^8)T^5 + 16x^6t^{10}T^6.
\end{aligned}$$

Running Maple12 on a modern laptop, the whole guessing computation requires about 80Mb of memory and takes less than 20 seconds. Once the candidate polynomial P is guessed, one could proceed to its empirical certification; this can be done in various ways, as explained in [5]. We do not need to do this here, since we are going to *prove* that $F(t;x,0)$ is a root of P.

2.2. **Proving Existence and Uniqueness.** Since $P(1,0,x)=0$ and $\frac{\partial P}{\partial T}(1,0,x) = -x$, the implicit function theorem implies that P admits a unique root $F_{\text{cand}}(t;x,0)$ in $\mathbb{Q}((x))[[t]]$. It follows that P has *at most one* root in $\mathbb{Q}[[x,t]]$ and that this root, if it exists, belongs to $\mathbb{Q}[x,x^{-1}][[t]]$.

Proving *the existence* of a root of P in $\mathbb{Q}[[x,t]]$ is less straightforward: this time, the equalities $P(1,0,0)=0$ and $\frac{\partial P}{\partial T}(1,0,0)=0$ prevent us from directly invoking the implicit function theorem. We are thus faced with a clumsy technical complication, since what we really need to prove is that the root $F_{\text{cand}}(t;x,0)$ actually belongs to $\mathbb{Q}[[x,t]]$: otherwise, the substitution of $U = F_{\text{cand}}(t;x,0)$ in Equation (K_{red}), used in Step (3) of the proof of Thm. 5, would not be legitimate.

To circumvent this complication, we exploit the fact that, when seen in $\mathbb{Q}(x)[T,t]$, the polynomial $P(T,t,x)$ defines a curve of genus zero over $\mathbb{Q}(x)$, which can thus be rationally parameterized. Precisely, using Maple's algcurves package, the rational functions $R_1(u,x)$ and $R_2(u,x)$ defined by:

$$R_1(u,x) = \frac{u(u+1)(1+2u+u^2+u^2x)^2}{h(u,x)},$$
$$R_2(u,x) = \frac{(u^4x^2+2u^2(u+1)^2x+1+4u+6u^2+2u^3-u^4)h(u,x)}{(1+u)^2(1+2u+u^2+u^2x)^4},$$

with

$$h(u,x) = u^6x^3+3u^4(u+1)^2x^2+3u^2(u+1)^4x+(u+1)^3(5u^3+3u^2+3u+1),$$

are found to share the following properties:

- $P(R_2(u,x),R_1(u,x),x)=0$;
- there exists a (unique) power series
  $$u_0(t,x) = t+t^2+(x+1)t^3+(2x+5)t^4+(2x^2+3x+9)t^5+\dots$$
  in $\mathbb{Q}[[x,t]]$ such that $R_1(u_0,x)=t$ and $u_0(0,x)=0$.

While the first property is easily checked by direct calculation, the second one is a consequence of the implicit function theorem, since $Q(u,t,x) = R_1(u,x)-t$ satisfies $Q(0,0,0)=0$ and $\frac{\partial Q}{\partial u}(0,0,0)=1$.

The existence proof of a power series solution of P is then completed using the following argument: R_2 having no pole at $u=0$, and the valuation of u_0 with respect to t being positive, the composed power series $R_2(u_0(t,x),x)$ is well defined in $\mathbb{Q}[[x,t]]$ and it satisfies

$$P(R_2(u_0,x),t,x) = P(R_2(u_0,x),R_1(u_0,x),x) = 0.$$

Therefore, $F_{\text{cand}}(t;x,0) = R_2(u_0(t,x),x)$ is the unique solution of P in $\mathbb{Q}[[x,t]]$.

2.3. **Proving compatibility with the reduced kernel equation.** We finally show that the series $F_{\text{cand}}(t;x,0)$ so defined satisfies (K_{red}). This can be done by resorting to closure properties for algebraic power series. These closure properties are performed by means of resultant computations, following Lemma 6 below.

