

# THE COMPLEXITY OF COMBINATORIAL OPTIMIZATION PROBLEMS ON $D$ -DIMENSIONAL BOXES\*

MIROSLAV CHLEBÍK<sup>†</sup> AND JANKA CHLEBÍKOVÁ<sup>‡</sup>

**Abstract.** The MAXIMUM INDEPENDENT SET problem in  $d$ -box graphs, i.e., in intersection graphs of axis-parallel rectangles in  $\mathbb{R}^d$ , is known to be NP-hard for any fixed  $d \geq 2$ . A challenging open problem is, how close the solution can be approximated by a polynomial time algorithm. For the restricted case of  $d$ -boxes with bounded aspect ratio a PTAS exists [12]. In general case no polynomial time algorithm with approximation ratio  $o(\log^{d-1} n)$  for a set of  $n$   $d$ -boxes is known.

In this paper we prove APX-hardness of the MAXIMUM INDEPENDENT SET problem in  $d$ -box graphs for any fixed  $d \geq 3$ . We give an explicit lower bound  $\frac{245}{244}$  on efficient approximability for this problem unless  $P = NP$ . Additionally, we provide a generic method how to prove APX-hardness for other graph optimization problems in  $d$ -box graphs for any fixed  $d \geq 3$ .

**Key words.** independent set, geometric intersection graphs, rectangle graphs

**AMS subject classifications.** 68Q25, 68R10

**1. Introduction.** Many optimization problems like MAXIMUM CLIQUE, MAXIMUM INDEPENDENT SET, and MINIMUM (VERTEX) COLORING are NP-hard in general graphs, but solvable in polynomial time in interval graphs [15]. However, many of the problems, e.g., MAXIMUM INDEPENDENT SET ([13], [17]) or MINIMUM COLORING ([22]), are known to be NP-hard already in 2-dimensional models of geometric intersection graphs as in unit disk graphs or in intersection graphs of axis-parallel rectangles in  $\mathbb{R}^d$  for any fixed  $d \geq 2$  (shortly,  $d$ -box intersection graphs or  $d$ -box graphs). Among basic NP-hard graph optimization problems only MAXIMUM CLIQUE is known to be solvable in polynomial time in  $d$ -box graphs ([4], [21], [26]). In most cases geometric restrictions on input instances allow to obtain better approximation algorithms for problems that are extremely hard to approximate in general graphs. On the other hand, geometric restrictions make the task to achieve hardness results more difficult.

The most studied problem in  $d$ -box intersection graphs, MAXIMUM INDEPENDENT SET (MAX-IS), can be formulated as follows: for a given set  $\mathcal{R}$  of  $n$  axis-parallel  $d$ -dimensional boxes (shortly,  $d$ -boxes) find a maximum cardinality subset  $\mathcal{R}^* \subseteq \mathcal{R}$  of pairwise disjoint boxes. The problem has attracted attention of many researchers (e.g., [1], [5], [6], [12], [16], [18], [25]) due to its applications in map labeling, data mining, VLSI design, image processing, and point location in  $d$ -dimensional Euclidean space. As the problem is NP-hard for any fixed  $d \geq 2$  ([13], [17]), attention is focused on efficient approximation algorithms. Let us describe briefly known approximability results for it; a more detailed overview of them can be found in [6]. The earliest result was a shifting grid method based PTAS by Hochbaum and Maass [16] in case of unit  $d$ -cubes. This method works for any collection of *fat* objects in  $\mathbb{R}^d$  of roughly the same size and it requires  $n^{O(k^{d-1})}$  time to guarantee an approximation factor of  $(1 + \frac{1}{k})$ . Moreover, this approach can be generalized to objects not necessarily fat, but whose projections to the last  $(d - 1)$  coordinates are fat and of roughly the same

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<sup>†</sup>Max Planck Institute for Mathematics in the Sciences, Inselstraße 22-26, D-04103 Leipzig, Germany, (chlebig@mis.mpg.de)

<sup>‡</sup>Faculty of Mathematics, Physics, and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, (chlebigova@fmph.uniba.sk). The author was supported by VEGA grant no. 1/3106/06.

size. This was essentially established by Agarwal et al. [1] in their work on unit-height rectangles in  $\mathbb{R}^2$ . Generalizing in another direction, Erlebach et al. ([12]) and Chan ([6]) obtained a PTAS for fat objects of possibly varying sizes, such as arbitrary  $d$ -cubes or bounded aspect ratio  $d$ -boxes. For *arbitrary*  $d$ -boxes, even for  $d = 2$ , the existence of a PTAS or a constant factor approximation is an open problem. As it has been observed in several papers ([1], [18]), a logarithmic approximation factor is possible in this case. For example, the results of Agarwal et al. [1] imply  $O(n \log_2^{d-1} n)$ -time algorithm with factor at most  $\lceil \log_2 n \rceil^{d-1}$ . Nielsen [25] independently described an algorithm with optimum-sensitive approximation factor  $(1 + \log_2(is(\mathcal{R})))^{d-1}$ , where  $is(\mathcal{R})$  is the maximum number of independent boxes of  $\mathcal{R}$ . Currently, no polynomial time algorithm is known with  $o(\log^{d-1} n)$ -approximation factor, although Berman et al. [5] have observed that a  $\log_2^{d-1} n$  bound can be reduced by arbitrary multiplicative constant. However, in spite of many efforts it remains open to understand the limits on the approximability of the MAXIMUM INDEPENDENT SET problem in intersection graphs of  $d$ -boxes.

**1.1. Our Results.** In this paper we present the proof of APX-hardness for the MAXIMUM INDEPENDENT SET problem in axis-parallel  $d$ -dimensional boxes for any fixed  $d \geq 3$ . It follows, in particular, that for any fixed  $d \geq 3$  the existence of a PTAS for the problem restricted to  $d$ -boxes with bounded aspect ratio ([12]) cannot be generalized to arbitrary axis-parallel  $d$ -boxes, unless  $P = NP$ .

