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The complex Monge–Ampère equation

by

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Chapter 1

1.1. Introduction

During the last twenty years the complex Monge–Ampère equation has been the subject of intensive studies.

In its classical form it is a fully non-linear equation of elliptic type:

$$\mathrm{MA}(u) := \det\left(\frac{\partial^2 u}{\partial z_j\,\partial \bar{z}_k}\right) = f, \tag{1.1.1}$$

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where the solution u is required to be a plurisubharmonic (psh) function in some open subset Ω of $\mathbf{C}^n$. E. Bedford and B. A. Taylor have shown in [BT1] the way the equation may be understood if $u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$. An essential ingredient of this generalization is the notion of a positive current introduced by P. Lelong. The right-hand side of (1.1.1) then becomes a Borel measure. Bedford and Taylor also solved the equation in the case of continuous f and continuous boundary data. The same authors developed in [BT2] the basic structure of pluripotential theory in which the Monge–Ampère operator MA plays a central role being the counterpart of the Laplacian in classical potential theory. The new theory has brought in a much better understanding of psh functions and it has given rise to many recent developments in the theory of extremal functions, the theory of polynomial approximation and complex dynamics. In a survey paper by E. Bedford, and in M. Klimek's book, the reader will find a detailed exposition of pluripotential theory and ample reference to the papers of other authors.

Differential geometry is another source of interest in the complex Monge–Ampère equation. The proofs of two conjectures of E. Calabi, one asserting the existence of a Kähler metric on a compact Kähler manifold which has the preassigned Ricci form, and another concerning the existence of Einstein–Kähler metrics, boil down to solving an appropriate Monge–Ampère equation (see (1.1.2) below) with the right-hand side depending also on the unknown function in the case of the latter conjecture. The equations were solved by T. Aubin [Au1] and S.-T. Yau [Y] (see Theorem 2.1.1 below) under suitable smoothness assumptions. Constructions of Einstein–Kähler metrics and Ricci-flat metrics on non-compact complex manifolds by means of solving the Monge–Ampère equation were carried out by S. Y. Cheng, S.-T. Yau in [CY] and G. Tian, S.-T. Yau in [TY1], [TY2]. The partial differential equations approach of those authors (and others: E. Calabi, L. Nirenberg, A. V. Pogorelov, to mention only a few) is analogous to the one applied in the case of the real Monge–Ampère equation which has a much longer history (see e.g. [GT]). It is based on the method of continuity and in any given situation requires laborious a priori estimates for the derivatives of the solution up to third order. In a similar vein L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck [CKNS] have proved regularity results for the Dirichlet problem associated to (1.1.1) in a strictly pseudoconvex domain.

This historical account is not meant to be complete but we would like to mention also very important works of J.-P. Demailly and L. Lempert. Demailly (see e.g. [D1], [D2] and [D3]) uses the Monge–Ampère operator techniques and the result of Yau to prove very deep facts of algebraic geometry. He usually deals with those unbounded psh functions for which $\mathrm{MA}(u)$ still makes sense. Lempert [Lem] solved the Monge–Ampère equation with pointwise singularity in a convex domain and this result has found striking applications in complex analysis.

In the present paper we seek solutions to the Monge–Ampère equation under possibly weak assumptions on its right-hand side. For this we use an approach based on the pluripotential theory. In particular, we exploit the specific properties of psh functions, like their quasicontinuity or the possibility of estimating the size of the sublevel sets in terms of pluricomplex capacities. The comparison principle of [BT2] is used at almost every stage.

Chapter 2 deals with the complex Monge–Ampère equation on a compact Kähler manifold M. The equation now takes the form

$$\det\left(g_{j\bar{k}}+\frac{\partial^2\phi}{\partial z_j\,\partial\bar{z}_k}\right)=f\det(g_{j\bar{k}}), \tag{1.1.2}$$

where $\sum g_{j\bar{k}}\,dz_j\otimes d\bar{z}_k$ is a Kähler metric on M and $f\geqslant 0$, $f\in L^1(M)$, suitably normalized, is given. Locally (1.1.2) is equivalent to (1.1.1) if we use a potential for the given metric. The absence of the boundary data in (1.1.2) accounts for the difference in treating the two equations. The equation (1.1.2) has been solved by S.-T. Yau [Y] in case $f\in C^3(M)$ and $f>0$ with improved regularity of the solution (see Theorem 2.1.1 below). Our aim is to generalize the existence part of this result admitting all non-negative f which belong to certain Orlicz spaces. In particular, we obtain continuous solutions to (1.1.2) for $f\in L^{\psi}(M)$ when $\psi(t)=|t|(\log(1+|t|))^n(\log(\log(1+|t|)))^m$, $m>n:=\dim M$. If we took $m<n$ here the assertion would no longer be true by a counterexample of L. Persson [P]. It readily follows that for $f\in L^p(M)$, $p>1$, the equation has a continuous solution. If $f\in C^{1,\alpha}$, $\alpha>0$, then the regularity of the solution is treated in [Au2].

In the last section of the second chapter we consider a special case of (1.1.2) taking $M=\mathbf{P}^n$ and $\sum g_{j\bar{k}}\,dz_j\otimes d\bar{z}_k$ equal to the Fubini–Study metric on $\mathbf{P}^n$. Then, applying the generalized Yau theorem one can solve the equation (1.1.1) in the class of functions of logarithmic growth in $\mathbf{C}^n$ under a fairly weak hypothesis on f. The equation has been treated in [BT3], [Be] and [CK2]. It is interesting because of its connections with the theory of extremal functions and complex dynamics.

The third chapter is devoted to the study of the Dirichlet problem for the equation (1.1.1) in a strictly pseudoconvex Ω with continuous boundary data. Again, in general f must be replaced by a Borel measure. We wish to determine which measures yield bounded (or continuous) solutions to (1.1.1). To this aim, for a given measure μ, we consider a special regularizing sequence μ_j weakly convergent to μ and such that (1.1.1) can be solved by using the Bedford–Taylor result or Cegrell's generalization [Ce2]. Denoting by u_j the corresponding solutions we define a candidate for the solution of the original problem putting $u=(\limsup u_j)^*$. Trying to verify whether u actually solves the equation we face two problems:

(a) Is the sequence u_j uniformly bounded?

(b) If so, is the convergence $u_j \to u$ good enough to entail the convergence of the corresponding measures μ_j to μ?

Before we state the sufficient conditions for those questions to have affirmative answers, let us consider a necessary condition which any measure μ leading to a bounded solution of (1.1.1) must satisfy:

$$\mu(K) \leqslant \text{const}\cdot\text{cap}(K,\Omega), \tag{1.1.3}$$

for any K a compact subset of Ω, where

$$\text{cap}(K,\Omega) = \sup\left\{\int_K \text{MA}(u) : u \in \text{PSH}(\Omega),\ -1 \leqslant u < 0\right\}$$

is the relative capacity introduced in [BT2]. As an example in [Ko1] shows, even in the one-dimensional case there are measures fulfilling (1.1.3) but yielding unbounded solutions of (1.1.1). So, searching for sufficient conditions we strengthen (1.1.3) by putting $F(\text{cap}(K,\Omega))$ on its right-hand side with a suitable function $F(x) \leqslant x$ when x is small. As shown in Theorem B of §3.1, under certain restrictions on F this leads to a positive answer to question (a). One may take, for instance, $F(x) = x(\log(1+x^{-1}))^{-(n+\varepsilon)}$, $\varepsilon > 0$. Consequently, if the density of μ with respect to the Lebesgue measure is in the Orlicz space mentioned above then (1.1.1) is solvable. The same counterexample from [P] as in the previous chapter shows that this result is almost sharp.

The answer to the second question is yes if μ satisfies the following local version of (1.1.3):

$$\mu(K) \leqslant \text{const}\cdot\text{cap}(K,B')\,\mu(B) \tag{1.1.4}$$

for any choice of $K \subset\subset B := B(x,r) \subset B' := B(x,4r) \subset \Omega$ (K compact).

Any measure fulfilling (1.1.4) and the hypothesis of Theorem B admits continuous solutions to the Monge–Ampère equation (1.1.1). However, (1.1.4) is not a necessary condition for the existence of bounded solutions—a relevant example is given in [Ko1]. Therefore we are still not able to characterize measures yielding bounded (or continuous) solutions to (1.1.1) by means of an inequality (or inequalities) like (1.1.3) or (1.1.4). The main result of the third chapter—Theorem C—says that if there exists a subsolution to (1.1.1) then the equation is solvable. Here the hypothesis is often easier to verify than those of the previous theorems.

The results of the second chapter are new. Chapter 3 contains the results of [Ko1], [Ko2], [Ko3] except the continuity part of Theorem B and Corollary 3.5.2. However, some technical parts of the proofs have been simplified.

I would like to thank Z. Błocki and U. Cegrell for their comments on the paper.

1.2. Preliminaries

In this section we recall fundamentals of pluripotential theory, focusing on notions and results which are used in the sequel. By now several texts on the subject have been written. We refer to books by U. Cegrell [Ce3] and M. Klimek [Kl], and extensive papers by E. Bedford, B. A. Taylor [BT2], [Be] and J. P. Demailly [D1], [D2] for the proofs and a thorough treatment of the theory or some of its aspects.

(A) *Positive currents.* A differential form with distribution coefficients on a complex n-dimensional manifold M given in local coordinates by

$$T=\sum_{\substack{|I|=p\\|J|=q}}{}' T_{IJ}\,dz_I\wedge d\bar{z}_J \tag{1.2.1}$$

($\sum'$ means that the sum is taken over increasing multiindices) is called a *current* of bidegree (p,q) (alternatively: of bidimension $(n-p,n-q)$). It is a continuous functional on the space of test forms $C^\infty_{0,(n-p,n-q)}(M)$.

The action of T on a test form

$$\omega=\sum_{\substack{|I|=n-p\\|J|=n-q}}{}' \omega_{IJ}\,dz_I\wedge d\bar{z}_J\in C^\infty_{0,(n-p,n-q)}(M)$$

is given by

$$(T,\omega)=\int_M T\wedge\omega=\int_M \sum_{\substack{|I|=n-p\\|J|=n-q}}{}' T_{I'J'}(\omega_{IJ})\,dz_{I'}\wedge d\bar{z}_{J'}\wedge dz_I\wedge d\bar{z}_J,$$

where I' (or J') complements I (or J) to $(1,2,...,n)$.

We say that T is a *positive current* of bidimension (p,p) if for any collection of $(1,0)$-forms $\alpha_1,\alpha_2,...,\alpha_p$,

$$T\wedge i\alpha_1\wedge\bar{\alpha}_1\wedge i\alpha_2\wedge\bar{\alpha}_2\wedge...\wedge i\alpha_p\wedge\bar{\alpha}_p$$

is a positive measure. Then its coefficients are complex measures and the action of T extends to the space of forms with continuous coefficients. The exterior differential of (positive) T is defined by

$$(dT,\omega)=-(T,d\omega),$$

where $\omega\in C^\infty_{0,(2n-2p-1)}(M)$. T is *closed* if $dT=0$. We often split d into differentials taken with respect to holomorphic and antiholomorphic coordinates, $d=\partial+\bar{\partial}$, and write $d^c:=i(\bar{\partial}-\partial)$.

(B) *Currents associated to psh functions. The Monge–Ampère operator.* If Ω is an open subset of M and $u\in \mathrm{PSH}(\Omega)$ then $dd^c u$ is a closed positive $(1,1)$-current. Conversely, if T is a positive closed current of bidegree $(1,1)$ defined in a neighbourhood of a closed ball then there exists a psh function inside the ball such that $dd^c u=T$ (see e.g. [LG]). Following [BT1] we can define wedge products of this sort of currents provided that the associated psh functions are locally bounded. Indeed, for $u\in \mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(\Omega)$ and a closed positive current T on Ω, the current uT is well defined and so is

$$dd^c u\wedge T := dd^c(uT).$$

Moreover, the latter current is also closed and positive.

This way, using induction, one may define closed positive currents

$$dd^c u_1\wedge dd^c u_2\wedge \ldots \wedge dd^c u_N,$$

for $u_j\in \mathrm{PSH}\cap L^\infty_{\mathrm{loc}}(\Omega)$. It is also possible to define

$$du_1\wedge d^c u_2\wedge dd^c u_3\wedge \ldots \wedge dd^c u_N,$$

with u_j as above (see [BT2], [Be]).

The Monge–Ampère operator MA acts on a C^2-smooth psh function u according to the formula

$$\mathrm{MA}(u) := 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j\,\partial \bar z_k}\right) d\lambda = (dd^c u)^n \quad (d\lambda \text{ denotes the Lebesgue measure}),$$

where the power on the right is taken with respect to the wedge product. As we have seen its action can be extended to all locally bounded psh functions. For $n=1$, MA is just the Laplacian multiplied by a constant. In general, the Monge–Ampère operator shares with the Laplacian some of its properties. The following basic result reflects an "elliptic" nature of the Monge–Ampère operator.

THEOREM 1.2.1 (comparison principle) [BT2]. *If*

$$u, v\in \mathrm{PSH}\cap L^\infty(\Omega) \quad \textit{and} \quad \liminf_{z\to\partial\Omega}(u(z)-v(z))\geqslant 0$$

then

$$\int_{\{u<v\}} (dd^c v)^n \leqslant \int_{\{u<v\}} (dd^c u)^n.$$

A psh function u is called *maximal* in Ω if $(dd^c u)^n=0$ in this set.

Given $u\in \mathrm{PSH}(\Omega)$ and a non-negative, radially symmetric function $\varrho\in C_0^\infty(B)$ (B stands for the unit ball in $\mathbf{C}^n$), where $\int \varrho\, d\lambda=1$, define a regularizing sequence

$u_j = u * \varrho_j$, with $\varrho_j(z) := j^{2n}\varrho(jz)$. The sequence decreases to u on any relatively compact subset of Ω. Convolutions are also used to regularize currents. Put $T_j := T * \varrho_j$, where, in the representation given in (1.2.1), we set $(T_j)_{I,J} = T_{I,J} * \varrho_j$. Then $T_j \to T$ in the sense of currents which, by definition, means that for any test form ω the sequence (T_j, ω) converges to (T, ω). Monotone convergence of psh functions implies the convergence of corresponding currents.

THEOREM 1.2.2 (convergence theorem) [BT2]. *Let $\{u_k^j\}_{j=1}^{\infty}$ be an increasing (or a decreasing) sequence of psh functions in Ω for $k=1,2,...,N$, and let $u_k^j \to u_k \in \mathrm{PSH} \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ almost everywhere as $j \to \infty$ for $k=1,2,...,N$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_N^j \to dd^c u_1 \wedge ... \wedge dd^c u_N$$

in the weak topology of currents.

One can relax the assumptions of the theorem a bit (see Theorem 1.2.12 below, [Be], [D1]) but there are counterexamples showing that, for instance, the convergence in L^p for any $p<\infty$ is not sufficient to get the statement (see [Ce1], [Le1]). The convergence results rely, in part, on the Chern–Levine–Nirenberg inequalities [CLN] or their generalizations [AT], [D1], [D2].

THEOREM 1.2.3 [CLN]. *If $\Omega' \subset\subset \Omega$ then for a constant $C = C(\Omega', \Omega)$ the following inequality holds:*

$$\int_{\Omega'} dd^c u_1 \wedge ... \wedge dd^c u_n \leqslant C \|u_1\|_{\Omega} ... \|u_n\|_{\Omega},$$

for any set of $u_k \in \mathrm{PSH} \cap L^{\infty}(\Omega)$, where $\|\cdot\|$ denotes the sup norm of a function.