One possibility is to first prove that the power series

$$S(t,x) = \frac{Y(t,x)}{t} - \frac{Y(t,x)}{x}F_{\text{cand}}(t;Y(t,x),0) \ \in \mathbb{Q}[x,x^{-1}][[t]]$$

is a root of the polynomial $P(T,t,x)$, and then to use the fact that P has only one root in $\mathbb{Q}[x,x^{-1}][[t]]$, namely $F_{\text{cand}}(t;x,0)$. This will imply that S and $F_{\text{cand}}(t;x,0)$ coincide, and thus that $F_{\text{cand}}(t;x,0)$ satisfies (K_{red}), as desired.

The main point of this approach is that, since the power series Y and $F_{\text{cand}}(t;x,0)$ are both algebraic, finding a polynomial which annihilates the series S can be done in an *exact* manner, without having to appeal to guessing routines. Moreover, the minimal polynomial of S can be determined by factoring an annihilating polynomial obtained through a resultant computation, and, if necessary, by matching the irreducible factors against the initial terms of the series S.

For the sake of completeness, we recall the following classical facts.

Lemma 6. *Let $\mathbb{K}$ be a field and let $P,Q \in \mathbb{K}[T,t,x]$ be annihilating polynomials of two algebraic power series A,B in $\mathbb{K}[x,x^{-1}][[t]]$. Then*

- *pA is algebraic for every $p \in \mathbb{K}(t,x)$, and it is a root of $p^{\deg_T P} P(T/p,t,x)$.*
- *$A \pm B$ is algebraic, and it is a root of $\operatorname{res}_z(P(z,t,x), Q(\pm(T-z),t,x))$.*
- *AB is algebraic, and it is a root of $\operatorname{res}_z(P(z,t,x), z^{\deg_T Q} Q(T/z,t,x))$.*
- *If $\operatorname{ord}_x B > 0$, then $A(t,B(t,x))$ is algebraic, and it is a root of $\operatorname{res}_z(P(T,t,z), Q(z,t,x))$.*

Since $z/t - z/xF_{\text{cand}}(t;z,0)$ is a root of (the numerator of) $P(x/z(z/t-T),t,z)$ and since Y is a root of $(x+T+x^2T^2)t - xT$, Lemma 6 suggests continuing our Maple session by constructing a polynomial in $\mathbb{Q}[T,t,x]$ which has S as a root:

```
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> res := resultant(numer(pol(x/z*(z/t-T),t,z)), ker(z,t,x), z):
> factor(primpart(res,T));
```

The output of the last line is $P(T,t,x)^2$, which proves that S is a root of $P(T,t,x)$. This completes the proof of Theorem 5.

2.4. **Consequences.** Setting x to 0 in P leads to the conclusion that the generating series $F(t;0,0)$ of Kreweras excursions is a root of the polynomial $64t^6T^3+16t^3T^2+T-72t^3T+54t^3-1$. An argument similar to that used in the proof of Corollary 2 then implies that the coefficients a_n of $F(t;0,0)$ satisfy the linear recursion

$$(n+6)(2n+9)a_{n+3} - 54(n+2)(n+1)a_n = 0, \qquad a_0 = 1,\ a_1 = 0,\ a_2 = 0,$$

which in turn provides an alternative proof of the classical fact [17, 12, 6] that the series $F(t;0,0)$ is both algebraic and hypergeometric, and it has the closed form

$$F(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ 3/2 & 2 & \end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

3. Gessel walks

For establishing the proof of Theorem 3, we apply essentially the same reasoning that was applied in the previous section for proving Theorem 5. The main difference is that the intermediate expressions get very big, so that they can only be handled by special purpose software (see the data provided on our website [4]). There are also some additional complications which require to vary the arguments slightly. In this section, we point out these complications, describe how to circumvent them, and we document our computations.

