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## THE COMPLEX BORDISM OF CYCLIC GROUPS

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**Introduction.** In their book, *Differentiable Periodic Maps* [2], P.E. Conner and E.E. Floyd initiated the study of cobordism groups of periodic maps and succeeded in determining the additive structure of the cobordism groups of free orientation-preserving  $\mathbf{Z}_p$ -actions on manifolds for odd primes  $p$  and of free  $\mathbf{Z}_p$ -actions preserving a stably almost-complex structure for arbitrary primes by calculating  $MSO_*(B\mathbf{Z}_p)$  and  $MU_*(B\mathbf{Z}_p)$  respectively. Kamata [5] obtained the same results for  $MU_*(B\mathbf{Z}_p)$  using slightly different methods. We extend these results to a determination of  $MU_*(BG)$  where  $G$  is an arbitrary cyclic group. The main result is Proposition 16:

$$MU_{2n+1}(B\mathbf{Z}_{p^s}) \cong \sum_{a=1}^s \sum_{b=p^{a-1}-1}^n \frac{\Gamma_{2(n-b)}(p^a)}{p^{\left[ \frac{b-p^{a-1}+1}{p^{a-1}(p-1)} \right] + s - a + 1} \Gamma_{2(n-b)}(p^a)}$$

where  $\Gamma_*(p^a) \simeq MU_* / \langle CP(p-1)^{p^a-1} \rangle$  and the square brackets indicate the greatest integer function. We show this by constructing an explicit set of generators coming from the  $K$ -theory of the generalized lens spaces  $L^n(p^s)$  and computing the order of the group they generate.

I would like to thank the referee for catching several embarrassing errors and suggesting ways of correcting them.

**Results.** We will have need of several homology and cohomology theories. Following J.F. Adams, let  $H$  be the Eilenberg-MacLane spectrum for the integers,  $K$  the  $BU$  spectrum, and  $MU$  the Thom spectrum for the unitary group. The resulting homology theories are denoted by  $H_*(\quad)$ ,  $K_*(\quad)$ , and  $MU_*(\quad)$ , and similarly in the case of cohomology theories. When we have need of unreduced theories, we write  $X^+$  for the disjoint union of  $X$  and a basepoint, so that  $H_*(X^+)$ , for example, is ordinary, unreduced, integral homology. In dealing with  $K$ -theory, we will be exclusively concerned with  $K^0(X)$  which we agree to write as  $K(X)$ , remembering that this is the reduced group, i.e., what is usually written

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as  $\tilde{K}(X)$ .

The following description of  $MU_*(X)$  will be very convenient. Consider the set of all continuous maps  $f: M^m \rightarrow X$  where  $M$  is a stably almost-complex manifold. Two such maps  $f_1$  and  $f_2$  are said to be equivalent if there is a stably almost-complex  $(m+1)$ -manifold  $W^{m+1}$  and a map  $f: W^{m+1} \rightarrow X$  such that the boundary of  $W$  is the disjoint union of  $M_1$  and  $M_2$  and  $f$  restricted to the boundary of  $W$  is the disjoint union of  $-f_1$  and  $f_2$ . Impose an addition on the set of resulting equivalence classes by the disjoint union of maps. It is a standard result that the resulting graded group is isomorphic to  $MU_*(X^+)$ .

Recall that the ring of coefficients  $MU_*$  is a polynomial ring over the integers on countably many generators, one in each positive, even dimension. There are many ways of choosing such generators, but it is convenient to have a standard set to work with. Following M. Hazewinkel [4] we proceed as follows:

Suppose  $S$  is a natural number. An ordered factorization of  $S$  is an ordered set  $(q_1, \dots, q_t, d)$  of natural numbers where each  $q_i$  is a positive power of a prime and  $d$  is not a power of a prime and  $S = q_1 \cdots q_t d$ . For example, the ordered factorizations of 12 are: (12), (2, 6), (4, 3, 1), (3, 4, 1), (2, 2, 3, 1), (2, 3, 2, 1), and (3, 2, 2, 1).

Associate to each ordered factorization  $(q_1, \dots, q_t, d)$  a positive integer  $b(q_1, \dots, q_t, d)$  as follows:

- 1)  $b(q_1, \dots, q_t, d) = b(q_1, \dots, q_t)$
- 2) If  $q_i = p_i^{t_i}$ , then  $b(q_1, \dots, q_t) = b(p_1, \dots, p_t)$
- 3)  $b(p) = 1$  and  $b(d) = 1$
- 4) If  $S = (p_1, \dots, p_t)$ , then

$$b(p_1, \dots, p_t) = \left\{ \prod_{p \in S} c(p, p_t) \right\} b(p_1, \dots, p_{t-1})$$

$$\text{where } c(p, q) = \begin{cases} 1 & \text{if } p = q \\ q^{p-1} & \text{if } p \neq q. \end{cases}$$

This suffices to give an inductive definition of  $b(q_1, \dots, q_t, d)$ . For example:  $b(2, 3, 2, 1) = b(2, 3, 2) = 12$ .

**Proposition 1** (Hazewinkel): *There exist elements  $v_i \in MU_{2i}$  such that*

- 1)  $MU_* = \mathbb{Z}[v_1, v_2, \dots]$
- 2) *If we set  $V_i = v_{i-1}$ , then, in  $MU_* \otimes \mathbb{Q}$ ,*

$$\frac{[CP(s-1)]}{s} = \sum_{((q_1, \dots, q_t, d))} \frac{b(q_1, \dots, q_t)}{p_1 \cdots p_t} V_{q_1}^{r_1} V_{q_2}^{r_2} \cdots V_{q_t}^{r_t-1} V_d^{r_t}$$

where  $q_i$  is a power of  $p_i$ ,  $r_i = q_1 \cdots q_i$ , and the sum is taken over all ordered factorizations of  $s$ .

