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THE COMPLETENESS AND COMPACTNESS OF A THREE-VALUED FIRST-ORDER LOGIC

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ABSTRACT. The strong completeness and the compactness of a three-valued first order predicate calculus with two distinguished truth-values are obtained. The system was introduced in *Sur un problème de Jaśkowski*, I.M.L. D'Ottaviano and N.C. A. da Costa, C.R. Acad.Sc. Paris 270A (1970),pp.1349-1353, and has several applications, especially in paraconsistent logics.

1. INTRODUCTION.

A theory T is said to be *inconsistent* if it has as theorems a formula and its negation; and it is said to be *trivial* if every formula of its language is a theorem.

A logic is *paraconsistent* if it can be used as the underlying logic for inconsistent but nontrivial theories.

Jaśkowski, motivated by some ideas of Łukasiewicz, was the first logician to construct a system of paraconsistent propositional logic (see [11], [12] and [13]). His principal motivations were the following: the problem of the systematization of theories which contain contradictions, as it occurs in dialectics; the study of theories in which there are contradictions caused by vagueness; the direct study of some empirical theories whose postulates or basic assumptions could be considered, under certain aspects, as contradictory ones (see [2] and [3]).

Jaśkowski proposed the problem of constructing a propositional calculus having the following properties:

i) an inconsistent system based on such a calculus should not be neccessarily trivial;

ii) the calculus should be sufficiently rich as to make posible most of the usual reasonings;

iii) the calculus should have an intuitive meaning.

Jaśkowski himself introduced a propositional calculus which he named "Discussive logic" and which was a solution to the problem. However he did recognize it was not the only solution (or even the best); in [11] he states:

"Obviously, these conditions do not univocally determine the solution, since they may be satisfied in varying degrees, the satisfaction of condition (iii) being rather difficult to appraise objectively".

In a previous paper (see [10]), we presented a propositional system, denoted by $\mathbf{J}_3$, which is another solution to Jaśkowski's problem. A characteristic of $\mathbf{J}_3$ is that it is a three-valued system with two distinguished truth-values. Furthermore, it reflects some aspects of certain types of modal logics.

In the same paper, we extended J_3 to the first-order predicate calculus with equality $\mathbf{J}_3^{*=}$.

Some of these results about J_3 were improved by J. Kotas and N.C.A. da Costa (see [15]).

Our aim here is to develop further the calculus $\mathbf{J}_3$.

In Sec. 2 we axiomatize $\mathbf{J}_3$ and establish relations between this calculus and several known logical systems like, for example, intuitionism. We especially emphasize the close analogy between J_3 and Łukasiewicz' three-valued propositional calculus $\mathcal{L}_3$.

Our solution to Jaśkowski's problem is discussed in the latter part of Sec 2.

In Sec. 3 we introduce the $\mathbf{L}_3$-Languages, among whose predicate symbols may appear in addition to identity other equalities. We axiomatize $\mathbf{J}_3$-theories, which are three-valued extensions of $\mathbf{J}_3^{*=}$, and we introduce a semantics for them.

In Sec. 4, after obtaining some theorems about first-order $\mathbf{J}_3$-theories, we define a strong equivalence which is compatible with the fact that the matrices defining $\mathbf{J}_3$ have more than one distinguished truth-value. This relation allows us to prove the Equivalence Theorems for $\mathbf{J}_3$-theories and the Reduction Theorem for non-Trivialization.

Finally, in Sec. 5, after giving a suitable definition of canonical structure, we present a Henkin-type proof for the Completeness Theorem and the Compactness Theorem.

In this paper, definitions, theorems and proofs, when analogous to the corresponding classical ones, will be omitted.

The Model-theory we developed for $\mathbf{J}_3$ allows us to obtain $\mathbf{J}_3$-versions of the following classical results: Model Extension Theorem, Łoś-Tarski Theorem, Chang-Łoś Susko Theorem, Tarski Cardinality Theorem, Löwenheim-Skolem Theorem, Quantifier Elimination Theorem and many of the usual theorems on categoricity.

Some of the above results about J_3 were also extended to $\mathbf{J}_n$-theories, $3 \leq n \leq \aleph_0$.

The mentioned results about J_n-theories and Model-theory will appear elsewhere.

2. THE CALCULUS J_3.

The *propositional calculus* J_3 is given by the matrix $M = \langle\{0,\frac{1}{2},1\}, \{\frac{1}{2},1\}, \vee, \nabla, \neg\rangle$, where $\vee$, ∇ and $\neg$ are defined as follows:

A∨B: A \ B	0	½	1
0	0	½	1
½	½	½	1
1	1	1	1

A	∇A
0	0
½	1
1	1

A	¬A
0	1
½	½
1	0

The *set of truth-values* and the *set of distinguished truth-values* are denoted by V and V_d respectively.

The *formulas* of J_3 are constructed as usually from the propositional variables, by means of $\vee$, ∇ and $\neg$, and parentheses. To write the formulas, schemas, etc. we use the conventions and notations of [14], with evident adaptations.

The concept of a *truth-function* is the usual one. The truth-functions defined by the tables above are denoted by $H_\vee$, H_∇, and $H_\neg$.

A *truth-valuation* ν for J_3 and the *truth-value* $\nu(A)$ for a formula A are defined in the standard way; and we observe that A is *valid* in M if, for every evaluation ν, $\nu(A)$ belongs to V_d (see, for example, [22]).

The following *abbreviations* will be used:

$$A \,\&\, B =_{def} \neg(\neg A \vee \neg B)$$
$$\Delta A =_{def} \neg\nabla\neg A$$
$$\neg^* A =_{def} \neg\nabla A$$
$$A \rightarrowtail B =_{def} \nabla\neg A \vee B$$
$$A \twoheadrightarrow B =_{def} (A \rightarrowtail B)\,\&\,(\neg B \rightarrowtail \neg A)$$
$$A \supset B =_{def} \neg\nabla A \vee B$$
$$A \equiv B =_{def} (A \supset B)\,\&\,(B \supset A)$$

$\neg$ is called *weak negation* or simply negation, $\neg^*$ is called *strong negation*, and $\supset$ *basic implication* of J_3.