The numbers $g(n,i,j)$ of Gessel walks of length n ending at $(i,j) \in \mathbb{Z}^2$ satisfy the recurrence equation

$$g(n+1;i,j) = g(n;i-1,j-1) + g(n;i+1,j+1) + g(n;i-1,j) + g(n;i+1,j)$$

for $n,i,j \geq 0$. Together with appropriate boundary conditions, this equation implies that the generating function

$$G(t;x,y) = \sum_{n=0}^{\infty}\Big(\sum_{i,j=0}^{\infty} g(n;i,j)x^i y^j\Big)t^n,$$

which we seek to prove algebraic, satisfies the equation

$$(\mathrm{K}^{\mathrm{G}}) \qquad ((1+y+x^2y+x^2y^2)t - xy)G(t;x,y) = (1+y)t\,G(t;0,y) + t\,G(t;x,0) - t\,G(t;0,0) - xy.$$

This is the starting point for the kernel method.

In this case, because of lack of symmetry with respect to x and y, there are two different ways to put the left hand side to zero, using the two substitutions

$$\begin{aligned} y \to Y(t,x) &:= -\left(tx^2 - x + t + \sqrt{(tx^2-x+t)^2 - 4t^2x^2}\right)/(2tx^2) \\ &= \tfrac{1}{x}t + \tfrac{x^2+1}{x^2}t^2 + \tfrac{x^4+3x^2+1}{x^3}t^3 + \tfrac{x^6+6x^4+6x^2+1}{x^4}t^4 + \cdots \\ \text{and} \quad x \to X(t,y) &:= \left(y - \sqrt{y(y-4t^2(y+1)^2)}\right)/(2ty(y+1)) \\ &= \tfrac{y+1}{y}t + \tfrac{(y+1)^3}{y^2}t^3 + \tfrac{2(y+1)^5}{y^3}t^5 + \tfrac{5(y+1)^7}{y^4}t^7 + \cdots \end{aligned}$$

They yield the equations

$$(\mathrm{K}^{\mathrm{G}}_{\mathrm{red}}) \qquad \begin{aligned} G(t;x,0) &= xY(t,x)/t + G(t;0,0) - (1+Y(t,x))G(t;0,Y(t,x)), \\ (1+y)G(t;0,y) &= X(t,y)y/t + G(t;0,0) - G(t;X(t,y),0), \end{aligned}$$

respectively. Note that the first equation is free of y while the second is free of x. If we rename y to x in the second equation, then all quantities belong to $\mathbb{Q}[x,x^{-1}][[t]]$. Note also that we can write $G(t;x,0) = G(t;0,0) + xU(t,x)$ and $G(t;0,x) = G(t;0,0) + xV(t,x)$ for certain power series $U,V \in \mathbb{Q}[[x,t]]$. In terms of U and V, the two equations above are then equivalent to

$$(\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}) \qquad \begin{aligned} xU(t,x) &= xY(t,x)/t - (1+Y(t,x))G(t;0,0) \\ &\qquad - Y(t,x)(1+Y(t,x))V(t,Y(t,x)), \\ (1+x)xV(t,x) &= X(t,x)x/t - (1+x)G(t;0,0) - X(t,x)U(t,X(t,x)). \end{aligned}$$

The two equations ($\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}$) correspond to the equation ($\mathrm{K}_{\mathrm{red}}$) in Section 2. The situation here is more complicated in two respects. First, we have two equations and two unknown power series U and V rather than a single equation with a single unknown power series $F(t;x,0)$; this difference originates from the lack of symmetry of $G(t;x,y)$ with respect to x and y, which itself comes from the asymmetry of the Gessel step set with respect to the main diagonal of $\mathbb{N}^2$. Second, the two equations for U and V still contain $G(t;0,0)$ while there is no term $F(t;0,0)$ present in ($\mathrm{K}_{\mathrm{red}}$); this difference originates from the fact that Gessel's step set contains the admissible step $\swarrow$, as opposed to Kreweras's step set. The occurrence of $G(t;0,0)$ in the equations ($\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}$) is not really problematic, as we know this power series explicitly thanks to Theorem 1. As for the other difference, we need the following variation of Lemma 4.