*Notation.* For a fixed prime  $p$ , let

$$d(s) = p^{s-1} + \dots + 1.$$

**DEFINITION.** By  $\Gamma_*(p^s)$  we mean  $MU_* / \langle v_{p-1}^{d(s)} - d^{(s-1)} \rangle$ . This definition bears a few words of explanation.  $\Gamma_*(p^s)$  as defined is a graded ring. We are interested, however, only in its structure as a graded abelian group. With this in mind, we will often write  $\Gamma_*(p) \subseteq \Gamma_*(p^2) \subseteq \dots \subseteq MU_*$  even though the inclusion is not true for the rings in question, only the groups. Each  $\Gamma_*(p^s)$  is, of course, a graded, free abelian group with a rather complicated number of generators in each dimension.

**Proposition 2.**  $MU_{2n}(BZ_{p^s})=0$  and  $MU_{2n+1}(BZ_{p^s})$  is a finite abelian group of order  $p^{s(n)}$ , where  $s(n) = \sum_{j=0}^n \pi(j)$  and  $\pi(n)$  is the number of partitions of  $n$ .

*Proof.* Consider the Atiyah-Hirzebruch spectral sequence, henceforth denoted AHSS.

$$E_{r,q}^2 = H_r(BZ_{p^s}; MU_q) = \begin{cases} 0 & \text{if } q \text{ is odd or } r \text{ is even} \\ (Z_{p^s})^{\alpha(q/2)} & \text{otherwise.} \end{cases}$$

For purely dimensional reasons there can be no non-zero differentials, so the spectral sequence collapses and  $E^2 = E^\infty$ . There is a filtration  $MU_t(BZ_{p^s}) = F_t \supseteq \dots \supseteq F_0 \supseteq F_{-1} = 0$  such that  $F_q / F_{q-1} = E_{q,t-q}^\infty$ . If  $t$  is even, then  $E_{q,t-q}^\infty = 0 \forall q$ . Therefore  $MU_t(BZ_{p^s}) = 0$ . If  $t$  is odd,  $E_{t-q,q}^\infty$  is zero for  $q$  odd and has order  $p^{s\alpha(q/2)}$  for  $q$  even. Q.E.D.

In order to get the precise structure of the odd dimensional groups, we need some information from  $K$ -theory.

There is a natural inclusion  $Z_{p^s} \rightarrow S^1$  given by  $1 \mapsto \exp(2\pi i/p^s)$  so that the standard free action of  $S^1$  on  $S^{2n+1}$  induces a free action of  $Z_{p^s}$  on  $S^{2n+1}$ . Denote the resulting  $(2n+1)$ -dimensional quotient manifold by  $L^n(p^s)$ , the  $(2n+1)$ -dimensional lens space. We then have the tower of fibrations.

$$\begin{array}{ccc} Z_{p^s} & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ S^1 = S^1/Z_{p^s} & \longrightarrow & L^n(p^s) \\ & & \downarrow \pi \\ & & CP(n) \end{array}$$

Let  $\xi_n$  be the canonical line bundle over  $CP(n)$ ,  $\eta_n = \pi^*(\xi_n)$  and  $(\eta_n - [1]) = x \in K(L^n(p^s))$ .

**Proposition 3:**  $K(L^n(p^s)) = \frac{Z[x]}{((1+x)p^s - 1, x^{n+1})}$ .

For the proof, see Atiyah [1], p. 105.

DEFINITION. For a given prime  $p$ , let  $m(j, s) = p^{\left[ \frac{j}{p^{s-1}(p-1)} \right] + 1}$ , where the square brackets indicate the greatest integer function.

**Proposition 4.** Consider  $K(L^n(p^s))$ . For every  $j \geq p^{s-1}$ , there is a sequence of integers  $\{b_i\}$  such that

$$m(n-j, s)x^j = pm(n-j, s)\left\{\sum_{i < j} b_i x^i\right\}$$

Proof. The proof is by induction on  $n$  and  $j$  for a fixed  $s$ .

The theorem is trivial for  $n < p^{s-1}$ , since in this case  $x^j = 0$ . Assume the theorem is true for  $n-1 \geq p^{s-1}-1$  and write

$$K(L^{n-1}(p^s)) = \frac{Z[y]}{((1+y)^{p^s}-1, y^n)}.$$

Mapping  $y^i \mapsto x^{i+1}$  induces a group homomorphism  $g: K(L^{n-1}(p^s)) \rightarrow K(L^n(p^s))$ . By induction,  $m(n-1-j, s)y^j = pm(n-1-j, s)\sum_{i < j} b_i y^i, j \geq p^{s-1}$ . Applying  $g$  to this equality, we obtain  $m(n-(j+1), s)x^{j+1} = pm(n-(j+1), s)\sum b_i x^{i+1}$ . Thus the theorem is true for  $n$  as long as  $j < p^{s-1}$ .

Suppose then that  $j = p^{s-1}$ . We know that  $\sum_{i=1}^{p^s} \binom{p^s}{i} x^i = 0$ . For  $1 \leq j \leq p^s-1$ ,  $\binom{p^s}{j}$  is divisible by  $p$  and for  $p < j < p^{s-1}$ ,  $\binom{p^s}{j}$  is divisible by  $p^2$ . Therefore  $m(n-p^s, s)\binom{p^s}{j} = km(n-p^{s-1}, s) = \bar{k} m(n-i, s)$ , for  $i \geq p^{s-1}$ . Multiply the above sum by  $m(n-p^s, s)$ . Then

$$0 = pm(n-p^{s-1}, s)\sum_{i=1}^{p^s-1} k_i x^i + km(n-p^{s-1}, s)x^{p^s-1} + \sum_{i=p^{s-1}+1}^{p^s} k_i m(n-i, s)x^i$$

where  $k \equiv 0 \pmod{p}$ .