We present the tables of some of the non-primitive connectives:

A	¬*A
0	1
½	0
1	0

A	ΔA
0	0
½	0
1	1

A↠B: A \ B	0	½	1
0	1	1	1
½	½	1	1
1	0	½	1

$A \supset B$

B / A	0	½	1
0	1	1	1
½	0	½	1
1	0	½	1

$A \equiv B$

B / A	0	½	1
0	1	0	0
½	0	½	½
1	0	½	1

In the following theorems, we mention only those results which are useful to the proofs of later theorems.

THEOREM 2.1. *The following schemas of* $\mathbf{J}_3$ *are valid in* M:

$\neg\neg A \equiv A$	$\nabla A \equiv A$
$\neg^* A \supset \neg A$	$\nabla A \equiv \nabla\nabla A$
$A \vee \neg A$	$\neg A \vee \nabla A$
$\neg(A \& \neg A)$	$A \& \neg A \equiv \neg A \& \nabla A$
$A \& (B \vee \neg B) \equiv A$	$A \vee \nabla A \equiv \nabla A$
$\neg(A \vee B) \supset \neg A \& \neg B$	$\neg\nabla A \supset (\nabla A \supset B)$
$A \vee B \equiv \neg(\neg A \& \neg B)$	$A \supset (\neg\nabla A \supset B)$
$\neg(A \& B) \equiv \neg A \vee \neg B$	$\nabla(A \& B) \equiv \nabla A \& \nabla B$
$\nabla A \equiv \neg\Delta\neg A$	$\nabla(A \vee B) \equiv \nabla A \vee \nabla B$
$(A \supset \neg A) \supset \neg A$	$A \rightarrowtail (B \rightarrowtail A)$
$(\neg A \supset A) \supset A$	$(\neg A \rightarrowtail \neg B) \rightarrowtail (B \rightarrowtail A)$
$\neg(\nabla A \vee \neg\nabla A) \supset B$	$(A \rightarrowtail B) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail (A \rightarrowtail C))$
$((A \supset B) \supset A) \supset A$	$(A \rightarrowtail \neg B) \rightarrowtail A) \rightarrowtail A$
$(A \supset B) \supset (A \rightarrowtail B)$	$\Delta(A \rightarrowtail B) \rightarrowtail \Delta(\Delta A \rightarrowtail \Delta B)$
$(A \rightarrowtail B) \supset (\neg B \rightarrowtail \neg A)$.	

THEOREM 2.2. *The following schemas are not valid in* $\mathbf{J}_3$:

$\neg A \supset (A \supset B)$	$(A \supset B) \supset (\neg B \supset \neg A)$
$A \supset (\neg A \supset B)$	$(\neg A \supset \neg B) \supset (B \supset A)$
$\neg A \supset (A \supset \neg B)$	$(A \supset \neg B) \supset (B \supset \neg A)$
$A \supset (\neg A \supset \neg B)$	$(\neg A \supset B) \supset (\neg B \supset A)$
$A \& \neg A \supset B$	$(A \equiv B) \supset (\neg A \equiv \neg B)$
$A \& \neg A \supset \neg B$	$A \vee (B \& \neg B) \equiv A$
$(A \equiv \neg A) \supset B$	$A \supset B \equiv \neg(A \& \neg B)$
$(A \equiv \neg A) \supset \neg B$	$A \supset B \equiv \neg A \vee B$
$(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$.	

It can be verified that, instead of $\vee$, ∇ and $\neg$ it is possible to use only $\neg$ and $\rightarrowtail$ as primitive connectives of $\mathbf{J}_3$, considering $A \vee B$ and ∇A as abbreviations respectively of $(A \rightarrowtail B) \rightarrowtail B$ and $\neg A \rightarrowtail A$.

So, there is a close analogy between $\mathbf{J}_3$ and Łukasiewicz' three-valued propositional calculus $\mathcal{L}_3$, defined by the matrix $M' = \langle\{0,\frac{1}{2},1\}, \{1\}, \neg, \rightarrowtail\rangle$, in which the Łukasiewicz-Tarski operators $\neg$ and $\rightarrowtail$ are given by the respective tables of $\mathbf{J}_3$ (see [4]).

$\mathbf{J}_3$ can be axiomatized by:

Axiom 1 : $\Delta(A \rightarrowtail (B \rightarrowtail A))$

Axiom 2 : $\Delta((A \rightarrowtail B) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail (A \rightarrowtail C)))$

Axiom 3 : $\Delta((\neg A \rightarrowtail \neg B) \rightarrowtail (B \rightarrowtail A))$

Axiom 4 : $\Delta(((A \rightarrowtail \neg A) \rightarrowtail A) \rightarrowtail A)$

Axiom 5 : $\Delta(\Delta(A \rightarrowtail B) \rightarrowtail \Delta(\Delta A \rightarrowtail \Delta B)$

Rule R1 : $\dfrac{A, \Delta(A \rightarrowtail B)}{B}$

Rule R2 : $\dfrac{\nabla A}{A}$

The completeness theorem for $\mathbf{J}_3$ is proved from the completeness of $\mathcal{L}_3$, due to Wajsberg (see [4] and [23]), using the following theorem.

THEOREM 2.3. *If* A *is a theorem of* $\mathcal{L}_3$, *then* ΔA *is a theorem of* $\mathbf{J}_3$.

Proof. As the axioms 1 to 4 are the axioms of $\mathcal{L}_3$ preceeded by Δ, if A is an axiom of $\mathcal{L}_3$, then ΔA is a theorem of $\mathbf{J}_3$.

Let A be obtained from B and $B \rightarrowtail A$ by the rule $\frac{B, B \rightarrowtail A}{A}$ of $\mathcal{L}_3$. By induction hypothesis, ΔB and $\Delta(B \rightarrowtail A)$ are theorems of $\mathbf{J}_3$. By axiom 5 and R_1 we obtain $\Delta(\Delta B \rightarrowtail \Delta A)$. Applying R_1, we have that ΔA is a theorem of $\mathbf{J}_3$.

THEOREM 2.4. (Completeness theorem for $\mathbf{J}_3$). *A formula* A *is a theorem of* $\mathbf{J}_3$ *if and only if* A *is valid in* M.

Proof. A straightforward induction shows that if A is a theorem of $\mathbf{J}_3$, then A is valid in M. On the other hand, if A is valid in M, then $\nu(\nabla A) = 1$ for every truth-valuation ν. By the axiomatization and completeness of $\mathcal{L}_3$, both ∇A and $\Delta(\nabla A \rightarrowtail \nabla A)$ are theorems of $\mathcal{L}_3$. By the above theorem and R_1, ∇A is a theorem of $\mathbf{J}_3$. By R_2, A is a theorem of $\mathbf{J}_3$.