Lemma 7. *Let* $A_1, A_2, B_1, B_2, Y_1, Y_2 \in \mathbb{Q}[x, x^{-1}][[t]]$ *be such that* $\operatorname{ord}_t B_1 > 0$, $\operatorname{ord}_t B_2 > 0$, $\operatorname{ord}_t Y_1 > 0$ *and* $\operatorname{ord}_t Y_2 > 0$. *Then there exists at most one pair* $(U_1, U_2) \in \mathbb{Q}[[x, t]]^2$ *with*

$$U_1(t, x) = A_1(t, x) + B_1(t, x)U_2(t, Y_1(t, x)),$$
$$U_2(t, x) = A_2(t, x) + B_2(t, x)U_1(t, Y_2(t, x)).$$

Proof. By linearity, it suffices to show that the only solution (U_1, U_2) in $\mathbb{Q}[[x, t]] \times \mathbb{Q}[[x, t]]$ of the homogeneous system

$$U_1(t, x) = B_1(t, x)U_2(t, Y_1(t, x)),$$
$$U_2(t, x) = B_2(t, x)U_1(t, Y_2(t, x))$$

is the trivial solution $(U_1, U_2) = (0, 0)$, for otherwise, if both U_1 and U_2 were non-zero, then the valuation of $B_1(t, x)U_2(t, Y_1(t, x))$ would be strictly greater than the valuation of U_2, and the valuation of $B_2(t, x)U_1(t, Y_2(t, x))$ would be strictly greater than the valuation of U_1, thus $\operatorname{ord}_t(U_1) > \operatorname{ord}_t(U_2) > \operatorname{ord}_t(U_1)$, a contradiction. Therefore, one of U_1, U_2 is zero, and the system now implies that both are zero. □

By a slightly more careful analysis, the lemma could be refined further such as to show that there is only one triple of power series (U, V, G) with $U, V \in \mathbb{Q}[[x, t]]$ and $G \in \mathbb{Q}[[t]]$ (free of x) which satisfies ($\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}$) with $G(t; 0, 0)$ replaced by G. In this version, the proof could be completed without reference to the independent proof of Thm. 1.

Either way, we can in principle proceed from this point as in Section 2. Out of convenience, we choose to regard $G(t; 0, 0)$ as known. Again, we divide the remaining task in three steps:

(1) *Guess* defining algebraic equations for U and V, by inspecting the initial terms of $G(t; x, 0)$, resp. of $G(t; 0, x)$.
(2) *Prove* that each of the guessed equations has a unique solution in $\mathbb{Q}[[x, t]]$, denoted $U_{\mathrm{cand}}(t; x, 0)$, resp. $V_{\mathrm{cand}}(t; x, 0)$.
(3) *Prove* that U_{cand} and V_{cand} indeed satisfy the two equations in ($\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}$).

Once this has been accomplished, Lemma 7 implies that the candidate series are actually equal to U and V, respectively, and so these series as well as $G(t; x, 0)$ and $G(t; 0, y)$ are in particular algebraic. Then equation (K^{G}) implies that $G(t; x, y)$ is algebraic, too. This then completes the proof of Thm. 3.

3.1. **Guessing.** In the beginning, we had no reason to suspect that $G(t; x, y)$ is algebraic, as even the specialization $G(t; 0, 0)$ was widely believed to be transcendental. Our goal was to find out whether $G(t; x, y)$ is D-finite, and so we searched in the beginning only for potential differential operators annihilating $G(t; x, 0)$ and $G(t; 0, y)$, respectively. Since no such operators could be found by the guessers implemented in packages like Maple's gfun or Mathematica's GeneratingFunctions, it was clear that if they existed, they would be large. We computed the first 1000 terms of $G(t; x, 0)$ and $G(t; 0, y)$ and proceeded as follows.