Now  $\sum_{i=p^{s-1}+1}^{p^s} k_i m(n-i, s)x^i =$

$$= p \sum_{i < p^{s-1}} \bar{k}_i m(n-p^{s-1}, s)x^i + p\bar{k}m(n-p^{s-1}, s)x^{p^s-1} + p \sum_{i=p^{s-1}+1}^{p^s-1} \bar{k}_i m(n-i, s)x^i$$

Therefore

$$0 = pm(n-p^{s-1}, s)\sum_{i < p^{s-1}} h_i x^i + (k+p\bar{k})m(n-p^{s-1}, s)x^{p^s-1} + \dots$$

$$\dots + \sum_{i=p^{s-1}+1}^{p^s-1} h_i m(n-i, s)x^i$$

Repeating this process as long as there are terms  $x^i$  for which  $i > p^{s-1}$ , we obtain

$$m(n-p^{s-1}, s)(k+pb)x^{p^s-1} = pm(n-p^{s-1}, s)\sum_{i < p^{s-1}} \bar{k}_i x^i.$$

Since  $(k+pb)$  is a unit mod  $p$ , this implies that

$$m(n - p^{s-1}, s)x^{p^{s-1}} = pm(n - p^{s-1}, s) \sum_{i < p^{s-1}} b_i x^i \text{ as claimed.} \quad \text{Q.E.D.}$$

**Corollary.** *In  $K(L^n(p^s))$  the order of the element  $x^{p^{s-1}} - p \sum_{i < p^{s-1}} b_i x^i$  is less than or equal to  $m(n - p^{s-1}, s)$ .*

Suppose  $\mathfrak{G}$  is any complex,  $n$ -plane bundle over a space  $X$ . The map  $cf_1: K(X) \rightarrow MU^2(X)$  which associates to  $[\mathfrak{G}] - n$  the first cobordism chern class of  $\mathfrak{G}$  is clearly an homomorphism and was shown by Conner and Floyd [3] to be the injection of a direct summand.

If the space  $X$  is an  $n$ -dimensional manifold which is  $MU$ -orientable and, in particular, if  $X$  is a  $U$ -manifold such as  $L^n(p^s)$ , then there is a Poincaré duality isomorphism

$$D: MU^k(X) \rightarrow MU_{n-k}(X).$$

**DEFINITION.** By  $X(k, s) \in MU_{2k+1}(BZ_{p^s})$  we mean the bordism element represented by the inclusion  $i: L^k(p^s) \rightarrow BZ_{p^s}$  of the  $(2k+1)$ -skeleton. When the context is clear, we will write  $X(k)$  for  $X(k, s)$ .

**Proposition 5.**  $i_*(D(cf_1(x)^k)) = X(n-k)$ .

*Proof.* This is Proposition 1.3 of [5].

In order to use the above information, it is necessary to understand some elementary results from the theory of formal groups which we now review.

**DEFINITION.** Suppose  $R$  is a commutative ring with unit. By a formal group over  $R$ , we mean a formal power series  $F(X_1, X_2) = \sum_{i,j \geq 0} a_{ij} X_1^i X_2^j$ ,  $a_{ij} \in R$  which satisfies

- 1)  $F(X_1, 0) = X_1$  and  $F(0, X_2) = X_2$
- 2)  $F(X_1, F(X_2, X_3)) = F(F(X_1, X_2), X_3)$

We are interested in the following formal group over  $MU^*$ . Recall that  $MU^*(BS^1) \cong MU^*[[X]]$  and  $MU^*(BS^1 \times BS^1) \cong MU^*[[X_1, X_2]]$ , the rings of formal power series in one and two variables respectively. The multiplication  $m: S^1 \times S^1 \rightarrow S^1$  in the group  $S^1$  induces a map  $Bm: BS^1 \times BS^1 \rightarrow BS^1$  which classifies the tensor product of line bundles. That is, if  $\pi_1, \pi_2: BS^1 \times BS^1 \rightarrow BS^1$  are the projections and  $\xi_1$  and  $\xi_2$ , the respective pullbacks of the universal line bundle  $\xi$  over  $BS^1$ , then  $Bm^*(\xi) = \xi_1 \otimes \xi_2$ . The standard result is that  $Bm^*(X)$  is a formal group over  $MU^*$ , being, in fact, a universal object for formal groups over an arbitrary (commutative) ring. We define elements  $a_{ik} \in MU^*$  by setting  $Bm^*(X) = X_1 + X_2 + \sum a_{ij} X_1^i X_2^j = F(X_1, X_2)$ .

If  $\mathfrak{G}$  is a line bundle, write  $\mathfrak{G}^2$  for the tensor product of  $\mathfrak{G}$  with itself. then

$\mathfrak{S}^2$  is classified by the map  $Bm \cdot \Delta: BS^1 \rightarrow BS^1 \times BS^1 \rightarrow BS^1$ , where  $\Delta$  is the diagonal map. Since  $X = cf_1(\xi)$ ,  $cf_1(\xi^2) = F(X, X)$  by naturality.

DEFINITION. Let  $[k]X \in MU^*(BS^1)$  be defined inductively as follows:

- 1)  $[1]X = X$
- 2)  $[k]X = F(X, [k-1]X)$ .

This definition is rigged, of course, to give us the result we really want, namely,  $cf_1(\xi^k) = [k]X$ .

Notation. We will write

$$[k]X = a(0, k)X + a(1, k)X^2 + \dots + a(m, k)X^{m+1} + \dots$$

with  $a(m, k) \in MU^* = MU_{-*}$ .

In general, it is somewhat difficult to give an explicit description of the  $a(m, k)$  as bordism classes of familiar manifolds. There is, however, the following result.

**Proposition 6.** *Given a prime  $p$ , the ideal in  $MU_*$  generated by  $\{a(m, p)\}$  is the ideal of all manifolds whose chern numbers are all divisible by  $p$ . This ideal is in fact generated by  $\{a(p^i - 1, p)\}$   $i=0, 1, \dots$*

Proof. See [2], Proposition 41.1.

**Proposition 7.**  *$a(p^s - 1, p^s) = cv_p^{\alpha(s)} + py$  where  $c \equiv 0 \pmod{p}$  and  $y \in MU_*$ . If  $j < p^s - 1$ ,  $a(j, p^s)$  is divisible by  $p$ .*

Proof. The proof is by induction on  $s$ . The case  $s=1$  is the above mentioned result of Conner and Floyd.