(C) *Capacities.* Capacities in $\mathbf{C}^n$, modelled on the capacities associated to subharmonic functions, prove to be very useful in the studies of psh functions (see e.g. [S]). In this paper we shall deal primarily with the relative capacity of E. Bedford and B. A. Taylor [BT2] and a capacity defined in terms of the global extremal function introduced by J. Siciak [S]. In particular, we shall make use of a result comparing those two capacities obtained by H. Alexander and B. A. Taylor [AT]. Given a compact subset K of a strictly pseudoconvex domain Ω in $\mathbf{C}^n$ we define *the relative extremal function* of K with respect to Ω and *the global extremal function* (the extremal function of logarithmic growth) (see [S]) of K by

$$u_K(z) = \sup\{u(z) : u \in \mathrm{PSH} \cap L^{\infty},\ u < 0 \text{ in } \Omega,\ u \leqslant -1 \text{ on } K\},$$
$$L_K(z) = \sup\{u(z) : u \in \mathrm{PSH}(\mathbf{C}^n),\ u(z) = \log(1+|z|) + O(1),\ u \leqslant 0 \text{ on } K\}.$$

The upper semicontinuous regularizations $u_K^*(z) := \overline{\lim}_{z' \to z} u_K(z')$ and L_K^* are psh functions. A compact set K is said to be *regular* if $u_K^* = u_K$ (equivalently: $L_K^* = L_K$). By means of extremal functions we define two capacities,

$$\operatorname{cap}(K, \Omega) = \sup\left\{ \int_K (dd^c u)^n : u \in \operatorname{PSH}(\Omega),\ -1 \leqslant u < 0 \right\}$$

and

$$T_R(K) := \exp(-\sup\{L_K^*(z) : |z| \leqslant R\})$$

for some fixed $R > 0$. The first one is called *relative capacity.* Both are Choquet and outer capacities (see [BT2], [S]). In particular, for an open set U,

$$\operatorname{cap}(U, \Omega) = \sup\{\operatorname{cap}(K, \Omega) : K \subset U,\ K \text{ compact}\}$$

and

$$T_R(U, \Omega) = \sup\{T_R(K, \Omega) : K \subset U,\ K \text{ compact}\}.$$

We shall need the following properties of relative capacity.

THEOREM 1.2.4 [BT2]. *Let K be a compact subset of a strictly pseudoconvex domain Ω. Then*

$$\operatorname{cap}(K, \Omega) = \int_K (dd^c u_K^*)^n = \int_\Omega (dd^c u_K^*)^n.$$

Moreover, u_K^* and L_K^* are maximal away from K.

THEOREM 1.2.5 [AT]. *Given three strictly pseudoconvex sets $\Omega'' \subset\subset \Omega' \subset\subset \Omega$, there exists a constant $A > 0$ such that for any compact subset $K \subset \Omega''$ we have*

$$\operatorname{cap}(K, \Omega) \leqslant \operatorname{cap}(K, \Omega') \leqslant A \operatorname{cap}(K, \Omega).$$

THEOREM 1.2.6 [BT2]. *Given an open set $\Omega' \subset\subset \Omega$, where Ω is strictly pseudoconvex, there exists a constant $C = C(\Omega', \Omega)$ such that*

$$\lambda(K) \leqslant C \operatorname{cap}(K, \Omega) \quad \textit{for } K \subset \Omega'.$$

The two capacities are comparable as the following result shows.

THEOREM 1.2.7 [AT]. *If $B := B(0, R)$ and $K \subset B(0, r)$, $r < R$, is compact, then*

$$\exp(-A(r)(\operatorname{cap}(K, B))^{-1}) \leqslant T_R(K) \leqslant \exp(-2\pi(\operatorname{cap}(K, B))^{-1/n}).$$

We close this section by giving a list of results where relative capacity is used to describe the behaviour of psh functions. They all come in handy in solving the Monge–Ampère equation.

THEOREM 1.2.8 [BT2]. *If* $u \in \mathrm{PSH}(\Omega)$, Ω *strictly pseudoconvex, then given* $\varepsilon > 0$ *one can find an open set* U *with* $\mathrm{cap}(U, \Omega) < \varepsilon$ *such that* $u|_{\Omega \setminus U}$ *is continuous.*

We say that plurisubharmonic functions are *quasicontinuous* because of the property given in the statement of Theorem 1.2.8.

THEOREM 1.2.9 [BT2]. *If* Ω *is strictly pseudoconvex and* $u, u_j \in \mathrm{PSH}(\Omega)$, $j = 1, 2, \ldots$, $u_j = u$ *in a neighbourhood of* $\partial\Omega$, $u_j \downarrow u$ *in* Ω, *then for any* $t > 0$ *we have*

$$\lim_{j\to\infty} \mathrm{cap}(\{u_j > u+t\}, \Omega) = 0.$$

COROLLARY 1.2.10. *If* Ω *is strictly pseudoconvex,* $t > 0$, $K \subset\subset \Omega$, $u \in \mathrm{PSH} \cap L^\infty(\Omega)$ *and* $u_j \in \mathrm{PSH} \cap L^\infty(\Omega)$ *with* $u_j \downarrow u$ *in* Ω, *then*

$$\lim_{j\to\infty} \mathrm{cap}(K \cap \{u_j > u+t\}, \Omega) = 0.$$

Proof. Adding a constant to all the functions we may suppose that $u_1 < -1$ on Ω. Fix a defining function ϱ for Ω which (after being multiplied by a constant) becomes less than u on K. Then $v := \max(u, \varrho)$, $v_j := \max(u_j, \varrho)$ fulfil the hypothesis of Theorem 1.2.9. Moreover, $K \cap \{u_j > u+t\} \subset \{v_j > v+t\}$ and thus the result follows.

THEOREM 1.2.11 [AT] (see also [Be]). *Given* $z_0 \in \Omega$ *and* K, *a compact subset of a strictly pseudoconvex domain* Ω, *there exists a constant* $A > 0$ *such that for any* $u \in \mathrm{PSH}(\Omega)$, $u < 0$, $u(z_0) > -1$, $s > 0$, *we have*

$$\mathrm{cap}(K \cap \{u < -s\}, \Omega) \leqslant A/s.$$

A sequence u_j of functions defined in Ω is said to *converge with respect to capacity* to u if for any $\delta > 0$ and $K \subset\subset \Omega$,

$$\lim_{j\to\infty} \mathrm{cap}(K \cap \{|u - u_j| > \delta\}, \Omega) = 0.$$

The Monge–Ampère operator is continuous with respect to sequences converging in this fashion.

THEOREM 1.2.12 [X]. *If* u_j *is a uniformly bounded sequence of psh functions in* Ω *converging with respect to capacity to* $u \in \mathrm{PSH}(\Omega)$ *then*

$$(dd^c u_j)^n \to (dd^c u)^n$$

in the sense of currents.

Chapter 2

2.1. The complex Monge–Ampère equation on a compact Kähler manifold. Results

Let us consider a compact n-dimensional Kähler manifold M equipped with the fundamental form $\omega=\frac{1}{2}i\sum_{k,j}g_{k\bar{j}}\,dz^k\wedge d\bar{z}^j$. By the definition of a Kähler manifold, $(g_{k\bar{j}})$ is a Hermitian matrix and $d\omega=0$. The volume form associated to the Hermitian metric is given by the nth wedge product of ω.

We shall study the Monge–Ampère equation

$$(\omega+dd^c\phi)^n=F\omega^n, \tag{2.1.1}$$

where ϕ is the unknown function such that $\omega+dd^c\phi$ is a non-negative (1,1)-form. The given non-negative function $F\in L^1(M)$ is normalized by the condition

$$\int_M F\omega^n=\int_M \omega^n=\mathrm{Vol}(M). \tag{2.1.2}$$

Since, by the Stokes theorem, the integral over M of the right-hand side of (2.1.1) is equal to $\mathrm{Vol}(M)$, this normalization is necessary for the existence of a solution.

The equation has been solved by S.-T. Yau in the case of smooth, positive F.

THEOREM 2.1.1 [Y]. *Let $F>0$, $F\in C^k(M)$, $k\geqslant 3$. Then there exists a solution to* (2.1.1) *belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0\leqslant\alpha<1$.*

By solving the Monge–Ampère equation Yau proved the Calabi conjecture which says that given a closed (1,1)-form R representing the first Chern class of M one can find a Kähler metric such that R is its Ricci form and the new fundamental form is in the same Chern class as R.

Using the Yau theorem we find continuous solutions $\phi\in C(M)$ when F is assumed to be non-negative and satisfying condition (A) below. This condition is quite weak as it admits all non-negative functions belonging to $L^p(M)$, $p>1$, and also functions from some more general subspaces of $L^1(M)$ (see §2.5).

By [Au2, Proposition 7.12], if $F>0$, $F\in C^{k,\alpha}(M)$, $k\geqslant 1$, $0<\alpha<1$, and a C^2-solution ϕ exists then $\phi\in C^{k+2,\alpha}(M)$.

We say that F satisfies *condition* (A) if there exist a sequence $F_j\in C^\infty(M)$, $F_j>0$, $F_j\to F$ in $L^1(M)$ and a covering of M by strictly pseudoconvex coordinate patches V_s such that for any compact, regular set $K\subset\subset V_s$ the following inequality holds:

$$\int_K F_j\omega^n\leqslant A\,\mathrm{cap}(K,V_s)[h((\mathrm{cap}(K,V_s))^{-1/n})]^{-1},\quad j,s=1,2,\ldots, \tag{2.1.3}$$

for some constant $A>0$ and some increasing function $h\colon(0,\infty)\to(1,\infty)$ satisfying

$$\int_1^\infty (yh^{1/n}(y))^{-1}\,dy<\infty.$$

Here cap stands for the Bedford–Taylor relative capacity (see §1.2).

If we take $h(x)=\max(1,x^a)$, $a>0$, in (2.1.3) then the right-hand side of the inequality simplifies to $A(\operatorname{cap}(K,V_s))^{1+a/n}$. With $a=0$ the condition would no longer be a sufficient one. In §2.5 we show that our assumptions are fairly sharp.

We shall apply the generalized version of the Yau theorem to solve the Monge–Ampère equation in the class of functions of logarithmic growth in $\mathbf{C}^n$. Let $\mathcal{L}_+$ denote the family of functions plurisubharmonic in $\mathbf{C}^n$ and differing from $v(z):=\log(1+|z|)$ by a bounded function (which may depend on the function). Given a Borel measure μ let us consider the equation

$$\begin{gathered}(dd^cu)^n=d\mu,\quad u\in\mathcal{L}_+,\\ \int_{\mathbf{C}^n}d\mu=(2\pi)^n.\end{gathered}\tag{2.1.4}$$

The normalizing condition is necessary due to a result of B. A. Taylor [Ta]. In [BT3] E. Bedford and B. A. Taylor proved that the solutions to (2.1.4) are unique up to an additive constant. E. Bedford discussed the problem of existence of a solution in his survey paper [Be]. He observed that applying Theorem 2.1.1 one obtains solutions for $d\mu=f\,d\lambda$ ($d\lambda$ denoting the Lebesgue measure), where $f(z)=\text{const}\cdot\exp(F(z))(1+|z|^2)^{-n-1}$ and F extends to a C^3-function in $\mathbf{P}^n$. In the paper of U. Cegrell and the author [CK2] it is shown that any C^3-smooth f bounded from above by $\text{const}\cdot(1+|z|^2)^{-n-1}$ admits solutions to (2.1.4) with $d\mu=f\,d\lambda$. In particular, the equation is solvable for measures with test function densities. On the other hand, some restriction on the growth of f is necessary as an example from the same paper shows.

In the present work we dispense with the smoothness assumption and relax the hypothesis on the growth of f. This result is sharp to the same extent as the generalized Yau theorem which is used to prove it (with $M=\mathbf{P}^n$). Again it is enough to assume that f belongs to a certain Orlicz space with respect to the volume form of $\mathbf{P}^n$.

Next we discuss the equation (2.1.4) for measures singular with respect to the Lebesgue measure. Then we consider μ_j, the standard regularizations of μ via convolution with a smoothing kernel, and we suppose that (2.1.3) still holds true with μ_j in place of $F_j\omega^n$. Instead of $\mu\in L^1(M)$ we now assume that μ is locally dominated by capacity (see §3.1). Under these assumptions (2.1.4) is solvable.

2.2. Weak solutions to the Monge–Ampère equation

Let us fix $F\in L^1(M)$, $F\geqslant 0$, satisfying (2.1.2). Consider an approximating sequence $F_j\in C^\infty(M)$, $F_j>0$, $F_j\to F$ in $L^1(M)$. Passing to a subsequence we can obtain

$$\|F-F_j\|_{L^1(M)}\leqslant \frac{1}{2^{j+1}}. \tag{2.2.1}$$

Multiplying F_j by a constant which tends to 1 as $j\to\infty$ we can also get

$$\int_M F_j\omega^n=\mathrm{Vol}(M).$$

By virtue of the Yau theorem (Theorem 2.1.1) one can find $\phi_j\in C^\infty(M)$ such that

$$(\omega+dd^c\phi_j)^n=F_j\omega^n.$$

LEMMA 2.2.1. *If the sequence ϕ_j is uniformly bounded then $\phi:=(\limsup_{j\to\infty}\phi_j)^*$ solves the equation* (2.1.1).

Proof. Let us introduce some auxiliary functions,

$$\phi_{kl}=\max_{k\leqslant j\leqslant l}\phi_j,\qquad g_k=\big(\lim_{l\to\infty}\uparrow\phi_{kl}\big)^*,$$
$$F_{kl}=\min_{k\leqslant j\leqslant l}F_j,\qquad G_k=\lim_{l\to\infty}\downarrow F_{kl}.$$

Since, locally, ω is representable by dd^cv, where v is a psh function, one can apply [BT1, Proposition 2.8] to get

$$(\omega+dd^c\phi_{kl})^n\geqslant F_{kl}\omega^n.$$

Hence, by the convergence theorem (Theorem 1.2.2),

$$G_k\omega^n\leqslant \lim_{l\to\infty}(\omega+dd^c\phi_{kl})^n=(\omega+dd^cg_k)^n, \tag{2.2.2}$$

where the convergence is understood in the weak* topology.

Note that $\phi=\lim_{k\to\infty}\downarrow g_k$, so one can apply the convergence theorem once more to get

$$(\omega+dd^cg_k)^n\to(\omega+dd^c\phi)^n. \tag{2.2.3}$$

From (2.2.1) we have $\|F-G_k\|_{L^1(M)}\leqslant 1/2^k$, so $G_k\to F$ in $L^1(M)$. Combine this conclusion with (2.2.2) and (2.2.3) to obtain

$$(\omega+dd^c\phi)^n\geqslant F\omega^n.$$

Since the integrals over M of both currents in the above inequality are equal to $\mathrm{Vol}(M)$ we finally arrive at

$$(\omega+dd^c\phi)^n=F\omega^n.$$

Thus the lemma follows.

In the next section we shall prove that the hypothesis of Lemma 2.2.1 is satisfied provided that F satisfies condition (A).

2.3. L^∞-estimates for the solutions

The following lemma and its proof will be used to prove the boundedness as well as the continuity of the solutions to the Monge–Ampère equation. It is a refined version of Theorem 1 in [Ko3].

LEMMA 2.3.1. *Let Ω be a strictly pseudoconvex subset of $\mathbf{C}^n$ and let $v\in \mathrm{PSH}\cap C(\Omega)$, $\|v\|<C$. Suppose that $u\in \mathrm{PSH}\cap L^\infty(\Omega)$ satisfies the following conditions: $u<0$, $u(0)>C'$ ($0\in\Omega$) and*

$$\int_K (dd^c u)^n \leqslant A\,\mathrm{cap}(K,\Omega)[h((\mathrm{cap}(K,\Omega))^{-1/n})]^{-1} \tag{2.3.1}$$

for any compact subset K of Ω, where $h\colon (0,\infty)\to(1,\infty)$ is an increasing function which fulfils the inequality

$$\int_1^\infty (yh^{1/n}(y))^{-1}\,dy<\infty.$$

If the sets $U(s):=\{u-s<v\}\cap\Omega''$ are non-empty and relatively compact in $\Omega''\subset\Omega'\subset\subset\Omega$ for $s\in[S,S+D]$ then $\inf_\Omega u$ is bounded from below by a constant depending on A,C,C', D,h,Ω',Ω, but independent of u,v,Ω''.