For several specific evaluation points $x_0 = 1, 2, 3, \ldots$ and several specific primes p, we used a Magma implementation of the (FFT-based) Beckermann-Labahn superfast algorithm for Hermite-Padé approximation [1] to construct some differential operators $\mathcal{L}^{(i)}_{p,x_0} \in \mathbb{Z}_p[t]\langle D_t \rangle$ $(i = 1, 2, \ldots)$ with the property

$$\mathcal{L}^{(i)}_{p,x_0}\, G(t; x_0, 0) = \mathrm{O}(t^{1000}) \bmod p.$$

Several operators of order 14 with coefficients of degree at most 43 in t could be found for each choice of x_0 and p. They can be regarded as the homomorphic images of certain (unknown) operators $\mathcal{L}^{(i)} \in \mathbb{Q}(x)[t]\langle D_t\rangle$ with $\mathcal{L}^{(i)}\,G(t;x,0) = \mathrm{O}(t^{1000})$ for which we expect $\mathcal{L}^{(i)}\,G(t;x,0) = 0$.

For every choice of x_0 and p, we next computed the greatest common right divisor in $\mathbb{Z}_p(t)\langle D_t\rangle$ of the operators $\mathcal{L}^{(i)}_{p,x_0}$ using an algorithm of Grigoriev [13]. This yielded, for every choice x_0 and p, a single operator $\mathcal{L}_{p,x_0}$ of order 11 with coefficients of degree at most 96 in t. This operator is likely to be the homomorphic image of the least order operator $\mathcal{L} \in \mathbb{Q}(x)[t]\langle D_t\rangle$ with $\mathcal{L}\,G(t;x,0) = 0$.

From the operators $\mathcal{L}_{p,x_0}$ for various choices x_0 and p we then constructed a candidate for the preimage $\mathcal{L}$ using standard algorithms in computer algebra for rational function interpolation and rational number reconstruction [11]. We obtained a convincing candidate operator of order 11 with degree 96 in t and degree 78 in x whose longest integer coefficient has 61 decimal digits. This is the operator posted on our website [4]. We post there also a candidate operator for $G(t;0,y)$ of order 11 and degree 68 in t and 28 in y containing integers with no more than 51 decimal digits, which was found in the same way. (Applying the reconstruction algorithms directly to the $\mathcal{L}^{(i)}_{p,x_0}$ instead of $\mathcal{L}_{p,x_0}$ is prohibitively expensive because the degree in x of their preimage is very large. In going via the gcrd, we gained a degree drop in x from >1500 to 28 payed by a moderate degree raise in t from 43 to 96.)

There are a number of tests which can be performed to experimentally sustain the evidence that a guessed differential operator s correct (see our paper [5] for a collection of such tests), and our operators successfully pass them all. One of the tests consists of checking whether the candidate operator $\mathcal{L}$ for $G(t;x,0)$ (and analogously for $G(t;0,y)$) is what is called *globally nilpotent* [10]. By definition, the order 11 operator $\mathcal{L}$ is globally nilpotent if it right-divides for almost all primes p the pure power D_t^{11p} in $\mathbb{Z}_p(x,t)\langle D_t\rangle$. When checking that this is the case for all primes $p < 100$, we observed that the $\mathcal{L}$ in fact even right-divide D_t^p for all primes p we checked. According to a famous conjecture of Grothendieck [22], this happens if and only if the operator in question has only algebraic solutions. So even though this conjecture is still wide open, there is definitely something interesting going on here: either $G(t;x,y)$ is algebraic or we have found operators which very much look like counterexamples to Grothendieck's conjecture.