COROLLARY (Modus Ponens Rule). *If both* A *and* $A \supset B$ *are theorems of* $\mathbf{J}_3$, *then* B *is a theorem of* $\mathbf{J}_3$.

However, contrary to $\mathcal{L}_3$, the Rule of Modus Ponens is not valid with respect to $\rightarrowtail$.

For some of the theorems that follow it will be convenient to assume that the language of $\mathbf{J}_3$ contains, as primitive symbols, all the connectives introduced so far. In particular we shall often identify $\mathbf{J}_3$ with the set of M-valid formulas in the expanded language.

The following theorems will be used in the proofs of many of the results about $\mathbf{J}_3$.

THEOREM 2.5. $\mathbf{J}_3$ *is a non-conservative extension of the classical positive propositional calculus with connectives* $\vee$, $\&$, $\supset$, $\equiv$.

THEOREM 2.6. $\mathbf{J}_3$ *is a conservative extension of the classical propositional calculus with connectives* $\neg^*$, $\vee$, $\&$, $\supset$ *and* $\equiv$

THEOREM 2.7. $\mathbf{J}_3$ *is a non-conservative extension of Łukaseewicz' three-valued logic* $\mathcal{L}_3$ *with connectives* $\neg, \rightarrowtail$.

THEOREM 2.8. $\mathbf{J}_3$ *is not functionally complete.*

Proof. It is not possible to define a connective, from the primitive conectives of $\mathbf{J}_3$, such that its truth-value is identically ½.

On the other hand, if we add the Słupecki T operador to the primitive connectives of $\mathbf{J}_3$, the calculus becomes functionally complete (see [21]).

By Theorem 2.4, the formulas $\neg A \supset (A \supset B)$, $A \supset (\neg A \supset B)$, $A \supset (\neg A \supset \neg B)$, $(A \& \neg A) \supset B$, $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$, $A \supset (B \& \neg B) \equiv A$, etc., are not theorems of $\mathbf{J}_3$. So, in $\mathbf{J}_3$, in general, it is not possible to deduce any formula whatsoever from a contradiction. Therefore, based on such a calculus we can construct nontrivial inconsistent deductive systems, in the sense of [11]. So, $\mathbf{J}_3$ satisfies condition (i) of Jaśkowski's problem.

By Theorem 2.5 to 2.8, J_3 is quite a strong system, which evidently satisfies Jaśkowski's condition (ii).

$\mathbf{J}_3$ admits intuitive interpretations. For instance, it can be used as the underlying logic of a theory whose preliminary formulation may involve certain contradictions, which should be eliminated in a later reformulation. This can be done as follows; among the truth-values of J_3, 0 can represent falsity, 1 truth, and ½ can represent the provisional value of a proposition A, so that both A and the negation of A are theorems of the theory, in its initial formulation; in a later reformulation, the truth-value ½ should be reduced, at least in principle, to 0 or to 1.

Therefore, J_3 is a solution to Jaśkowski's problem.

$\mathbf{J}_3$ can also be used as a foundation for paraconsistent systems, in the sense of da Costa (see [5], [6], [7] and [8]). In this case, the value 0 represents falsity, 1 truth, and ½ represents the logic value of a formula that is simultaneously true and false.

Finally, as the calculus J_3 was constructed from $\mathcal{L}_3$, it is possible to obtain similar calculi J_n, from Łukasiewicz n-valued calculi $\mathcal{L}_n$, $3 \leq n < \aleph_0$.

3. SEMANTICS FOR FIRST-ORDER J_3-THEORIES.

The *symbols* of a first-order $\mathbf{L}_3$-language are the individual variables, the function symbols, the predicate symbols, the primitive connectives $\neg$, $\vee$ and ∇, the quantifies $\exists$ and $\forall$, and the parentheses.

The *identity* = must be among the predicate symbols. Other equalities can be especified among the predicate symbols.

We use x,y,z and w as syntactical variables for individual variables; f and g, for function symbols; p and q, for predicate symbols, and c for constants.

The definitions of *term*, *atomic formula* and *formula* are the usual ones; a, b,c, etc. are syntactical variables for terms and A,B,C, etc. for formulas.

By an $\mathbf{L}_3$-*language* we understand a first-order language whose logical symbols include the ones mentioned above.

The symbols &, $\rightarrowtail$, $\twoheadrightarrow$, $\supset$, $\equiv$, Δ and $\neg^*$ are defined in the $\mathbf{L}_3$-languages, as in $\mathbf{J}_3$.

Free occurrence of a variable, open formula, closed formula, variable-free term and *closure of a formula* are used as in [22].

The definition of a is *substitutible for* x *in* A is also the usual one.

We let $b_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$ be the term obtained from b by replacing all occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively; and we let $A_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$ be the formula obtained from A by replacing free occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively.

Whenever either of these is used, it will be implicitly assumed that $x_1,\ldots,x_n$ are distinct variables and that, in the case of $A_{x_1,\ldots,x_n}[a_1,\ldots,a_n]$, a_i is substitutible for x_i, $i = 1,\ldots,n$.

In the following definitions, let L be an $\mathbf{L}_3$-language.

DEFINITION 3.1. A *structure* $\mathfrak{A}$ for a first-order $\mathbf{L}_3$-language L consists of:

i) a nonempty set $|\mathfrak{A}|$, called universe of $\mathfrak{A}$;

ii) for each n-ary function symbol f of L, a function $\underset{\sim}{f}$ from $|\mathfrak{A}|^n$ to $|\mathfrak{A}|$;

iii) for each n-ary predicate symbol p of L, other than =, an n-ary predicate $p_{\mathfrak{A}}$, such that $p_{\mathfrak{A}}$ is a mapping from $|\mathfrak{A}|\times\ldots\times|\mathfrak{A}|$ to $\{0,\frac{1}{2},1\}$.

As in [22], we construct the language $L(\mathfrak{A})$; define $\mathfrak{A}(a)$ for each variable free term of $L(\mathfrak{A})$, and define $\mathfrak{A}$-*instance* of a formula A.

We use i and j as syntactical variable for the names of individuals of $\mathfrak{A}$.