The idea of our proof is based on the following two results:

- (i) In Section 3 we observe that MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, and some other graph optimization problems are APX-hard even in certain subdivisions of graphs with low maximum degree. For example, for any fixed integer  $k \geq 0$  the MAXIMUM INDEPENDENT SET problem is APX-hard in graphs obtained from 3-regular graphs by  $2k$  subdivision of each edge.
- (ii) In Section 2 we prove that each graph obtained from another one by at least 2-subdivision of each edge is an intersection graph of axis-parallel  $d$ -boxes for any fixed  $d \geq 3$ . Moreover, a  $d$ -box intersection representation of such graphs can be provided in polynomial time.

Both results (i) and (ii) are very general and can be of independent interest. Using them we provide a method how to achieve approximation hardness results in  $d$ -box graphs for other graph optimization problems, e.g., for covering and domination problems. The method used allows to provide also explicit lower bounds on efficient approximability. This is demonstrated on the problems MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER in  $d$ -box graphs (for any fixed  $d \geq 3$ ) proving NP-hardness to achieve an approximation factor of  $1 + \frac{1}{244}$  and  $1 + \frac{1}{249}$ , respectively. One can notice that the best known approximation algorithms for graph optimization problems in  $d$ -box graphs assume that an intersection representation of an input graph by  $d$ -boxes is given. Therefore it should be emphasized that our hardness results apply to this setting as well. Moreover, they hold for instances, in which no point of  $\mathbb{R}^d$  is simultaneously covered by more than two  $d$ -boxes and each  $d$ -box intersects at most three others.

**1.2. Definitions and Notations.** Recall that a  $d$ -dimensional box (shortly,  $d$ -box) is a subset of  $\mathbb{R}^d$  that is a Cartesian product of  $d$  intervals in  $\mathbb{R}$ . For convenience, terms an *interval* and a *rectangle* are used for a 1-box and a 2-box, respectively.

**DEFINITION 1.1.** *The intersection graph of a family of sets  $S_v$ ,  $v \in V$ , is a graph with vertex set  $V$  such that for any  $u, v \in V$  a vertex  $u$  is adjacent to a vertex  $v$  if and*

only if  $S_u \cap S_v \neq \emptyset$ . The family  $\{S_v, v \in V\}$  is an intersection representation of the intersection graph. The intersection graphs of families of axis-parallel  $d$ -dimensional boxes are called  $d$ -box intersection graphs, or simply  $d$ -box graphs.

DEFINITION 1.2. Let  $G$  be a simple graph with vertex set  $V$  and edge set  $E$ . If  $G$  contains a cycle, then the girth of  $G$  is the length of its shortest cycle. A vertex  $v \in V$  is said to cover itself, all edges incident with  $v$ , and all vertices adjacent to  $v$ . An edge  $\{u, v\} \in E$  is said to cover itself, vertices  $u$  and  $v$ , and all edges incident with  $u$  or  $v$ . Two elements of  $V \cup E$  are independent if neither covers the other.

For a graph  $G$ , a vertex cover is a subset of  $V$  that covers all edges  $E$ , a dominating set is a subset of  $V$  that covers all vertices  $V$ , and an edge dominating set is a subset of  $E$  that covers all edges  $E$ .

The goal of the MAXIMUM INDEPENDENT SET problem is to find an independent set of maximum cardinality in a graph  $G$ , let  $is(G)$  denote its cardinality. The MINIMUM VERTEX COVER problem (MIN-VC) asks to find a vertex cover of minimum cardinality in  $G$ , let  $vc(G)$  denote its optimum value. The problems MINIMUM DOMINATING SET (MIN-DS), MINIMUM INDEPENDENT DOMINATING SET (MIN-IDS), and MINIMUM EDGE DOMINATING SET (MIN-EDS), ask for a dominating set, an independent dominating set, and an edge dominating set of minimum size in  $G$ , respectively. Let  $ds(G)$ ,  $ids(G)$ , and  $eds(G)$  stand, respectively, for the corresponding minima.

DEFINITION 1.3. Let  $G = (V, E)$  be a given graph. For an integer  $k \geq 0$ , a  $k$ -subdivision of an edge  $e = \{u, v\} \in E$  in  $G$  is defined as a replacement of  $e$  by a path with endvertices  $u$  and  $v$ , and with  $k$  new internal vertices. A  $k$ -subdivision of  $G$ , denoted by  $\text{div}_k(G)$ , is a graph obtained from  $G$  by a  $k$ -subdivision of each edge  $e$  from  $E$ . (All added paths are pairwise disjoint.)

We will consider also subdivisions of  $G = (V, E)$  that are not uniform, but edge dependent. In such case an edge function  $s := s_G$  from  $E$  to nonnegative integers will be given and the resulting graph will be obtained by  $s(e)$ -subdivision of each edge  $e \in E$ .

For the basic optimization terminology we refer the reader to Ausiello et al. [3]. For any NPO optimization problem  $Q$ ,  $I_Q$  is the set of instances of  $Q$ ,  $\text{sol}_Q(x)$  is the set of feasible solutions for  $x \in I_Q$ , and for each pair  $(x, y)$  such that  $x \in I_Q$  and  $y \in \text{sol}_Q(x)$ ,  $m_Q(x, y)$  is the value of a feasible solution  $y$ . The optimal value for an instance  $x \in I_Q$  is denoted by  $\text{OPT}_Q(x)$ .

DEFINITION 1.4. Let  $Q$  and  $Q'$  be two NPO problems and  $f$  be a polynomial time computable function that maps instances of  $Q$  to instances of  $Q'$ . Then  $f$  is said to be an  $L$ -reduction from  $Q$  to  $Q'$ , if there are constants  $\alpha, \beta \in (0, \infty)$  and a polynomial time computable function  $g$  such that for every  $x \in I_Q$  (i)  $\text{OPT}_{Q'}(f(x)) \leq \alpha \text{OPT}_Q(x)$ , (ii) for every  $y' \in \text{sol}_{Q'}(f(x))$ ,  $g(x, y') \in \text{sol}_Q(x)$  so that  $|\text{OPT}_Q(x) - m_Q(x, g(x, y'))| \leq \beta |\text{OPT}_{Q'}(f(x)) - m_{Q'}(f(x), y')|$ .

To show APX-completeness of a problem  $Q \in \text{APX}$  it is enough to show that there is an  $L$ -reduction from some APX-complete problem to  $Q$ .