This finally suggested to search for potential polynomial equations satisfied by the power series $U, V \in \mathbb{Q}[[x,t]]$ defined by $G(t;x,0) = G(t;0,0) + xU(t,x)$ and $G(t;0,x) = G(t;0,0) + xV(t,x)$. Using again guessing techniques based on fast modular Hermite-Padé approximation, combined with an interpolation scheme, we discovered two polynomials $P_1(T,t,x) \in \mathbb{Q}[T,t,x]$ and $P_2(T,t,y) \in \mathbb{Q}[T,t,y]$ satisfying

$$P_1(U(t,x),t,x) = 0 \bmod t^{1200} \quad \text{and} \quad P_2(V(t,y),t,y) = 0 \bmod t^{1200}.$$

These polynomials are posted on our website [4]. The polynomial P_1 has degrees 24, 44, and 32 with respect to T, t, and x, respectively, and involves integers with no more than 21 decimal digits. The polynomial P_2 has degrees 24, 46, and 56 with respect to T, t, and y, respectively, and involves integers with no more than 27 decimal digits. Spelled out on paper, they would together fill about thirty pages. (For comparison: the differential operators $\mathcal{L}$ would fill more than 500 pages.)

We now proceed to prove that the guessed polynomials P_1 and P_2 are correct.

3.2. **Proving Existence and Uniqueness.** As in the case of Kreweras's walks, the implicit function theorem does not apply to these polynomials, but unlike in the Kreweras case, an existence proof using a suitable rational parameterization is not possible either, because the polynomials at hand define curves of positive genus, and therefore a rational parameterization does not exist.

In order to obtain a proof in this situation, we proceeded as follows:

- First we used Theorem 3.6 of McDonald [19] to obtain the existence of a series solution
$$\sum_{p,q\in\mathbb{Q}} c_{p,q} t^p x^q$$
with $c_{p,q} = 0$ for all (p, q) outside a certain halfplane $H \subseteq \mathbb{Q}^2$.
- Next, we computed a system of bivariate recurrence equations with polynomial coefficients that the coefficients $c_{p,q}$ must necessarily satisfy. This can be done in principle by software packages such as Chyzak's `mgfun` [9] or Koutschan's `HolonomicFunctions.m` [16]. However, for reasons of efficiency we used our own implementation of the respective algorithms.
- The form of the recurrences together with the shape of the halfplane H imply that the coefficients $c_{p,q}$ of any solution can be nonzero only in a finite union of cones $v + \mathbb{N}u + \mathbb{N}w$ with vertices $v \in \mathbb{Q}^2$ and basis vectors $u, w \in \mathbb{Q}^2$ that can be computed explicitly. If $c_{p,q} \neq 0$ for some index (p, q) in such a cone, then also the coefficient at the cone's vertex is nonzero.
- Applying McDonald's generalization of Puiseux's algorithm, we determined the first coefficients of series solutions to an accuracy that all further coefficients belong to some translate of H which contains no vertices.
- As one of these partial solutions contained no terms with fractional powers, it was possible to conclude that the entire series contains no terms with fractional exponents. Reference to u and w implied that this partial solution could also not contain any terms with negative integral exponents, so the only remaining possibility was that the solution is in fact a power series.

A full description of the argument requires a somewhat lengthy discussion of a number of technical details, which we prefer to avoid here. In a supplement to this article provided on our website [4] we carry out existence proofs in full detail that both P_1 and P_2 admit some power series solutions U_{cand} and V_{cand}, respectively.