DEFINITION 3.2. The *truth-value* $\mathcal{O}\!\mathcal{l}(A)$ for each closed formula A in $L(\mathcal{O}\!\mathcal{l})$ is given by:

i) if A is a = b, then $\mathcal{O}\!\mathcal{l}(A) = 1$ iff $\mathcal{O}\!\mathcal{l}(a) = \mathcal{O}\!\mathcal{l}(b)$; otherwise, $\mathcal{O}\!\mathcal{l}(A) = 0$;

ii) if A is $p(a_1,\dots,a_n)$, where p is not =, then $\mathcal{O}\!\mathcal{l}(A) = \dot{p}_{\mathcal{O}\!\mathcal{l}}(\mathcal{O}\!\mathcal{l}(a_1),\dots\mathcal{O}\!\mathcal{l}(a_n))$;

iii) if A is $\neg B$, then $\mathcal{O}\!\mathcal{l}(A)$ is $H_{\neg}(\mathcal{O}\!\mathcal{l}(B))$;

iv) if A is ∇B, then $\mathcal{O}\!\mathcal{l}(A)$ is $H_{\nabla}(\mathcal{O}\!\mathcal{l}(B))$;

v) if A is $B \vee C$, then $\mathcal{O}\!\mathcal{l}(A)$ is $H_{\vee}(\mathcal{O}\!\mathcal{l}(B), \mathcal{O}\!\mathcal{l}(C))$;

vi) if A is a $\exists xB$, then $\mathcal{O}\!\mathcal{l}(A) = \max\{\mathcal{O}\!\mathcal{l}(B_x[i]) / i \in L(\mathcal{O}\!\mathcal{l})\}$;

vii) if A is a $\forall xB$, then $\mathcal{O}\!\mathcal{l}(A) = \min\{\mathcal{O}\!\mathcal{l}(B_x[i]) / i \in L(\mathcal{O}\!\mathcal{l})\}$.

DEFINITION 3.3. (1) A formula B of $L(\mathcal{O}\!\mathcal{l})$ is *true* in $\mathcal{O}\!\mathcal{l}$ (or $\mathcal{O}\!\mathcal{l}$ is a model of B) iff $\mathcal{O}\!\mathcal{l}(B) \in V_d$.

(2) A formula A of L is *valid* in $\mathcal{O}\!\mathcal{l}$ iff for every $\mathcal{O}\!\mathcal{l}$-instance A' of A, A' is true in $\mathcal{O}\!\mathcal{l}$.

A *first-order predicate calculus* $\mathbf{J}_3^*$= is the formal system whose language is an $\mathbf{L}_3$ plus the following, with the usual restrictions (see [14]):

Axiom 6 : $\forall x(x = x)$

Axiom 7 : $x = y \supset (A[x] \equiv A[y])$

Axiom 8 : $A_x[a] \supset \exists xA$

Axiom 9 : $\forall xA \supset A_x[a]$

Axiom 10: $\exists xA \equiv \neg\forall x\neg A$

Axiom 11: $\forall xA \equiv \neg\exists x\neg A$

Axiom 12: $\neg\exists xA \equiv \forall x\neg A$

Axiom 13: $\neg\forall xA \equiv \exists x\neg A$

Axiom 14: $\nabla\exists xA \equiv \exists x\nabla A$

Axiom 15: $\nabla\forall xA \equiv \forall x\nabla A$

Rule R3 : ($\exists$-*introduction rule*): $\dfrac{A \supset C}{\exists xA \supset C}$

Rule R4 : ($\forall$-*introduction rule*): $\dfrac{C \supset A}{C \supset \forall xA}$

THEOREM 3.1. $\mathbf{J}_3^*$= *is a conservative extension of* $\mathbf{J}_3$.

Proof. We apply the Hilbert-Bernays theorem of k-transforms, that can be extended to this case.

THEOREM 3.2. $\mathbf{J}_3^*$= *is an extension of the classical predicate calculus, with connectives* $\neg^*$, $\vee$, $\&$, $\supset$, $\equiv$, $\exists$ *and* $\forall$.

DEFINITION 3.4. A *first-order* J_3-*theory* is a formal system T such that:

i) the language of T, L(T), is an $\mathbf{L}_3$-language;

ii) the axioms of T are the axioms of $J_3^{*=}$, called the logical axioms of T, and certain further axioms, called the non-logical axioms;

iii) the rules of T are those of $J_3^{*=}$.

A is a *theorem of* T, in symbols: $\vdash_T A$, and B is a *semantical consequence* of a set Γ of formulas of L(T) are defined in the standard way. If B is a semantical consequence of Γ, then we shall also say that "B is valid in Γ".

THEOREM 3.3. (Validity Theorem): *Every theorem of a* J_3*-theory* T *is valid in* T.

4. SOME THEOREMS IN FIRST-ORDER J_3-THEORIES AND THE EQUIVALENCE THEOREM.

DEFINITION 4.1. A J_3-theory T is *finitely trivializable* if there exists a fixed formula F such that, for any formula A, $F \supset A$ is a theorem of T (see [2]).

THEOREM 4.1. *The* J_3*-theories are finitely trivializable.*

Proof. Any formula $\neg(\neg\nabla A \vee \nabla A)$ trivializes a J_3-theory.

The following results hold in any J_3-theory T:

Generalization Rule: If $\vdash_T A$, then $\vdash_T \forall xA$.

Substitution Rule: Is $\vdash_T A$ and A' is an instance of A, then $\vdash_T A'$.

Substitution Theorem: a) $\vdash_T A_{x_1,\dots,x_n}[a_1,\dots,a_n] \supset \exists x_1 \dots x_n A$;

b) $\vdash_T \forall x_1 \dots \forall x_n A \supset A_{x_1,\dots,x_n}[a_1,\dots,a_n]$

Distribution Rule: If $\vdash_T A \supset B$, then $\vdash_T \exists xA \supset \exists xB$ and $\vdash_T \forall xA \supset \forall xB$.

Closure Theorem: If A' is the closure of A, then $\vdash_T A$ if and only if $\vdash_T A'$.

Theorem on Constants: If T' is a J_3-theory obtained from T by adding new constants (but no new nonlogical axioms), then for every formula A of T and every sequence $e_1,\dots,e_n$ of new constants, $\vdash_T A$ if and only if $\vdash_{T'} A_{x_1,\dots,x_n}[e_1,\dots,e_n]$.