REMARK 1.1. Let us recall that all problems MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET are APX-complete in bounded degree graphs. Their inclusion in APX follows from easy counting arguments, when restricted to graphs of degree at most  $B$ ,  $B \geq 3$ ,  $is(G) \geq ids(G) \geq ds(G) \geq \frac{|V|}{B+1}$ ,  $vc(G) \geq \frac{|V|}{B+1}$ , and  $eds(G) \geq \frac{|V|}{2B}$ . (For some of these inequalities it is necessary to confine ourself to graphs without isolated vertices.) Hence for any of the above *min*-

imization problems in bounded degree graphs any feasible solution approximates the optimal one within a constant. For MAXIMUM INDEPENDENT SET, the lower bounds given above apply to any inclusionwise maximal independent set. This provides a constant factor approximation in all cases. In most cases the proof of APX-hardness even in 3-regular graphs is known (see [2], [14], [24], and reference therein).

**2. Intersection Graphs of Axis-Parallel Boxes.** Roberts [27] proved that each graph can be realized as an intersection graph of axis-parallel  $d$ -dimensional boxes for some  $d$  depending on the graph. For any fixed  $d \geq 2$ , the recognition of  $d$ -box graphs is NP-hard ([19], [28]), and hence the reconstruction of their representation by  $d$ -boxes is NP-hard as well. In this section we prove that highly non-trivial subclasses of general graphs are  $d$ -box graphs for any  $d \geq 3$ . Namely, each graph obtained from another one by at least 2-subdivision of each edge is an intersection graph of  $d$ -boxes for any fixed  $d \geq 3$  and its intersection representation can be found in polynomial time.

**THEOREM 2.1.** *Let  $G = (V, E)$  be a graph, and let an integer  $s(e) \geq 2$  be given for each edge  $e \in E$ . Denote by  $G'$  a graph obtained from  $G$  by a  $s(e)$ -subdivision of each edge  $e$ . Then for any fixed integer  $d \geq 3$ , the graph  $G'$  can be realized as an intersection graph of a set of axis-parallel  $d$ -dimensional boxes. Moreover, such realization can be done in time polynomial in  $|V| + \sum_e s(e)$ .*

*Proof.* Let  $G = (V, E)$ ,  $s : E \rightarrow \{2, 3, \dots\}$ , and  $G'$  be given as above. First, we describe the realization of  $G'$  as an intersection graph of a set  $\{R_1, R_2, \dots, R_N\}$  of axis-parallel boxes in  $\mathbb{R}^3$ , where  $N = |V| + \sum_e s(e)$ .

We can assume that  $V = \{1, 2, \dots, |V|\}$ , and assign each edge  $e \in E$  a number  $n_e$  using a bijection  $e \in E \mapsto n_e \in \{1, 2, \dots, |E|\}$  between  $E$  and  $\{1, 2, \dots, |E|\}$ . Each vertex  $i \in \{1, 2, \dots, |V|\}$  will be represented by a 3-box  $R_i = [2i - 1, 2i] \times [2i - 1, 2i] \times [1, 2|E|]$  (see Fig. 2.1).

The graph  $G'$  is obtained from  $G$  replacing each edge  $e = \{i, j\} \in E$  (assume  $i < j$ ) by a path with vertices  $i, A_e^1, A_e^2, \dots, A_e^{s(e)}, j$ . Now we define the boxes  $R_e^1, \dots, R_e^{s(e)}$  representing vertices  $A_e^1, A_e^2, \dots, A_e^{s(e)}$ , respectively. The projection on the third coordinate axis is chosen to be  $[2n_e - 1, 2n_e]$  to ensure that no two boxes  $R_e^i$  and  $R_{e'}^j$ , which correspond to distinct edges  $e$  and  $e'$ , intersect. More precisely, define  $R_e^1 := [2i - 1, 2j] \times [2i - 1, 2i] \times [2n_e - 1, 2n_e]$ , and put further  $R_e' := [2j - 1, 2j] \times [2i - 1, 2j] \times [2n_e - 1, 2n_e]$  (see Fig. 2.1). If  $s(e) = 2$ , one can simply put  $R_e^2 := R_e'$ . If  $s(e) \geq 3$ , then boxes  $R_e^2, R_e^3, \dots, R_e^{s(e)}$  will be taken as subboxes of  $R_e'$  of the form  $R_e^l := [2j - 1, 2j] \times [c_l, d_l] \times [2n_e - 1, 2n_e]$ , for  $l = 2, 3, \dots, s(e)$ , where  $c_l, d_l$  are rationals such that  $c_2 = 2i - 1$ ,  $d_{s(e)} = 2j$ ,  $2i < c_3 < d_2$ ,  $c_{s(e)} < d_{s(e)-1} < 2j - 1$  and, if  $s(e) \geq 4$ ,  $c_{l+1} < d_l < c_{l+2}$  whenever  $2 \leq l \leq s(e) - 2$  (see Fig. 2.1). One can easily check that the intersection graph of the set  $\{R_1, R_2, \dots, R_{|V|}\} \cup \bigcup_{e \in E} \{R_e^1, R_e^2, \dots, R_e^{s(e)}\}$  of axis-parallel boxes in  $\mathbb{R}^3$  is (isomorphic to)  $G'$ . Moreover, the time complexity of this construction is polynomial in  $|V| + \sum_e s(e)$ .

To obtain the corresponding realization in  $\mathbb{R}^d$  for  $d > 3$ , one can take the set  $\{R_i \times [0, 1]^{d-3} : i = 1, 2, \dots, N\}$  of  $d$ -boxes.  $\square$

**REMARK 2.1.** The graph  $G'$  from Theorem 2.1 is of girth at least 9. In any realization of  $G'$  by axis-parallel  $d$ -dimensional boxes no point of  $\mathbb{R}^d$  is simultaneously covered by more than two boxes. For the 2-dimensional case a  $4K$ -approximation algorithm is known for finding a maximum weighted independent set in a given set  $\mathcal{R}$  of weighted axis-parallel rectangles, where  $K$  is the maximum number of rectangles in  $\mathcal{R}$  that simultaneously cover a point in  $\mathbb{R}^2$  [23].