3.3. **Proving compatibility with the reduced kernel equation.** It remains to show that these solutions U_{cand} and V_{cand} satisfy the system ($\mathrm{K}^{\mathrm{G},2}_{\mathrm{red}}$). Because of $X(t, Y(t, x)) = x$, the substitution $x \to Y(t, x)$ transforms the second equation of that system to the first. Therefore, it suffices to prove the second equation:

$$(1+x)xV_{\text{cand}}(t; x, 0) = X(t, x)x/t - (1+x)G(t; 0, 0) - X(t, x)U_{\text{cand}}(t; X(t, x), 0). \tag{2}$$

If we define $G_1(t, x) = G(t; 0, 0) + xU_{\text{cand}}(t; x, 0)$ and $G_2(t, x) = G(t; 0, 0) + xV_{\text{cand}}(t; x, 0)$, the last equation is equivalent to

$$(1 + x)G_2(t, x) - G(t; 0, 0) = xX(t, x)/t - G_1(t, X(t, x)). \tag{3}$$

By Corollary 2 and Lemma 6, the power series

$$(1 + x)G_2(t, x) - G(t; 0, 0) \quad \text{and} \quad xX(t, x)/t - G_1(t, X(t, x))$$

are algebraic and we can compute their minimal polynomials—at least in theory. Now the polynomials P_1 and P_2 are so big that the required resultant computations cannot be carried out by Maple or Mathematica.

There are efficient special purpose algorithms available for the particular kind of resultants at hand [3] and our Magma implementation of these algorithms is able to perform the necessary computations. It turns out that the minimal polynomials for both power series are identical. It is provided electronically on the website to this article. After determining a suitable number of initial terms of both series and observing that they match, it can be concluded that Equations (3) and (2) hold. This completes the proof of Theorem 3.

Note that we have shown that $G(t;x,y)$ is algebraic without actually constructing its minimal polynomial. This polynomial will, in fact, be pretty large. From the sizes of the minimal polynomials of $G(t;x,0)$ and $G(t;0,y)$, which we know explicitly, it can be deduced that the minimal polynomial $p(T,t,x,y)$ of $G(t;x,y)$ will have degrees 72, 141, 263, and 287 with respect to T, t, x, and y, respectively, and thus consist of more than 750 Mio terms.

3.4. **Consequences.** The fact that $G(t;x,y)$ is algebraic has some immediate consequences which are of combinatorial interest, including the following:

Corollary 8.
- *The series* $G(t;1,1), G(t;0,1), G(t;1,0)$ *are algebraic.*
- *For every specific point* (i,j)*, the series* $G_{i,j} := \sum_{n=0}^{\infty} g(n;i,j)t^n$ *is algebraic.*
- *The series* $G(t;x,y)$ *is D-finite with respect to any of the variables* x, y, t*.*
- *For every fixed* i *and* j*, the number* $g(n;i,j)$ *can be computed with* $\mathrm{O}(n)$ *arithmetic operations.*
- *For every fixed* x *and* y*, the coefficient* $\langle t^n\rangle G(t;x,y)$ *can be computed with* $\mathrm{O}(n)$ *arithmetic operations.*

The second item confirms the conjecture of Petkovšek and Wilf [21] that $g(n;0,j)$ is P-finite in n for every fixed j, and it refutes their conjecture that $g(n;2,0)$ is not P-finite.

Acknowledgments. We thank Frédéric Chyzak, Philippe Flajolet, Christoph Koutschan, Mireille Bousquet-Mélou, Preda Mihăilescu, Peter Paule, Tanguy Rivoal, Bruno Salvy, Josef Schicho and Doron Zeilberger, for stimulating discussions.

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Appendix (by Mark van Hoeij)

The following formula holds:

$$G(t;x,y) = \frac{\frac{64(u(v+1)-2v)v^{3/2}}{x(u^2-v(u^2-8u+9-v))^2} - \frac{(w-1)^4(wy+1)yv^{-3/2}}{t(y+1)(w+1)(w^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)} \tag{4}$$

where v, u, and w are solutions of

$$(v-1)(v+3)^3 - 256v^3t^2 = 0$$

$$x(v^2-1)u^3 - 2v(3x+5xv-8vt)u^2 - xv(v^2-24v-9)u + 2v^2(xv-9x-8vt) = 0$$

and

$$y(v-1)w^3 + y(v+3)w^2 + (v+3)w + v - 1 = 0.$$

The only ingredients that were used to compute this formula were: (i) the minimal polynomials P_1 and P_2 for $G(t;x,0)$ and $G(t;0,y)$, as copied from [4], (ii) techniques, explained below, to simplify an algebraic expression that is given by a large minimal polynomial, and (iii) equation (K^G) from Section 3.