In the case of classical logic, the equivalence ≡ behaves as a congruence relation with respect to the other logical symbols. Unfortunately this is not the case in J_3-theories, for it is possible to have $\vdash_T A \equiv B$ and $\nvdash_T \neg A \equiv \neg B$.

However we can introduce a stronger equivalence, $\equiv^*$, which is a $J_3^=$-congruence relation and thus allow us to prove a J_3-version of the equivalence theorem (see [22]).

DEFINITION 4.2. $A \equiv^* B =_{def} (A \equiv B) \& (\neg A \equiv \neg B)$.

THEOREM 4.2. *If* T *is a* J_3*-theory and* $\vdash_T A \equiv^* B$, *then* $\vdash_T A$ *if and only if* $\vdash_T B$.

THEOREM 4.3. (Equivalence Theorem). *Let* T *be a* J_3*-theory and let* A' *be obtained from* A *by replacing some occurrences of* $B_1,\dots,B_n$ *by* $B'_1,\dots,B'_n$ *respectively. If* $\vdash_T B_1 \equiv^* B'_1, \dots, \vdash_T B_n \equiv^* B'_n$, *then* $\vdash_T A \equiv^* A'$.

Proof. After considering the special case when there is only one such occurrence and it is all of A, we use induction on the length of A.

For A *atomic*, the result is obvious.

A *is* $\neg C$ *and* A' *is* $\neg C'$, where C' results from C by replacements of the type described in the theorem. By induction hypothesis, $\vdash_T C \equiv^* C'$, that is, $\vdash_T C \equiv C'$ and $\vdash_T \neg C \equiv \neg C'$. As by Theorem 2.4, $\vdash_T C \equiv \neg\neg C$ and $\vdash_T C' \equiv \neg\neg C'$, we have $\neg\neg C \equiv \neg\neg C'$. So $\neg C \equiv^* \neg C'$.

A *is* ∇C *and* A' *is* $\nabla C'$, with $\vdash_T C \equiv^* C'$. From $\vdash_T C \equiv C'$, it follows that $\vdash_T \neg^* C \equiv \neg^* C'$, by Theorem 2.6. Also from $\vdash_T C \equiv C'$ it follows that $\vdash_T \nabla C \equiv \nabla C'$, since $\vdash_T \nabla C \equiv C$ by Theorem 2.4. Therefore, $\vdash_T \nabla C \equiv^* \nabla C'$.

A *is* $C \vee D$ *and* A' *is* $C' \vee D'$, with $\vdash_T C \equiv^* C$ and $\vdash_T D \equiv^* D'$. As by theorem 2.6,

$$\vdash_T ((C \equiv C') \& (D \equiv D')) \supset ((C \vee D) \equiv (C' \vee D'))$$

and

$$\vdash_T ((\neg C \equiv \neg C') \& (\neg D \equiv \neg D')) \supset ((\neg C \& \neg D) \equiv (\neg C' \& \neg D'))$$

we have that $\vdash_T C \vee D \equiv C' \vee D'$ and $\vdash_T \neg(C \vee D) \equiv \neg(C' \vee D')$.

A *is* $\exists x C$ *and* A' *is* $\exists x C'$, with $C \equiv^* C'$. By Distribution Rule, $\vdash_T \exists x C \equiv \exists x C'$ and $\vdash_T \forall x \neg C \equiv \forall x \neg C'$. Using Axiom 12 we complete the proof.

If A *is* $\forall x C$ *and* A' *is* $\forall x C'$, with $\vdash_T C \equiv^* C'$, the proof is similar.

In the spirit of the equivalence theorem, we have the following corollaries and remark.

COROLLARY 1. *In a* J_3*-theory* T, *it is possible to replace:*

i) $\neg\neg A$ by A;

ii) $\neg^*\neg^* A$ by $\neg\neg^* A$;

iii) $\neg(A \vee B)$ by $\neg A \& \neg B$;

iv) $\neg^*(A \vee B)$ by $\neg^* A \& \neg^* B$;

v) $\forall x A$ by $\neg\exists x \neg A$;

vi) $\neg\exists x A$ by $\forall x \neg A$;

vii) $\neg\forall x A$ by $\exists x \neg A$;

viii) $\nabla\exists x A$ by $\exists x \nabla A$;

ix) $\nabla\forall x A$ by $\forall x \nabla A$.

Proof. It is enough to verify that $\vdash_T \neg\neg A \equiv^* A$, $\vdash_T \neg^*\neg^* A \equiv^* \neg\neg^* A$, etc.

COROLLARY 2. *In a* $\mathbf{J}_3$*-theory* T, *if* $\vdash_T x = y$, *then, for every formula* A, A(x) *can be replaced by* A(y).

REMARK. Although $\vdash_T \neg^*\neg^* A \equiv A$, it is not possible, in general, to replace $\neg^*\neg^* A$ by A.

DEFINITION 4.3. A formula A' is a *variant of* A just in case A' has been obtained from A by renaming bound variables.

THEOREM 4.4. (Variant Theorem). *If* A' *is a variant of* A, *then* $\vdash_T A \equiv^* A'$.

Proof. In view of Theorem 4.3 and Corollary 1, it is enough to observe that $\vdash_T \exists xB \equiv^* \exists y B_x[y]$.

Let T[Γ] be the $\mathbf{J}_3$-theory whose non-logical axioms are those of T plus the formulas of the set Γ.

THEOREM 4.5. (Reduction Theorem). *Let* Γ *be a set of formulas in the* $\mathbf{J}_3$*-theory* T *and let* A *be a formula of* T. A *is a theorem of* T[Γ] *if, and only if, there is a theorem of* T *of the form* $B_1 \supset \ldots \supset B_n \supset A$, *where each* B_i *is the closure of a formula in* Γ.

Given a non-empty set Γ of formulas we let:

$\Gamma_{\vee\neg\forall\nabla}$ = {B | B is a disjunction of negations of closures of formulas of the type ∇A, with A ∈ Γ}

$\Gamma_{\vee\neg\nabla\forall}$ = {C | C is a disjunction of negations of formulas of the type ∇A', where A' is the closure of a formula of Γ}

THEOREM 4.6. (Reduction Theorem for non-trivialization). *Let* Γ *be a non-empty set of formulas in a* $\mathbf{J}_3$*-theory* T. *Then the extension* T[Γ] *is trivial, if and only if, there is a theorem of* T *which belongs to* $\Gamma_{\vee\neg\forall\nabla}$.