Assume by induction that  $a(p^{s-1} - 1, p^{s-1}) = c_1 v_p^{\alpha(s-1)} + py_1$  and that for  $j < p^{s-1} - 1$ ,  $a(j, p^{s-1})$  is divisible by  $p$ . Now,

$$[p^s]X = [p]([p^{s-1}]X) = \sum_{k \geq 0} a(k, p) \{[p^{s-1}]X\}^{k+1}.$$

Therefore

$$a(p^s - 1, p^s) = \sum_{k \geq 0} \sum_{\langle i_0, \dots, i_k \rangle} a(k, p) a(i_0, p^{s-1}) \dots a(i_k, p^{s-1}).$$

where  $i_0 + \dots + i_k = p^s - 1 - k$ .

Suppose  $k < p - 1$ . Then  $a(k, p)$  is divisible by  $p$ . Similarly, if  $i_j < p^{s-1} - 1$ , then  $a(i_j, p^{s-1})$  is divisible by  $p$ .

If  $k \geq p - 1$  and  $i_j \geq p^{s-1} - 1$  for all  $j$ , then, since  $k + i_0 + \dots + i_k = p^s - 1$ ,  $k = p - 1$  and  $i_j = p^{s-1} - 1$  for all  $j$ . But  $a(p - 1, p) a(p^{s-1} - 1, p^{s-1})^p = (c_0 v_{p-1} + py_0)(c_1 v_p^{\alpha(s-1)} + py_1)^p = c_0 c_1 v_p^{\alpha(s)} + py$  and  $c_0 c_1 \equiv 0 \pmod{p}$ . Q.E.D.

**Proposition 8.** For each integer  $n \geq p^{s-1} - 1$ , there is an element  $Y(n, s) \in MU_{2n+1}(BZ_{p^s})$  which satisfies

- 1)  $Y(n, s) = v_{p-1}^{\alpha(s-1)} X(n - p^{s-1} + 1) + \sum_k w_{k,s} X(n - k)$   
 where  $w_{k,s} \in MU_* / \langle v_{p-1}^{\alpha(s-1)} \rangle = N_*$
- 2)  $m(n - p^{s-1} + 1, s) Y(n, s) = 0$ .

Proof. Induction on  $n$ . We showed that in  $K(L^n(p^s))$ ,

$$m(n - p^{s-1}, s) x^{p^{s-1}} = pm(n - p^{s-1}, s) \sum_{j < p^{s-1}} b_j x^j.$$

Recall that  $x = \eta_n - 1$  and  $x^k = \sum_{i=0}^k (-1)^i \binom{k}{i} \eta_n^i$ .

Apply the map  $cf_1$ , yielding

$$\begin{aligned} & m(n - p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} [i](cf_1(x)) \\ &= pm(n - p^{s-1}, s) \sum_{j < p^{s-1}} b_j \left\{ \sum_{i'=1}^j (-1)^{i'} \binom{j}{i'} [i'](cf_1(x)) \right\}. \end{aligned}$$

Equivalently, applying  $i_* \circ D$ ,

$$\begin{aligned} & m(n - p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} \left\{ \sum_{k=0}^{n-1} a(k, i) X(n - k - 1) \right\} \\ &= pm(n - p^{s-1}, s) \sum_{j < p^{s-1}} b_j \left\{ \sum_{i'=1}^j (-1)^{i'} \binom{j}{i'} \left\{ \sum_{k'=0}^{n-1} a(k', i') X(n - k' - 1) \right\} \right\}. \end{aligned}$$

Note that for  $* < 2(p^{s-1} - 1) MU_* = N_*$  and we see that

$$\begin{aligned} & m(n - p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} \left\{ \sum_{k \geq p^{s-1}-1}^{n-1} a(k, i) X(n - k - 1) \right\} \\ & - pm(n - p^{s-1}, s) \sum_{j < p^{s-1}} b_j \left\{ \sum_{i'=1}^j (-1)^{i'} \binom{j}{i'} \left\{ \sum_{k' \geq p^{s-1}-1}^{n-1} a(k', i') X(n - k' - 1) \right\} \right\} \end{aligned}$$

has the form  $m(n - p^{s-1}, s) \sum w_{k,s} X(n - k)$ ,  $w_{k,s} \in N_*$ .

Now suppose that  $k \geq p^{s-1}$ . If we expand the  $a(k, i)$  in terms of our chosen basis, we will get sums of monomials in the  $v_i$ . If a monomial contains no factor  $v_{p-1}^{\alpha(s-1)}$ , then the product of that monomial and  $X(n - k - 1)$  has the required form. Suppose that the monomial has the form  $\beta v_{p-1}^{\alpha(s-1)} X(n - k - 1)$ . Since  $k \geq p^{s-1}$ , the degree of  $X(n - k - 1)$  is strictly less than that of  $X(n - p^{s-1})$ . Therefore, by induction,

$$m(n - p^{s-1}, s) \beta v_{p-1}^{\alpha(s-1)} X(n - k - 1) = m(n - p^{s-1}, s) \beta \sum_k w_{k,s} X(n - k)$$

with  $w_{k,s} \in N_*$ .

Repeating the induction if necessary, we have that



$$m(n-p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} \{a(p^{s-1}-1, i)X(n-p^{s-1})\} \\ - pm(n-p^{s-1}, s) \sum_{j < p^{s-1}} b_j \left\{ \sum_{i'=1}^j (-1)^{i'} \binom{j}{i'} a(p^{s-1}-1, i') X(n-p^{s-1}) \right\}$$

has the form  $m(n-p^{s-1}, s) \sum w_{k,s} X(n-k)$ ,  $w_{k,s} \in N_*$ . Utilizing Proposition 7, since for  $i < p^{s-1}$ ,  $\binom{p^{s-1}}{i}$  is divisible by  $p$ , we have that

$$m(n-p^{s-1}, s) \{ (c+pd)v_{p-1}^{a(s-1)} + w \} X(n-p^{s-1}) \\ - pm(n-p^{s-1}, s) \sum_{j < p^{s-1}} \bar{b}_j a(p^{s-1}-1, j) X(n-p^{s-1}), \quad c \not\equiv 0 \pmod p,$$

has the same form. Expanding the  $\bar{b}_j a(p^{s-1}-1, j)$  in terms of our chosen basis as  $\bar{b}_j a(p^{s-1}-1, j) = -c_j v_{p-1}^{a(s-1)} + \dots$ , we see that

$$m(n-p^{s-1}, s) (c+pd+p \sum_{i < p^{s-1}} c_i) v_{p-1}^{a(s-1)} X(n-p^{s-1})$$

has the same form. But  $c+p(d+\sum_{i < p^{s-1}} c_i)$  is a unit mod  $p$ , so

$$m(n-p^{s-1}, s) v_{p-1}^{a(s-1)} X(n-p^{s-1}) = m(n-p^{s-1}, s) \sum_k w_{k,s} X(n-k), \\ w_{k,s} \in N_*.$$