Proof. Let us introduce the notation

$$a(s):=\mathrm{cap}(U(s),\Omega),\quad b(s)=\int_{U(s)}(dd^c u)^n.$$

Then

$$t^n a(s)\leqslant b(s+t)\quad \text{for } 0<t<S+D-s. \tag{2.3.2}$$

Indeed, consider a compact regular set $K\subset U(s)$, the psh function $w:=(u-s-t)/t$ and the set $V:=\{w<u_K+v/t\}\cap\Omega''$, where u_K denotes the relative extremal function of K with respect to Ω. Let us first verify the inclusions $K\subset V\subset U(s+t)$.

Take $x\in K\subset U(s)$. Then $u(x)-s<v(x)$ and so

$$w(x)=(u(x)-s-t)/t\leqslant u_K(x)+v(x)/t,$$

which means that $x\in V$. To see the latter inclusion, note that if $x\in V$ then

$$(u(x)-s-t)/t\leqslant u_K(x)+v(x)/t\leqslant v(x)/t$$

since u_K is negative.

Once we have the inclusions we can apply the comparison principle and Theorem 1.2.4 to the effect that

$$\begin{aligned}\mathrm{cap}(K,\Omega)&\leqslant\int_K[dd^c(u_K+v/t)]^n\leqslant\int_V[dd^c(u_K+v/t)]^n\leqslant\int_V(dd^c w)^n\\ &\leqslant t^{-n}\int_V(dd^c u)^n\leqslant t^{-n}\int_{U(s+t)}(dd^c u)^n=t^{-n}b(s+t).\end{aligned}$$

In this way (2.3.2) follows.

Next we define an increasing sequence $s_0, s_1, ..., s_N$, setting $s_0 := S$ and

$$s_j := \sup\{s : a(s) \leqslant \lim_{t\to s_{j-1}+} da(t)\}$$

for $j=1,2,...,N$, where d is a fixed number such that $1<d<2$. Then

$$\lim_{t\to s_j-} a(t) \leqslant \lim_{t\to s_{j-1}+} da(t)$$

and

$$a(s_j) \geqslant da(s_{j-2}). \tag{2.3.3}$$

The integer N is chosen to be the greatest one satisfying $s_N \leqslant S+D$. Then

$$a(S+D) \leqslant \lim_{t\to s_N+} da(t).$$

From the last inequality, (2.3.1) and (2.3.2), it follows that for any $t\in(s_N, S+D)$ we have

$$\begin{aligned}(S+D-t)^n a(t) &\leqslant b(S+D) \leqslant Aa(S+D)h^{-1}([a(S+D)]^{-1/n})\\ &\leqslant Ada(t)h^{-1}([a(S+D)]^{-1/n}).\end{aligned}$$

Hence

$$S+D-s_N \leqslant (Ad)^{1/n}h^{-1/n}([a(S+D)]^{-1/n}) = (Ad)^{1/n}L_1. \tag{2.3.4}$$

Now we shall estimate $s_N - S$. Consider two numbers $S<s'<s<S+D$ such that $a(s)\leqslant da(s')$, and set $t := s-s'$. Then by (2.3.1) and (2.3.2) we have

$$a(s') \leqslant t^{-n}b(s) \leqslant At^{-n}a(s)h^{-1}([a(s)]^{-1/n}) \leqslant Adt^{-n}a(s')h^{-1}([a(s)]^{-1/n}).$$

Hence

$$t \leqslant (Ad)^{1/n}h_1(a(s)),$$

where $h_1(x) := [h(x^{-1/n})]^{-1/n}$. Letting $s\to s_{j+1}-$ and $s'\to s_j+$ we thus get

$$t_j := s_{j+1} - s_j \leqslant (Ad)^{1/n}h_1(a(s_{j+1})).$$

Using this inequality, (2.3.3) and the fact that the function $h_2(x) := h_1(d^x) = h^{-1/n}(d^{-x/n})$ is increasing one can estimate as follows:

$$\begin{aligned}\sum_{j=0}^{N-1} t_j &\leqslant (Ad)^{1/n}\sum_{j=0}^{N-1} h_2(\log_d a(s_{j+1}))\\ &\leqslant (Ad)^{1/n}\left[\sum_{j=1}^{N-2}\int_{\log_d a(s_j)}^{\log_d a(s_{j+2})} h_2(x)\,dx + 2h_2(\log_d a(s_N))\right]\\ &\leqslant 2(Ad)^{1/n}\left[\int_{\log_d a(S)}^{\log_d a(S+D)} h_2(x)\,dx + h_2(\log_d a(S+D))\right].\end{aligned}$$

The change of variable $y{=}d^{-x/n}$ leads to the following transformation of the above integral:

$$\int_{\log_d a(S)}^{\log_d a(S+D)} h_2(x)\,dx = \int_{\log_d a(S)}^{\log_d a(S+D)} [h(d^{-x/n})]^{-1/n}\,dx$$
$$= \frac{n}{\ln d}\int_{[a(S+D)]^{-1/n}}^{[a(S)]^{-1/n}} [(h(y))^{1/n}y]^{-1}\,dy.$$

Hence finally,

$$s_N - S \leqslant (Ad)^{1/n} L_2, \tag{2.3.5}$$

where

$$L_2 := \frac{2n}{\ln d}\int_{[a(S+D)]^{-1/n}}^{[a(S)]^{-1/n}} [yh^{1/n}(y)]^{-1}\,dy + 2[h(a(S+D)^{-1/n})]^{-1}.$$

Note that, due to our hypothesis on h, both L_1 and L_2 tend to 0 as $a(S+D)\to 0$.

By Theorem 1.2.11,

$$\lim_{s\to\infty} \operatorname{cap}(\{u<-s\}\cap\Omega',\Omega)=0$$

and the convergence is uniform with respect to u as long as u satisfies the assumptions of the lemma. Since $U(S+D)\subset\{u<C+S+D\}$ one concludes from these remarks, (2.3.4) and (2.3.5) that for $S<S_0=S_0(A,C,C',D,h,\Omega',\Omega)$ the following inequalities hold:

$$S+D-s_N \leqslant (Ad)^{1/n}L_1 \leqslant 2A^{1/n}L_1 < \tfrac{1}{2}D,$$
$$s_N - S \leqslant (Ad)^{1/n}L_2 \leqslant 2A^{1/n}L_2 < \tfrac{1}{2}D.$$

Combined, they yield a contradiction. This shows that S is controlled from below by a constant depending only on $A, C, C', D, h, \Omega', \Omega$, and thus the statement follows.

From now on we suppose that F satisfies condition (A) (see §2.1). Consider the sequence ϕ_j of solutions to the Monge–Ampère equation from the previous section. They are determined up to a constant, so we need to impose some kind of normalization to obtain a finite limit ϕ. We choose ϕ_j so that

$$\sup_M \phi_j = 0.$$

It is no loss of generality to assume that M is connected. From [H, Theorem 4.1.9] applied in coordinate patches V_s to psh functions $v_s+\phi_j$, where $dd^c v_s=\omega$ in V_s, we conclude that either $\phi_j\to-\infty$ uniformly on compact subsets of V_s or there exists a subsequence of $\{\phi_j\}$ converging in $L^1(V_s)$. If the former possibility occurs then, since M is connected, $\phi_j\to-\infty$ uniformly on M, contrary to the normalizing condition above. Thus upon passing to a subsequence and using the diagonal procedure with respect to coordinate patches, one can assume that $\phi_j\to\phi$ in $L^1(M)$.

Before proceeding further we shall fix a special covering of M by coordinate patches. Let us observe that for any $x \in M$ there exist a neighbourhood U_x and a potential $v \in \mathrm{PSH}(U_x)$ satisfying $dd^c v = \omega$ and having local minimum at x. Indeed, take any $\tilde{v} \in \mathrm{PSH}(U_x)$ such that $dd^c \tilde{v} = \omega$. The Taylor expansion of $\tilde{v}$ at x in local coordinates has the form

$$\tilde{v}(x+h) = \operatorname{Re} P(h) + H(h) + o(|h|^2),$$

where P is a complex polynomial and H the complex Hessian of $\tilde{v}$. Since H is positive definite it is easy to see that $v := \tilde{v} - \operatorname{Re} P(\cdot - x)$ has a local minimum at x.

Once we know this, we can find, using compactness of M, positive constants r, R such that $6r < R$ and for any $x \in M$ there exist a coordinate chart $\varrho_x : U'_x \to B(0,R)$ and $v_x \in \mathrm{PSH}(U'_x)$ satisfying $dd^c v_x = \omega$, $v_x \leqslant 0$ and

$$\sup_{U_x} v_x < \inf_{\partial U''_x} v_x,$$

where $U_x := \varrho_x^{-1}(B(0,r))$, $U''_x := \varrho_x^{-1}\big(B\big(0, \tfrac{1}{3}R\big)\big)$ with $B(0,r)$ and $B\big(0, \tfrac{1}{3}R\big)$ denoting open balls in $\mathbf{C}^n$ centered at the origin of radius r and $\tfrac{1}{3}R$ respectively. One may choose $\{U_x\}$ to be subordinate to $\{V_s\}$.

We fix a finite covering $U_s := U_{x_s}$, $s = 1, 2, \ldots, N$, of M and write for brevity $U'_s = U'_{x_s}$, $U''_s = U''_{x_s}$, $\varrho_s = \varrho_{x_s}$, $v_s = v_{x_s}$. Then there exists $c_0 > 0$ such that

$$\sup_{U_s} v_s < \inf_{\partial U''_s} v_s - c_0. \tag{2.3.6}$$

Since, due to our choice of ϕ_j's, the integrals $\int_{U_s} \phi_j \omega^n$ are bounded from below by a constant c_1 independent of s and j, we infer that also

$$\sup_{U_s} \phi_j > c_1, \quad j = 1, 2, \ldots, \quad s = 1, 2, \ldots, N. \tag{2.3.7}$$

Now we shall see how to apply Lemma 2.3.1 to derive that ϕ_j is uniformly bounded. Given j fix $a'_j \in M$ such that $\phi_j(a'_j) = \min \phi_j$, then choose U_{s_j} containing a'_j and a point $a_j \in \bar{U}_{s_j}$ satisfying $\phi_j(a_j) = \sup_{\bar{U}_{s_j}} \phi_j$. Take $\varrho_{s_j}^{-1}\big(B\big(\varrho_{s_j}(a_j), \tfrac{2}{3}R\big)\big)$, $\varrho_{s_j}^{-1}\big(B\big(\varrho_{s_j}(a_j), \tfrac{1}{2}R\big)\big)$ and U''_{s_j} to play the role of Ω, Ω' and Ω'' respectively, in the lemma. Since for every j thus defined the sets Ω and Ω' can be identified with the fixed balls $B\big(0, \tfrac{2}{3}R\big)$ and $B\big(0, \tfrac{1}{2}R\big)$ we may consider these sets to be independent of j. As u and v in the lemma we take $v_{s_j} + \phi_j$ and 0 respectively. Clearly, v_s are uniformly bounded from below, so from (2.3.7) one concludes that

$$(v_{s_j} + \phi_j)(a_j) \geqslant c_2, \quad j = 1, 2, \ldots,$$

which settles the assumption "$u(0)>C'$".

By the choice of a'_j and (2.3.6) we have

$$(v_{s_j}+\phi_j)(a'_j) \leqslant \inf_{\partial U''_{s_j}} (v_{s_j}+\phi_j)-c_0, \quad j=1,2,\ldots.$$

Thus the constant c_0 can be taken as D in the lemma. Both c_2 and c_0 do not depend on j.

To verify (2.3.1) we use the fact that $\{U'_s\}$ is subordinate to $\{V_s\}$. It is enough to observe that the right-hand side of (2.3.1) is increasing in $\operatorname{cap}(K,\Omega)$ and that if $\Omega\subset\Omega'$ then $\operatorname{cap}(K,\Omega)\geqslant\operatorname{cap}(K,\Omega')$ for $K\subset\Omega$ (see Theorem 1.2.5).

Applying Lemma 2.3.1 in the way described above we conclude that $v_{s_j}+\phi_j$ are uniformly bounded on U_{s_j} by a constant independent of j. Since v_s are uniformly bounded and ϕ_j assumes its infimum in U_{s_j} it follows that ϕ_j are uniformly bounded. Thus, by Lemma 2.2.1 the bounded function $\phi=(\limsup\phi_j)^*$ solves the Monge–Ampère equation (2.1.1).

2.4. Continuity of the solution

Suppose that ϕ were not continuous. Then $d:=\sup(\phi-\phi_*)>0$. Since $\phi-\phi_*$ is upper semicontinuous and bounded (by §2.3), the supremum is actually attained at some point $x_0\in M$. One may choose x_0 so that

$$\phi(x_0)=\min_{\{\phi-\phi_*=d\}}\phi.$$

Such x_0 exists since $F=\{\phi-\phi_*=d\}$ is closed and if $x_j\in F$ with $\phi(x_j)\to\inf_F\phi$ then for any accumulation point x_0 of the sequence x_j we have $\phi(x_0)=\inf_F\phi$, otherwise $(\phi-\phi_*)(x_0)$ would exceed d, contrary to the definition of d.

Let us fix a coordinate chart ψ onto $B:=B(0,1)\subset\mathbf{C}^n$, and a potential function $v\in\mathrm{PSH}\cap C^\infty(B)$ such that $\psi(x_0)=0$, $\psi^*dd^cv=\omega$ and

$$\inf_S v-v(0):=b>0, \tag{2.4.1}$$

where $S:=\partial B(0,r)$, $r<1$.

The function $u:=v+\phi\circ\psi^{-1}\in\mathrm{PSH}\cap L^\infty(B)$ satisfies condition (2.3.1). After adding a constant one can assume that $u>0$ on B and $A:=u(x_0)>d$. (Here again we have used the result of the previous section.)

We wish to apply the proof of Lemma 2.3.1 to show that the hypothesis "ϕ discontinuous" leads to a contradiction. To this end we choose a sequence of smooth psh functions $u_j \downarrow u$ which are defined in a neighbourhood B'' of the closure of $B' := B(0, r)$. Our first objective is to prove that for some $a_0 > 0$, $t > 1$ the sets

$$W(j, c) := \{w + c < u_j\}, \quad \text{where } w := tu + d - a_0,$$

are non-empty and relatively compact in B' for c belonging to an interval which does not depend on $j > j_0$.

Obviously $E := \{u - u_* = d\} \cap \bar{B}' = \{\phi \circ \psi^{-1} - (\phi \circ \psi^{-1})_* = d\} \cap \bar{B}'$ and $0 \in E$. For $0 < a < d$ we denote by $E(a)$ the set $\{u - u_* \geqslant d - a\} \cap \bar{B}'$. Those sets are closed and $E(a) \downarrow E$ as $a \to 0$. Hence, by semicontinuity of ϕ and the choice of x_0 one gets

$$\limsup_{a \to 0} c(a) \leqslant 0, \quad c(a) := \phi \circ \psi^{-1}(0) - \min_{E(a)} \phi \circ \psi^{-1}.$$

Indeed, suppose that for some $\gamma > 0$ and $x_j \in E(a_j)$, $a_j \to 0$, we had $(\phi \circ \psi^{-1})(x_j) < (\phi \circ \psi^{-1})(0) - \gamma$. Then any accumulation point x of x_j belongs to E and so $(\phi \circ \psi^{-1})(x) \geqslant (\phi \circ \psi^{-1})(0)$. Thus

$$\limsup (\phi \circ \psi^{-1})(x_j) \leqslant (\phi \circ \psi^{-1})(x) - \gamma \quad \text{and} \quad \lim [(\phi \circ \psi^{-1})(x_j) - (\phi \circ \psi^{-1})_*(x_j)] = d.$$

From these two formulas we obtain $(\phi \circ \psi^{-1})(x) - (\phi \circ \psi^{-1})_*(x) \geqslant d + \gamma$, a contradiction.

Fix a_0 satisfying the two conditions

$$\begin{aligned} &0 < a_0 < \min\left(\tfrac{1}{3}b, d\right), \\ &c(a) < \tfrac{1}{3}b \quad \text{for } a \leqslant a_0. \end{aligned} \tag{2.4.2}$$

Next, choose $t > 1$ satisfying the inequalities

$$(t-1)(A-d) < a_0 < (t-1)\left(A - d + \tfrac{2}{3}b\right). \tag{2.4.3}$$

We shall need the following version of the Hartogs lemma.