Proof. The corollary to the replacement theorem gives us that every formula of $\Gamma_{\vee\neg\forall\nabla}$ is strongly equivalent to a formula of $\Gamma_{\vee\neg\nabla\forall}$. The proof of the theorem can be completed using the properties of strong negation.

COROLLARY. *If* A' *is the closure of* A, *then the formula* A *is a theorem of* T *if, and only if,* T[$\neg^*$A'] *is trivial.*

5. THE COMPLETENESS AND THE COMPACTNESS THEOREMS FOR J_3-THEORIES

We study certain aspects of the J_3-theories and present a Henkin-type proof of the completeness theorem for this type of many-valued theories.

DEFINITION 5.1. If T is a J_3-theory containing a constant, and if a and b are variable-free terms of T, then:

i) $a \sim b =_{def} \vdash_T a = b$;
ii) $a^o = \{b \mid a \sim b\}$.

DEFINITION 5.2. A canonical structure for the J_3-theory T is the structure $\mathfrak{A}$:

i) whose universe $|\mathfrak{A}|$ is the set of all equivalence classes under $\sim$;
ii) $f_{\mathfrak{A}}(a_1^o,\dots,a_n^o) = (f(a_1,\dots,a_n))^o$;
iii) $p_{\mathfrak{A}}(a_1^o,\dots,a_n^o)$ is in V_d iff $\vdash_T p(a_1,\dots,a_n)$.

Observe that (iii) could have been replaced by

$p_{\mathfrak{A}}(a_1^o,\dots,a_n^o) = 0$ iff $\nvdash_T p(a_1,\dots,a_n)$.

THEOREM 5.1. *If* $\mathfrak{A}$ *is a canonical structure for* T *and* $p(a_1,\dots,a_n)$ *is a variable-free atomic formula in* L(T), *then:*

i) $\mathfrak{A}(p(a_1,\dots,a_n)) = 0$ *iff* $\nvdash_T p(a_1,\dots,a_n)$
ii) $\mathfrak{A}(p(a_1,\dots,a_n)) = \frac{1}{2}$ *iff* $\vdash_T p(a_1,\dots,a_n)$ *and* $\vdash_T \neg p(a_1,\dots,a_n)$;
iii) $\mathfrak{A}(p(a_1,\dots,a_n)) = 1$ *iff* $\vdash_T p(a_1,\dots,a_n)$ *and* $\nvdash_T \neg p(a_1,\dots,a_n)$.

Proof. ii) If $\mathfrak{A}(p(a_1,\dots,a_n)) = \frac{1}{2}$ then $\mathfrak{A}(\neg p(a_1,\dots,a_n)) = \frac{1}{2}$. By the last definition, $\vdash_T p(a_1,\dots,a_n)$ and $\vdash_T \neg p(a_1,\dots,a_n)$.

On the other hand, if $\vdash_T p(a_1,\dots,a_n)$ and $\vdash_T \neg p(a_1,\dots,a_n)$, also by Definition 5.2, $\mathfrak{A}(p(a_1,\dots,a_n)$ and $\mathfrak{A}(\neg p(a_1,\dots,a_n))$ belong to V_d. Then, $\mathfrak{A}(p(a_1,\dots,a_n)) = \frac{1}{2}$.

iii) If $\mathfrak{A}(p(a_1,\dots,a_n)) = 1$, then $\mathfrak{A}(\neg p(a_1,\dots,a_n)) = 0$; then, $\vdash_T p(a_p,\dots,a_n)$ and $\nvdash_T \neg p(a_1,\dots,a_n)$.

On the other hand, if $\vdash_T p(a_1,\dots,a_n)$ and $\nvdash_T \neg p(a_1,\dots,a_n)$, we have that $\mathfrak{A}(p(a_1,\dots,a_n))$ belongs to V_d and $\mathfrak{A}(\neg p(a_1,\dots,a_n))$ does not belong to V_d; if $\mathfrak{A}(p(a_1,\dots,a_n) = \frac{1}{2}$ then $\mathfrak{A}(\neg p(a_1,\dots,a_n)) = \frac{1}{2}$ and, so, $\vdash_T \neg p(a_1,\dots,a_n)$. Then, $\mathfrak{A}(p(a_1,\dots,a_n)) = 1$.

Now, (i) is immediate.

As a consequence of the theorem we obtain that there is exactly one canonical structure for a J_3-theory. Furthermore, as in the calssical case, in order for a canonical structure to characterize the theorems of a theory, the theory must be in some sense maximal, for there may be a closed formula A such that $\nvdash_T A$, $\nvdash_T \neg A$ and $\nvdash_T \neg^* A$.

DEFINITION 5.3. A formula A of a J_3-theory T is *undecidable* in T if neither A nor $\neg^* A$ is a theorem of T. Otherwise, A is *decidable* in T.

DEFINITION 5.4. A $\mathbf{J}_3$-theory T is *complete* if it is non-trivial and if every closed formula of T is decidable in T.

THEOREM 5.2. *A* $\mathbf{J}_3$*-theory* T *is complete if, and only if,* T *maximal in the class of nontrivial theories.*

DEFINITION 5.5. A $\mathbf{J}_3$-theory T is a *Henkin* $\mathbf{J}_3$*-theory* if for every closed formula $\exists xA$ of T, there is a constant e such that $\exists xA \supset A_x[e]$ is a theorem of T.

THEOREM 5.3. *If* T *is a Henkin* J_3*-theory, then for every closed formula* $\forall xA$ *in* T *there is a constant* e *such that* $A_x[e] \supset \forall xA$ *is a theorem of* T.

Proof. As T is a Henkin $\mathbf{J}_3$-theory, there is e, such that $\vdash_T \exists x \neg^* A \supset \neg^* A_x[e]$. We obtain the desired result, by successive applications of Theorem 2.6.

THEOREM 5.4. *If* T *is a complete Henkin* $\mathbf{J}_3$*-theory and* $\mathfrak{A}$ *is the canonical structure for* T, *then for all closed formulas* A *of* L[T]:

i) $\mathfrak{A}(A) = 0$ *iff* $\nvdash_T A$

ii) $\mathfrak{A}(A) = \frac{1}{2}$ *iff* $\vdash_T A$ *and* $\vdash_T \neg A$

iii) $\mathfrak{A}(A) = 1$ *iff* $\vdash_T A$ *and* $\nvdash_T \neg A$.