Set  $Y(n-1, s) = v_{p-1}^{a(s-1)} X(n-p^{s-1}) - \sum_k w_{k,s} X(n-k)$ . This clearly satisfies 1) and 2). Q.E.D.

**Proposition 9.** *For each integer  $a \leq s$  and each integer  $n \geq p^{a-1}-1$ , there is an element  $Y(n, a) \in MU_{2n+1}(BZ_{p^s})$  which satisfies:*

- 1)  $Y(n, a) = v_{p-1}^{a(a-1)} X(n-p^{a-1}+1) + \sum_k w_k X(n-k)$  with  $w_k \in MU_* / \langle v_{p-1}^{a(a-1)} \rangle$ .
- 2)  $p^{s-a} m(n-p^{a-1}+1, a) Y(n, a) = 0$ .

*Proof.* Induction on  $s$ . The case  $s=1$  follows immediately from Proposition 8. Suppose we have defined such elements for  $s-1$ . For each  $a < s$ , let

$$Y(n, a) = v_{p-1}^{a(a-1)} X(n-p^{a-1}+1, s) + \sum_k w_{k,a} X(n-k, s).$$

According to [2], page 101, if  $i: BZ_{p^{s-1}} \rightarrow BZ_{p^s}$ , then

$$pi_*(X(n, s-1)) = p^2 X(n, s). \quad \text{Therefore, since } p^{s-a} m(n-p^{a-1}+1, a) \text{ is} \\ \text{divisible by } p^2, p^{s-a} m(n-p^{a-1}+1, a) \{ v_{p-1}^{a(a-1)} X(n-p^{a-1}+1, s) + \\ + \sum_k w_{k,s} X(n-k, s) \} = i_*(p^{s-1-a} m(n-p^{a-1}+1, a) \{ v_{p-1}^{a(a-1)} X(n-p^{a-1}+1, s-1) + \\ + \sum_k w_{k,s} X(n-k, s-1) \}) = 0.$$

Clearly the elements  $Y(n, a)$  have the form prescribed by 1).

The case  $a=s$  is precisely the substance of Proposition 8. Q.E.D.

**Proposition 10.** *In  $MU_{2p^s+1}(BZ_{p^s})$ , the element  $v_{p-1}^{\alpha(s)}X(0)$  is divisible by  $p$ .*

Proof. Notice that  $\eta_n p^s = 1$ . Therefore  $cf_1(\eta_n p^s) = 0$  or, equivalently,  $\sum_j a(j-1, p^s)X(n-j) = 0$ .

By Proposition 7,  $a(j-1, p^s)$  is divisible by  $p$  for  $j < p^s$ . Therefore  $a(p^s-1, p^s)X(0)$  is divisible by  $p$ . But again by Proposition 7,  $a(p^s-1, p^s) = cv_{p-1}^{\alpha(s)} + pW$ , where  $c$  is a unit mod  $p$ . Therefore  $v_{p-1}^{\alpha(s)}X(0)$  is divisible by  $p$ . Q.E.D.

We are now in a position to set up the result we wish to prove. Fix an integer  $n$ .

DEFINITION. By  $T(a, b)$  we mean

$$\frac{\Gamma_{2(n-b)}(p^a)}{p^{s-a}m(b-p^{a-1}+1, a)\Gamma_{2(n-b)}(p^a)}$$

By  $T$  we mean  $\sum_{a=1}^s \sum_{b=p^{a-1}-1}^n T(a, b)$ .

Construct a map  $f(a, b): T(a, b) \rightarrow MU_{2n+1}(BZ_{p^s})$ ,  $f(a, b): w_{n-b} \mapsto w_{n-b}Y(b, a)$  for every  $w_{n-b} \in \Gamma_{2(n-b)}(p^a)$ . By Proposition 9, this map is a well-defined homomorphism. Let  $f = \sum_{a,b} f(a, b): T \rightarrow MU_{2n+1}(BZ_{p^s})$ . Our aim is to show that  $f$  is an isomorphism. To accomplish this, we will first show that  $f$  is an epimorphism and then that the orders of  $T$  and  $MU_{2n+1}(BZ_{p^s})$  are equal.

In order to show that  $f$  is an epimorphism, we will consider the groups  $MUZ_{p^*}(BZ_{p^s})$ , that is, complex bordism with  $Z_p$  coefficients. For this purpose, let  $R$  be a  $Z_p$  Moore spectrum and define  $MUR_*(X) = S_*(MU \wedge R \wedge X^+) = MU_*(R \wedge X^+)$ . The result is a generalized homology theory.

**Proposition 11.**  $MUR_{2n+1}(BZ_{p^s}) \cong MU_{2n+1}(BZ_{p^s}) \otimes Z_p$ .

Proof. There is a Künneth short exact sequence in complex bordism.

$$\begin{aligned} 0 \rightarrow MU_m(BZ_{p^s}) \otimes Z_p &\rightarrow MUR_m(BZ_{p^s}) \rightarrow \\ &\rightarrow \text{Tor}_2^1(MU_{m-1}(BZ_{p^s}), Z_p) \rightarrow 0. \end{aligned}$$

Since  $MU_{2n}(BZ_{p^s}) = 0$ , the result follows.