PROPOSITION 2.4.1. *If $u - tu_* < c$ on a compact set $K \subset \bar{B}'$ then for some $j_0 \in \mathbf{N}$ we have*

$$u_j < tu + c \quad \text{on } K,$$

where u, u_j are the functions we are dealing with.

Proof. By the assumption and the semicontinuity of u one can find for any $x \in K$ a neighbourhood V of x and $c' < c$ such that

$$tu > \sup_{\bar{V}} u - c' \quad \text{on } \bar{V}.$$

The Hartogs lemma provides then an integer j_0 such that

$$u_j \leqslant \sup_{\overline{V}} u+(c-c') < tu+c$$

on $\overline{V}$ if $j>j_0$. Since K is compact the above inequality extends to the whole set K after increasing j_0 if necessary. Thus the proposition follows.

Consider now $y\in S\cap E(a_0)$. Then by (2.4.1) and (2.4.2) one gets

$$u_*(y) \geqslant v(0)+b+\phi\circ\psi^{-1}(y)-d \geqslant v(0)+b+\phi\circ\psi^{-1}(0)-c(a_0)-d \geqslant A-d+\tfrac{2}{3}b.$$

Hence by (2.4.3),

$$(t-1)u_*(y) > a_0,$$

which implies that

$$u(y) \leqslant u_*(y)+d < tu_*(y)+d-a_0. \tag{2.4.4}$$

Since the left-hand side of this inequality is upper semicontinuous and the right-hand side is lower semicontinuous it extends to $\overline{V}$, where V is a neighbourhood of $S\cap E(a_0)$. Applying Proposition 2.4.1 we thus obtain

$$u_j < tu+d-a_0 \quad \text{on } \overline{V} \text{ if } j>j_1. \tag{2.4.5}$$

Since $E(a_0)\cap(S\setminus V)=\varnothing$ the inequality

$$u-u_* < d-a_0$$

holds on $S\setminus V$. Applying Proposition 2.4.1 once more and increasing j_1 if necessary we get

$$u_j \leqslant u+d-a_0 < tu+d-a_0 \tag{2.4.6}$$

on $S\setminus V$ if $j>j_1$.

From the first inequality in (2.4.3) it follows that for some $a_1>0$

$$tu_*(0)+d-a_0 < u(0)-a_1 < u_j(0)-a_1. \tag{2.4.7}$$

Putting $w:=tu+d-a_0$ we see from (2.4.5) and (2.4.6) that the sets $W(j,c)=\{w+c<u_j\}$ are relatively compact in B' for $c>0$, $j>j_1$. Furthermore, (2.4.7) implies that for $c\in(0,a_1)$ some point near 0 belongs to $W(j,c)$.

Now we can apply the proof of Lemma 2.3.1 with $w, u_j, B', B'', 0, a_1$ in place of $u, v, \Omega'', \Omega, S, D$. We have just verified that the hypothesis of the lemma is then satisfied. Thus from (2.3.4) and (2.3.5) we obtain that

$$(L_1+L_2)A^{-1/n} \geqslant \tfrac{1}{2}D,$$

which by definition of L_1 and L_2 gives a positive lower bound for $\mathrm{cap}(W(j,a_1),B'')\geqslant a_2>0$ not depending on j. This leads to a contradiction since we have $\{w+a_1<u_j\}\subset\{u+(d-a_0+a_1)<u_j\}$, where $d-a_0+a_1>0$ and the capacity of the intersection of the latter set with B' tends to 0 as $j\to\infty$ by Corollary 1.2.10.

In this way we have proved

THEOREM 2.4.2. *If* $F\in L^1(M)$ *satisfies* (2.1.2) *and condition* (A) *then there exists a continuous solution to the equation* (2.1.1).

2.5. Functions satisfying condition (A)

In this section we are going to identify some Orlicz spaces of functions which fulfil condition (A). The spaces $L^p(M)$, $p>1$, are among them. At the end of the section we give an example indicating that our assumption is fairly sharp. We adopt here the results of [Ko3] giving more detailed exposition.

Condition (A) is given in terms of the relative capacity whereas the Orlicz spaces we shall be dealing with are defined with respect to the Lebesgue measure. The following lemma exhibits a relation between the capacity and the measure.

LEMMA 2.5.1. *Suppose* $u\in\mathrm{PSH}(\Omega)\cap C(\bar\Omega)$, $u=0$ *on* $\partial\Omega$, $\int(dd^cu)^n\leqslant 1$. *Then the Lebesgue measure* $\lambda(U_s)$ *of the set* $U_s:=\{u<s\}$ *is bounded from above by* $c\exp(-2\pi|s|)$, *where* c *does not depend on* u.

Proof. Assume $\bar\Omega$ to be contained in a ball $B=B(0,R)$. We denote by λ_k the Lebesgue measure in $\mathbf{C}^k$. Let us write the coordinates of a point $z\in\mathbf{C}^n$ in the form $z=(z_1,z')\in\mathbf{C}\times\mathbf{C}^{n-1}$, and denote by B_1 (or B') the balls $\{z\in\mathbf{C}:|z|<R\}$ (or $\{z\in\mathbf{C}^{n-1}:|z|<R\}$). Consider the slices of the set U_s,

$$U_s(z'):=\{z_1\in\mathbf{C}:(z_1,z')\in U_s\}.$$

For fixed s, the extremal function of logarithmic growth of $U_s(z')$ in $\mathbf{C}$ (or of U_s in $\mathbf{C}^n$) will be denoted by $V_{z'}$ (or V). We shall use a capacity which corresponds to the global extremal function (see §1.2),

$$T_R(E)=\exp\big(-\sup_{B(0,R)}V_E\big),\quad E\subset B(0,R).$$

For $n=1$ the set function T_R dominates the logarithmic capacity multiplied by a constant depending on R. Hence by classical potential theory (see [Ts])

$$\lambda_1(U_s(z'))\leqslant C_1T_R(U_s(z')),$$

where C_1 is an independent constant. Thus, making use of the Fubini theorem we can estimate as follows:

$$\begin{aligned}\lambda(U_s) &= \int \lambda_1(U_s(z'))\,d\lambda_{n-1}(z') \leqslant C_1 \int T_R(U_s(z'))\,d\lambda_{n-1}(z') \\ &\leqslant C_1 \int \exp\bigl(-\sup_{|z_1|<R} V(z_1,z')\bigr)\,d\lambda_{n-1}(z').\end{aligned} \tag{2.5.1}$$

By a result of Alexander [A] there exists an independent constant C_2 such that

$$\sup_{B_1\times B'} V_E < \int V_E\,dS - C_2, \quad E\subset B,$$

where dS is the normalized Lebesgue measure on $S=\partial(B_1\times B')$ and V_E denotes the global extremal function of a Borel set E. We deduce from this inequality that the dS-measure of the set

$$\bigl\{V_E \leqslant \sup_S V_E - a\bigr\}$$

tends to 0 as $a\to\infty$. Therefore, by taking C_3 large enough we get

$$\lambda_{n-1}\bigl(\bigl\{z'\in B' : \sup_{B_1} V_E(z_1,z') \geqslant \sup_{B_1\times B'} V_E - C_3\bigr\}\bigr) \geqslant \tfrac{1}{2}\lambda_{n-1}(B').$$

Thus the right-hand side of (2.5.1) is dominated by

$$C_4 \exp\bigl(-\sup_{B_1\times B'} V(z)\bigr) \leqslant C_4 T_R(U_s).$$

From Theorem 1.2.7 it follows that

$$T_R(U_s) \leqslant \exp[-2\pi(\operatorname{cap}(U_s,B))^{-1/n}] \leqslant \exp[-2\pi(\operatorname{cap}(U_s,\Omega))^{-1/n}].$$

So, continuing the estimate (2.5.1) we finally arrive at

$$\lambda(U_s) \leqslant C_4 \exp[-2\pi(\operatorname{cap}(U_s,\Omega))^{-1/n}].$$

To complete the proof it remains to show that

$$\operatorname{cap}(U_s,\Omega) \leqslant |s|^{-n}. \tag{2.5.2}$$

Fix $t>1$ and a regular compact set $K\subset U_s$. Then by the comparison principle we have

$$\operatorname{cap}(K,\Omega) = \int_K (dd^c u_K)^n = \int_{\{-ts^{-1}u<u_K\}} (dd^c u_K)^n \leqslant t^n|s|^{-n} \int_\Omega (dd^c u)^n \leqslant t^n|s|^{-n}.$$

Thus (2.5.2) and the lemma follow.

We now proceed to find Orlicz spaces satisfying the condition (A). The background material on Orlicz spaces can be found in [M]. Let $\psi: [0,\infty)\to[0,\infty)$ be an increasing function fulfilling the inequalities

$$2t \leqslant \psi(2t) \leqslant a\psi(t), \tag{2.5.3}$$

for some $a>0$ and any $t>t_0$. Then the Orlicz space corresponding to ψ and ω^n is defined by

$$L^\psi(M) = \left\{ f \in L^1(M) : \int_M \psi(|f|)\,\omega^n < \infty \right\}.$$

Since condition (A) is expressed in coordinate patches we shall work in $L^\psi(V_s, d\lambda)$ ($d\lambda$ denotes here the pullback of the Lebesgue measure via the coordinate chart) rather than in $L^\psi(M)$. Obviously $f\in L^\psi(M)$ if and only if $f\in L^\psi(V_s, d\lambda)$ for all s. Under the above assumptions on ψ, given $f\in L^\psi(M)$ there exists a sequence $f_j\in C^\infty(M)$ such that $f_j\to f$ in $L^1(M)$ and the integrals $\int_M \psi(|f_j|)\omega^n$ are uniformly bounded.

Suppose that

$$\psi(t) = |t|(\log(1+|t|))^n h(\log(1+|t|)),$$

with $h(x)\leqslant x$, $x>1$, satisfying the hypothesis of condition (A), and that ψ fulfils (2.5.3). We are going to verify the inequality (2.1.3) for a sequence F_j of smooth positive functions, uniformly bounded in $L^\psi(M)$ and converging in $L^\psi(M)$ to F. Fix $\Omega=V_s$. We need to show that for some $A>0$ and any compact regular set $K\subset\Omega$ the following inequalities hold:

$$\int_K F_j\,d\lambda \leqslant A\,\mathrm{cap}(K,\Omega)[h((\mathrm{cap}(K,\Omega))^{-1/n})]^{-1}. \tag{2.5.4}$$

First, let us note that (2.5.4) follows from

$$\int_\Omega |v|^n h(|v|) F_j\,d\lambda \leqslant A, \quad j=1,2,..., \tag{2.5.5}$$

where $v\in\mathrm{PSH}(\Omega)$ is of the form $v=\mathrm{cap}^{-1/n}(K,\Omega)u_K$, with u_K the relative extremal function of K with respect to Ω. Indeed, from (2.5.5) we have

$$\begin{aligned} A &\geqslant \int_\Omega |v|^n h(|v|) F_j\,d\lambda \geqslant \int_K |v|^n h(|v|) F_j\,d\lambda \\ &\geqslant \mathrm{cap}^{-1}(K,\Omega)\, h((\mathrm{cap}(K,\Omega))^{-1/n}) \int_K F_j\,d\lambda, \end{aligned}$$

which proves (2.5.4). To prove (2.5.5) we shall use Young's inequality applied to $G(r) = g(\log(1+r)) = (\log(1+r))^n h(\log(1+r))$ and its inverse. Then

$$\begin{aligned} g(|v(x)|)F_j(x) &\leqslant \int_0^{F_j(x)} g(\log(1+r))\,dr + \int_0^{g(|v(x)|)} [\exp(g^{-1}(t))-1]\,dt \\ &\leqslant F_j(x)g(\log(1+F_j(x))) + \int_0^{|v(x)|} e^s g'(s)\,ds \\ &\leqslant \psi(F_j(x)) + g(|v(x)|)e^{|v(x)|}. \end{aligned}$$

Since, by the choice of F_j, the integrals $\int_\Omega \psi(|F_j|)\,d\lambda$ are bounded by some constant $A_0 < \infty$, we obtain by integrating the above inequality over Ω

$$\int_\Omega |v|^n h(|v|)F_j\,d\lambda \leqslant A_0 + \int_\Omega g(|v(x)|)e^{|v(x)|}\,d\lambda.$$

It remains to find a uniform bound (independent of K) for the last term. To do this we make use of Lemma 2.5.1:

$$\begin{aligned} \int_\Omega g(|v(x)|)e^{|v(x)|}\,d\lambda &= \sum_{s=0}^{\infty} \int_{\{-s-1<v<-s\}} g(|v(x)|)e^{|v(x)|}\,d\lambda \\ &\leqslant \sum_{s=0}^{\infty} (s+1)^n h(s+1)e^{s+1}\lambda(\{v<-s\}) \\ &\leqslant c\sum_{s=0}^{\infty} (s+1)^n h(s+1)e^{1+s(1-2\pi)} \\ &\leqslant c\left[h(1) + \sum_{s=1}^{\infty} (s+1)^{n+1} e^{1+s(1-2\pi)}\right] \leqslant \text{const} < \infty. \end{aligned}$$

The extra assumption $h(x) \leqslant x$ which has been used above is unrestrictive since by decreasing h (and ψ) we extend the space L^ψ. Thus we have proved

THEOREM 2.5.2. *If h is the function from condition* (A) *and*

$$\psi(t) = |t|(\log(1+|t|))^n h(\log(1+|t|))$$

satisfies (2.5.3) *then for any $F \in L^\psi(M)$ the Monge–Ampère equation* (2.1.1) *has a continuous solution.*

Example 1. Take $\psi(t) = |t|(\log(1+|t|))^n (1+\log(1+\log(1+|t|)))^m$, $m>n$. Then it is straightforward that this function satisfies the assumptions of Theorem 2.5.2. Now if $\chi(t) = |t|(\log(1+|t|))^m$, $m<n$, then by a result of L. Persson [P], the Monge–Ampère equation admits unbounded solutions with pointwise singularities for some radially symmetric densities from L^χ. This shows how sharp is the hypothesis of Theorem 2.5.2.

Example 2. For any $p>1$ we have $L^p(M) \subset L^\psi(M)$, where ψ is the function from the previous example. Thus for $F \in L^p$, $p>1$, the equation (2.1.1) is solvable in the domain of continuous psh functions.

2.6. Solutions of the Monge–Ampère equation belonging to $\mathcal{L}_+$

The class of functions plurisubharmonic in $\mathbf{C}^n$ of logarithmic growth,

$$\mathcal{L}_+ := \{u \in \mathrm{PSH}(\mathbf{C}^n) : |u(z) - \log(1+|z|)| < c_u\},$$

plays an important role in the study of polynomials and the theory of extremal functions in particular. The total Monge–Ampère mass $\int_{\mathbf{C}^n} (dd^c u)^n$ of a function u belonging to $\mathcal{L}_+$ is always equal to $(2\pi)^n$ (see [Ta]). In the present section we shall deal with the following problem. Given a Borel measure μ with

$$\int_{\mathbf{C}^n} d\mu = (2\pi)^n,$$

find $u \in \mathcal{L}_+$ satisfying

$$(dd^c u)^n = d\mu. \tag{2.6.1}$$

If μ is not singular with respect to the Lebesgue measure then the generalizations of the Yau theorem obtained so far can be applied directly. Indeed, let us specify $M = \mathbf{P}^n$ and $\omega = \omega_0$, where $\omega_0 = \frac{1}{2} dd^c \log |Z|^2$ is the Fubini–Study form with Z denoting the homogeneous coordinates in $\mathbf{P}^n$. In $\mathbf{C}^n$, embedded in $\mathbf{P}^n$ in the usual way, the Fubini–Study form is equal to $dd^c v_0$, $v_0 := \frac{1}{2} \log(1+|z|^2)$. Straightforward computation leads to

$$\omega_0^n(z) = \frac{n!}{(1+|z|^2)^{n+1}}\, d\lambda.$$

Applying Theorem 2.4.2 we get

COROLLARY 2.6.1. *If $d\mu = F\omega_0^n$, where $F \in L^1(\mathbf{P}^n)$, satisfies condition* (A) *then* (2.6.1) *is solvable and the solution is continuous.* (*It is unique up to an additive constant by* [BT3].)