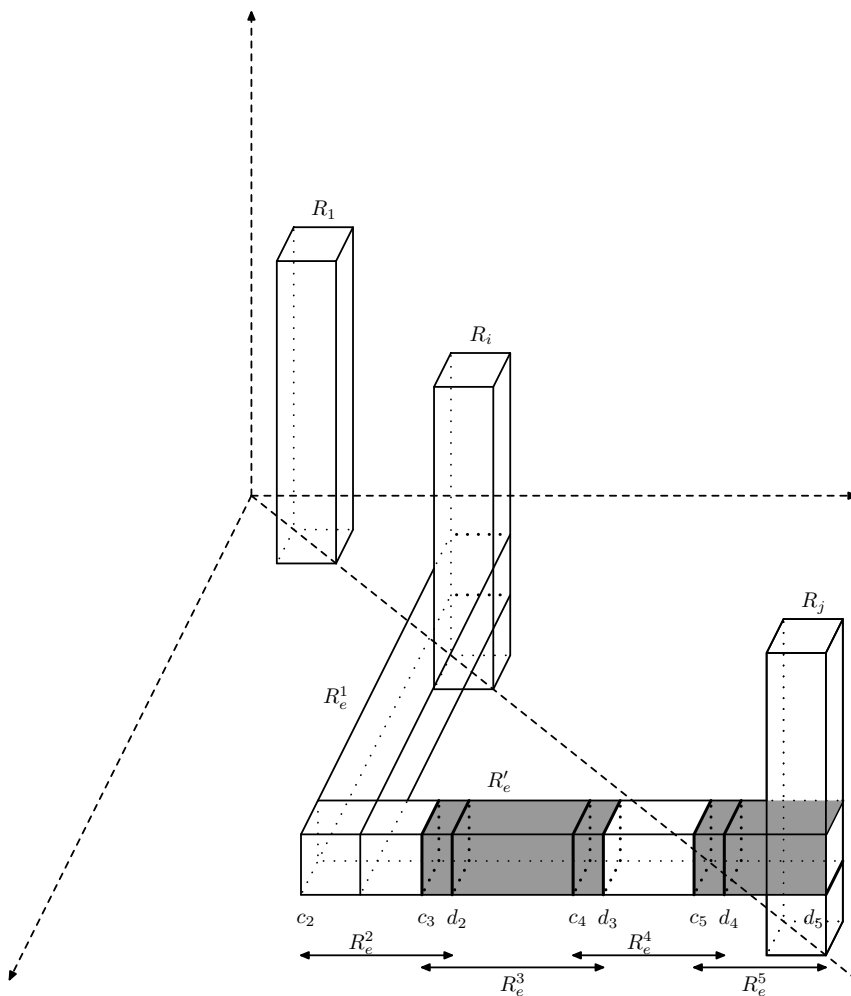


FIG. 2.1.

Theorem 2.1 shows that for every fixed  $d \geq 3$  the intersection graphs of sets of  $d$ -boxes are from topological point of view as complex as general graphs. It is far from being clear whether 2-box graphs have much simpler topological structure. However, the complexity of intersection graphs of axis-parallel lines in dimensions 2 and 3 significantly differs one from another. In the following theorem we show that, similarly as in case of axis-parallel boxes, highly nontrivial subclasses of general graphs are already intersection graphs of sets of axis-parallel lines in  $\mathbb{R}^d$  for any  $d \geq 3$ . On the other hand, for the 2-dimensional case intersection graphs of axis-parallel lines are exactly complete bipartite graphs, for which classical optimization problems are easily solvable.

**THEOREM 2.2.** *Let  $G = (V, E)$  be a given graph. Suppose that for each edge  $e \in E$  an integer  $s(e)$  with  $s(e) \in \{2, 3\} \cup \{k : k \geq 5\}$  is given; denote by  $G'$  a graph obtained from  $G$  by a  $s(e)$ -subdivision of each edge  $e$ . Then the graph  $G'$  can be realized as an intersection graph of a set of axis-parallel lines in  $\mathbb{R}^3$ . Moreover, such realization can be done in time polynomial in  $|V| + \sum_e s(e)$ .*

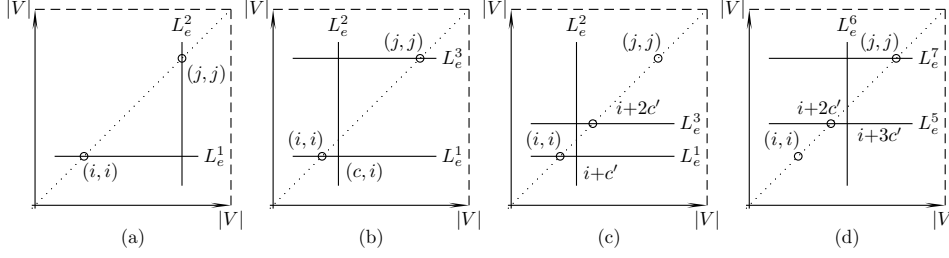


FIG. 2.2. The realization of lines belonging to an edge  $e = \{i, j\}$  for different values of  $s(e)$ . The lines parallel to  $z$ -axes are displayed as a circle in the corresponding vertex. (a)  $s(e) = 2$ , the cross-section with the plane  $z = n_e$ . (b)  $s(e) = 3$ , the cross-section with the plane  $z = n_e$ . (c) and (d)  $s(e) = 7$ , the cross-section with the planes  $z = n_e$  and  $z = n_e + c'$ , respectively.

*Proof.* Let a graph  $G = (V, E)$ ,  $s : E \rightarrow \{2, 3, 5, 6, \dots\}$ , and  $G'$  be given as above. In what follow we describe the realization of  $G'$  as an intersection graph of a set of  $N$  axis-parallel lines in  $\mathbb{R}^3$ , where  $N = |V| + \sum_e s(e)$ . Assume that  $V = \{1, 2, \dots, |V|\}$ , and number the edges from  $E$  by a bijection  $e \mapsto n_e$  between  $E$  and  $\{1, 2, \dots, |E|\}$ . Each vertex  $i \in \{1, 2, \dots, |V|\}$  will be represented as the line  $L_i = (i, i, \cdot)$  parallel to  $z$ -axis.