For further calculations, we need the existence of cap products in the AHSS. The following proposition may be garnered from a paper of R. Kultze [6].

**Proposition 12.** *Suppose  $h^* \otimes k_* \rightarrow k_*$  is a pairing of coefficient groups of theories  $h^*(\ )$  and  $k_*(\ )$ . The cap product  $H^*(X; h^*) \otimes H_*(X; k_*) \rightarrow H_*(X; k_*)$  induces a cap product  $\cap_2$  on the  $E^2$  terms of the corresponding Atiyah-Hirzebruch spectral sequences which satisfies:*

- 1)  $\cap_2$  induces cap products  $\cap_r: E^r \otimes E_r \rightarrow E_r$
- 2) Each differential  $d^r$  is a (graded) derivation with respect to  $\cap_r$ ; i.e.  $d^r(a \cap_r b) = d_r(a) \cap_r b \pm a \cap_r d^r(b)$ .

We will generally write  $\cap$  for  $\cap_r$ .

We will apply this proposition to the module pairing arising from  $MU \wedge MUR \rightarrow MUR$ .

REMARK. The more natural thing to do would be to use a ring spectrum pairing  $MUR \wedge MUR \rightarrow MUR$  here. Unfortunately, the general perversity of the universe demands that  $MUZ_2$  not be a ring spectrum. Such is life.

The map of spectra  $MU \rightarrow MUR$  induces a map  $t: MU_*(BZ_{p^s}) \rightarrow MUR_*(BZ_{p^s})$ . Let  $t(X(n)) = Z(2n+1)$ .

**Proposition 13.**  $MUR_{2n+1}(BZ_{p^s})$  is additively generated by elements of the form  $w_j Z(2(n-j)+1)$  where  $w_j \in MU_* / \langle v_{p-1}^{a(s)} \rangle$ .

Proof. Consider the AHSS for  $MUR_*(BZ_{p^s})$  in which

$$E_{i,q}^2 = H_i(BZ_{p^s}; MUR_q) = \begin{cases} 0 & q \text{ odd} \\ (Z_p)^{\pi(q/2)} & \text{otherwise.} \end{cases}$$

Let  $r \geq 2$  be the smallest integer such that  $E^r \neq E^{r+1}$ . Since  $E_{p,q}^2 = 0$  for  $q$  odd and  $d^r$  has bidegree  $(r, r-1)$ ,  $r$  must be odd.

Let  $\bar{E}$  be the AHSS for  $MU_*(BZ_{p^s})$ . The map  $t$  is induced on the  $E^2$  level by the reduction  $\bar{t}: H_*(BZ_{p^s}; MU_*) \rightarrow H_*(BZ_{p^s}; MUR_*)$ . Since  $H_{2n}(BZ_{p^s}; Z) = 0$ , the universal coefficient theorem says that  $\bar{t}$  is an epimorphism in odd dimensions. Therefore  $MUR_{2n+1}(BZ_{p^s})$  is at least generated by the elements  $b_j Z(2(n-j)+1)$ , as  $b_j$  ranges over  $MU_*$ .

- 1) Claim  $d^r(Z(2j+1) \otimes b_k) = 0 \forall j \geq 0$  and  $b_k \in MU_*$ . In fact we have already noticed that the spectral sequence  $\bar{E}$  is trivial for dimensional reasons. Therefore

$$d^r(Z(2j+1) \otimes b_k) = d^r(\bar{t}(X(j) \otimes b_k)) = \bar{t}(d^r(X(j) \otimes b_k)) = 0.$$

- 2) Claim  $d^r: E_{r+1,0}^r \rightarrow E_{1,r-1}^r$  is non-zero. For there is an integer  $j$  and a  $b_k \in MU_*$  such that

$0 \neq d^r(Z(2j) \otimes b_k) = d^r(Z(2j)) \otimes b_k$ . Then  $d^r(Z(2j)) \neq 0$ . There is a class  $u \in H^2(BZ_{p^s}; Z)$  which gives the periodicity of  $H_*(BZ_{p^s}; Z_p)$  via cap products, i.e.  $H_m(BZ_{p^s}; Z_p) = Z_p$  on a generator  $w_m$  and  $w_m = u \cap w_{m+2}$ . A similar periodicity holds for  $H_*(BZ_{p^s}; Z)$  with respect to the same  $u$ . Denote also by  $u$  the corresponding generator in  $\bar{E}_2^{2,0}$  of the AHSS for  $MU_*(BZ_{p^s})$ . Then

$$\begin{aligned} d^r(Z(2j-2)) &= d^r(u \cap Z(2j)) = d_r(u) \cap Z(2j) \pm u \cap d^r(Z(2j)) = \\ &= \pm u \cap d^r(Z(2j)). \end{aligned}$$

But, for  $2j \geq r+3$ ,  $u \cap_r = u \cap_2$  is an isomorphism. Therefore, in this range  $d^r(Z(2j-2)) \neq 0$ . By induction  $d^r(Z(r+1)) \neq 0$  as claimed.

- 3) Claim  $d^r(Z(r+1)) = Z(1) \otimes v_{p-1}^{(r-1)/2(p-1)}$ . For, since  $d^r(Z(r+1)) \neq 0$ , there is a  $b_k \in MU_*$ ,  $b_k \neq 0$ , such that  $d^r(Z(r+1)) = Z(1) \otimes b_k$ . Then, for any  $b_j \in MU_*$ ,  $b_j \neq 0$ ,  $d^r(Z(r+1) \otimes b_j) = Z(1) \otimes b_j b_k \neq 0$ . In  $Z(1) \otimes b_k = d^r(Z(r+1)) = d^r(u \cap (Z(r+3))) = u \cap d^r(Z(r+3))$ . But  $u \cap (Z(3) \otimes b_k) = Z(1) \otimes b_k$  and  $u \cap$  is an isomorphism. Therefore  $d^r(Z(r+3)) = Z(3) \otimes b_k$ . Arguing inductively  $d^r(Z(r+2j+1) \otimes b_j) = Z(2j+1) \otimes b_j b_k$ .