To make the hypothesis more explicit one can use Theorem 2.5.2.

COROLLARY 2.6.2. *If $\psi(t) = |t|(\log(1+|t|))^n h(\log(1+|t|))$ satisfies* (2.5.3), *h fulfils the hypothesis of condition* (A) *and the function f given by*

$$d\mu(z) = f(z) \frac{1}{(1+|z|^2)^{n+1}}\, d\lambda$$

satisfies

$$\int_{\mathbf{C}^n} \psi(|f(z)|) \frac{1}{(1+|z|^2)^{n+1}}\, d\lambda < \infty,$$

then there exists a continuous solution of the Monge–Ampère equation (2.6.1).

Example 1. Putting $h_0(x) = (1+x)^m$, $m>0$; $h_1(x) = 1 + (\log(1+x))^m$, $m>n$; $h_2(x) = 1 + (\log(1+x))^n (\log(\log(1+x)))^m$, $m>n$, in place of h in Corollary 2.6.2 we get some particular classes of measures admitting solutions to (2.6.1). We may continue to decrease h and thereby extend the corresponding class defined in the corollary. Again we observe that our result cannot be substantially improved since, on the one hand, for

$$f(z)=\frac{|z|^2}{(\log(1+|z|))^p}, \quad p>n+1,$$

we have a solution in $\mathcal{L}_+$; on the other hand if $n=1$ and

$$f(z)=\frac{|z|^2}{(\log(1+|z|))^2} \quad (p=n+1\,(!))$$

then the solution of the Poisson equation

$$dd^c u = f(z)\frac{1}{(1+|z|^2)^2}\,d\lambda$$

is no longer in $\mathcal{L}_+$. The first part of this statement follows from Corollary 2.6.2 with $h=h_0$ as defined above and $m<p-1-n$. Then the convergence of the integral $\int \psi(|f|)\,\omega_0^n$ is equivalent to the convergence of

$$\int_1^\infty \frac{1}{x(\log(1+x))^{p-n-m}}\,dx<\infty.$$

As for the solution u of the Poisson equation above, observe that

$$\int_{\mathbf{C}} \log(1+|z|)\,dd^c u(z)=+\infty,$$

which implies that $u\notin\mathcal{L}_+$ (see e.g. [CKL, Proposition 1.4]). Using the integral formulas for radial psh functions, obtained in [P] by L. Persson, one can draw the same conclusion in higher-dimensional case.

While dealing with psh functions one very often comes across functions having Monge–Ampère mass singular with respect to the Lebesgue measure. The standard operation of taking maximum of a finite number of psh functions gives rise to such objects. Extremal functions are of this type as well.

Trying to cope with the equation (2.6.1) for general Borel measures we cannot use Lemma 2.2.1 any more, so we need to find a replacement for this result with possibly weak assumptions on μ. In the case of the Dirichlet problem for the Monge–Ampère equation in a strictly pseudoconvex domain Ω with continuous boundary data we have such a theorem (Theorem A in §3.1). There we consider measures satisfying the condition

$$\mu(E)\leqslant A\,\mathrm{cap}(E,B')\,\mu(B), \tag{2.6.2}$$

where $E \subset B = B(x,r) \subset B' = B(x,4r) \subset \Omega$ and the inequality holds for any choice of such sets with an independent constant $A>0$.

Then we regularize μ via convolution, $\mu_j = \varrho_j * \mu$ (with $\varrho \in C_0^\infty(B(0,1)$ a radial non-negative function such that $\int \varrho \, d\lambda = 1$ and $\varrho_j(z) = j^{2n}\varrho(jz)$, $j=1,2,\dots$), and solve the Dirichlet problem for μ_j. If the resulting sequence of solutions u_j is uniformly bounded then $u = (\limsup u_j)^*$ is the desired solution for μ.

To solve the equation (2.6.1) we repeat this procedure. Let $\mu_j = \varrho_j * \mu$. Assume that with μ_j in place of $F_j\omega^n$ (2.1.3) is still satisfied and that for μ (2.6.2) holds true. By Corollary 2.6.1 we find $u_j \in \mathcal{L}_+$ such that $(dd^c u_j)^n = d\mu_j$. Applying the results of §§ 2.3 and 2.4 we conclude that $u = (\limsup u_j)^*$ is continuous and belongs to $\mathcal{L}_+$. Adding a constant one may suppose that $u_j > 0$, $j=1,2,\dots$. It remains to prove that $(dd^c u)^n = d\mu$.

Let us fix $R>0$ and denote by $u_{j,R}$, $j=1,2,\dots$, the psh solution of the following Dirichlet problem in the ball $B(0,R)$:

$$\begin{cases} (dd^c v)^n = d\mu_j|_{B(0,R-1)}, \\ v = u \quad \text{on } \partial B(0,R). \end{cases} \tag{2.6.3}$$

Applying Theorem A in §3.1 we conclude that $v_R := (\limsup u_{j,R})^*$ satisfies

$$(dd^c v_R)^n = d\mu|_{B(0,R-1)} \quad \text{in } B(0,R). \tag{2.6.4}$$

Since all u_j differ from $\log(1+|z|)$ by a constant independent of j (because the corresponding solutions of (2.1.1) in $\mathbf{P}^n$ are uniformly bounded due to Lemma 2.3.1) one can find for any $t>1$ a radius R_0 such that

$$t^{-1}u_j \leqslant u_{j,R} \leqslant t u_j \quad \text{in } B(0,R),\ R > R_0. \tag{2.6.5}$$

(Here we use the comparison principle in $B(0,R)$ to verify the first inequality and we do the same in $B(0,R-1)$ to prove the other one.) Hence passing to the sup limits one obtains $t^{-1}u \leqslant v_R \leqslant tu$. Since t was arbitrary exceeding 1, we deduce that $v_R \to u$ locally uniformly as $R \to \infty$. Then the convergence theorem implies that $(dd^c u)^n = d\mu$. Let us state the result we have just proved.

THEOREM 2.6.3. *If the measure μ from* (2.6.1) *satisfies* (2.6.2) *and if* (2.1.3) *holds true with $F_j\omega^n$ replaced by $\mu_j = \varrho_j * \mu$ then the equation* (2.6.1) *has a continuous solution.*

The condition (2.6.2) is not necessary for the existence of a continuous solution. In [Ko1] we gave an example of a continuous subharmonic function in $\mathbf{C}$ with the Laplacian not satisfying (2.6.2).

Chapter 3

3.1. The complex Monge–Ampère equation in a strictly pseudoconvex domain. Results

In this chapter we shall study the Dirichlet problem for the complex Monge–Ampère equation in a strictly pseudoconvex domain Ω. Given $\varphi \in C(\partial\Omega)$ and a non-negative Borel measure $d\mu$ we look for a plurisubharmonic (psh) function u satisfying

$$\begin{cases} u \in \mathrm{PSH} \cap L^\infty(\Omega), \\ (dd^c u)^n = d\mu, \\ \lim_{z\to x} u(z) = \varphi(x) \quad \text{for } x \in \partial\Omega. \end{cases} \tag{$*$}$$

So far the set of Borel measures for which there exists a bounded (continuous) solution has not been characterized. We believe that the results presented here give a fairly accurate description of this set.

In 1976 E. Bedford and B. A. Taylor proved the following fundamental result.

THEOREM 3.1.1 [BT1]. *If $d\mu = f\,d\lambda$ (λ denotes the Lebesgue measure), $f \in C(\bar\Omega)$, then $(*)$ has a unique continuous solution.*

U. Cegrell [Ce2] generalized this theorem to the case of bounded f, and then, together with L. Persson [CP], solved $(*)$ for $f \in L^2(\Omega)$. There are examples (see [CS], [P]) indicating that one cannot do the same for $f \in L^1(\Omega)$. For measures equicontinuous with a rotation-invariant measure in a ball.the equation has been solved in [CK1] under the condition that a subsolution exists.

There is a number of results (see e.g. [CKNS], [CY]) showing that under additional assumptions on smoothness of f and φ, and non-degeneracy of f, one may obtain smooth solutions to $(*)$. In particular, we have the following regularity theorem.

THEOREM 3.1.2 [CKNS]. *If $d\mu = f\,d\lambda$, $f \in C^\infty(\bar\Omega)$, $f > 0$ and $\partial\Omega$ is C^∞-smooth, then $(*)$ has a unique solution $u \in C^\infty(\bar\Omega)$.*

In this paper we focus on solving $(*)$ under possibly weak assumptions on $d\mu$. From the very definition of the relative capacity of Bedford and Taylor,

$$\mathrm{cap}(K,\Omega) := \sup\left\{ \int_K (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 < u < 0 \right\},$$

it follows that a bounded solution to $(*)$ exists only if the measure μ is dominated by capacity,

$$\mu(K) \leqslant A\,\mathrm{cap}(K,\Omega). \tag{3.1.1}$$

This condition, however, is not sufficient even in the case of the Poisson equation (see [Kol]). In the first of our main theorems (Theorem A below) we use a local version of this condition. We say that μ is *locally dominated by capacity* if for any cube I and the ball B_I of radius equal to $2\operatorname{diam} I$, concentric with the cube and contained in Ω, the following inequality holds:

$$\mu(E) \leqslant A \operatorname{cap}(E, B_I)\, \mu(I), \tag{3.1.2}$$

where A is an independent constant and E is a Borel subset of I. One may weaken this condition requiring only that the inequality holds away from a set of arbitrarily small measure μ, but even then it does not become a necessary condition (see [Kol]).

One may strengthen (3.1.1) in yet another way by putting $F(\operatorname{cap}(K,\Omega))$, with some $F(x) \leqslant x$, in place of $\operatorname{cap}(K,\Omega)$ on the right-hand side of (3.1.1). Condition (A) from §2.1 is of this form. As we have already seen, at least for $d\mu = f\,d\lambda$, $f \in L^1(\Omega, d\lambda)$, this condition seems to be close to giving a characterization of measures leading to continuous solutions.

Given a Borel measure μ let us consider an approximating sequence $\mu_j = f_j\,d\lambda$, where f_j is constant on small cubes constituting the jth subdivision of Ω and $\mu_j(I) = \int_I f_j\,d\lambda$ for any such cube. Using Cegrell's result [Ce2] we find solutions u_j of $(*)$ with μ replaced by μ_j.

THEOREM A [Kol]. *If a Borel measure μ, compactly supported in Ω, is locally dominated by capacity and the sequence u_j defined above is uniformly bounded then $u := (\limsup u_j)^*$ solves the Monge–Ampère equation $(*)$.*

One may use some other way of approximating μ (for instance, in [Kol] we use convolutions with a smoothing kernel) to get the same result. However, not any weakly convergent sequence $\mu_j \to \mu$ would do (see [CK1]).

Once we have Theorem A a natural question arises: When is u_j uniformly bounded? As in Chapter 2, condition (A) takes care of that.

THEOREM B. *Let Ω be a strictly pseudoconvex domain in $\mathbf{C}^n$ and let μ be a Borel measure in Ω such that $\int_\Omega d\mu \leqslant 1$. Consider an increasing function $h: \mathbf{R} \to (1,\infty)$ satisfying*

$$\int_1^\infty (y h^{1/n}(y))^{-1}\,dy < \infty.$$

If μ satisfies the inequality

$$\mu(K) \leqslant A \operatorname{cap}(K,\Omega)\, h^{-1}((\operatorname{cap}(K,\Omega))^{-1/n})$$

for any $K \subset \Omega$ compact and regular, then the norm $\|u\|_{L^\infty}$ of a solution of the Dirichlet problem $(*)$ is bounded by a constant $B = B(h, A)$ which does not depend on μ. Moreover, u is continuous.

Applying the results of §2.5 we obtain the following consequence of Theorem B.

COROLLARY 3.1.3. *Let* $L^{\phi}(\Omega, d\lambda)$ *denote the Orlicz space corresponding to*

$$\phi(t)=|t|(\log(1+|t|))^{n}h(\log(1+|t|)),$$

with h *satisfying the hypothesis of Theorem B. If* $f\in L^{\phi}(\Omega, d\lambda)$ *then* $(*)$ *is solvable with* $d\mu=f\,d\lambda$*, and the solution is continuous.*

In particular, the corollary provides continuous solutions to the Monge–Ampère equation for any $f\in L^p$, $p>1$. Let us note that for $n=1$, when we deal with the Poisson equation, the result is similar to the classical one of A. P. Calderón and A. Zygmund [CZ] (see also [GT, §9.4]). The method of the proof is completely different as we make no appeal to the Newtonian potential.

Combining Theorems A and B one gets

COROLLARY 3.1.4. *If a measure* μ *in* Ω *is locally dominated by capacity and satisfies the hypothesis of Theorem B with* h *such that*

$$h(ax)\leqslant bh(x),\quad x>0,$$

for some $a>1$, $b>1$, *then there exists a continuous solution to* $(*)$.

This result allows us to solve $(*)$ also in the case of measures singular with respect to the Lebesgue measure.

Even in the situation of the above corollaries it is not easy to verify the assumptions of Theorems A and B. Perhaps the following theorem is of more use when it comes to solving $(*)$. We say that a bounded psh function v is a *subsolution* for $(*)$ if $(dd^c v_s)^n\geqslant d\mu$ and $\lim_{z\to x} v(z)=\varphi(x)$, $x\in\partial\Omega$.

THEOREM C. *If there exists a subsolution for the Dirichlet problem* $(*)$ *then the problem is solvable.*

This statement remains true also in the case of weakly pseudoconvex domains under the necessary hypothesis that there exists a maximal function with given boundary data. An interesting consequence of Theorem C is that for any collection of $u_j\in \mathrm{PSH}\cap L^{\infty}(\Omega)$, $j=1,2,\dots,n$, the current $dd^c u_1\wedge dd^c u_2\wedge\dots\wedge dd^c u_n$ is equal to $(dd^c u)^n$ for some bounded psh function u.

The results presented here give answers to some of the problems posed in the survey paper by E. Bedford [Be]. We refer also to M. Klimek's monograph [Kl] for background material in pluripotential theory and to Z. Błocki's paper [Bl] for a simplified proof of Theorem 3.1.1.

In the next section we make some preparations for the proofs of Theorems A and C. This unified approach simplifies the exposition of the results originally published in [Ko1] and [Ko2]. Then, in §3.3 we fairly easily complete the proof of Theorem A. To finish the other proof (§3.4) requires much more effort. §3.5 is closely related to §2.5. The proof of Theorem B follows almost directly from Lemma 2.3.1. The corollary following it has also been proved in the previous chapter. The assumption on the measure μ in Corollary 3.1.4 slightly differs from that of condition (A) in §2.1, so to prove it we show that the assumptions are equivalent. Then we apply Corollary 3.1.4 for some particular classes of measures.

3.2. Preliminaries for the proofs of Theorems A and C

In this section we define a candidate for the solution of the equation $(*)$ in terms of the solutions corresponding to a sequence of measures approximating μ. Then we formulate a condition (3.2.1) which is proved to guarantee the solution of $(*)$. In the following sections we shall verify (3.2.1) under the assumptions of Theorem A and Theorem C. We close this section by showing a lemma which is an essential ingredient of the proofs that follow.

Let us first state some additional assumptions and observe that by doing this we do not affect the generality of the proofs. So, in Theorem C it is enough to consider only measures μ which have compact support. Then, given a non-compactly supported measure μ one can find solutions corresponding to $\chi_j\mu$, where χ_j is a non-decreasing sequence of cut-off functions, $\chi_j \uparrow 1$ on Ω. The solutions will be bounded from below by the given subsolution (due to Theorem 1.2.1) and they will decrease to the solution for μ by virtue of the convergence theorem (Theorem 1.2.2).