Proof. By induction on the height of A. For *atomic* A, the result follows from Theorem 5.1.

Case: A *is* $\neg B$. **i)** If $\mathfrak{A}(A) = 0$, then $\mathfrak{A}(B) = 1$. Thus $\nvdash_T \neg B$, that is $\nvdash_T A$. On the other hand if $\nvdash_T A$, then since T is complete $\vdash_T \neg^* A$, and then $\vdash_T \neg A$, $\vdash_T \neg\neg B$, $\vdash_T B$. Thus we have that $\vdash_T B$ and $\nvdash_T \neg B$, from which if follows that $\mathfrak{A}(B) = 1$ and that $\mathfrak{A}(A) = 0$.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$, then $\mathfrak{A}(B) = \frac{1}{2}$. Thus $\vdash_T B$ and $\vdash_T \neg B$, from which it follows that $\vdash_T \neg A$ and $\vdash_T A$, the converse is analogous.

iii) If $\mathfrak{A}(A) = 1$, then $\mathfrak{A}(B) = 0$ and thus $\nvdash_T B$. Since T is complete, $\vdash_T \neg^* B$ and thus $\vdash_T \neg B$. Since $\nvdash_T B$, we obtain that $\vdash_T \neg\neg B$, in other words, we have that $\vdash_T A$ and $\nvdash_T \neg A$.

Assume next that $\nvdash_T A$ and $\vdash_T A$, that is, $\nvdash_T \neg\neg B$ and $\vdash_T \neg B$. Then $\nvdash_T B$, and so by induction $\mathfrak{A}(B) = 0$, from which it follows that $\mathfrak{A}(A) = 1$.

Case: A *is* $B \vee C$. **i)** If $\mathfrak{A}(A) = 0$ then $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = 0$. Hence $\nvdash_T C$ and $\nvdash_T B$, from which it follows, since T is complete, that $\nvdash_T B \vee C$. The converse is analogous.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$, then either: $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = \frac{1}{2}$,
or $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 0$,
or $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = \frac{1}{2}$.

Let us only consider the situation when $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 0$ (the others are analogous). The induction hypothesis gives us that

$$\vdash_T B, \quad \vdash_T \neg B, \quad \nvdash_T C.$$

Since T is complete we obtain that $\vdash_T \neg^* C$ and $\vdash_T \neg C$. From $\vdash_T B$ we get $\vdash_T B \vee C$, and from $\vdash_T \neg B$ and $\vdash_T \neg C$ we may conclude that $\vdash_T \neg(B \vee C)$.

Conversely, suppose that $\vdash_T B \vee C$ and $\vdash_T \neg(B \vee C)$. The latter gives us that $\vdash_T \neg B$ and $\vdash_T \neg C$. From the former, since T is complete, we obtain that either $\vdash_T B$ or $\vdash_T C$. The induction hypothesis allows us then to conclude that $\mathfrak{A}(B \vee C) = \frac{1}{2}$.

iii) If $\mathfrak{A}(A) = 1$, then either:
$\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = 0$,
or $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = \frac{1}{2}$,
or $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = 1$,
or $\mathfrak{A}(B) = 0$ and $\mathfrak{A}(C) = 1$,
or $\mathfrak{A}(B) = \frac{1}{2}$ and $\mathfrak{A}(C) = 1$.

We will only consider the case when $\mathfrak{A}(B) = 1$ and $\mathfrak{A}(C) = \frac{1}{2}$. The induction hypothesis gives us that

$$\vdash_T B, \quad \nvdash_T \neg B, \quad \vdash_T C, \quad \vdash_T \neg C.$$

From the first we obtain that $\vdash_T (B \vee C)$. Suppose on the other hand that $\vdash_T \neg(B \vee C)$. Then $\vdash_T (\neg B \wedge \neg C)$, from which it would follow that $\vdash_T \neg B$, contradicting that $\nvdash_T \neg B$. Thus $\nvdash_T \neg(B \vee C)$.

On the other hand, suppose that $\vdash_T (B \vee C)$ and $\nvdash_T \neg(B \vee C)$. Then from the completeness of T we obtain that either

$$\vdash_T B \quad \text{or} \quad \vdash_T C.$$

From $\nvdash_T \neg(B \vee C)$, we obtain that

$$\nvdash_T \neg B \text{ and } \nvdash_T \neg C.$$

The induction hypothesis then gives us that $\mathfrak{A}(B \vee C) = 1$.

Case: A *is* ∇B. **i)** If $\mathfrak{A}(\nabla B) = 0$. Then $\mathfrak{A}(B) = 0$. Thus $\nvdash_T B$; from which it follows that $\nvdash_T \nabla B$. Converse, analogous.

ii) $\mathfrak{A}(\nabla B)$ is never $\frac{1}{2}$.

iii) $\mathfrak{A}(\nabla B) = 1$ then either $\mathfrak{A}(B) = \frac{1}{2}$ or $\mathfrak{A}(B) = 1$.

Subcase: $\mathfrak{A}(B) = \frac{1}{2}$. Then $\vdash_T B$ and $\vdash_T \neg B$, from which we obtain $\vdash_T \nabla B$ and $\vdash_T \nabla \neg B$. Using that T is complete we conclude $\vdash_T \nabla B$, and $\vdash_T \neg \nabla B$.

Subcase: $\mathfrak{A}(B) = 1$. Then $\vdash_T B$ and $\nvdash_T \neg B$. Suppose that $\vdash_T \neg \nabla B$. Then since $\vdash_T B$, we should obtain that T is trivial, which we are assuming it is not. Thus

$\not\vdash_T \neg\nabla B$ and $\vdash_T \nabla B$. On the other hand, suppose that $\vdash_T A$ and $\not\vdash_T \neg A$. That is suppose that

$$\vdash_T \nabla B \quad \text{and} \quad \not\vdash_T \neg\nabla B.$$

Then $\vdash_T B$, and either $\vdash_T \neg B$ or $\not\vdash_T B$. In one case the induction hypothesis gives that $\mathfrak{A}(B) = \frac{1}{2}$, and in the other that $\mathfrak{A}(B) = 1$. Thus $\mathfrak{A}(\nabla B) = 1$ in both. That is $\mathfrak{A}(A) = 1$.