This has two consequences. First,  $d_{2j,*}^r$  is a monomorphism for  $2j \geq r+1$ , so that  $E_{2j,*}^{r+1} = 0$  and  $d_{2j,*}^i = 0$  for all  $i \geq r+1$ . Therefore  $E_{2j+1,*}^{r+1} = E_{2j-1,*}^\infty$ . Secondly, for  $j \geq 0$ ,  $Z(2j+1) \otimes b = 0$  in  $E^\infty$  if and only if  $b$  is in the ideal  $\langle b_k \rangle$  generated by  $b_k$ . For suppose  $b = b_k a \in MU_*$ . Then  $Z(2j+1) \otimes b = d^r(Z(2j+r+1) \otimes a)$ , so that  $Z(2j+1) \otimes b = 0$  in  $E_{2j+1,*}^{r+1} = E_{2j+1,*}^\infty$ . On the other hand, if  $b \neq b_k a$ , then  $Z(2j+1) \otimes b$  cannot be the image of any  $d^r$  and we have shown that  $d_{2i,*}^i = 0$  for all  $i \geq r+1$ . Therefore, in this case  $Z(2j+1) \otimes b \neq 0$ .

Now  $v_{p-1}^{\alpha(s)} X(0)$  is divisible by  $p$  by Proposition 10. Therefore  $t(v_{p-1}^{\alpha(s)} X(0)) = v_{p-1}^{\alpha(s)} Z(1) = 0$  in  $MUR_{2p^s-1}(BZ_{p^s}) = MU_{2p^s-1}(BZ_{p^s}) \otimes Z_p$ , so that  $Z(1) \otimes v_{p-1}^{\alpha(s)} = 0$  in  $E^\infty$ . Thus  $v_{p-1}^{\alpha(s)} \in \langle b_k \rangle$ , i.e.  $b_k$  is a power of  $v_{p-1}$ . For dimensional reasons

$$b_k = v_{p-1}^{\frac{r-1}{2(p-1)}}.$$

REMARK. It turns out that  $r = 2p^s - 1$  and  $b_k = v_{p-1}^{\alpha(s)}$ , but this is not necessary for the proof.

Now, the only non-zero groups appearing in the associated graded of  $MUR_{2n+1}(BZ_{p^s})$  are of the form  $E_{2j+1, 2(m-j)}^\infty$ . But we have just shown these groups to be generated by the elements  $Z(2j+1) \otimes b$  with  $b \in MU_* - \langle b_k \rangle \subseteq MU_* - \langle v_{p-1}^{\alpha(s)} \rangle$ .  
 Q.E.D.

**Proposition 14.** *The map  $f: T \rightarrow MU_{2n+1}(BZ_{p^s})$  is an epimorphism.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} T & \longrightarrow & T \otimes Z_p \\ f \downarrow & & \downarrow f \otimes Z_p \\ MU_{2n+1}(BZ_{p^s}) & \xrightarrow{t} & MUR_{2n+1}(BZ_{p^s}), \end{array}$$

and suppose that  $t \circ f$  were an epimorphism. Then  $f \otimes Z_p$  would be an epimorphism. Since both  $T$  and  $MU_{2n+1}(BZ_{p^s})$  are finite abelian  $p$ -groups,  $f$  would also be an epimorphism.

We must show, therefore, that  $t \circ f$  is an epimorphism. This is equivalent to showing that image  $f \supseteq \{b_k X(n-k): b_k \in MU_* - \langle v_{p-1}^{\alpha(s)} \rangle\}$ . Consider the in-

creasing sequence of groups  $MU_* - \langle v_{p-1}^{a(1)} \rangle \subseteq MU_* - \langle v_{p-1}^{a(2)} \rangle \subseteq \dots \subseteq MU_* - \langle v_{p-1}^{a(s)} \rangle$  and suppose that  $a \in MU_* - \langle v_{p-1}^{a(i+1)} \rangle$ ,  $a \notin MU_* - \langle v_{p-1}^{a(i)} \rangle$ . Then  $a = v_{p-1}^{a(i)} \cdot v_{p-1}^c \cdot b_j$  where  $c < d(i+1) - d(i)$  and  $b_j \in \Gamma_{2j}(p)$ . In other words  $v_{p-1}^c \cdot b_j \in \Gamma_*(p^{i+1})$ . Recall that

$$Y(n-j-c(p-1), i) = v_{p-1}^{i(i)} X(n-j-c(p-1)-p^i+1) + \sum a_k X(n-j-c(p-1)-k)$$

where  $a_k \in MU_* / \langle v_{p-1}^{a(i)} \rangle$ .

Since we may assume by induction on the power of  $v_{p-1}$  appearing in a given monomial that  $v_{p-1}^c \cdot b_j \cdot a_k X(n-j-c(p-1)-k)$  is in the image of  $f$ , it follows that  $aX(n-|a|/2) = v_{p-1}^c \cdot b_j \cdot Y(n-j-c(p-1))$  modulo the image of  $f$ . Therefore  $aX(n-|a|/2)$  is in the image of  $f$ . Q.E.D.

**DEFINITION.** By  $\pi(n; m, r)$  we mean the number of partitions of  $n$  which contain no more than  $m$  terms equal to  $r$ .

**EXAMPLE.** Let  $m=1, r=2$ . Then  $(3,2)$  is an allowable partition of 5, but  $(2,2,1)$  is not.  $\pi(5; 1, 2)=6$  and  $\pi(5; 2, 1)=5$ .