Then, the subsolution v given by the hypothesis of Theorem C can be modified so that $\lim_{z\to x} v(z)=0$ for any $x\in\partial\Omega$. Furthermore, using the balayage procedure, one can make the support of $d\nu:=(dd^c v)^n$ compact.

To limit the number of independent constants we also assume that

(a) $\Omega\subset I_0:=[0,1]^{2n}\subset \mathbf{C}^n$,

(b) $\mu(\Omega)\leqslant\nu(\Omega)<1$,

(c) when a sequence u_j or v_j defined below is uniformly bounded by a constant then this constant is taken to be 1,

(d) the boundary values φ in $(*)$ are negative,

(e) $-1<v<0$.

Now we define a regularizing sequence for μ. Let us consider a sequence $\mathcal{B}_s$ of subdivisions of I_0 into 3^{2sn} congruent open cubes of equal size which are pairwise disjoint but whose closures cover I_0. It is no restriction to assume that for each s we have $\nu(\bigcup_{I\in\mathcal{B}_s}\partial I)=0$. Set

$$\mu_j := f_j\,d\lambda,\quad f_j(x) := \frac{\mu(I\cap\Omega)}{\lambda(I\cap\Omega)}\quad \text{if } x\in I,$$
$$\nu_j := g_j\,d\lambda,\quad g_j(x) := \frac{\nu(I\cap\Omega)}{\lambda(I\cap\Omega)}\quad \text{if } x\in I$$

(for $x\in\partial I$ we put $f_j(x)=g_j(x)=0$.)

Using Cegrell's result [Ce2] one can solve the Dirichlet problems

$$\begin{cases} u_j\in \mathrm{PSH}(\Omega)\cap C(\bar\Omega),\\ (dd^c u_j)^n = f_j\,d\lambda,\\ u_j(x)=\varphi(x)\quad \text{for } x\in\partial\Omega, \end{cases}$$

and

$$\begin{cases} v_j\in \mathrm{PSH}(\Omega)\cap C(\bar\Omega),\\ (dd^c v_j)^n = g_j\,d\lambda,\\ v_j=0\quad \text{on } \partial\Omega. \end{cases}$$

Passing to a subsequence (after renumbering we stick to the original notation) one may suppose that u_j and v_j are convergent in L^1_{loc} (see [H, Theorem 4.1.9]). Set $u:=(\limsup u_j)^*$. It is meant to be the solution of $(*)$. One should keep in mind that if we again pass to a subsequence of u_j (as we shall do in the sequel) the function u remains unchanged.

PROPOSITION 3.2.1. *The function u defined above solves the Dirichlet problem $(*)$ provided that for any $a>0$ and any compact $K\subset\Omega$ we have*

$$\lim_{j\to\infty}\int_{E_j(a)\cap K}(dd^c u_s)^n = 0,\quad \textit{where } E_j(a) := \{u-u_j\geqslant a\}. \tag{3.2.1}$$

Proof. Indeed, if (3.2.1) holds then for any s one can find $j(s)$ such that

$$\int_{E_j(1/s)\cap K}(dd^c u_s)^n < \frac{1}{s},\quad j\geqslant j(s).$$

Set $\varrho_s:=\max(u_{j(s)}, u-1/s)$. Then $(dd^c\varrho_s)^n=(dd^c u_{j(s)})^n$ on the interior of $K\setminus E_{j(s)}(1/s)$, and so the above inequality implies that any accumulation point of $\{(dd^c\varrho_s)^n\}$ is $\geqslant d\mu$ on $\operatorname{int} K$. On the other hand, by the definition of ϱ_s and a version of the Hartogs lemma given in [H, Theorem 4.1.9], $\varrho_s\to u$ uniformly on any compact E such that $u|_E$

is continuous. So it follows from Theorem 1.2.8 that ϱ_s converge to u with respect to capacity. Therefore applying Theorem 1.2.12 we obtain $(dd^c\varrho_s)^n \to (dd^c u)^n$, and further

$$(dd^c u)^n \geqslant d\mu. \tag{3.2.2}$$

To get the reverse inequality note that $\varrho_s = u_{j(s)}$ on a neighbourhood of $\partial\Omega$ since all the u_j's (and therefore u as well) are bounded from above by the solution of the homogeneous Monge–Ampère equation with the same boundary data, and this solution is continuous in the closure of Ω. Hence, due to the Stokes theorem, $\int_\Omega (dd^c\varrho_s)^n = \int_\Omega (dd^c u_{j(s)})^n$. By the construction, the last integral is equal to $\int_\Omega d\mu$, so the measures in (3.2.2) must be equal.

We shall prove Theorem A and Theorem C by verifying (3.2.1).

The following lemma is a key element of the proofs that follow.

LEMMA 3.2.2 [Ko1]. *Given $z_0 \in \Omega$ and two numbers $M>1$, $R_0>0$ such that*

$$B_M = \{|z-z_0| < e^M R_0\} \subset\subset \Omega,$$

and given $v \in \mathrm{PSH}(\Omega)\cap C(\Omega)$, $-1<v<0$, denote by $E=E(\delta)$ the set

$$\{z \in B_0 : (1-\delta)v(z) \leqslant \sup_{B_0} v\},$$

where $\delta \in (0,1)$ and $B_0 = \{|z-z_0|<R_0\}$.

Then

$$\operatorname{cap}(E, B_1) < \frac{C_0}{M\delta},$$

where $B_1 = \{|z-z_0|<eR_0\}$ and C_0 is an independent constant.

Proof. From the logarithmic convexity of the function $r \to \sup_{|z-z_0|<r} v(z)$ it follows that for $z \in B_M \setminus B_0$ and $a_0 := \sup_{B_0} v$ we have

$$v(z) \leqslant a_0\left(1 - \frac{1}{M}\log\frac{|z-z_0|}{R_0}\right).$$

Hence

$$a_1 := \sup_{B_1} v \leqslant a_0\left(1-\frac{1}{M}\right).$$

Let $u = u_{E,B_1} := \sup\{w \in \mathrm{PSH}(B_1) : w<0,\ w \leqslant -1 \text{ on } E\}$ be the relative extremal function of E with respect to B_1 (see §1.2). From the inequality $v(z) \leqslant a_0/(1-\delta)$, $z \in E$, one obtains

$$\frac{1}{a_1 - a_0/(1-\delta)}(v - a_1) \leqslant u.$$

So, for some $z_1 \in \partial B_0$ the following inequalities hold:

$$u(z_1) \geqslant \frac{a_0-a_1}{a_1-a_0/(1-\delta)} \geqslant \frac{\delta-1}{(M-1)\delta+1}.$$

Note that $E \subset \{|z-z_0|<2R_0\} \subset B_1$. Therefore Theorem 1.2.11 and the above estimate yield

$$\operatorname{cap}(E, B_1) = \int_E (dd^c u)^n \leqslant \frac{C_0}{M\delta}.$$

COROLLARY 3.2.3. *If* $\delta>0$ *and* $\Omega' \subset\subset \Omega$ *then*

$$\lim_{\varepsilon\to 0} \sup_{I,v} \frac{\lambda(I\cap E(v,I,\delta))}{\lambda(I)} = 0,$$

where $E(v,I,\delta) := \{z\in I : v(z) < \sup_I v - \delta\}$ *and* sup *is taken over all cubes* $I\subset\Omega'$ *of diameter* $<\varepsilon$ *and all* $v\in \mathrm{PSH}\cap C(\Omega)$ *satisfying* $-1<v<0$.

Proof. Fix $a>0$. Let B_0, B_1, B_M be defined as in Lemma 3.2.2 for some z_0, R_0. Then the constant C in Theorem 1.2.6 corresponding to (B_0, B_1) in place of (Ω', Ω) does not depend on z_0 and R_0 provided that we normalize the measure replacing $\lambda(K)$ by $\lambda(K)/\lambda(B_0)$ in this formula.

For $\varepsilon>0$ small enough there exists $M>0$ so large that

$$\frac{C_0 C}{M\delta} < a,$$

where C_0 comes from the statement of Lemma 3.2.2, and furthermore, for any cube $I\subset\Omega'$ of diameter $<\varepsilon$ one can find z_0, R_0 such that $\operatorname{diam} I = 2R_0$ and $I\subset B_0\subset B_M\subset\Omega$. Then, by Theorem 1.2.6 and Lemma 3.2.2 we get

$$\frac{\lambda(I\cap E(v,I,\delta))}{\lambda(I)} \leqslant C\frac{\lambda(B_0)}{\lambda(I)} \operatorname{cap}(I\cap E(v,I,\delta), B_1) \leqslant \frac{CC_0\lambda(B_0)}{M\delta\lambda(I)} \leqslant a\frac{\lambda(B_0)}{\lambda(I)}.$$

By the choice of B_0 the quotient $\lambda(B_0)/\lambda(I)$ depends only on the dimension of the space, and thus the result follows.

3.3. Proof of Theorem A

To finish the proof we need one more lemma.

LEMMA 3.3.1. *Suppose that a compactly supported measure* μ *on* Ω *is locally dominated by capacity (i.e. it satisfies* (3.1.2)*). Then there exist* $A_1>0$, $s_0\in\mathbf{N}$ *such that for* *any $s>s_0$ and any $j>s+n$ we have*

$$\mu_j(E)<A_1\mu_j(I')\operatorname{cap}(E,B_{I'}),$$

for $E\subset I\subset\subset I'$, $I\in\mathcal{B}_s$, $I'\in\mathcal{B}_{s-1}$.

Proof. Let us note that I' is the unique cube from $\mathcal{B}_{s-1}$ which is concentric with I. Since μ has compact support one may choose s_0 so that if $\operatorname{supp}\mu\cap I\neq\varnothing$ for some $I\in\mathcal{B}_{s_0}$ then $B_{I'}\subset\Omega$. First we shall estimate μ_j in terms of $\hat{\mu}_j:=\mu*\varrho_j$, where ϱ_j is a smoothing kernel defined as follows. Denote by d_j the diameter of a cube belonging to $\mathcal{B}_j$. We require that a radially symmetric non-negative function $\varrho_j\in C_0^\infty(B(0,2d_j))$ be constant on $B(0,d_j)$ and furthermore that

$$\int_{B(0,d_j)}\varrho_j\,d\lambda=\tfrac12 \quad\text{and}\quad \int_{B(0,2d_j)}\varrho_j\,d\lambda=1.$$

Then for any $x\in I\in\mathcal{B}_j$ we have $I\subset B(x,d_j)$, and so

$$\hat{\mu}_j(x)=\int\varrho_j(x-y)\,d\mu(y)\geqslant\frac{\mu(I)}{2\lambda(B(x,d_j))}=\frac{\lambda(I)}{2\lambda(B(x,d_j))}\mu_j(x)=:A_2^{-1}\mu_j(x). \tag{3.3.1}$$

By the choice of ϱ_j,

$$\hat{\mu}_j(E)<\sup_{|x|<2d_j}\mu(E-x), \tag{3.3.2}$$

where $E-x:=\{z:z=y-x,\ y\in E\}$. Finally, if $j>s+n$ then for any $I\in\mathcal{B}_s$,

$$I+B(0,2d_j)\subset I'.$$

Hence, applying Theorem 1.2.5 we find an independent constant A_3 such that

$$\operatorname{cap}(E-x,B_{I'})<A_3\operatorname{cap}(E,B_{I'}),\quad E\subset I,\ |x|<2d_j. \tag{3.3.3}$$

Combining the estimates (3.3.1), (3.3.2), (3.3.3) and (3.1.2) one obtains

$$\begin{aligned}\mu_j(E)&\leqslant A_2\hat{\mu}_j(E)\leqslant A_2\sup_{|x|<2d_j}\mu(E-x)\\&\leqslant AA_2\sup_{|x|<2d_j}\mu(I')\operatorname{cap}(E-x,B_{I'})\leqslant 3^{2n}AA_2A_3\mu_j(I)\operatorname{cap}(E,B_{I'}),\end{aligned}$$

which is the desired conclusion.

We proceed to complete the proof of Theorem A. Recall that it is enough to show (3.2.1). Given $\varepsilon>0$ apply Lemma 3.2.2 to find s so large that

$$\operatorname{cap}\bigl(E(v,\tfrac12a)\cap I,B_I\bigr)<\varepsilon, \tag{3.3.4}$$

whenever $I\in\mathcal{B}_s$ and $E(v,\frac{1}{2}a):=\bigcup_{I\in\mathcal{B}_s}\{z\in I: v(z)\leqslant \sup_I v-\frac{1}{2}a\}$, where $v\in \mathrm{PSH}(\Omega)$ is such that $-1<v<0$.

Put $b(I):=\sup_I u$, $b_j(I):=\sup_I u_j$ and choose $j_0>s+n$ such that

$$b(I)\leqslant b_j(I)+\tfrac{1}{2}a,\quad I\in\mathcal{B}_s,\ j>j_0.$$

Then for $j>j_0$,

$$E_j=E_j(a)=\{u_j\leqslant u-a\}\subset F_j:=\bigcup_{I\in\mathcal{B}_s}\{z\in I: u_j(z)\leqslant b_j(I)-\tfrac{1}{2}a\}. \tag{3.3.5}$$

Hence by (3.3.4), $\operatorname{cap}(F_j,B_I)<\varepsilon$, and further, by applying Lemma 3.3.1 we get

$$\mu_j(F_j\cap I)\leqslant A_1\mu_j(I')\operatorname{cap}(F_j,B_I)<A_1\varepsilon\mu_j(I').$$

If we sum up these inequalities over all $I\in\mathcal{B}_s$ then, recalling that $\mu(\Omega)<1$ and using (3.3.5), we finally arrive at

$$\mu_j(E_j(a))\leqslant\mu_j(F_j)<3^{2n}A_1\varepsilon.$$

This gives (3.2.1) and completes the proof.

3.4. Proof of Theorem C

To avoid some technicalities we shall work under an extra assumption that v_j is uniformly bounded. How to get the general case is explained at the end of the proof. Let us first show the following stability result.

THEOREM 3.4.1. $v=(\limsup v_j)^*$.

The theorem will follow from the following two lemmata.

LEMMA 3.4.2. *For any $t>1$ and $a>0$,*

$$\lim_{k\to\infty}\int_{V(k,a,t)}(dd^cv_k)^n=0,$$

where $V(k,a,t):=\{v_k<tv-a\}$.

LEMMA 3.4.3. *For any $t<1$ and $a>0$,*

$$\lim_{k\to\infty}\int_{V(k,a,t)}(dd^cv)^n=0,$$

where $V(k,a,t):=\{tv<v_k-a\}$.

Proof of Lemma 3.4.2. We fix t and write for brevity $V(k,a):=V(k,a,t)$. Note that if the statement holds for some a_0 then it is also true for $a>a_0$. So we may feel free to decrease a if necessary. Keeping this in mind one can fix $\varepsilon>0$ and $b>a$ so close to a that

$$\int_{V(k,a)\setminus V(k,b)}(dd^cv_k)^n\leqslant\varepsilon \tag{3.4.1}$$

for infinitely many k from a preassigned subsequence.

Using Theorem 1.2.8 choose an open set U such that $v|_{\Omega\setminus U}$ is continuous and

$$\int_U (dd^c v_s)^n \leqslant \varepsilon \quad \text{and} \quad \int_U (dd^c v_k)^n \leqslant \varepsilon \tag{3.4.2}$$

for any k.

Denote by $V'(k,a)$ the union of cubes belonging to $\mathcal{B}_k$ and contained in $V(k,a)\cup U$. Let us first prove that

$$\lim_{k\to\infty} \int_{V(k,a)\setminus V'(k,a)} (dd^c v_k)^n = 0. \tag{3.4.3}$$

Let $E(\delta,k) := \bigcup_{I\in\mathcal{B}_k} \{z\in I : \sup_I v - v(z) \geqslant \delta\}$. We shall see that if $I\in\mathcal{B}_k$ and $I\not\subset V'(k,a)$ then

$$I\cap V(k,a) \subset [V(k,a)\setminus V(k,b)]\cup E(\delta,k)\cup U \tag{3.4.4}$$

for some $\delta>0$ and k large enough.