The graph  $G'$  is obtained from  $G$  replacing each edge  $e = \{i, j\} \in E$  (assume  $i < j$ ) by a path with vertices  $i, A_e^1, A_e^2, \dots, A_e^{s(e)}, j$ . Keeping one such  $e$  fixed, we define lines  $L_e^1, L_e^2, \dots, L_e^{s(e)}$  representing vertices  $A_e^1, A_e^2, \dots, A_e^{s(e)}$ , respectively.

(a) Assume first that  $s(e) \in \{2, 3, 7\}$ . In all these three cases, we take as  $L_e^1$  the line  $(\cdot, i, n_e)$ . If  $s(e) = 2$ , we put  $L_e^2 := (j, \cdot, n_e)$  (see Fig. 2.2(a)). If  $s(e) = 3$ , then we take  $L_e^2 := (c, \cdot, n_e)$  and  $L_e^3 := (\cdot, j, n_e)$  for some  $i < c < i + 1$  (see Fig. 2.2(b)). In case  $s(e) = 7$ , let  $L_e^2 := (i + c', \cdot, n_e)$ ,  $L_e^3 := (\cdot, i + 2c', n_e)$ ,  $L_e^4 := (i + 2c', i + 2c', \cdot)$ ,  $L_e^5 := (\cdot, i + 2c', n_e + c')$ ,  $L_e^6 := (i + 3c', \cdot, n_e + c')$ ,  $L_e^7 := (\cdot, j, n_e + c')$  for some  $0 < c' < \frac{1}{4}$  (see Fig. 2.2(c)-(d)).

(b) Assume now that  $s(e) = a + 3m$  for some  $a \in \{2, 3, 7\}$  and  $m \geq 1$ . We will proceed in two steps. In the first one we realize  $3m$  subdivision of  $e$ , which reduces the task to the above case of  $a$ -subdivision, where  $a \in \{2, 3, 7\}$ . Choose  $i^{(1)} < i^{(2)} < \dots < i^{(m)}$  from  $(i, i + 1)$  (we can ensure that for distinct edges  $e, e'$  these sets are disjoint), and  $n_e < n^{(1)} < n^{(2)} < \dots < n^{(m)} < n_e + 1$ . Take  $L_e^1 := (\cdot, i, n_e)$ ,  $L_e^2 := (i^{(1)}, \cdot, n_e)$ ,  $L_e^3 := (i^{(1)}, i^{(1)}, \cdot)$ ,  $L_e^4 := (\cdot, i^{(1)}, n^{(1)})$ ,  $L_e^5 := (i^{(2)}, \cdot, n^{(1)})$ ,  $L_e^6 := (i^{(2)}, i^{(2)}, \cdot)$ ,  $\dots$ ,  $L_e^{3m-2} := (\cdot, i^{(m-1)}, n^{(m-1)})$ ,  $L_e^{3m-1} := (i^{(m)}, \cdot, n^{(m-1)})$ ,  $L_e^{3m} := (i^{(m)}, i^{(m)}, \cdot)$ . Now, in the second step, it suffices to insert  $a$  lines, where  $a \in \{2, 3, 7\}$ . The construction is the same as in (i), but the role of  $(i, i, \cdot)$  and  $n_e$  is now played by  $(i^{(m)}, i^{(m)}, \cdot)$  and  $n^{(m)}$ , respectively. It is also easy to see that parameters can be chosen in such way that the intersection graph of set  $\{L_1, L_2, \dots, L_{|V|}\} \cup \bigcup_{e \in E} \{L_e^1, L_e^2, \dots, L_e^{s(e)}\}$  is (isomorphic to)  $G'$ . Moreover, time complexity of the construction is polynomial in  $|V| + \sum_e s(e)$ .  $\square$

**3. Approximation Hardness Results in Subdivisions of Graphs.** Let  $\mathcal{C}$  denote the collection of the following problems: MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET. Each problem from  $\mathcal{C}$  is well known to be APX-complete when restricted to graphs of degree at most 3 or even to 3-regular graphs (see Remark 1.1). Moreover, explicit NP-hard gap type results and

explicit lower bounds on their efficient approximability are known for several of them ([10], [7], [8]). In this section we show APX-completeness for each problem from  $\mathcal{C}$  even when restricted to certain subdivisions of low-degree graphs.

First we prove that for MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER the optimum value for a graph and for its certain subdivisions are in a simple relation.

**LEMMA 3.1.** *Let  $G = (V, E)$  be a graph, and let  $e \in E$  be a given edge. Denote by  $G'$  a graph obtained from  $G$  by a 2-subdivision of the edge  $e$ . Then  $vc(G') = vc(G) + 1$  and  $is(G') = is(G) + 1$ .*

*Proof.* Suppose that the edge  $e = \{u, v\}$  is replaced by a path  $u, u', v'$ , and  $v$  with new vertices  $u'$  and  $v'$ . For every vertex cover  $C$  in  $G$  either  $C \cup \{u'\}$  or  $C \cup \{v'\}$  is the vertex cover in  $G'$ , hence  $vc(G') \leq vc(G) + 1$ .

Now we prove the opposite inequality  $vc(G) \leq vc(G') - 1$ . Let  $C'$  be a vertex cover in  $G'$ . We can modify it to a vertex cover  $C$  in  $G$  with  $|C| \leq |C'| - 1$  as follows. If  $C' \cap \{u, v\} \neq \emptyset$ , we take  $C := C' \setminus \{u', v'\}$ . If  $C' \cap \{u, v\} = \emptyset$ , then clearly both  $u', v' \in C'$  and we take  $C := \{u\} \cup C' \setminus \{u', v'\}$ .

The claim for independent sets follows in a straightforward way.  $\square$

It is easy to see that the proof of Lemma 3.1 is constructive and that the corresponding algorithm applies to all feasible solutions and not only to optimal ones. Applying iteratively its steps we can obtain the following theorem.