Case: A *is* $\exists xB$. **i)** If $\mathfrak{A}(A) = 0$, then for every variable-free term b, $\mathfrak{A}(B_x[b]) = 0$, and by induction hypothesis this is equivalent to $\not\vdash_T B_x[b]$. As T is a Henkin theory this gives us that $\not\vdash_T \exists xB$. The converse does not need to use that T is a Henkin theory.

ii) If $\mathfrak{A}(A) = \frac{1}{2}$. Then for all b we have that $\mathfrak{A}(B_x[b]) \leq \frac{1}{2}$. The induction hypothesis then tells us that

(1) for those b such that $\mathfrak{A}(B_x[b]) = \frac{1}{2}$ (and there is at least one such): $\vdash_T B_x[b]$ and $\vdash_T \neg B_x[b]$.

(2) for the remaining b's: $\not\vdash_T B_x[b]$ and (because T is complete) $\vdash_T \neg B_x[b]$. Thus we have that for all constants b: $\vdash_T \neg B_x[b]$; from which it follows that $\vdash_T \forall x \neg B$, i.e. $\vdash_T \exists xB$. From (1) we obtain $\vdash_T \exists xB$.

Conversely, suppose that $\vdash_T A$ and $\vdash_T \neg A$; that is $\vdash_T \exists xB$ and $\vdash_T \neg\exists xB$. Using that T is a Henkin theory and induction, we obtain an e such that $\vdash_T B_x[e]$, $\vdash_T \neg B_x[e]$, and thus $\mathfrak{A}(B_x[e]) = \frac{1}{2}$. A proof by contradiction shows that there is no b such that $\mathfrak{A}(B_x[b]) = 1$. Hence $\mathfrak{A}(\exists xB) = \frac{1}{2}$.

iii) If $\mathfrak{A}(A) = 1$, then there is at least one b such that $\mathfrak{A}(B_x[b]) = 1$. From the induction hypothesis, we obtain that $\vdash_T B_x[b]$ and $\not\vdash_T \neg B_x[b]$. From the former, we obtain that $\vdash_T \exists xB$. Suppose next contrary to what we want to show, that $\vdash_T \neg\exists xB$. Then $\vdash_T \exists x \neg B$ and thus $\vdash_T \neg B_x[b]$, a contradiction. Thus $\not\vdash_T \neg\exists xB$.

COROLLARY 1. *Let* T *be a complete Henkin* J_3*-theory,* $\mathfrak{A}$ *the canonical structure for* T *and* A *a closed formula of* T; *then,* $\mathfrak{A}(A)$ *belongs to* V_d *if and only if* A *is a theorem of* T.

COROLLARY 2. *If* T *is a complete Henkin* J_3*-theory, then the canonical structure for* T *is a model of* T.

By the above corollary, to prove the completeness of a J_3-theory T, as in the classical case, it is enough to show that it is possible to extend T to a complete Henkin J_3-theory.

Thus, given a nontrivial J_3-theory T, we will first extend it, conservatively, to a Henkin J_3-theory T_c, and then extend it to a complete Henkin J_3-theory T_c'.

Given a J_3-theory T with language $\mathbf{L}$, we proceed as in [22] and define the special constants of level n, the language L_c with the special constants, and introduce the special axioms for the special constants.

DEFINITION 5.6. Let T be a $\mathbf{J}_3$-theory with language L. Then T_c is the Henkin $\mathbf{J}_3$-theory whose language is L_c and whose nonlogical axioms are the nonlogical axioms of T plus the special axioms for the special constants of L_c.

THEOREM 5.5. T_c *is a conservative extension of* T.

Proof. By Theorem 4.4 and by Theorem 5.3, the proof is similar to the classical one.

THEOREM 5.6. (Lindenbaum's Theorem). *If* T *is a nontrivial* $\mathbf{J}_3$*-theory, then* T *admits a complete simple extension.*

Finally, we can obtain the completeness theorem for $\mathbf{J}_3$-theories.

THEOREM 5.7. (Completeness Theorem). *A* $\mathbf{J}_3$*-theory* T *is nontrivial if, and only if, it has a model.*

Proof. If $\mathfrak{A}$ is a model of T and A is a closed formula in T, then $\mathfrak{A}(A \& \neg^* A) = 0$. So, by the validity Theorem, $A \& \neg^* A$ is not a theorem in T. Then T is nontrivial.

If T is nontrivial, then we extend T to T_c, which is a non-trivial Henkin $\mathbf{J}_3$-theory. By Lindenbaum's Theorem, we can extend T_c to a complete Henkin $\mathbf{J}_3$-theory T_c'. By Corollary 2 to Theorem 5.4, T_c' has a model $\mathfrak{A}$. Therefore, $\mathfrak{A} \restriction L(T)$ is a model of T.

THEOREM 5.8. (Gödel's Completeness Theorem). *A formula* A *in the* $\mathbf{J}_3$*-theory* T *is a theorem in* T *if, and only if, it is valid in* T.

Proof. By supposing that the closed formula A is a theorem in T and using the above Completeness Theorem, we shall show that there is no model of T in which A is not valid.

Therefore, suppose that the closed formula A is a theorem in T.

By the corollary to the Reduction Theorem for non-Trivialization, $\vdash_T A$ if and only if $T[\neg\nabla A]$ is trivial; which, by Theorem 5.7, is equivalent to $T[\neg\nabla A]$ not having a model.

On the other hand, a model of $T[\neg\nabla A]$ is a model $\mathfrak{A}$ of T in which $\neg\nabla A$ is valid, that is, a structure $\mathfrak{A}$ such $\mathfrak{A}(\neg\nabla A) = 1$. This is equivalent to $\mathfrak{A}(\nabla A) = 0$, and so $\mathfrak{A}(A) = 0$.

Therefore, $\vdash_T$ A if and only if A is valid in T.

COROLLARY 3. *If* T *and* T' *are* $\mathbf{J}_3$*-theories with the same language, then* T' *is an extension of* T *if, and only if, every model of* T' *is a model of* T.

THEOREM 5.9. (Compactness Theorem). *A formula* A *in a* $\mathbf{J}_3$*-theory is valid in* T *if, and only if, it is valid in some finitely axiomatized part of* T.

COROLLARY 4. *A* J_3*-theory* T *has a model if, and only if, every finitely axiomatized part of* T *has a model.*

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