Let $F = \mathbb{Q}(t,x)$ and consider the tower of algebraic extensions $F \subset F(v) \subset F(v,u) \subset F(v,u,\sqrt{v})$. Now $G(t;x,0)$ is a primitive element of $F(v,u,\sqrt{v})$ over F, and as for most primitive elements, its minimal polynomial P_1 is large.

Since primitive elements lead to large expressions, the reverse process (if $F \subset K$ is given by a single minimal polynomial, find a tower of subfields $F \subset \cdots \subset K$) can then be expected to lead to smaller expressions. Moreover, there exist algorithms for computing subfields (e.g. by Klüners). In light of this it seemed very likely that a smaller expression for $G(t; x, 0)$ could be computed.

It is quite possible that there exist implementations that can directly simplify (i.e. subfields or some other type of simplification) a polynomial relation the size of P_1. Nevertheless, it does not hurt to speed up the computation by first searching for some simple simplifications that reduce the bitsize of this polynomial relation. For instance, one can do the following:

(1) If $P(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots$ then one tries $T \mapsto T - a_{n-1}/(n a_n)$. If this makes the bitsize smaller, then use it, otherwise, do not use it.
(2) If $\sigma : T \mapsto -T$, and if $\mathrm{monic}(P) = \mathrm{monic}(\sigma(P))$ then $T \mapsto T^{1/2}$ will send P to a polynomial in T of half the degree. Here "monic" is a procedure that divides a polynomial by its leading coefficient.
(3) If P is a bivariate polynomial in x, t with coefficients in some field $L = \mathbb{Q}(T)$, so $P \in L[x, t] \subset L(x, t)$, and if P is invariant under $\sigma : (x, t) \mapsto (-x, -t)$ then P is also an element of the fixed field of σ, which in this example is $L(x^2, tx)$. So in this case, we can introduce new variables x', t' to represent x^2 and tx, and rewrite P in terms of x', t'.
(4) Suppose that $P(T) = \sum a_i T^i$ and suppose that many of the a_j/a_i have a factor f with multiplicity close to $j - i$. Then $\mathrm{monic}(P(T/f))$ is likely to be smaller than $P(T)$.
(5) Suppose that $P(x, t)$ is invariant under $\sigma : (x, t) \mapsto (t, x)$. As before, compute the invariant subfield of $L(x, t)$ to find $L(x + t, xt)$ and introduce new variables to rewrite P in terms of $x + t$ and xt.

By taking steps (1), (2), (3), (4), (4), (5), (1), (2), (1), (1)+(4), the size (measured by Maple's length command) of the polynomial relation P_1 was reduced from 58375 to: 21254, 21073, 20349, 21005, 20493, 10977, 6442, 6369, 6139, 5505 respectively (some of the steps were meant to reduce the degree instead of the bitsize). Next, a subfield was computed. This was done by substituting values for one parameter (the other parameter was not involved in this first field extension), then computing subfields, and then using reconstruction, although a more direct approach might have worked as well. By reversing the above mentioned steps, the computed subfield becomes $F \subset F(v)$. The minimal polynomial P_1 must then be reducible over $F(v)$. The same holds for the reduced polynomial obtained by the steps listed above, and factoring reduces the size from 5505 to 322. A call to Maple's parameterization code leads to a further reduction. Reversing all these steps produces $G(t; x, 0)$, and a similar computation was done for $G(t; 0, y)$. The steps of the computation are given in detail on the website [4] of this article.

To verify correctness of formula (4), one can use resultants, as was done at [4].

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