**THEOREM 3.2.** *Let  $G = (V, E)$  be a graph, and let for each edge  $e \in E$  an integer  $s(e) \geq 0$  be given. Denote by  $G'$  a graph obtained from  $G$  by a  $2s(e)$ -subdivision of each edge  $e \in E$ . Let  $Q$  be either the problem MINIMUM VERTEX COVER, or MAXIMUM INDEPENDENT SET. Then*

- (A)  $\text{OPT}_Q(G') = \text{OPT}_Q(G) + \sum_e s(e)$ ;
- (B) every  $y \in \text{sol}_Q(G)$  can be transformed in polynomial time (in size of  $G$  and  $\sum_e s(e)$ ) to  $y' \in \text{sol}_Q(G')$  such that  $|y'| = |y| + \sum_e s(e)$ ;
- (C) every  $y' \in \text{sol}_Q(G')$  can be transformed in polynomial time to  $y \in \text{sol}_Q(G)$  such that  $|y'| - \sum_e s(e) \leq |y|$  if  $Q$  is maximization problem; respectively  $|y| \leq |y'| - \sum_e s(e)$  if  $Q$  is minimization problem.

*Proof.* We can assume that  $Q$  is the MINIMUM VERTEX COVER problem (for MAXIMUM INDEPENDENT SET we can argue analogously).

Let  $K := \sum_e s(e)$  and assume that  $K > 0$ . We can find a sequence of graphs  $G_0 := G, G_1, \dots, G' := G_K$  such that for each  $i = 1, 2, \dots, K$  the graph  $G_i$  is created from  $G_{i-1}$  as in Lemma 3.1 (by a 2-subdivision of one of its edge). To prove the property (B), consider a vertex cover  $C$  in  $G$ . Put  $C_0 := C$  and as in the proof of Lemma 3.1 find, for each  $i = 1, 2, \dots, K$ , a vertex cover  $C_i$  in  $G_i$  with  $|C_i| = |C_{i-1}| + 1$ . Then  $C' := C_K$  is a vertex cover in  $G'$  with  $|C'| = |C| + K$ . This also shows that  $vc(G') \leq vc(G) + K$ . To prove the property (C), consider a vertex cover  $C'$  in  $G'$ . Now as in the proof of Lemma 3.1 find, for each  $i = K, K-1, \dots, 2, 1$ , a vertex cover  $C_{i-1}$  in  $G_{i-1}$  with  $|C_{i-1}| \leq |C_i| - 1$ . Then  $C := C_0$  is a vertex cover in  $G$  with  $|C| \leq |C'| - K$  and hence  $vc(G) \leq vc(G') - K$ . Consequently,  $vc(G') = vc(G) + K$ , and the property (A) is proved as well.  $\square$

Also the optimum of several other graph optimization problems behaves well under certain subdivision operations, similarly as for MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER. We will demonstrate that for MINIMUM DOMINATING SET and MINIMUM EDGE DOMINATING SET.

**LEMMA 3.3.** *Let  $G = (V, E)$  be a given graph. Denote by  $G'$  a graph obtained from  $G$  by a 3-subdivision of an edge  $e \in E$ . Then (i)  $ds(G') = ds(G) + 1$ , and (ii)*

$$eds(G') = eds(G) + 1.$$

*Proof.* Let  $G'$  be a graph obtained from  $G$  by a 3-subdivision of the edge  $e = \{u, v\}$ , i.e., replacing  $e$  by a path  $u, u', w, v', v$  with new vertices  $u', w$ , and  $v'$ .

(i) To prove  $ds(G') \leq ds(G) + 1$ , consider a dominating set  $D$  in  $G$ . Adding one of vertices  $u', w, v'$  to  $D$  we can obtain a dominating set  $D'$  in  $G'$  with  $|D'| = |D| + 1$  as follows: (I) If  $(u \in D \ \& \ v \in D)$  or  $(u \notin D \ \& \ v \notin D)$  we take  $D' := D \cup \{w\}$ . (II) If  $(u \in D \ \& \ v \notin D)$  we take  $D' := D \cup \{v'\}$ . (III) If  $(v \in D \ \& \ u \notin D)$  we take  $D' := D \cup \{u'\}$ .

Notice that  $D \subset D'$  and that the restriction of  $D'$  to the path  $u, u', w, v', v$  is an independent set. This observation will be used later in the proof of Theorem 3.4.

To prove  $ds(G') \geq ds(G) + 1$ , consider a dominating set  $D'$  in  $G'$ . We can modify it to a dominating set  $D$  in  $G$  with  $|D| \leq |D'| - 1$  as follows. If  $D' \cap V$  is a dominating set in  $G$ , we take  $D := D' \cap V$ . If  $D' \cap V$  is not dominating set in  $G$  then clearly  $u, v \notin D'$ ,  $|D' \cap \{u', w, v'\}| \geq 2$ , and we take  $D := \{u\} \cup D' \cap V$ .

(ii) To prove  $eds(G') \leq eds(G) + 1$ , consider an edge dominating set  $M$  in  $G$  and denote  $V(M)$  the set of end vertices of edges in  $M$ . We modify  $M$  to an edge dominating set  $M'$  in  $G'$  with  $|M'| = |M| + 1$  as follows: (I) If  $u \notin V(M)$  we take  $M' := M \cup \{\{u', w\}\}$ . (II) If  $v \notin V(M)$  and  $u \in V(M)$  we take  $M' := M \cup \{\{v', w\}\}$ . (III) If  $u, v \in V(M)$  and  $e \notin M$  we take  $M' := M \cup \{\{u', w\}\}$ . (IV) If  $e \in M$  we take  $M' := M \setminus \{e\} \cup \{\{u, u'\}, \{v, v'\}\}$ .

To prove  $eds(G') \geq eds(G) + 1$ , consider an edge dominating set  $M'$  in  $G'$  and put  $M_0 := M' \cap \{\{u, u'\}, \{u', w\}, \{v', w\}, \{v, v'\}\}$ . Clearly  $M_0 \neq \emptyset$  and if  $|M_0| = 1$  then either  $\{u', w\} \in M'$  or  $\{v', w\} \in M'$ . We can modify  $M'$  to an edge dominating set  $M$  in  $G$  with  $|M| \leq |M'| - 1$  as follows. If  $|M_0| \geq 2$  we take  $M := M' \setminus M_0 \cup \{e\}$ . If  $|M_0| = 1$  we take  $M := M' \setminus M_0$ .  $\square$

Using steps of the proof of the previous lemma we can obtain the following theorem.