**Proposition 15.** 
$$\sum_{k=0}^n \pi(k) = \sum_{j=0}^n \left( \left\lfloor \frac{n-j}{(m+1)r} \right\rfloor + 1 \right) \pi(j; m, r)$$

**Proof.** First notice that the number of partitions of  $n$  containing exactly  $m$  terms equal to  $r$  is equal to the number of unrestricted partitions of  $n-mr$ . Furthermore,  $\pi(k)$  is equal to the sum of the number of partitions of  $k$  containing no terms equal to  $r$ , those with exactly one  $r$ , and so forth. Therefore

$$\pi(k) = \pi(k; 0, r) + \pi(k-r; 0, r) + \pi(k-2r; 0, r) + \dots$$

Similarly,

$$\pi(k; m, r) = \pi(k; 0, r) + \pi(k-r; 0, r) + \dots + \pi(k-mr; 0, r).$$

Therefore

$$\begin{aligned} \pi(k; m, r) &= \pi(k) - \pi(k-(m+1)r) \text{ and} \\ \pi(k) &= \pi(k; m, r) + \pi(k-(m+1)r; m, r) + \pi(k-2(m+1)r; m, r) + \dots \end{aligned}$$

Summing over  $k$ ,

$$\begin{aligned} \sum_{k=0}^n \pi(k) &= \sum_{k=0}^n \sum_a \pi(k-a(m+1)r; m, r) \\ &= \sum_j (\max\{a: j = k-a(m+1)r\} + 1) \pi(j; m, r) \\ &= \sum_j \left( \left\lfloor \frac{n-j}{(m+1)r} \right\rfloor + 1 \right) \pi(j; m, r) \qquad \text{Q.E.D.} \end{aligned}$$

**Proposition 16.**

$$MU_{2n+1}(BZ_{p^s}) \cong \sum_{a=1}^s \sum_{b=p^{a-1}-1}^n \frac{\Gamma_{2(n-b)}(p^a)}{p^{\left\lfloor \frac{b-p^{a-1}+1}{p^{a-1}(p-1)} \right\rfloor + s - a - 1} \Gamma_{2(n-b)}(p^a)}$$

Proof. The proposition states that the map  $f: T \rightarrow MU_{2n+1}(BZ_{p^s})$  is an isomorphism. Since we have already shown it to be an epimorphism, it suffices to verify that the two groups involved have the same order.

According to Proposition 2, the order of  $MU_{2n+1}(BZ_{p^s})$  is  $p^{A(s)}$  where  $A(s) = \sum_{k=0}^n s\pi(k)$ . The order of  $T$  on the other hand is clearly  $p^{B(s)}$  where

$$B(s) = \sum_{a=1}^s \sum_{b=p^{a-1}-1}^n \left\{ \left\lfloor \frac{b-p^{a-1}+1}{p^{a-1}(p-1)} \right\rfloor + s - a + 1 \right\} \pi(n-b; p^{a-1}-1, p-1).$$

We must show  $A(as) = B(s)$ .

Proceed by induction on  $s$ . The case  $s=1$  is an example of Proposition 15. Write  $\pi(n; m)$  for  $\pi(n; m, p-1)$ . Now

$$\begin{aligned} B(s) &= B(s-1) + \sum_{a=1}^{s-1} \sum_{b=p^{a-1}-1}^n \pi(n-b; p^{a-1}-1) + \dots \\ &\dots + \sum_{b=p^{s-1}-1}^n \left\{ \left\lfloor \frac{b-p^{s-1}+1}{p^{s-1}(p-1)} \right\rfloor + 1 \right\} \pi(n-b; p^{s-1}-1). \end{aligned}$$

By Proposition 15,

$$\begin{aligned} &\sum_{b=p^{s-1}-1}^n \left\{ \left\lfloor \frac{b-p^{s-1}+1}{p^{s-1}(p-1)} \right\rfloor + 1 \right\} \pi(n-b; p^{s-1}-1) = \\ &= \sum_{b=0}^{n-p^{s-1}+1} \left\{ \left\lfloor \frac{b}{p^{s-1}(p-1)} \right\rfloor + 1 \right\} \pi(n-b-p^{s-1}+1; p^{s-1}-1) \\ &= \sum_{b=0}^{n-p^{s-1}+1} \pi(b). \end{aligned}$$

Remember from the proof of Proposition 15 that

$$\pi(k) = \sum_{a \geq 0} \pi(k - ap^{a-1}(p-1); p^{a-1}-1).$$

Therefore

$$\sum_{b=p^{a-1}-1}^n \pi(n-b; p^{a-1}-1) = \sum_{b=n-p^a+2}^{n-p^{a-1}+1} \pi(b)$$

and so

$$B(s) - B(s-1) = \sum_{b=0}^{n-p^{s-1}+1} \pi(b) + \sum_{a=1}^{s-1} \sum_{b=n-p^a+2}^{n-p^{a-1}+1} \pi(b) = \sum_{b=0}^n \pi(b) = A(s) - A(s-1).$$

Since  $A(1) = B(1)$ , induction shows that  $A(s) = B(s)$  for all  $s$ . Q.E.D.

**Proposition 17.** *Suppose  $r$  and  $s$  are relatively prime.*

$$\text{Then } MU_*(BZ_{rs}^+) \simeq MU_*(BZ_r^+) \otimes_{MU_*} MU_*(BZ_s^+).$$

Proof. This proposition follows almost immediately from a theorem of Landweber [8] to the effect that if  $X$  and  $Y$  are  $CW$ -complexes such that the AHSS for  $MU_*(X)$  is trivial, then there is a natural short exact sequence

$$\begin{aligned} 0 \rightarrow MU_*(X^+) \otimes MU_*(Y^+) &\rightarrow MU_*(X^+ \wedge Y^+) \rightarrow \\ &\rightarrow \text{Tor}_1^{MU_*}(MU_*(X^+), MU_*(Y^+)) \rightarrow 0. \end{aligned}$$

Since the AHSS for  $MU_*(BZ_r)$  collapses for dimensional reasons and the torsion of  $MU_*(BZ_r)$  and  $MU_*(BZ_s)$  are of relatively prime order,  $\text{Tor}_1^{MU_*}(MU_*(BZ_r^+), MU_*(BZ_s^+))=0$ . Q.E.D.

**Corollary.** *If  $r$  and  $s$  are relatively prime, then*

$$MU_{2n+1}(BZ_{rs}) = MU_{2n+1}(BZ_r) \oplus MU_{2n+1}(BZ_s).$$

Taken in conjunction, these last two propositions clearly suffice to give the complex bordism of any (finite) cyclic group.

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