Indeed, for some k_1 and any $x,y\in I\setminus U$, $I\in\mathcal{B}_k$, $k>k_1$, we have by continuity of $v|_{\Omega\setminus U}$

$$|v(x)-v(y)| < \delta := \frac{b-a}{t+1}. \tag{3.4.5}$$

Suppose now that $I\in\mathcal{B}_k$, $k>k_1$, and that there exists $z_0\in I\setminus(U\cap V(k,a))$. Take any $z\in I\setminus[E(\delta,k)\cup U]$. To verify (3.4.4) we need to show that $z\notin V(b,k)$. Indeed, since $z\notin E(\delta,k)$, we have $v_k(z)\geqslant v_k(z_0)-\delta$ and, due to (3.4.5), $v(z)\leqslant v(z_0)+\delta$. Therefore

$$v_k(z)-tv(z) \geqslant v_k(z_0)-tv(z_0)-\delta(t+1) \geqslant -a-\delta(t+1) = -b,$$

and so (3.4.4) follows.

Increasing k_1 if necessary and applying Corollary 3.2.3 one obtains

$$\int_{I\cap E(\delta,k)} (dd^c v_k)^n < \varepsilon \int_I (dd^c v_k)^n, \quad k>k_1. \tag{3.4.6}$$

Since by (3.4.4),

$$[V(k,a)\setminus V'(k,a)] \subset [V(k,a)\setminus V(k,b)]\cup E(\delta,k)\cup U,$$

(3.4.3) follows from (3.4.1), (3.4.2) and (3.4.6).

Recall that, by the construction of $(dd^c v_k)^n$, we have

$$\int_{V'(k,a)} (dd^c v_s)^n = \int_{V'(k,a)} (dd^c v_k)^n.$$

From this and the comparison principle one infers that

$$t^n\int_{V(k,a)}(dd^cv_s)^n\leqslant\int_{V(k,a)}(dd^cv_k)^n\leqslant\int_{V'(k,a)}(dd^cv_k)^n+\int_{V(k,a)\setminus V'(k,a)}(dd^cv_k)^n$$
$$=\int_{V'(k,a)}(dd^cv_s)^n+\int_{V(k,a)\setminus V'(k,a)}(dd^cv_k)^n.$$

Hence

$$(t^n-1)\int_{V'(k,a)\setminus U}(dd^cv_k)^n-t^n\int_U(dd^cv_s)^n\leqslant(t^n-1)\int_{V(k,a)}(dd^cv_s)^n$$
$$\leqslant\int_{V(k,a)\setminus V'(k,a)}(dd^cv_k)^n.$$

By (3.4.3) the right-hand side tends to 0 as $k\to\infty$, which forces the left-hand side to have non-positive upper limit. Thus by invoking (3.4.2) and (3.4.3) once more the lemma follows.

Proof of Lemma 3.4.3. The proof is similar to the preceding one. Formula (3.4.1) can be replaced by

$$\int_{V(k,a)\setminus V(k,b)}(dd^cv_s)^n\leqslant\varepsilon. \tag{3.4.1$'$}$$

Instead of (3.4.4) we now prove

$$I\setminus[E(\delta,k)\cup U]\subset V(k,a) \tag{3.4.4$'$}$$

for any $I\in\mathcal{B}_k$ such that $(I\setminus U)\cap V(k,b)\neq\varnothing$. The proof is analogous to the one of (3.4.4).

So, if $V'(k)$ denotes the union of cubes from $I\in\mathcal{B}_k$ satisfying $(I\setminus U)\cap V(k,b)\neq\varnothing$ then $V'(k)\subset V(k,a)\cup U\cup E(\delta,k)$ and, obviously,

$$V(k,a)\subset V'(k)\cup[V(k,a)\setminus V(k,b)]\cup U.$$

These inclusions combined with (3.4.1′), (3.4.2) and (3.4.6) lead to

$$\begin{aligned}\int_{V(k,a)}(dd^cv_k)^n&\geqslant\int_{V'(k)}(dd^cv_k)^n-\int_U(dd^cv_k)^n-\int_{E(\delta,k)}(dd^cv_k)^n\\&\geqslant\int_{V'(k)}(dd^cv_k)^n-2\varepsilon=\int_{V'(k)}(dd^cv_s)^n-2\varepsilon\\&\geqslant\int_{V(k,a)}(dd^cv_s)^n-\int_{[V(k,a)\setminus V(k,b)]}(dd^cv_s)^n-\int_U(dd^cv_s)^n-2\varepsilon\\&\geqslant\int_{V(k,a)}(dd^cv_s)^n-4\varepsilon.\end{aligned}\tag{3.4.7}$$

On the other hand by the comparison principle,

$$\int_{V(k,a)} (dd^c v_k)^n \leqslant t^n \int_{V(k,a)} (dd^c v_s)^n, \quad t<1. \tag{3.4.8}$$

The inequalities (3.4.7) and (3.4.8) are contradictory for sufficiently small ε unless $\lim_{k\to\infty} \int_{V(k,a)} (dd^c v_s)^n = 0$, which gives the result.

Proof of Theorem 3.4.1. First, let us verify $(\limsup v_k)^* \leqslant v$. Set $w=(\limsup v_k)^*$. Suppose that the statement were not true. Then for some $z_0 \in \Omega$ and $\varepsilon>0$ we would have $v(z_0)<w(z_0)-3\varepsilon$. Since the inequality is valid on a set of positive Lebesgue measure in any neighbourhood of z_0, and the set where $w>\limsup v_k$ has measure zero one may also suppose that $w(z_0)=\limsup v_k(z_0)$. By the upper semicontinuity of v we have $v<v(z_0)+\frac{1}{2}\varepsilon$ on a ball $B=B(z_0,r)$. Apply the comparison principle to obtain

$$\int_{G_k} (dd^c(v_k+\phi))^n \leqslant \int_{G_k} (dd^c v_s)^n,$$

where $G_k=\{v<v_k+\phi\}$ and ϕ is a strictly psh negative function in $\bar\Omega$ with $\phi>-\frac{1}{2}\varepsilon$ and $dd^c\phi>\delta\beta$, $\delta>0$, $\beta(z):=dd^c|z|^2$. Then $G_k \subset\subset \Omega$. Since $v\leqslant v(z_0)+\frac{1}{2}\varepsilon$ on B we get by the Hartogs lemma $v_k<v_k(z_0)+\varepsilon$ for $k>k_0$. Then by the mean value inequality for subharmonic functions $v_k\geqslant v_k(z_0)-\varepsilon$ on a subset F_k of B such that $2\lambda(F_k)\geqslant\lambda(B)$.

For $z\in F_k$ and k large enough,

$$v_k(z) \geqslant v_k(z_0)-\varepsilon > w(z_0)-2\varepsilon > v(z_0)+\varepsilon > v(z)+\tfrac{1}{2}\varepsilon.$$

Thus $F_k\subset G_k$ and so the left-hand side integral exceeds

$$\int_{F_k} (dd^c\phi)^n \geqslant \delta^n \int_{F_k} \beta^n \geqslant \tfrac{1}{2}\delta^n \int_B \beta^n = \text{const} > 0.$$

This leads to a contradiction as the right-hand side tends to 0 when $k\to\infty$, by Lemma 3.4.3.

The proof of the reverse inequality is analogous with the roles of v_k and v interchanged. To get $v_k<v(z_0)+\varepsilon$ in a neighbourhood of z_0 the Hartogs lemma is used. To draw the final conclusion we now apply Lemma 3.4.2. The details are given in [Ko2].

REMARK. *In the proof of Theorem* 3.4.1 *we do not really need to know that* v_k *is uniformly bounded.*

Proof. In the general case only the inequality (3.4.2) requires an explanation.

Let $V(k,s)$ denote the set $\{v_k<-s\}$ and put $\mathcal{B}_k(s):=\{I\in\mathcal{B}_k : \sup_I v_k \geqslant -s\}$. Then for $I\in\mathcal{B}_k(s)$ we get by applying Theorem 1.2.6 and Theorem 1.2.11

$$\nu_k(I\cap V(k,s^2)) \leqslant C_1\nu_k(I)\,\mathrm{cap}(I\cap V(k,s^2),B_I) \leqslant \frac{C_2\nu_k(I)}{s}, \tag{3.4.9}$$

where C_1, C_2 are independent constants and ν_k stands for $(dd^c v_k)^n$.

On the other hand, for $I \in \mathcal{B}_k \setminus \mathcal{B}_k(s)$ we have $I \subset V(k,s)$, and using Theorem 1.2.11 one finds a constant C_3 such that

$$\operatorname{cap}(K \cap V(k,s), \Omega) < \frac{C_3}{s},$$

where $K := \operatorname{supp}(dd^c v_s)^n$. Hence (recall that $-1<v<0$),

$$\sum_{I \in \mathcal{B}_k \setminus \mathcal{B}_k(s)} \int_I (dd^c v_k)^n = \sum_{I \in \mathcal{B}_k \setminus \mathcal{B}_k(s)} \int_I (dd^c v_s)^n$$
$$\leqslant \int_{K \cap V(k,s)} (dd^c v_s)^n \leqslant \operatorname{cap}(K \cap V(k,s), \Omega) \leqslant \frac{C_3}{s}.$$

This combined with (3.4.9) provides s so large that

$$\int_{\{v_k < -s\}} (dd^c v_k)^n < \tfrac{1}{2}\varepsilon$$

for all k. Then, choosing U with $\operatorname{cap}(U,\Omega) < \varepsilon/2s^n$ one obtains, by the definition of the relative capacity,

$$\int_{U \cap \{v_k \geqslant -s\}} \left(dd^c \frac{v_k}{s}\right)^n < \operatorname{cap}(U,\Omega) < \frac{\varepsilon}{2s^n},$$

and so

$$\int_U (dd^c v_k)^n = \int_{U \cap \{v_k \geqslant -s\}} (dd^c v_k)^n + \int_{U \cap \{v_k < -s\}} (dd^c v_k)^n < \varepsilon.$$

COROLLARY 3.4.4. *Let T be a current of the form*

$$dd^c \varrho_1 \wedge dd^c \varrho_2 \wedge \ldots \wedge dd^c \varrho_n, \quad \varrho_s \in \operatorname{PSH}(\Omega),\ -1 < \varrho_s < 0.$$

Then

$$\lim_{k \to \infty} \int_{V(k,a,t) \cap K} T = 0$$

for $V(k,a,t) = \{v_k < tv - a\}$, $t>1$, $a>0$.

Proof. For fixed a and t set $\varrho = a(\varrho_1 + \varrho_2 + \ldots + \varrho_n - 1)$ and $G_k = \{v_k < tv + \varrho\}$. Then, by the hypothesis, $-(1+n)a < \varrho < -a$. Therefore $V(k,(n+1)a,t) \subset G_k \subset V(k,a,t)$. Apply the comparison principle to obtain

$$\int_{V(k,(n+1)a,t)} T \leqslant \int_{V(k,(n+1)a,t)} (dd^c(tv+\varrho))^n$$
$$\leqslant \int_{G_k} (dd^c(tv+\varrho))^n \leqslant \int_{V(k,a,t)} (dd^c v_k)^n.$$

The statement now follows from Lemma 3.4.2, which says that the right-hand side of the above inequality tends to zero as $k \to \infty$.

Now we are in a position to prove the crucial lemma.

LEMMA 3.4.5. *Suppose that* (3.2.1) *is not true, and so, after passing to a subsequence, we have*

$$\int_{E_j(a_0)} (dd^c u_j)^n > A_0, \quad A_0 > 0,\, a_0 > 0.$$

Then there exist $a_m > 0$, $A_m > 0$, $k_1 > 0$ *such that*

$$\int_{E_j(a_m)} (dd^c v_j)^{n-m} \wedge (dd^c v_k)^m > A_m, \quad k > k_1,\, j > j(k). \tag{3.4.10}$$

Proof. We shall proceed by induction over m. For $m=0$ the statement holds by the hypothesis. We assume that (3.4.10) is true for some fixed $m<n$ and now we shall prove it for $m+1$.

Let us observe that by the Chern–Nirenberg–Levine inequalities there exists $C>0$ such that

$$\int_\Omega T \leqslant C \tag{3.4.11}$$

for currents T which are wedge products of $dd^c v_j, dd^c u$ and $dd^c v_{jk}$ (defined below). Indeed, all the functions u, v_j, v_{jk} are bounded from below by -1 on a compact subset of Ω and maximal away from this set. Take a defining function for Ω which does not exceed -1 on the compact set. We can now extend u, v_j, v_{jk} by this function to a neighbourhood of $\bar{\Omega}$, and thus by Theorem 1.2.3 the inequality (3.4.11) follows.

Let us denote by $T=T(j,k,m)$ the current $(dd^c v_j)^{n-m-1} \wedge (dd^c v_k)^m$ and set $v_{jk} := \max(v_j, v_k - 2\varepsilon)$ for some fixed $\varepsilon > 0$.

Using quasicontinuity of u and v (Theorem 1.2.8) we choose an open set U such that

$$\operatorname{cap}(U, \Omega) < \frac{\varepsilon}{3^n}, \tag{3.4.12}$$

and both u and v are continuous on $\Omega \setminus U$. Then for $j > j_0$ and $k > k_0$ we have

$$\max(v_j, v_k) \leqslant v + \varepsilon \quad \text{and} \quad u_j \leqslant u + \varepsilon, \tag{3.4.13}$$

on $\Omega \setminus U$. Indeed, the inequalities are valid in a neighbourhood of $\partial\Omega$ because all u_j (or v_j) are bounded from above by the maximal function in Ω with boundary data φ (or 0). On the remaining part of $\Omega \setminus U$ one obtains (3.4.13) by the Hartogs lemma, since due to Theorem 3.4.1, $v = (\limsup v_j)^*$. Set

$$J'(j,k) := \int_\Omega (u - u_j)\, dd^c v_{jk} \wedge T,$$

$$J(j,k) := \int_\Omega (u - u_j)\, dd^c v_k \wedge T, \quad j > j_0,\, k > k_0.$$

Using the inequalities $v_k - v_{jk} \leqslant 2\varepsilon$, $|v_k| \leqslant 1$, (3.4.11) and integration by parts we can estimate the difference of those integrals in the following way:

$$\begin{aligned} J'(j,k) - J(j,k) &= \int_\Omega (v_{jk} - v_k)\, dd^c(u - u_j) \wedge T \\ &= \int_\Omega (v_{jk} - v_k)\, dd^c u \wedge T + \int_\Omega (v_k - v_{jk})\, dd^c u_j \wedge T \\ &\leqslant \int_\Omega (v_{jk} - v_k)\, dd^c u \wedge T + 2\varepsilon \int_\Omega dd^c u_j \wedge T \\ &\leqslant \int_{\{v_k < v_j - 2\varepsilon\}} \|v_k\|\, dd^c u \wedge T + 2\varepsilon \int_\Omega dd^c (u + u_j) \wedge T \\ &\leqslant \int_{\{v_k < v_j - 2\varepsilon\} \setminus U} dd^c u \wedge T + \int_U dd^c u \wedge T + 4\varepsilon C. \end{aligned}$$

The second term on the right-hand side is bounded from above by

$$\int_U (dd^c(u + v_j + v_k))^n \leqslant 3^n \operatorname{cap}(U, \Omega) < \varepsilon.$$

As for the first one, we shall make use of Corollary 3.4.4. We need to know that $\{v_k < v_j - 2\varepsilon\} \setminus U \subset V(k, a, t)$ for some $t > 1$, $a > 0$. Recalling (3.4.13) one obtains $v_j - 2\varepsilon < v - \varepsilon < \left(1 + \frac{1}{2}\varepsilon\right) v - \frac{1}{2}\varepsilon$ on $\Omega \setminus U$. Thus $\{v_k < v_j - 2\varepsilon\} \setminus U \subset V\left(k, \frac{1}{2}\varepsilon, 1 + \frac{1}{2}\varepsilon\right)$. Applying Corollary 3.4.4 one can find $k_1 > k_0$ such that

$$J'(j,k) - J(j,k) \leqslant 4(C+1)\varepsilon. \tag{3.4.14}$$

In the next step we shall estimate $J'(j,k)$ from below. Using the second inequality of (3.4.13) we have

$$\begin{aligned} J'(j,k) &\geqslant a_m \int_{E_j(a_m)} dd^c v_{jk} \wedge T - \varepsilon \int_{\Omega \setminus U} dd^c v_{jk} \wedge T - \int_U dd^c v_{jk} \wedge T \\ &\geqslant a_m \int_{E_j(a_m)} dd^c v_{jk} \wedge T - \varepsilon (C+1). \end{aligned} \tag{3.4.15}$$