**THEOREM 3.4.** *Let  $G = (V, E)$  be a graph, and let for each edge  $e \in E$  an integer  $s(e) \geq 0$  be given. Denote by  $G'$  a graph obtained from  $G$  by a  $3s(e)$ -subdivision of each edge  $e \in E$ . Then the properties (A)–(C) from Theorem 3.2 are fulfilled for both problems MINIMUM DOMINATING SET and MINIMUM EDGE DOMINATING SET.*

*Moreover, if  $s(e) > 0$  for each  $e \in E$ , then  $ids(G') = ds(G')$  and every dominating set  $D$  in  $G$  can be transformed in polynomial time to an independent dominating set  $D'$  in  $G'$  with  $|D'| = |D| + \sum_e s(e)$ .*

*Proof.* We provide the proof for the MINIMUM DOMINATING SET problem, the proof for the second problem is analogous using the corresponding part of the proof of Lemma 3.3. Let  $G'$  be a graph obtained from  $G$  by a  $3s(e)$ -subdivision of each edge  $e$ , i.e., replacing the edge  $e = \{u, v\}$  by a path with endvertices  $u, v$ , and  $3s(e)$  new vertices (the paths are pairwise disjoint). Let  $K := \sum_e s(e)$ . We can assume that  $K > 0$ , and find  $G_0 := G, G_1, \dots, G_K := G'$  as in the proof of Theorem 3.2.

To prove the property (B), consider a dominating set  $D$  in  $G$ . Put  $D_0 := D$  and as in the proof of Lemma 3.3 find, for each  $i = 1, 2, \dots, K$ , a dominating set  $D_i$  in  $G_i$  such that  $|D_i| = |D_{i-1}| + 1$ ,  $D_{i-1} \subset D_i$ , and the restriction of  $D_i$  to the path used to create  $G_i$  from  $G_{i-1}$  is an independent set. Then  $D' := D_K$  is a dominating set in  $G'$  with  $|D'| = |D| + K$ . This also shows that  $ds(G') = ds(G) + K$ . Moreover, if  $s(e) > 0$  for every  $e \in E$ , then the set  $D'$  is an independent dominating set in  $G'$ , and  $ds(G') = ids(G') \leq ds(G) + K$  in this case. To prove the property (C), consider a dominating set  $D'$  in  $G'$  and put  $D_K := D'$ . As in the proof of Lemma 3.3 find, for each  $i = K, K - 1, \dots, 2, 1$ , a dominating set  $D_{i-1}$  in  $G_{i-1}$  with  $|D_{i-1}| \leq |D_i| - 1$ .



Then  $D := D_0$  is a dominating set in  $G$  with  $|D| \leq |D'| - K$ . This also shows that  $ds(G) \leq ds(G') - K$ . Consequently,  $ds(G') = ds(G) + K$ . If  $s(e) > 0$  for every  $e \in E$ , then as it follows from the proof, there is a minimum dominating set in  $G'$ , which is also independent, hence  $ids(G') = ds(G')$ .  $\square$

REMARK 3.1. In Theorem 3.4, if  $s(e)$  is an *odd* integer for each edge  $e$  then the graph  $G'$  is bipartite.

Now using Theorems 3.2 and 3.4 we can easily prove APX-completeness of each of the basic optimization problems MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET even when restricted to certain subdivisions of graphs degree at most 3.

THEOREM 3.5. (i) *The problems MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER are APX-complete when restricted to  $2k$ -subdivisions of 3-regular graphs for any fixed integer  $k \geq 0$ .*

(ii) *The problems MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET are APX-complete when restricted to  $3k$ -subdivisions of degree at most 3 graphs for any fixed integer  $k \geq 0$ .*

*Proof.* Let  $k \geq 0$  be a fixed integer. Without loss of generality we can consider only graphs without isolated vertices. As it was mentioned in Remark 1.1 all considered problems are in APX when restricted to graphs of degree at most 3. Hence, to prove APX-completeness of each of the problems MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET restricted to certain subdivisions of low-degree graphs, it is enough to show that such subdivision operations are in bounded degree graphs in fact  $L$ -reductions to the same problems.

(i) Let us start with MAX-IS and a  $2k$ -subdivision operation. To verify the first condition of an  $L$ -reduction we have to check that there is a constant  $c$  such that  $is(\text{div}_{2k}(G)) \leq c \cdot is(G)$  for every graph  $G$  of maximum degree  $B$ ,  $B \geq 3$ . As follows from Theorem 3.2,

$$is(\text{div}_{2k}(G)) = is(G) + |E|k. \quad (3.1)$$

Recall that for a graph  $G = (V, E)$  of maximum degree  $B$  the following inequalities hold:  $|E| \leq \frac{|V|}{2}B$  and  $is(G) \geq \frac{|V|}{B+1}$ . Now one can see that the choice  $\alpha := 1 + \frac{B(B+1)k}{2}$  will do. The second condition from the definition of an  $L$ -reduction is satisfied with  $\beta = 1$  by Theorem 3.2. Hence the operation that transforms a graph to its  $2k$ -subdivision is an  $L$ -reduction that self-reduces MAX-IS restricted to graphs of maximum degree  $B$ .

We can argue similarly for MIN-VC using Theorem 3.2 and simple lower bound  $vc(G) \geq \frac{|V|}{B+1}$ .

(ii) The same approach as in (i) can be used also for problems MIN-DS, and MIN-EDS, to prove that a  $3k$ -subdivision is an  $L$ -reduction for them, when restricted to graphs of maximum degree  $B$ . It is enough to consider Theorem 3.4 together with lower bounds  $is(G) \geq ids(G) \geq ds(G) \geq \frac{|V|}{B+1}$ , and  $eds(G) \geq \frac{|V|}{2B}$ . Moreover, a  $3k$ -subdivision for  $k > 0$  reduces MIN-DS to MIN-IDS and it is again an  $L$ -reduction when restricted to graphs of maximum degree  $B$ .  $\square$

REMARK 3.2. Notice that the theorem above shows hardness results for graphs with low maximum degree and large girth. The part (ii) for  $k$  odd claims APX-completeness results in bipartite graphs of maximum degree 3 and of girth at least  $9k + 3$ .

For the later applications, we formulate also the explicit NP-hard gap type results for MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER restricted to certain subdivisions of low-degree graphs.