Furthermore,

$$\begin{aligned} \int_{E_j(a_m)} dd^c v_{jk} \wedge T &\geqslant \int_{E_j(a_m) \cap \{v_j > v_k - 2\varepsilon\}} dd^c v_{jk} \wedge T \\ &= \int_{E_j(a_m) \cap \{v_j > v_k - 2\varepsilon\}} dd^c v_j \wedge T \\ &\geqslant \int_{E_j(a_m)} dd^c v_j \wedge T - \int_U dd^c v_j \wedge T \\ &\quad - \int_{(E_j(a_m) \cap \{v_j \leqslant v_k - 2\varepsilon\}) \setminus U} dd^c v_j \wedge T. \end{aligned} \tag{3.4.16}$$

To estimate the last integral we again apply Corollary 3.4.4. Using (3.4.13) we get $\{v_j \leqslant v_k - 2\varepsilon\}\setminus U \subset \{v_j < v - \varepsilon\} \subset \{v_j < (1+\frac{1}{2}\varepsilon)v - \frac{1}{2}\varepsilon\}$. Therefore, given k one can find $j(k)$ such that for $j > j(k)$

$$\int_{(E_j(a_m)\cap\{v_j\leqslant v_k-2\varepsilon\})\setminus U} dd^c v_j \wedge T \leqslant \varepsilon.$$

Hence, according to the induction hypothesis one obtains from (3.4.16)

$$\int_{E_j(a_m)} dd^c v_{jk} \wedge T \geqslant A_m - 2\varepsilon, \quad j > j(k).$$

Plug it into (3.4.15) to get

$$J'(j,k) \geqslant a_m(A_m - 2\varepsilon) - \varepsilon(C+1), \quad j > j(k).$$

Thus, if we start with ε small enough, we may conclude from (3.4.14) and the above inequality that

$$J(j,k) \geqslant \tfrac{1}{2} a_m A_m, \quad k > k_1,\ j > j(k). \tag{3.4.17}$$

Fixing $d > 0$ one can estimate $J(j,k)$ from above:

$$J(j,k) \leqslant \int_{\{u_j<u-d\}} dd^c v_k \wedge T + d\int_\Omega dd^c v_k \wedge T \leqslant \int_{\{u_j<u-d\}} dd^c v_k \wedge T + dC.$$

Setting $a_{m+1} := d = a_m A_m/4C$ in the last formula and combining it with (3.4.17) we finally arrive at

$$\int_{E_j(a_{m+1})} dd^c v_k \wedge T \geqslant \tfrac{1}{4} a_m A_m := A_{m+1}, \quad k > k_1,\ j > j(k),$$

which concludes the proof of the inductive step. Thus the lemma follows.

Now we shall prove Theorem C reasoning by contradiction. So, suppose that the hypothesis of Lemma 3.4.5 is valid. Then using its statement for $m = n$ we can fix $k > k_1$ such that

$$\int_{E_j(a_n)} (dd^c v_k)^n > A_n \quad \text{if } j > j(k).$$

Since, by the construction, $(dd^c v_k)^n \leqslant M_k \beta^n$ for some $M_k > 0$, one infers from the last inequality that

$$\lambda(E_j(a_n)) \geqslant M_k^{-1} \int_{E_j(a_n)} (dd^c v_k)^n > \frac{A_n}{M_k}, \quad j > j(k),$$

which contradicts the fact that $u_j \to u$ in L^1_{loc}. This completes the proof of Theorem C modulo the additional assumption. To dispense with the extra hypothesis one should replace v_k in Lemma 3.4.5 by functions $w_k := \max(v_k, Cv)$ with $Cv < \inf_\Omega v$ on supp $(dd^c v)^n$. Lemma 3.4.5 still holds true in that case, but the argument completing the proof of Theorem C needs to be modified since w_k are not smooth. Fortunately, due to Corollary 3.4.4 we have $(dd^c w_k)^n = (dd^c v_k)^n$ away from a set of arbitrarily small measure $(dd^c v_k)^n$ provided k is large enough. We refer to [Ko2] for details.

REMARK. *Theorem C remains valid in a pseudoconvex domain, provided that there exists a solution to the homogeneous Dirichlet problem for the Monge–Ampère equation with the given boundary data.*

Proof. Let us consider an exhaustion sequence of smooth strictly pseudoconvex sets $\Omega_j \uparrow \Omega$. Fix Ω_j and a decreasing sequence φ_{jk}, $k=1,2,...$, of continuous functions on $\partial\Omega_j$ such that $\lim_{k\to\infty} \varphi_{jk} \downarrow v$, where v again denotes the given subsolution. Theorem C now provides u_{jk} solving $(*)$ in Ω_j with boundary data equal to φ_{jk}. By the convergence theorem $u_j := \lim\downarrow u_{jk}$ solves $(dd^c u_j)^n = d\mu$ in Ω_j and hence, via the comparison principle, $u_j \geqslant v$ in Ω_j. In particular, the last inequality holds on $\partial\Omega_{j-1}$. So, again applying the comparison principle, $u_j \geqslant u_{j-1}$ on Ω_{j-1}. Thus we have shown that u_j is (locally) increasing. By the convergence theorem, $u=(\sup u_j)^*$ is the solution of $(*)$ in Ω. The extra hypothesis ensures that u satisfies the boundary condition. (I overlooked that point in [Ko2].)

COROLLARY 3.4.6. *If Ω is strictly pseudoconvex and $v_1, ..., v_n \in \mathrm{PSH} \cap L^\infty(\Omega)$ then there exists $u \in \mathrm{PSH} \cap L^\infty(\Omega)$ matching any prescribed continuous boundary data and such that $(dd^c u)^n = dd^c v_1 \wedge ... \wedge dd^c v_n$.*

Proof. The Monge–Ampère mass of $v_1 + ... + v_n$ obviously exceeds the given measure μ. Take a sequence of cut-off functions $\chi_j \uparrow 1$ in Ω and solve $(*)$ for $\chi_j\, d\mu$ and given boundary data φ. The solutions u_j produced in this way decrease to a psh function u which is bounded from below by $-(\|v\|+\|\varphi\|)$ and solves $(*)$.

The Monge–Ampère operator is well defined for psh functions which are locally bounded outside a compact subset of Ω (see [D1], [D2], [P]).

COROLLARY 3.4.7. *If $v \in \mathrm{PSH}(\Omega)$, $v \in L^\infty_{\mathrm{loc}}(\Omega \setminus E)$, $E \subset\subset \Omega$ with $\lim_{z\to x} v(z) = \varphi(x)$ for $x \in \partial\Omega$, and if a Borel measure μ satisfies $\mu(E)=0$, $d\mu \leqslant (dd^c v)^n$, then there exists $u \geqslant v$ solving*

$$\begin{cases} u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega \setminus E), \\ (dd^c u)^n = d\mu, \\ \lim_{z\to x} u(z) = \varphi(x) \quad \text{for } x \in \partial\Omega. \end{cases}$$

Proof. Denote by U_s the set $\{v < -s\}$, $s=1,2,...$, and solve the Dirichlet problem $(*)$ for measures $\mu_s := \mu|_{\Omega\setminus U_s}$. The solutions are denoted by u_s. Then by the comparison principle, $u_s \geqslant \max(v, -s)$, and so $u = \lim\downarrow u_s$ is the desired solution.

3.5. Measures admitting solutions to the Monge–Ampère equation

Theorem B follows easily from Lemma 2.3.1.

Proof of Theorem B. Due to Lemma 2.3.1, $\|u\|_{L^\infty}$ is bounded by a constant depending on A and h. The change of the normalizing condition $u(0)>-1$ into $\int d\mu<1$ is harmless, since by shifting the origin if necessary one can obtain the lower bound for $u(0)$ also in that case.

We need to prove the continuity of u. To do this we shall apply Lemma 2.3.1 again. Since φ is continuous one can find for any given $d>0$ a compact $K\subset\Omega$ such that $u_j<u+d$ on ∂K, where $u_j:=u*\omega_j$ is the standard regularization sequence for u. Then the sets $\{u_j>u+2d\}$ must be empty for j large enough. Otherwise, by the formulas (2.3.4) and (2.3.5), their capacity would be bounded away from zero in contradiction to Corollary 1.2.10.

Corollary 3.1.3 has been proved in §2.5.

Proof of Corollary 3.1.4. First we shall prove the statement under the assumption that μ has compact support in Ω. Define a regularizing sequence of measures μ_j by fixing $\omega\in C_0^\infty(B)$, a radially symmetric non-negative function with $\int\omega\,d\lambda=1$ (here B is the unit ball in $\mathbf{C}^n$), and setting

$$\mu_j=\omega_j*\mu,$$

where

$$\omega_j(z)=j^{2n}\omega(jz).$$

By Theorems A and B it is enough to find $j_0>0$, $A>0$ such that for any compact set $K\subset\Omega$ the following inequalities hold:

$$\mu_j(K)\leqslant A\operatorname{cap}(K,\Omega)h^{-1}((\operatorname{cap}(K,\Omega))^{-1/n}),\quad j>j_0. \tag{3.5.1}$$

PROPOSITION 3.5.1. *If $E\subset\subset\Omega$ is regular then for any $d>1$ there exists t_0 such that*

$$\operatorname{cap}(K_y,\Omega)\leqslant d\operatorname{cap}(K,\Omega),\quad |y|<t_0,$$

where $K\subset E$ is regular and $K_y:=\{x:x-y\in K\}$.

Proof of Proposition 3.5.1. For $K\subset E$ define $w_y(x):=u_{K_y}(x+y)$, where u_{K_y} is the relative extremal function of K_y. For any c such that $0<c<\frac{1}{2}$ define $\Omega_c=\{u_E<-c\}$. By continuity of u_E one can fix $t_0>0$ such that if $|y|\leqslant t_0$ and $x\in\Omega_{c/2}$ then $x+y\in\Omega$. Then

$$g(x):=\begin{cases}\max(w_y-c,(1+2c)u_E)(x), & x\in\Omega_{c/2},\\ (1+2c)u_E(x), & x\notin\Omega_{c/2},\end{cases}$$

is a well-defined plurisubharmonic function in Ω. Since $K\subset E$ and $w_y=-1$ on K one concludes that $g=w_y-c$ in a neighbourhood of K. Hence by Theorem 1.2.4,

$$\begin{aligned}\operatorname{cap}(K,\Omega) &\geqslant (1+2c)^{-n}\int_K (dd^c g)^n = (1+2c)^{-n}\int_K (dd^c w_y)^n \\ &= (1+2c)^{-n}\int_{K_y} (dd^c u_{K_y})^n = (1+2c)^{-n}\operatorname{cap}(K_y,\Omega).\end{aligned}$$

Thus the proposition is proved.

To complete the proof of Corollary 3.1.4 let us fix a set E and a positive number j_0 such that the above proposition holds with $E:=\bigcup_{j>j_0}\operatorname{supp}\mu_j\subset\subset\Omega$, $j_0>1/t_0$, and $d=a^n$. By the assumptions there exists $A_0>0$ such that

$$\mu(K)\leqslant A_0\operatorname{cap}(K,\Omega)\,h^{-1}((\operatorname{cap}(K,\Omega))^{-1/n}).$$

Hence for $j>j_0$ we have by Proposition 3.5.1 and the extra assumption on h

$$\begin{aligned}\mu_j(K) &\leqslant \sup_{|y|<1/j}\mu(K_y)\leqslant A_0\sup_{|y|<1/j}\operatorname{cap}(K_y,\Omega)\,h^{-1}((\operatorname{cap}(K_y,\Omega))^{-1/n}) \\ &\leqslant A_0 d\operatorname{cap}(K,\Omega)\,h^{-1}((d\operatorname{cap}(K,\Omega))^{-1/n}) \\ &\leqslant A_0 db\operatorname{cap}(K,\Omega)\,h^{-1}((\operatorname{cap}(K,\Omega))^{-1/n}).\end{aligned}$$

Setting $A:=A_0a^nb$ we verify in this way that μ_j satisfy (3.5.1) for $j>j_0$, with the constant A independent of j. Thus by Theorem B the family of solutions of $(*)$ for μ_j, $j>j_0$, is uniformly bounded. So one can apply Theorem A to get the conclusion.

To verify the statement for an arbitrary measure μ note that by the above argument the solutions exist for $\chi_j\,d\mu$, where χ_j is a non-decreasing sequence of smooth cut-off functions, $\chi_j\uparrow 1$ in Ω. Moreover, the L^∞-norms of those solutions are uniformly bounded by a constant depending only on A. Hence the result follows by applying the convergence theorem.

Let us point out some families of measures that fulfil the hypothesis of Corollary 3.1.4. Recall that the p-Hausdorff content of a set $E\subset\mathbf{R}^n$ is given by

$$\widehat{\mathcal{H}}_p(E)=\inf\sum_{j\in J} r_j^p,$$

where the infimum is taken over all coverings of E by unions of balls $B(a_j,r_j)$.

COROLLARY 3.5.2. *Let μ be a Borel measure in Ω satisfying the inequality*

$$\mu(E)\leqslant C\int \widehat{\mathcal{H}}_p(E(z'))\,d\lambda_{n-1}(z'),\quad p>0,$$

where C is an independent constant, $z=(z_1,z')\in\mathbf{C}\times\mathbf{C}^{n-1}$ and

$$E(z'):=\{z_1\in\mathbf{C}:(z_1,z')\in E\}.$$

Then one can solve the Monge–Ampère equation $(*)$ *for* μ.

Proof. It follows from Corollary 1.3.1 via the following proposition which shows that both assumptions on μ are satisfied in this case.

PROPOSITION 3.5.3. *If μ is as in Corollary* 3.5.2 *then for any cube $I\subset B_I\subset\Omega$ (B_I as defined in* §3.1) *we have*

$$\mu(E)\leqslant C_0(\operatorname{cap}(E,B_I))^2,\quad E\subset I.$$

Proof of Proposition 3.5.3. By the hypothesis,

$$\mu(E)\leqslant C\int\widehat{\mathcal{H}}_p(E(z'))\,d\lambda(z')\leqslant C\int\widehat{\mathcal{H}}^{2n}_{p/2n}(E(z'))\,d\lambda(z').$$

Applying [Ts, proof of Theorem III.19] and recalling that for $n=1$ the capacity T_R and the logarithmic capacity are equivalent (see e.g. [Ta]), one can estimate $\widehat{\mathcal{H}}_{p/2n}(E(z'))$ by the capacity $T_R(E(z'))$ ($R=$ radius of B_I):

$$\widehat{\mathcal{H}}_{p/2n}(E(z'))\leqslant\frac{C_1}{-\log T_R(E(z'))}.\tag{3.5.2}$$

Now, following the argument from the proof of Lemma 2.5.1 we get the conclusion.

REMARK 1. *We are free to choose the coordinate system in Ω to meet the requirements on μ in Corollary* 3.5.2.

REMARK 2. *The surface measure of a smooth compact real hypersurface satisfies the assumptions of the last corollary.*

The method of the proof of Corollary 3.5.2 works also for some other measures as long as their one-dimensional slices are dominated by capacity. This is the case, for instance, when a measure μ is upper bounded by the Lebesgue measure of the totally real part $\mathbf{R}^n$ of $\mathbf{C}^n$:

$$\mu(E)\leqslant\text{const}\cdot\lambda_{\mathbf{R}^n}(E\cap\mathbf{R}^n).$$

Then instead of (3.5.2) we use the well-known inequality between the length of a subset of the real axis and its logarithmic capacity (see [Ts]).

In §2.5 we gave further examples of measures for which $(*)$ is solvable (clearly, M should now be replaced by Ω).

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