**THEOREM 3.6.** *It is NP-hard to approximate*

- (i) MAXIMUM INDEPENDENT SET in 2-subdivisions of 3-regular graphs within  $1 + \frac{1}{387}$ , and in 2-subdivisions of 4-regular graphs within  $1 + \frac{1}{244}$ ;
- (ii) MINIMUM VERTEX COVER in 2-subdivisions of 3-regular graphs within  $1 + \frac{1}{390}$ , and in 2-subdivisions of 4-regular graphs within  $1 + \frac{1}{249}$ .

*Proof.* (i) We will use the corresponding NP-hard gap results from [10] for MAXIMUM INDEPENDENT SET in  $B$ -regular graphs,  $B \geq 3$ . For any  $\varepsilon > 0$  it is NP-hard to decide in  $B$ -regular graphs  $G = (V, E)$  of whether  $is(G) < \frac{|V|}{2} \left(1 - 3\delta_B + \varepsilon\right)$ , or  $is(G) > \frac{|V|}{2} \left(1 - 2\delta_B - \varepsilon\right)$ , where  $\delta_B$  is a constant for  $B$ -regular graphs,  $\delta_3 \approx 0.0103305$  and  $\delta_4 \approx 0.020242915$ . Using the formula (3.1) we see that this translates to the following NP-hardness result for 2-subdivisions of  $B$ -regular graphs: for any  $\varepsilon > 0$  it is NP-hard to decide of whether  $is(\text{div}_2(G)) < \frac{|V|}{2} (1 + B - 3\delta_B + \varepsilon)$ , or  $is(\text{div}_2(G)) > \frac{|V|}{2} (1 + B - 2\delta_B - \varepsilon)$ . Consequently, the approximation within any constant smaller than  $1 + \frac{\delta_B}{1+B-3\delta_B}$  is NP-hard.

(ii) We can argue similarly for MINIMUM VERTEX COVER using NP-hard gap results for it in  $B$ -regular graphs,  $B \geq 3$  ([10]). For any  $\varepsilon > 0$  it is NP-hard to decide in  $B$ -regular graphs  $G = (V, E)$  of whether  $vc(G) < \frac{|V|}{2} \left(1 + 2\delta_B + \varepsilon\right)$ , or  $vc(G) > \frac{|V|}{2} \left(1 + 3\delta_B - \varepsilon\right)$ , where  $\delta_3$  and  $\delta_4$  are as above.  $\square$

**4. Approximation Hardness Results in  $d$ -box Graphs.** Theorem 2.1 shows that any graph obtained from another one by at least 2-subdivision of each edge is a  $d$ -box graph for any  $d \geq 3$ . This immediately implies that many optimization problems in intersection graphs of  $d$ -boxes are as hard to approximate as in general graphs. It is rather easy to make this conclusion for such problems as MINIMUM STEINER TREE or MINIMUM TRAVELING SALESMAN. For these problems replacing edges by pairwise disjoint paths (and splitting edge weights properly) cannot make the problem easier to approximate. But for some optimization problems the algorithms with better approximation ratios have been designed in  $d$ -box graphs than in general graphs.

In this section we prove APX-hardness and hence non-existence of a PTAS (unless  $P = NP$ ) for some basic graph optimization problems in  $d$ -box graphs for any  $d \geq 3$ . Moreover, all our hardness results apply as well to the setting when a representation by  $d$ -boxes is given as an input, not merely its intersection graph. This makes hardness results stronger, as the problem to find a  $d$ -box intersection representation of a graph is known to be NP-hard.

**THEOREM 4.1.** *Let  $d \geq 3$  be a fixed integer. Each of the problems MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, MINIMUM EDGE DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET, is APX-hard when restricted to intersection graphs of sets of axis-parallel  $d$ -dimensional boxes, and hence does not admit PTAS unless  $P = NP$ . These hardness results apply also to instances whose intersection graph is simultaneously of maximum degree 3, of girth at least  $k$  (for any prescribed constant  $k$ ), and, except MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER, bipartite as well.*

*Proof.* The proof is straightforward using Theorems 2.1 and 3.5.  $\square$

These results could be stated as explicit NP-hard gap type results and provide

explicit lower bounds on its approximability. This is demonstrated on MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER to show how large explicit values can be obtained with the current methods.

**THEOREM 4.2.** *For any fixed  $d \geq 3$  it is NP-hard to approximate the MAXIMUM INDEPENDENT SET problem within  $1 + \frac{1}{244}$  and the MINIMUM VERTEX COVER problem within  $1 + \frac{1}{249}$  in sets of axis-parallel  $d$ -dimensional boxes.*

*Proof.* We provide the proof for MAXIMUM INDEPENDENT SET, the proof for MINIMUM VERTEX COVER is analogous. Let  $d \geq 3$  be a fixed integer. Assume that  $G' = (V', E')$  be a 2-subdivision of a 4-regular graph  $G = (V, E)$ . As follows from Theorem 2.1,  $G'$  is an intersection graph of a set  $\mathcal{R}$  of  $d$ -boxes and an intersection realization of  $\mathcal{R}$  can be found in polynomial time. Due to Theorem 3.6 it is NP-hard to decide whether the maximum number of pairwise disjoint  $d$ -boxes of  $\mathcal{R}$  is less than  $0.49392715|V'|$  or greater than  $0.495951417|V'|$  (under promise that one of these two cases occurs). Consequently, it is NP-hard to approximate MAXIMUM INDEPENDENT SET within  $1 + \frac{1}{244}$  in  $d$ -boxes for  $d \geq 3$ .  $\square$

**REMARK 4.1.** The results of Theorems 4.1 and 4.2 hold also for intersection graphs of sets of axis-parallel lines for any fixed  $d \geq 3$ . The proofs are the same, only Theorem 2.2 is used instead of Theorem 2.1.

The method of this paper is rather general and can provide inapproximability results also for other combinatorial optimization problems on sets of  $d$ -boxes for any  $d \geq 3$  (see [9] for more details). The question of approximation hardness of these problems in the 2-dimensional case is open. However, as shown in [9], using subdivisions of planar graphs provides a generic method of proving NP-hardness of all these problems on sets of axis-parallel rectangles (even unit squares) in the plane. Similar methods how to prove NP-hardness for problems in geometric intersection graphs of planar objects have been already used in [11] for unit disk graphs, and in [20] for intersection graphs of line segments.

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